SELF-INTERSECTION LOCAL TIME FOR $\mathcal{S}'(\mathbb{R}^d)$-ORNSTEIN-UHLENBECK PROCESSES ARISING FROM IMMIGRATION SYSTEMS

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Abstract.

Fluctuation limits of an immigration branching particle system and an immigration branching measure-valued process yield different types of $\mathcal{S}'(\mathbb{R}^d)$-valued Ornstein-Uhlenbeck processes whose covariances are given in terms of an excessive measure for the underlying motion in $\mathbb{R}^d$, which is taken to be a symmetric α-stable process. In this paper we prove existence and path continuity results for the self-intersection local time of these Ornstein-Uhlenbeck processes. The results depend on relationships between the dimension d and the parameter α.

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1. Introduction.

Immigration branching particle systems and their (measure-valued) superprocess limits have been investigated by several authors, e.g. Dawson [D], Dynkin [Dy2], Gorostiza and López-Mimbela [GLM], Konno and Shiga [KS], Li [L1, L2, L3], Li and Shiga [LS]. In the papers of Li a theory for immigration systems has been developed by introducing and using the definition of skew-convolution semigroups, which are also connected with the generalized Mehler semigroups investigated by Bogachev, Röckner and Schmuland [BRS]. Fluctuation limits of immigration systems lead to $\mathcal{S}'(\mathbb{R}^d)$-valued Ornstein-Uhlenbeck (OU) processes [GL1, GL2]. Concerning the properties of these processes the following question arises: for which space dimensions d do they have self-intersection local time (SILT)? Questions of this type were investigated originally by Adler, Feldman and Lewin [AFL], and Adler and Rosen [AR] for the special case of density processes, which do not involve branching or immigration.

In this paper we study existence and path continuity of SILT for the two $\mathcal{S}'(\mathbb{R}^d)$-$OU$ processes found in [GL1, GL2]. The techniques are refinements of those developed in [BG3]. We will put this problem in a general context, explain why the methods of [BG3] are not applicable in the present cases, and carry out the necessary analytical extensions to deal with them.

In [BG3] we studied existence and path continuity of SILT for a class of $\mathcal{S}'(\mathbb{R}^d)$-$OU$ processes related to inhomogeneous fields. The covariances of those processes are expressed, in one way or another, in terms of some measures on $\mathbb{R}^d$. In several of the examples that have served as motivations and test cases these measures were related to the intensity of the spatial configuration of a system of particles. In [BG3] the main point regarding those measures was that the random fields on $\mathbb{R}^d$ associated with them were not necessarily homogeneous, and this required developing new techniques for studying SILT. In addition the measures were assumed to be finite and this played a significant role in the proofs. There are examples of $\mathcal{S}'$-OU processes whose convariance structures are similar to those in [BG3] (and in previous papers referred to therein) and are also related to inhomogeneous random fields, but whose associated measures are not finite. Hence a natural question is if the methods of [BG3] can be extended to cover this setting. Our aim in the present paper is to give an answer to this question. While it seems difficult to obtain results of such generality as in [BG3], it is possible to extend the methods to investigate existence and continuity of SILT for the two above mentioned processes [GL1, GL2], which cannot be treated with the previous techniques or simple extensions of them. In both cases there is an underlying process in $\mathbb{R}^d$ (representing particle motion), which is the spherically symmetric α-stable process. The measures on $\mathbb{R}^d$ that arise in both cases are of the form

$$\gamma = \int_0^\infty \nu T_t dt,$$

where (T_t) is the semigroup of the motion process and ν is a finite (non-zero) measure. It is clear that the measure γ is not finite. Measures of this form are excessive measures for (T_t) and they constitute a basic part of the structures of the two $\mathcal{S}'$-*OU* processes.

We will focus on the stationary cases for both processes. However, the techniques are useful for non-stationary cases as well. Stationary $\mathcal{S}'$-*OU* processes are a natural an important class of processes [BJ], and in our previous work on SILT relatively little care was devoted to them (in fact, only the simplest density process was a stationary one). We also found it worthwhile to exhibit an interesting phenomenon that for the stationary processes we are dealing with it is the convolution integral part of the process, and not the one coming from the stationary distribution, which determines whether the process has SILT or not (see e.g. Proposition 3.3 and Remark 3.18).

Our objective is to determine the range of space dimensions d for which the SILT's of the two $\mathcal{S}'(\mathbb{R}^d)$-processes exist and to show that they have continuous paths. Although the basic ideas of the proofs are similar to those in [BG3], the main technical problem is that some of the analytical estimations used in that paper are no longer useful. Some of estimations need to be more subtle and others have to be replaced where extensions of the previous method fail.

We refer the reader to [BG3] for our motivations for studying this kind of problem and for the definitions, notation and technical background, which we will use freely in this paper, as well as references (here we give only a minimal bibliography which we need to refer to).

2. Two stationary $\mathcal{S}'$-*OU* processes related to immigration branching systems.

Recall that $\mathcal{S}'(\mathbb{R}^d)$-*OU* processes are of the form

$$X_t = T'_t X_0 + \int_0^t T'_{t-s} dW_s, \qquad t > 0, \tag{2.1}$$

(we will restrict to $t \in [0,1]$), where X_0 is an $\mathcal{S}'(\mathbb{R}^d)$-valued centered Gaussian random variable, (W_t) is an $\mathcal{S}'(\mathbb{R}^d)$-Wiener process independent of X_0, (T_t) is a semigroup on $\mathcal{S}(\mathbb{R}^d)$, and T'_t is the adjoint of T_t. Here (T_t) is the semigroup of the spherically symmetric α-stable process in $\mathbb{R}^d$, $\alpha \in (0,2]$ (Brownian motion corresponds to $\alpha = 2$). The infinitesimal generator of (T_t) is denoted by Δ_α. The $\mathcal{S}'$-*OU* process X in (2.1) is the solution of the Langevin equation

$$dX_t = \Delta'_\alpha X_t dt + dW_t,$$

where Δ'_α is the adjoint of Δ_α. The processes $(T'_t X_0)_{t\in[0,1]}$ and $(\int_0^t T'_{t-s} dW_s)_{t\in[0,1]}$ on the right–hand side of (2.1) are called the *flow process* and the *convolution integral process*, respectively. See [BG3] for existence of these $\mathcal{S}'$-processes and the sense in which (2.1) is the solution of the Langevin equation.

We denote the convariance functionals of X_0 and (W_t), respectively, by

$$K_{X_0}(\varphi,\psi) = \text{ Cov } (\langle X_0,\varphi\rangle, \langle X_0,\psi\rangle)$$

and

$$K_W(s,\varphi;t,\psi) = \text{Cov } (\langle W_s,\varphi\rangle, \langle W_t,\psi\rangle), \qquad s,t\in[0,1],$$

where $\varphi,\psi\in\mathcal{S}(\mathbb{R}^d)$.

We consider the measure γ on $\mathbb{R}^d$ defined by

$$\gamma = \int_0^\infty \nu T_t dt, \tag{2.2}$$

where ν is a finite measure. Note that γ is an excessive measure for (T_t) (e.g. [Dy1]). It is easy to see that, for $d>\alpha, \gamma$ is a well-defined tempered measure and it has a density given by

$$\gamma(x) = \kappa\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-\alpha}}\nu(dy), \tag{2.3}$$

where $\kappa>0$ is a constant.

Example 1: K_{X_0} and K_W are given by

$$K^{(1)}_{X_0}(\varphi,\psi) = \int_0^\infty \langle\gamma,(T_t\varphi)(T_t\psi)\rangle dt \tag{2.4}$$

and

$$K^{(1)}_W(s,\varphi;t,\psi) = (s\wedge t)q^{(1)}(\varphi,\psi), \tag{2.5}$$

where

$$q^{(1)}(\varphi,\psi) = \langle\gamma,\varphi\psi\rangle. \tag{2.6}$$

This example refers to a measure-valued immigration process with α-stable motion in $\mathbb{R}^d$, critical branching mechanism and immigration corresponding to the entrance law νT_t. See [L1] for a detailed description of a general class of such processes. The particular entrance law νT_t corresponds to immigration according to a space-time Poisson random measure with intensity $\nu\otimes dt$ (for the underlying branching particle system). In [GL1] the fluctuations of this process around γ are considered, and the limit in law of the fluctuations as the branching intensity tends to zero is obtained. This limit yields an $\mathcal{S}'(\mathbb{R}^d)$-*OU* process which is Gaussian in the case of finite variance branching. This process has a stationary distribution and it is the stationary process which is characterized by (2.4), (2.5), (2.6). In the non-stationary case obtained in [GL1] the initial condition is $X_0=0$, so the *OU* process then coincides with the convolution integral process.

Example 2: K_{X_0} and K_W are given by

$$K^{(2)}_{X_0}(\varphi,\psi) = \langle\gamma,\varphi\psi\rangle + c\int_0^\infty \langle\gamma,(T_t\varphi)(T_t\psi)\rangle dt \tag{2.7}$$

and

(2.8) $$K_W^{(2)}(s,\varphi;t,\psi)=\langle s\wedge t)q^{(2)}(\varphi,\psi),$$

where

(2.9) $$q^{(2)}(\varphi,\psi)=\langle\gamma,c\varphi\psi-\varphi\Delta_\alpha\psi-\psi\Delta_\alpha\varphi\rangle,$$

and $c\geq 0$ is a constant. This example comes from an immigration particle system in $\mathbb{R}^d$ with α-stable particle motion, critical finite variance branching and immigration according to the entrance law νT_t. The meaning of this entrance law is as in the previous example. The constant c is related to the second moment of the branching law. In [GL2] it is shown that the fluctuations of this system around the mean converge in law as the density of particles tends to infinity, and the limit is an $\mathcal{S}'(\mathbb{R}^d)$-$OU$ process. This process has a stationary distribution and the stationary process is characterized by (2.7), (2.8), (2.9). In the non-stationary case obtained in [GL2] only the term $\langle\gamma,\varphi\psi\rangle$ appears in $K_{X_0}^{(2)}$.

Example 2 is similar to Example 4.4 in [BG3], but now the non-finite measure γ plays a basic role, whereas in [BG3] we had only finite measures. We shall see that a necessary and sufficient condition for existence (and continuity) of SILT of the $\mathcal{S}'(\mathbb{R}^d)$-$OU$ process is $\alpha<d<2\alpha$ (Theorem 3.16, Remarks 3.17 and 3.18). This means that the α-stable particle motion is transient ($d>\alpha$) and the paths intersect ($d<2\alpha$) [Ta]. The second part of this condition is consistent with the "particle picture" interpretation of existence of SILT for Example 4.4 in [BG3] (see the comments in the Introduction of [BG3]). Note that $\alpha<d<2\alpha$ holds in the following cases: $d=1$ and $\frac{1}{2}<\alpha<1$, $d=2$ and $1<\alpha\leq\frac{3}{2}$, $d=2$ and 3 and $1<\alpha\leq\frac{3}{2}$, $d=3$ and $\alpha=2$. The value $\alpha=1$ (in the interval $(\frac{1}{2},2]$), which corresponds to the Cauchy process, is excluded.

Remark. In [GL1, GL2] the particle motion in $\mathbb{R}^d$ was assumed to have a differential inifinitesimal generator for simplicity (instead of Δ_α). The results can be extended to the α-stable case with some additional technical work.

In all cases the covariance K_F of the flow process is given by

(2.10) $$K_F(s,\varphi;t,\psi)=K_{X_0}(T_s\varphi,T_t\psi),$$

the covariance K_{CI} of the convolution integral process is given by

(2.11) $$K_{CI}(s,\varphi;t,\psi)=\int_0^{s\wedge t}q(T_{s-r}\varphi,T_{t-r}\psi)dr,$$

and the covariance K_X of the OU-process X, denoted by

$$K_X(s,\varphi;t,\psi)=\text{ Cov }(\langle X_s,\varphi\rangle,\langle X_t,\psi\rangle),$$

is given by

(2.12) $$K_X=K_F+K_{CI}.$$

In both examples the covariance K_X has some of the general structures considered in [BG1], but there is a significant difference which is the source of the new technical difficulties: the density $\gamma(x)$ given by (2.3) is in general not a bounded fuction. This function plays the role of $m(x)$ in the covariances in Section 3.2 of [BG1], but there it was essential for the proofs of sufficient conditions for existence of SILT that this function be bounded. It turns out that the methods of [BG3] can be extended to cover this new situation, but in the present cases the analysis of finiteness or infiniteness of the integrals, which are analogous to those in [BG3], is considerably more intricate. Moreover, although the measure γ plays the role of the measures denoted by θ, μ or ν in [BG3], it does not have the same interpretation as in the examples of [BG3].

3. Results.

We denote $\mathcal{S}_d \equiv \mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'_d \equiv \mathcal{S}'(\mathbb{R}^d)$. In what follows $\alpha \in (0,2]$ is fixed and $d > \alpha$. Recall that $\widetilde{\varphi}$ and $\widetilde{\mu}$ denote the Fourier transforms of a function $\varphi \in \mathcal{S}_d$ and a tempered measure μ on $\mathbb{R}^d$, respectively.

We start with Example 1.

3.1. Lemma. *$K^{(1)}_{X_0}$ given by (2.4) is a well-defined, continuous inner product in $\mathcal{S}_d$, and one has*

$$K^{(1)}_{X_0}(\varphi,\psi) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} \frac{\overline{\widetilde{\nu}(z+z')}}{(|z|^\alpha + |z'|^\alpha)|z+z'|^\alpha} \widetilde{\varphi}(z)\widetilde{\psi}(z') dz dz'. \tag{3.1}$$

Formula (3.1) permits to derive the following theorem.

3.2. Theorem. *The following conditions are sufficient for the existence of SILT for*

(a) *the flow process associated with $K^{(1)}_{X_0}$:*

$$\int_{\mathbb{R}^{4d}} \frac{|\widetilde{\nu}(z+z')|}{(1+|z|^\alpha)(1+|z'|^\alpha)(|z|^\alpha+|z'|^\alpha)|z+z'|^\alpha} \frac{|\widetilde{\nu}(y+y')|}{(1+|y|^\alpha)(1+|y'|^\alpha)(|y|^\alpha+|y'|^\alpha)|y+y'|^\alpha} \cdot |\varphi(z+y)||\varphi(z'+y')| dz dz' dy dy' < \infty; \tag{3.2}$$

(b) *the convolution integral process associated with $q^{(1)}$:*

$$\int_{\mathbb{R}^{4d}} \frac{|\widetilde{\nu}(z+z')|}{(1+|z|^\alpha)(1+|z'|^\alpha)|z+z'|^\alpha} \frac{|\widetilde{\nu}(y+y')|}{(1+|y|^\alpha)(1+|y'|^\alpha)|y+y'|^\alpha} \cdot |\varphi(z+y)||\varphi(z'+y')| dz dz' dy dy' < \infty; \tag{3.3}$$

(c) *the corresponding OU process:*

$$(3.4)\quad \int_{\mathbb{R}^{4d}} \frac{|\widetilde{\nu}(z+z')|(2+|z|^\alpha+|z'|^\alpha)}{(1+|z|^\alpha)(1+|z'|^\alpha)(|z|^\alpha+|z'|^\alpha)|z+z'|^\alpha} \frac{|\widetilde{\nu}(y+y')|(2+|y|^\alpha+|y'|^\alpha)}{(1+|y|^\alpha)(1+|y'|^\alpha)(|y|^\alpha+|y'|^\alpha)|y+y'|^\alpha} \cdot|\varphi(z+y)||\varphi(z'+y')|dzdz'dydy' < \infty.$$

All these integrals should be finite for each $\varphi \in \mathcal{S}_d$.

A relationship between these conditions is given in the next proposition.

3.3. Proposition. *Condition (3.3) is equivalent to (3.4) and it implies (3.2).*

Conditions (3.2)–(3.4) are not easy to check. The following criterion is more convenient.

3.4. Theorem. *Let*

$$(3.5)\quad p_0 = p_0(d) = \inf\left\{p \geq 1 : \int_{|z|>1} \frac{|\widetilde{\nu}(z)|^{2p}}{|z|^{2\alpha p}} dz < \infty\right\}.$$

(a) If

$$(3.6)\quad \alpha < d < \frac{6p_0}{2p_0 - 1}\alpha,$$

then the flow process associated with $K^{(1)}_{X_0}$ *has SILT, which is a continuous process in* $\mathcal{S}'_d$.

(b) If

$$(3.7)\quad \alpha < d < \frac{4p_0}{2p_0 - 1}\alpha,$$

then the convolution integral process associated with $q^{(1)}$ *as well as the corresponding OU process have SILT's which are continuous processes in* $\mathcal{S}'_d$.

3.5. Remark. We have not formulated analogues to Theorems 3.14 and 3.17 of [BG3] concerning continuity of SILT, though, as will be seen in the proof of Theorem 3.4, such analogues do hold. As it was recently shown by A. Talarczyk [T], our Theorem 3.17 [BG3] and Proposition 4.4 [BG2] on continuity of SILT of the convolution integral process can be improved (after rather hard work).

3.6. Corollary.

(a) (i) If $\alpha < d < 4\alpha$, *then the flow process associated with* $K^{(1)}_{X_0}$ *has (continuous) SILT for each finite measure* ν.

(ii) If $\alpha < d < 6\alpha$, *then the flow process has (continuous) SILT for each finite measure* ν *having an* L^2*-density.*

(b) (i) If $\alpha < d < 3\alpha$, then both the convolution integral process associated with $q^{(1)}$ and the corresponding OU process have (continuous) SILT's for each finite measure ν.

(ii) If $\alpha < d < 4\alpha$, then these process have (continuous) SILT's for each finite measure ν having an L^2-density.

Observe that the maximal upper bound on the dimension d resulting from (3.6) is 6α, while the corresponding bound resulting from (3.7) is 4α. The next proposition shows that the latter bound is the right one.

3.7. Proposition. *The condition $\alpha < d < 4\alpha$ is necessary for existence of SILT for the convolution integral process associated with $q^{(1)}$, as well as for the OU process.*

We do not have an analogous result for the flow process (an obvious candidate is 6α).

Combining Corollary 3.6 (b) (ii) and Proposition 3.7 we obtain immediately the following corollary.

3.8. Corollary. *If ν has on L^2-density, then a necessary and sufficient condition for existence of SILT for the convolution integral process associated with $q^{(1)}$, as well as for the corresponding OU process, is $\alpha < d < 4\alpha$.*

Also "the worst" cases (4α for flow, 3α for convolution integral) can occur. Namely, we have the following proposition.

3.9. Proposition. *If $\nu = \delta_a$ for a fixed $a \in \mathbb{R}^d$, then the necessary and sufficient condition for existence of SILT for the flow process associated with $K^{(1)}_{X_0}$ is $\alpha < d < 4\alpha$, and an analogous condition for the convolution integral process and the OU process is $\alpha < d < 3\alpha$.*

Finally, for use in the next example we discuss briefly the flow process associated with $q^{(1)}$, i.e. the process whose convariance is $\langle \gamma, (T_s\varphi)(T_t\psi)\rangle$.

3.10. Proposition.

(a) A sufficient condition for existence and continuity of SILT of the flow process associated with $q^{(1)}$ is given by condition (3.7) of Theorem 3.4.

(b) A necessary condition is $\alpha < d < 4\alpha$.

(c) The assertion of Corollary 3.6 (b) applies to this process as well.

We now pass to Example 2.

Note that the covariance $K^{(2)}_{X_0}$ of the stationary distribution has the form

$$K^{(2)}_{X_0}(\varphi,\psi) = q^{(1)}(\varphi,\psi) + cK^{(1)}_{X_0}(\varphi,\psi). \tag{3.8}$$

Hence the flow process associated with $K_{X_0}^{(2)}$ presents no problem. If we compare Theorem 3.4 (a) and Proposition 3.10 we see that in the convariance $K_{X_0}^{(2)}(T_s\varphi, T_t\psi)$ of the flow the summand corresponding to $q^{(1)}$ prevails, so we obtain the following corollary.

3.11. Corollary. *Proposition 3.10 remains true for the flow process associated with $K_{X_0}^{(2)}$.*

To investigate the convolution integral and OU processes we need the following lemma.

3.12. Lemma. *$q^{(2)}$ given by (2.9) is a well defined, continuous inner product in $\mathcal{S}_d$, and one has*

$$q^{(2)}(\varphi,\psi) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} \frac{c+|z|^\alpha + |z'|^\alpha}{|z+z'|^\alpha} \overline{\widetilde{\nu}(z+z')\widetilde{\varphi}(z)\widetilde{\psi}(z')}dzdz'. \tag{3.9}$$

Then, analogously as before (Theorem 3.2) we obtain

3.13. Proposition. *If*

$$\int_{\mathbb{R}^{4d}} \frac{|\widetilde{\nu}(z+z')|}{(1+|z|^\alpha)(1+|z'|^\alpha)} \frac{c+|z|^\alpha+|z'|^\alpha}{|z+z'|^\alpha} \frac{|\widetilde{\nu}(y+y')|}{(1+|y|^\alpha)(1+|y'|^\alpha)} \frac{c+|y|^\alpha+|y'|^\alpha}{|y+y'|^\alpha} \cdot|\varphi(z+y)||\varphi(z'+y')|dzdz'dydy' < \infty \tag{3.10}$$

for each $\varphi \in \mathcal{S}_d$, then the convolution integral process associated with $q^{(2)}$ has SILT.

Note that condition (3.10) is obviously satisfied if

$$\int_{\mathbb{R}^{4d}} \frac{c+|z|^\alpha+|z'|^\alpha}{(1+|z|^\alpha)(1+|z'|^\alpha)|z+z'|^\alpha} \frac{c+|y|^\alpha+|y'|^\alpha}{(1+|y|^\alpha)(1+|y'|^\alpha)|y+y'|^\alpha} \cdot|\varphi(z+y)||\varphi(z'+y')|dzdz'dydy' < \infty. \tag{3.11}$$

It turns out that in fact (3.10) and (3.11) are equivalent, and we can determine exactly when (3.10) holds.

3.14. Proposition. *Condition (3.10) holds for each $\varphi \in \mathcal{S}_d$ if and only if $\alpha < d < 2\alpha$. In particular, if $\alpha < d < 2\alpha$, then the convolution integral process associated with $q^{(2)}$ has (continuous) SILT.*

We are not able to prove in full generality that SILT for the convolution integral does not exist if $d \geq 2\alpha$. We have only the following corollary.

3.15. Corollary. *The condition $\alpha < d < 2\alpha$ is necessary (and sufficient) for existence of SILT for the convolution integral process associated with $q^{(2)}$ if either $\gamma = \delta_a$ for some $a \in \mathbb{R}^d$, or $\widetilde{\nu} > 0$.*

So we see (taking, for instance, ν to be a symmetric Gaussian measure) that in the present example smoothness of the measure ν has no effect on the range of dimensions for which SILT exists. Therefore no result involving integrability properties like Theorem 3.4

should be expected. For the stationary OU process corresponding to this example such a theorem is not even needed since for this process the problem of existence of SILT turns out to have a complete solution.

3.16. Theorem. *The stationary OU process has (continuous) SILT if and only if* $\alpha < d < 2\alpha$.

3.17. Remark. It will be seen from the proof that this result is the same for the non-stationary OU process obtained in [GL2].

3.18. Remark. We have seen that in our two examples the existence of SILT for the stationary OU process was related to the existence of SILT of the convolution integral rather than to that of the corresponding flow process (SILT for the flow may exist for a wider range of dimensions). It is worthwhile to note that the same situation occurs in the case of homogeneous fields. Indeed, assume that

$$K_W(\varphi, \psi) = \int_{\mathbb{R}^d} \widetilde{\varphi}(z)\overline{\widetilde{\psi}(z)}\sigma(dz),$$

where σ is a symmetric tempered measure in $\mathbb{R}^d$ (see [BG2, BJ]). It is shown in [BJ] that there exists an invariant distribution for the corresponding Langevin equation if and only if

$$\int_{|z|<1} \frac{\sigma(dz)}{|z|^\alpha} < \infty,$$

and then the invariant distribution has covariance

$$K_{X_0}(\varphi, \psi) = \frac{1}{2}\int_{\mathbb{R}^d} \widetilde{\varphi}(z)\overline{\widetilde{\psi}(z)}\frac{\sigma(dz)}{|z|^\alpha}.$$

On the other hand, by [BG2] (Theorem 4.2) it is known that the convolution integral process associated with K_W has SILT if and only if

$$\int_{\mathbb{R}^{2d}} |\varphi(z+z')|\frac{\sigma(dz)}{1+|z|^{2\alpha}}\frac{\sigma(dz')}{1+|z'|^{2\alpha}} < \infty$$

for any $\varphi \in S_d$, and the flow process associated with K_{X_0} has SILT if and only if

$$\int_{\mathbb{R}^{2d}} \varphi(z+z')|\frac{\sigma(dz)}{|z|^\alpha(1+|z|^{2\alpha})}\frac{\sigma(dz')}{|z'|^\alpha(1+|z|^{2\alpha})} < \infty.$$

So we see that, once again, if the convolution integral has SILT then so does the flow process, and by [BG2] (Theorem 4.3) the stationary OU process has SILT if and only if the convolution integral process has SILT.

4. Proofs.

$C, C_1, C_2, \ldots$ will denote generic positive constants, and we put dependencies in parenthesis.

We will make use several times of the following lemma.

4.1. Lemma.

(a) The integral

$$I=\int_{\substack{|z|\le 1\\ |z'|\le 1}}\frac{1}{(|z|^\alpha+|z'|^\alpha)|z+z'|^\alpha}dzdz'$$

is finite if and only of $d>\alpha$.

(b) The integral

$$I(\varphi,\psi)=\int_{\mathbb{R}^{2d}}\frac{1}{(|z|^\alpha+|z'|^\alpha)|z+z'|^\alpha}\varphi(z)\psi(z')dzdz'$$

is finite for each $\varphi,\psi\in\mathcal{S}_d$ if and only if $d>\alpha$. If $d>\alpha$, it is a continuous functional on $\mathcal{S}_d\times\mathcal{S}_d$.

Proof. (a) Assume $d>\alpha$. We have

$$I\le\int_{\substack{|z|\le 1\\ |z'|\le 1}}\frac{1}{|z|^{\alpha/2}|z'|^{\alpha/2}|z+z'|^\alpha}dzdz'=\int_{\mathbb{R}^d}g(z)(g*g_1)(z)dz,$$

where

$$g(z)=|z|^{-\alpha/2}1_{[0,1]}(|z|),\quad g_1(z)=|z|^{-\alpha}1_{[0,2]}(|z|).$$

The latter integral is finite since $g\in L^2(\mathbb{R}^d)$ and $g_1\in L^1(\mathbb{R}^d)$ (so $g*g_1\in L^2(\mathbb{R}^d)$).

Now suppose that $d\le\alpha$. We have

$$\begin{aligned}I\ge C\int_{\substack{|z|\le 1\\ |z'|\le 1}}(|z|^{2\alpha}+|z'|^{2\alpha})^{-1}dzdz' &= C_1\int_{[0,1]^2}(r^{2\alpha}+(r')^{2\alpha})^{-1}r^{d-1}(r')^{d-1}drdr'\\ &= C_2\int_{[0,1]^2}(s+s')^{-1}s^{\frac{d-2\alpha}{2\alpha}}(s')^{\frac{d-2\alpha}{2\alpha}}dsds'.\end{aligned}$$

Straightforward calculations show that the latter integral is infinite if $d=\alpha$, and it is clear that it is then infinite for $d>\alpha$ as well.

(b) Necessity follows from part (a). If $d>\alpha$ we write

$$I(\varphi,\psi)=I_1+I_2+I_3,$$

where

$$I_1=\int_{\substack{|z+z'|\le 2\\ |z|\le 1}},\qquad I_2=\int_{\substack{|z+z'|\le 2\\ |z|>1}},\qquad I_3=\int_{|z+z'|>2}.$$

We then have

$$
\begin{aligned}
|I_1| &\le C\sup_z |\varphi(z)| \sup_{z'} |\psi(z')| \quad \text{by (a)},\\
|I_2| &\le \int_{\mathbb{R}^d} |\varphi(z)|dz \sup_{z'} |\psi(z')| \int_{|y|\le 2} |y|^{-\alpha} dy,\\
|I_3| &\le \int_{\mathbb{R}^d} |\varphi(z)|dz \int_{\mathbb{R}^d} |\psi(z')|dz',
\end{aligned}
$$

hence the assertion follows. □

Let $\mu_t = \nu T_t$. We have $\widetilde{\mu}_t(z) = e^{-t|z|^\alpha}\widetilde{\nu}(z)$. As in [BG3] we denote by p_t the standard α-stable density. To prove Lemma 3.1 we need the following simple lemma.

4.2. Lemma. *For the measure μ_t equality (2.4.4) of [BG3] holds for each $\varphi, \psi \in L^2(\mathbb{R}^d)$. Explicitly,*

$$
\int_{\mathbb{R}^d} \varphi(x)\psi(x) \int_{\mathbb{R}^d} p_t(x-y)\nu(dy)dx = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} \widetilde{\varphi}(z)\widetilde{\psi}(z') e^{-t|z+z'|^\alpha} \overline{\widetilde{\nu}(z+z')} dzdz'.
$$

Proof. If suffices to use the fact that $\mathcal{S}_d$ is dense in $L^2(\mathbb{R}^d)$, that μ_t has a bounded density, that the Fourier transform is a continuous mapping $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, and that the convolution is a continuous mapping $L^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. □

Proof of Lemma 3.1. It is well known that $T_t\varphi \in L^2(\mathbb{R}^d)$ for $\varphi \in \mathcal{S}_d$, so we can apply Lemma 4.2 to $T_s\varphi, T_s\psi$ to obtain

$$
\begin{aligned}
(4.1) \qquad & \int_{\mathbb{R}^d} (T_s\varphi)(x)(T_s\psi)(x)\mu_t(dx)\\
&= \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} \widetilde{\varphi}(z)\widetilde{\psi}(z') e^{-s(|z|^\alpha+|z'|^\alpha)} e^{-t|z+z'|^\alpha} \overline{\widetilde{\nu}(z+z')} dzdz'.
\end{aligned}
$$

Integrating $\int_0^\infty \int_0^\infty \dots dtds$ and using Lemma 4.1 we see that the right-hand side of (3.1) is well defined and continuous in φ, ψ. To complete the proof it suffices to observe (taking first $\varphi, \psi \ge 0$) that

$$
\int_0^\infty \int_{\mathbb{R}^d} (T_s\varphi)(x)(T_s\psi)(x)\mu_t(dx)dt = \int_{\mathbb{R}^d} (T_s\varphi)(x)(T_s\psi)(x)\gamma(dx).
$$

□

Remark. By Lemma 4.1 it is seen that there is no possibility to extend the definition of $K^{(1)}_{X_0}$ to lower dimensions ($d \le \alpha$) by means of formula (3.1).

Proof of Theorem 3.2. By Lemma 3.1 the flow process associated with $K^{(1)}_{X_0}$ has covariance

$$
\begin{aligned}
(4.2)\ K^{(1)}_F(s,\varphi;t,\psi) &= K^{(1)}_{X_0}(T_s\varphi, T_t\psi)\\
&= \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} \widetilde{\varphi}(z)\widetilde{\psi}(z') e^{-s|z|^\alpha} e^{-t|z'|^\alpha} \frac{\overline{\widetilde{\nu}(z+z')}}{(|z|^\alpha + |z'|^\alpha)|z+z'|^\alpha} dzdz'.
\end{aligned}
$$

Applying formula (4.1) and repeating the argument of the poof of Lemma 3.1 we see that the convolution integral process associated with $q^{(1)}$ has covariance

$$(4.3)\ K_{CI}^{(1)}(s,\varphi;t,\psi) = \int_0^{s\wedge t}\int_{\mathbb{R}^d}(T_{s-\tau}\psi)(x)(T_{t-\tau}\psi(x))\gamma(dx)d\tau$$
$$= \frac{1}{(2\pi)^{2d}}\int_0^{s\wedge t}\int_{\mathbb{R}^{2d}}\widetilde{\varphi}(z)\widetilde{\psi}(z')e^{-(s-\tau)|z|^\alpha}e^{-(t-\tau)|z'|^\alpha}\frac{\overline{\widetilde{\nu}(z+z')}}{|z+z'|^\alpha}dzdz'd\tau.$$

After straightforward calculations we then obtain the covariance of the correspondig *OU* process:

$$(4.4)\ K_X^{(1)}(s,\varphi;t,\psi) = K_F^{(1)}(s,\varphi;t,\psi) + K_{CI}^{(1)}(s,\varphi;t,\psi)$$
$$= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^{2d}}\widetilde{\varphi}(z)\widetilde{\psi}(z')e^{-|t-s|(|z'|^\alpha 1_{\{t\geq s\}}+|z|^\alpha 1_{\{t<s\}})}\frac{\overline{\widetilde{\nu}(z+z')}}{(|z|^\alpha+|z'|^\alpha)|z+z'|^\alpha}dzdz'.$$

It is clear that all three processes exist.

Now, sufficiency of the conditions (3.2) and (3.3) is obtained form (4.2) and (4.3) in an analogous way as in [BG3], using (2.1.1), (2.1.2), (2.1.3), (5.6), (5.7) of [BG3] and replacing the function h in that paper by $\frac{1}{(|z|^\alpha+|z'|^\alpha)|z+z'|^\alpha}$ in the case of the flow, and by $\frac{1}{|z+z'|^\alpha}$ in the case of the convolution integral. The same argument can be applied to derive (3.4), using (4.4). One should additionally observe that

$$\int_{[0,1]^2}e^{-|t-s|(|z'|^\alpha 1_{\{t\geq s\}}+|z|^\alpha 1_{\{t<s\}})}dtds \leq C\left(\frac{1}{1+|z|^\alpha}+\frac{1}{1+|z'|^\alpha}\right).$$

□

We shall need the following generalization of Lemma 5.1 of [BG3].

4.3. Lemma. *If $\Psi_1 \in L^p(\mathbb{R}^{2d})$ and $\Psi_2 \in L^q(\mathbb{R}^{2d})$, where $1 \leq q \leq \dfrac{p}{p-1}$, then the integral*

$$(4.5)\qquad \int_{\mathbb{R}^{4d}}\Psi_1(y,y')\Psi_2(z,z')|\varphi(y+z)||\varphi(y'+z')|dzdz'dydy'$$

is finite for each $\varphi \in \mathcal{S}_d$.

Proof. Put $r = \frac{pq}{2pq-p-q}$. By assumption, $0 \leq 2-\frac{1}{p}-\frac{1}{q} \leq 1$, hence $r \geq 1$. We have $\varphi\otimes\varphi \in L^r(\mathbb{R}^{2d})$. So, by the Young inequality, $\Psi_2 * |\varphi\otimes\varphi| \in L^{p'}(\mathbb{R}^{2d})$, where $\frac{1}{p'} = \frac{1}{q}+\frac{1}{r}-1 = \frac{p-1}{p}$. Hence the assertion follows because (4.5) has the form $\int\Psi_1(\Psi_2^* * (|\varphi|\otimes|\varphi|))$, where $\Psi_2^*(z,z') := \Psi_2(-z,-z')$. □

Proof of Proposition 3.3. It is obvious that (3.4) implies (3.2) and (3.3), so it suffices to prove that (3.3) implies (3.4).

The integral in (3.4) is always finite over the set $\{|z| \leq 1, |z'| \leq 1, |y| \leq 1, |y'| \leq 1\}$ by Lemmas 4.1 and 4.3. On the set $\{|z| > 1, |y| > 1\}$, as well as on the other sets of this form, (3.4) is clearly equivalent to (3.3). Hence it remains to consider the sets of the form $\{|z| \leq 1, |z'| \leq 1, |y| > 1\}$.

Denote

$$\varphi_1(y) = \sup_{|z|\leq 1} |\varphi(z+y)|, \quad \varphi_2(y) = \inf_{|z|\leq 1} |\varphi(z+y)|.$$

(3.3) implies

$$\int_{\substack{|z|\leq 1\\ |z'|\leq 1}} \frac{|\nu(z+z')|}{(1+|z|^\alpha)(1+|z'|^\alpha)|z+z'|^\alpha} dzdz' \int_{|y|>1} \frac{|\widetilde{\nu}(y+y')|}{(1+|y|^\alpha)(1+|y'|^\alpha)|y+y'|^\alpha} \varphi_2(y)\varphi_2(y')dydy'$$
$$< \infty$$

for each $\varphi \in \mathcal{S}_d$. On the other hand, the integral in (3.4) on the considered set is bounded above by

$$C \int_{\substack{|z|\leq 1\\ |z'|\leq 1}} \frac{1}{(|z|^\alpha + |z'|^\alpha)|z+z'|^\alpha} dzdz' \int_{|y|>1} \frac{|\widetilde{\nu}(y+y')|}{(1+|y|^\alpha)(1+|y'|^\alpha)|y+y'|^\alpha} \varphi_1(y)\varphi_1(y')dydy'.$$

The first factor is finite by Lemma 4.1. To complete the proof is suffices to observe that for any $\varphi \in \mathcal{S}_d$ there exists $\psi \in \mathcal{S}_d$ such that $|\varphi(x)| \leq \psi_2(x)$ for $x \in \mathbb{R}^d$, since then we shall have also that for any $\varphi \in \mathcal{S}_d$ there exists $\psi \in \mathcal{S}_d$ with the property $\varphi_1(x) \leq |\psi(x)|$, $x \in \mathbb{R}^d$. This fact can be proved, e.g., by repeating the argument at the beginning of the proof of Lemma 6.2 in [BG2]. We define $\psi(x) = \sum_{n=0}^{\infty} a_n \lambda_n(x)$, where $\lambda_n \in C^\infty(\mathbb{R}^d)$, $\lambda_n \geq 0$, $\text{supp}\lambda_n \subset \{x \in \mathbb{R}^d : n \leq |x| \leq n+2\}, n = 0,1,2,\ldots, \sum_{n=0}^{\infty} \lambda_n \equiv 1$, and $a_n = \sup_{n-1\leq|x|\leq n+2}|\varphi(x)|$ for $n = 1,2,\ldots$, $a_0 = \sup_{|x|\leq 2}|\varphi(x)|$. □

Proof of Theorem 3.4.

Step 1. Observe that if $\alpha < d \leq 2\alpha$, then $p_0 = 1$ (see (3.5)) and (3.6), (3.7) are obviously satisfied. In this step we prove that if $\alpha < d \leq 2\alpha$ then all three processes have SILT.

By Proposition 3.3 it suffices to show that condition (3.3) is satisfied. To this end it is enough to prove that

$$\int_{\mathbb{R}^{4d}} \frac{1}{(1+|z|^\alpha)(1+|z'|^\alpha)|z+z'|^\alpha} \frac{1}{(1+|y|^\alpha)(1+|y'|^\alpha)|y+y'|^\alpha} \cdot|\varphi(z+y)||\varphi(z'+y')|dzdz'dydy' < \infty. \tag{4.6}$$

for each $\varphi \in \mathcal{S}_d$, if $\alpha < d \leq 2\alpha$.

The latter integral can be written as $I_1 + I_2 + I_3 + I_4$, where

$$I_1 = \int_{\substack{|z+z'|\leq 1\\ |y+y'|\leq 1}}, \quad I_2 = \int_{\substack{|z+z'|>1\\ |y+y'|\leq 1}}, \quad I_3 = \int_{\substack{|z+z'|\leq 1\\ |y+y'|>1}}, \quad I_4 = \int_{\substack{|z+z'|>1\\ |y+y'|>1}}.$$

I_1 has the form (4.5) with

$$\Psi_1(z,z') = \Psi_2(z,z') = \frac{1}{(1+|z|^\alpha)(1+|z'|^\alpha)|z+z'|^\alpha} 1_{[0,1]}(|z+z'|).$$

Fix p such that $1 < p < \frac{d}{\alpha}$ (≤ 2). We have

$$\begin{aligned}\int_{\mathbb{R}^{2d}} \Psi_1^p(z,z')dzdz' &\leq C\int_{\mathbb{R}^{2d}} \frac{1}{(1+|z|^{\alpha p})(1+|z'|^{\alpha p})|z+z'|^{\alpha p}} 1_{[0,1]}(|z+z'|)dzdz' \\ &= C\int_{\mathbb{R}^d} g_1(z)(g_1 * g_2)(z)dz,\end{aligned}$$

where

$$g_1(z) = \frac{1}{1+|z|^{\alpha p}}, \quad g_2(z) = \frac{1}{|z|^{\alpha p}} 1_{[0,1]}(|z|).$$

Now, $g_1 \in L^2$ since $p > 1 \geq d/2\alpha$, and $g_2 \in L^1$ because $\alpha p < d$. Hence $g_1 * g_2 \in L^2$ and we obtain that $\Psi_1 \in L^p(\mathbb{R}^{2d})$. By Lemma 4.3 applied with $q = p$ $(\leq \frac{p}{p-1}$ since $p \leq 2)$ we conclude that $I_1 < \infty$.

In I_2 we take Ψ_1 as before and

$$\Psi_2(z,z') = \frac{1}{(1+|z|^{\alpha})(1+|z'|^{\alpha})|z+z'|^{\alpha}} 1_{(1,\infty)}(|z+z'|). \tag{4.7}$$

We prove that $\Psi_2 \in L^2(\mathbb{R}^d)$. Indeed, we have

$$\int_{\mathbb{R}^{2d}} \Psi_2(z,z')dzdz' \leq C\int_{\mathbb{R}^d} g_1(z)(g_1 * g_2)(z)dz,$$

where this time

$$g_1(z) = \frac{1}{1+|z|^{2\alpha}}, \quad g_2(z) = \frac{1}{|z|^{2\alpha}} 1_{(1,\infty)}(|z|).$$

It is clear that $g_1, g_2 \in L^p$ for each $p > 1$ since $2\alpha p > 2\alpha \geq d$. Fix $p \in (1,2)$ and let $q = \frac{p}{2p-2}$ (> 1). As $g_1 \in L^p$ and $g_2 \in L^q$, the Young inequality implies that $g_1 * g_2 \in L^{p'}$ with $\frac{1}{p'} = \frac{1}{p} + \frac{1}{q} - 1 = \frac{p-1}{p}$, so

$$\int_{\mathbb{R}^d} g_1(z)(g_1 * g_2)(z)dz < \infty.$$

We now apply Lemma 4.3 with $q = 2$ (since $p \leq 2 \leq \frac{p}{p-1}$) and we obtain $I_2 < \infty$.

$I_3 < \infty$ by symmetry.

In I_4 we take $\Psi_1 = \Psi_2$ given by (4.7) and we apply Lemma 4.3 with $p = q = 2$ to obtain $I_4 < \infty$.

Thus (4.6) is proved.

Step 2. We prove that if (3.6) is satisfied, then the flow process has SILT. By Step 1 we can additionaly assume that $d > 2\alpha$.

By Lemma 4.3 with $p = q = 2$, condition (3.2) is satisfied if

$$\int_{\mathbb{R}^{2d}} \frac{1}{(1+|z|^{\alpha})^2(1+|z'|^{\alpha})^2(|z|^{\alpha}+|z'|^{\alpha})^2} \frac{|\widetilde{\nu}(z+z')|^2}{|z+z'|^{2\alpha}} dzdz' < \infty. \tag{4.8}$$

Let us consider this integral on the sets $\{|z| \le 1, |z'| \le 1\}$, $\{|z| > 1, |z'| \le 1, |z - z'| \le 2\}$, $\{|z| > 1, |z'| \le 1, |z - z| > 2\}$, $\{|z| \le 1, |z'| > 1\}$, $\{|z| > 1, |z'| > 1\}$, denoting the corresponding integrals by $I_1, I_2', I_2'', I_3, I_4$.

Now,

$$I_1 \le C \int_{\substack{|z|\le 1\\ |z'|\le 1}} \frac{1}{(|z|^{2\alpha} + |z'|^{2\alpha})|z+z'|^{2\alpha}} dr dr' < \infty$$

by Lemma 4.1, since $d > 2\alpha$. $I_2' < \infty$ by the same reasoning (we have $|z| \le 3$).

Let

$$m(z) = \frac{|\widetilde{\nu}(z)|^2}{|z|^{2\alpha}}. \tag{4.9}$$

We have

$$I_2'' \le \int_{|z'|\le 1} \Big(\int_{\substack{|z|>1\\ |z+z'|>2}} \frac{1}{|z|^{4\alpha}} m(z+z') dz \Big) dz'. \tag{4.10}$$

We can find $p \ge p_0$ such that $m1_{\{|z|>2\}} \in L^p(\mathbb{R}^d)$ and $d < \frac{6p}{2p-1}\alpha$. But $\frac{6p}{2p-1}\alpha < \frac{4p}{p-1}\alpha$, hence the function $z \mapsto \frac{1}{|z|^{4\alpha}} 1_{\{|z|>1\}}$ belongs to $L^{p/(p-1)}(\mathbb{R}^d)$. So the inner integral on the right-hand side of (4.10) is a bounded function as the convolution of a function in L^p with a function in $L^{p/(p-1)}$. Hence $I_2'' < \infty$.

$I_3 < \infty$ by symmetry.

Finally

$$I_4 \le \int_{\substack{|z|>1\\ |z'|>1}} \frac{1}{|z|^{3\alpha}|z'|^{3\alpha}} m(z) dz dz'.$$

We have the same situation as in Proposition 3.3 of [BG3] with $\frac{3}{2}\alpha$ instead of α (it is not important here that $\frac{3}{2}\alpha$ may be bigger than 2, also the form of m is irrelevant). In consequence we obtain that $I_4 < \infty$ provided that $d < \frac{4p_0}{2p_0-1}\frac{3}{2}\alpha = \frac{6p_0}{2p_0-1}\alpha$.

Thus (4.8) is proved.

Step 3. We prove that if (3.7) is satisfied then both the convolution integral and OU processes have SILT's. Again, by Step 1 we may additionally assume that $d > 2\alpha$. By Proposition 3.3 it suffices to prove that (3.3) is satisfied, and this will be shown again by Lemma 4.3 with $p = q = 2$, if we prove that

$$\int_{\mathbb{R}^{2d}} \frac{1}{(1+|z|^\alpha)^2(1+|z'|^\alpha)^2} m(z+z') dz dz' < \infty.$$

Finiteness of this integral is obtained by the same argument as in the proof of Proposition 3.3 of [BG3] as the form of the function m is irrelevant here.

Step 4. It remains to prove continuity of the SILT's. We can repeat the arguments of the proof of Theorem 3.14 of [BG3] with $h(z,z') = \frac{1}{(|z|^\alpha + |z'|^\alpha)|z+z'|^\alpha}$ for the flow and $h(z,z') = \frac{1}{|z+z'|^\alpha}$ for the convolution integral. In consequence we obtain that if there exists $\varepsilon \in (0,1)$ such that

$$(4.11)\ \int_{\mathbb{R}^{4d}} \frac{|\widetilde{\nu}(z+z')|}{(1+|z|^{\alpha(1-\varepsilon)})(1+|z'|^{\alpha(1-\varepsilon)})(|z|^\alpha+|z'|^\alpha)|z+z'|^\alpha}$$

$$\cdot\frac{|\widetilde{\nu}(y+y')|}{(1+|y|^{\alpha(1-\varepsilon)})(1+|y'|^{\alpha(1-\varepsilon)}(|y|^\alpha+|y'|^\alpha)|y+y'|^\alpha}|\varphi(z+y)||\varphi(z'+y')|dzdz'dydy' < \infty$$

for each $\varphi \in \mathcal{S}_d$, then the SILT of the flow process is continuous, and if there exists $\varepsilon \in (0,1)$ such that

$$(4.12)\quad \int_{\mathbb{R}^{4d}} \frac{|\widetilde{\nu}(z+z')|}{(1+|z|^{\alpha(1-\varepsilon)})(1+|z'|^{\alpha(1-\varepsilon)})|z+z'|^\alpha}\frac{|\widetilde{\nu}(y+y')|}{(1+|y|^{\alpha(1-\varepsilon)})(1+|y'|^{\alpha(1-\varepsilon)})|y+y'|^\alpha}$$

$$\cdot|\varphi(z+y)||\varphi(z'+y')|dzdz'dydy' < \infty$$

for each $\varphi \in \mathcal{S}_d$, the the SILT of the convolution integral process is continuous.

Looking carefully at the previous steps of the proof, we see that if (3.6) and (3.7) hold we can find $\varepsilon \in (0,1)$ such that (4.11) and (4.12) are satisfied, respectively.

Continuity of the SILT of the *OU* process can be proved analogously as in [BG3] (Corollary 3.20), using the bounds derived for the flow and convolution integral processes. □

Proof of Corollary 3.6. (a), (b), (i): By Theorem 3.4 we know that if $\alpha < d \leq 2\alpha$, then the SILT's exists. Assume $d > 2\alpha$. If ν makes no contribution in (3.5), then $p_0 = d/2\alpha$, so in (3.6) we obtain $d < \frac{3d\alpha}{d-\alpha}$ (hence $d < 4\alpha$), and in (3.7) we have $d < \frac{2d\alpha}{d-\alpha}$ (hence $d < 3\alpha$).

(a), (b), (ii): The argument is identical with $p_0 = 1$. □

Proof of Proposition 3.7. Recall that the covariance of the convolution integral process is

$$(4.13)\quad K_{CI}^{(1)}(s,\varphi;t,\psi) = \int_0^{s\wedge t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{2d}} p_{s-\tau}(x,z)p_{t-\tau}(x,z')\varphi(z)\psi(z')dzdz'\gamma(dx)d\tau$$

(see (4.3)), so it has the same form as the covariance in [BG3] (see the formula immediately after (5.11)), with γ instead of μ_τ. Therefore we can repeat the proof of Theorem 3.11 of [BG3] provided that we know that the function

$$(z,z') \mapsto \int_{\mathbb{R}^d} p_s(x,z)p_t(x,z')\gamma(dx)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{p_s(x,z)p_t(x,z')}{|x-y|^{d-\alpha}}dx\nu(dy)$$

is bounded continuous for s,t fixed. To show boundedness it suffices to prove that the function

$$F(y,z,z') = \int_{\mathbb{R}^d}\frac{p_s(x,z)p_t(x,z')}{|x-y|^{d-\alpha}}dx$$

is bounded, because ν is a finite measure. We have

$$\begin{aligned} F(y,z,z') &= \int_{|x-y|\le 1} + \int_{|x-y|>1} \\ &\le C(s,t)\int_{|x-y|\le 1}\frac{1}{|x-y|^{d-\alpha}}dx + \int_{\mathbb{R}^d} p_s(x,z)p_t(x,z')dx \\ &\le C_1(s,t) + p_{s+t}(z,z') \le C_2(s,t), \end{aligned}$$

so the boundedness is proved. Continuity can be shown in a similar way.

Finally, non-existence of SILT for the OU process if $d > 4\alpha$ follows from the definition of SILT, from the fact that $K^{(1)}_{X_0}(s,\varphi;t,\psi) \ge 0$ and $K^{(1)}_{CI}(s,\varphi;t,\psi)\ge 0$ for $\varphi,\psi\ge 0$, and from the non-existence proof for the convolution integral process. □

Proof of Proposition 3.9. We start from the convolution integral and OU processes. By Corollary 3.6 (b)(i) and by the argument at the end of the previous proof it suffices to show that if $d \ge 3\alpha$ then the convolution integral process does not have SILT.

By (4.13) it is clear that $J_{s,r,u,v}$ (see (2.1.1)–(2.1.3) and the notation before Definition 2.1.1 of [BG3]) has the form (5.6) of [BG3] with $h(z,z') = \frac{1}{|z+z'|^\alpha}$, $\mu_\tau = \nu = \delta_a$. Fix $f \in \mathcal{F}$ such that $\widetilde{f} > 0$, and $\varphi \in \mathcal{S}_d$ such that $\widetilde{\varphi}_a > 0$, where $\varphi_a(x) := \varphi(x+a)$. Analogously as in the proof for Example 4.1 (b) of [BG3] we have

$$\begin{aligned} & J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}\Phi^f_{\varepsilon,\varphi}) \\ &= \frac{1}{(2\pi)^{4d}}\int_0^{s\wedge u}\int_0^{r\wedge v}\int_{\mathbb{R}^{4d}} e^{-(s-\tau)|z|^\alpha}e^{-(u-\tau)|z'|^\alpha}e^{-(r-\tau')|y|^\alpha}e^{-(v-\tau')|y'|^\alpha} \\ &\quad \cdot\frac{1}{|z+z'|^\alpha}\frac{1}{|y+y'|^\alpha}\widetilde{f}_\varepsilon(z)\widetilde{f}_\varepsilon(z')\widetilde{\varphi}_a(z+y)\widetilde{\varphi}_a(z'+y')dzdz'dydy'd\tau'd\tau \\ &+ \text{ the other (similar) term.} \end{aligned}$$

If the SILT existed we would have, by Fatou's lemma,

$$\begin{aligned} (4.14) \qquad \infty &> \liminf_{\varepsilon\to 0}\int_{[0,1]^4} J_{s,r,u,v}(\Phi_{\varepsilon,\varphi},\Phi^f_{\varepsilon,\varphi})dsdrdudv \\ &\ge \frac{1}{(2\pi)^{4d}}\int_{\mathbb{R}^{4d}}\Psi(z,z')\Psi(y,y')\widetilde{\varphi}_a(z+y)\widetilde{\varphi}_a(z'+y')dzdz'dydy', \end{aligned}$$

where

$$\Psi(z,z') = \int_{[0,1]^2}\int_0^{s\wedge u} e^{-(s-\tau)|z|^\alpha}e^{-(u-\tau)|z'|^\alpha}\frac{1}{|z+z'|^\alpha}d\tau duds.$$

Using (5.4) of [BG3] we obtain

$$\Psi(z,z') \ge C|z|^{-\alpha}|z'|^{-\alpha}|z+z'|^{-\alpha}$$

on the set $\{|z| > 1, |z'| > 1\}$, so we are led to the same situation as in the proof for Example 4.1 (d) of [BG3]. Hence we know that the integral on the right-hand side of (4.14) is infinite if $d \geq 3\alpha$.

For the flow process the argument is analogous and we omit it. We only point out that everything reduces to showing that

$$\int_{\substack{|z|>1\\|z'|>1\\|z+z'|>1}} \frac{1}{|z|^{2\alpha}|z'|^{2\alpha}(|z|^\alpha + |z'|^\alpha)^2|z+z'|^{2\alpha}} dzdz' = \infty$$

if $d \geq 4\alpha$. By similar transformations as in the proof for Example 4.1 (d) of [BG3] this integral can be proved to be bounded from below by

$$C \int_{\substack{v>1\\u^{1/2\alpha}-v^{1/2\alpha}>1}} u^{\frac{d-4\alpha}{2\alpha}} v^{\frac{d-4\alpha}{2\alpha}} (u+v)^{-2} dudv \geq C \int_{\substack{v>1\\u^{1/2\alpha}-v^{1/2\alpha}>1}} (u+v)^{-2} dudv = \infty$$

if $d \geq 4\alpha$. □

Proof of Proposition 3.10. The covariance functional of the flow process associated with $q^{(1)}$ is

$$(4.15) \quad \int_{\mathbb{R}^d} (T_s\varphi)(T_t\psi)d\gamma = \int_{\mathbb{R}^{3d}} p_s(x,z)p_t(x,y)\varphi(z)\psi(y)\gamma(x)dzdydx = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} e^{-s|z|^\alpha} e^{-t|z'|^\alpha} \widetilde{\varphi}(z)\widetilde{\psi}(z') \frac{\overline{\widetilde{\nu}(z+z')}}{|z+z'|^\alpha} dzdz',$$

where the second equality is obtained analogously as before, using Lemma 4.2. Note that the integral in the middle of (4.15) has the form (3.2.3) of [BG1], but we cannot use the sufficiency criterion of [BG1] because $\gamma(\cdot)$ given by (2.3) is not a bounded function. However, the necessity criterion does apply because in order to use Proposition 3.2.1 (b) of [BG1] we only need that $\gamma(x) \geq 0$ and $\gamma(x) > \varepsilon > 0$ for some ε and for x in a set of positive Lebesgue measure. These requirements are obviously satisfied in our case, so we know that SILT does not exists if $d \geq 4\alpha$.

The second equality in (4.15) permits to derive, by the same argument as in the proof of Theorem 3.2, the following sufficient SILT existence condition:

$$\int_{[0,1]^4} \int_{\mathbb{R}^{4d}} e^{-s|z|^\alpha - u|z'|^\alpha - r|y|^\alpha - v|y'|^\alpha} \frac{|\widetilde{\nu}(z+z')||\widetilde{\nu}(y+y')|}{|z+z'|^\alpha|y+y'|^\alpha} \cdot |\widetilde{\varphi}(z+y)||\widetilde{\varphi}(z'+y')| dzdz'dydy'dsdrdudv < \infty$$

for each $\varphi \in \mathcal{S}_d$. This condition is clearly equivalent to (3.3) (since $d > \alpha$). Hence all the remaining assertions of Proposition 3.10 follow. □

Proof of Lemma 3.12. Formula (3.9) is derived by an argument as in the proof of Lemma 3.1. Note, however, that while it is clear that $q^{(2)}$ is well defined, bilinear and continuous (by (3.9)), the fact that $q^{(2)}(\varphi,\varphi) \geq 0$ does not seem evident. This property relies on the fact that γ is an excessive measure. We refer to [GL2] for the proof of this property of $q^{(2)}$.

Proof of Proposition 3.14. Assume $\alpha < d < 2\alpha$ and fix $\varphi \in \mathcal{S}_d$. We will prove that (3.11) holds. In the proof we will apply several times Lemma 4.3, but we will also have to use some estimations more subtle than those resulting from that lemma.

(3.11) will be proved if we show that (we omit the differentials)

$$\int_{\substack{|z|\leq 1\\ |z'|\leq 1}} \int_{\substack{|y|\leq 1\\ |y'|\leq 1}} \frac{1}{|z+z'|^\alpha} \frac{1}{|y+y'|^\alpha} |\varphi(z+y)||\varphi(z'+y')| < \infty, \tag{4.16}$$

$$\int_{\substack{|z|\leq 1\\ |z'|\leq 1}} \int_{\substack{|y|>1\\ |y'|\leq 1}} \frac{1}{|z+z'|^\alpha} \frac{1}{|y+y'|^\alpha} |\varphi(z+y)||\varphi(z'+y')| < \infty, \tag{4.17}$$

$$\int_{\substack{|z|>1\\ |z'|\leq 1}} \int_{\substack{|y|>1\\ |y'|\leq 1}} \frac{1}{|z+z'|^\alpha} \frac{1}{|y+y'|^\alpha} |\varphi(z+y)||\varphi(z'+y')| < \infty, \tag{4.18}$$

$$\int_{\substack{|z|\leq 1\\ |z'|\leq 1}} \int_{\substack{|y|>1\\ |y'|>1}} \frac{1}{|z+z'|^\alpha} \frac{1}{|y|^\alpha|y+y'|^\alpha} |\varphi(z+y)||\varphi(z'+y')| < \infty, \tag{4.19}$$

$$\int_{\substack{|z|\leq 1\\ |z'|>1}} \int_{\substack{|y|>1\\ |y'|>1}} \frac{1}{|z+z'|^\alpha} \frac{1}{|y|^\alpha|y+y'|^\alpha} |\varphi(z+y)||\varphi(z'+y')| < \infty, \tag{4.20}$$

$$\int_{\substack{|z|>1\\ |z'|>1}} \int_{\substack{|y|>1\\ |y'|>1}} \frac{1}{|z'|^\alpha|z+z'|^\alpha} \frac{1}{|y'|^\alpha|y+y'|^\alpha} |\varphi(z+y)||\varphi(z'+y')| < \infty. \tag{4.21}$$

Apparently we have not exhausted all cases, but some of them are immediate by symmetry and others will be clarified during the proof.

It is convenient to introduce the following notation:

$$\begin{aligned}
g_1(z) &= \frac{1}{|z|^\alpha} 1_{[0,1]}(|z|),\\
g_2(z) &= \frac{1}{|z|^\alpha} 1_{(1,\infty)}(|z|),\\
g_3(z,z') &= \frac{1}{|z'|^\alpha|z+z'|^\alpha} 1_{(1,\infty)}(|z|)1_{(1,\infty)}(|z'|),\\
\varphi_1(z) &= \sup_{|y|\leq 1} |\varphi(z+y)|.
\end{aligned}$$

We have $g_1 \in L^1(\mathbb{R}^d)$ and

$$g_2 \in L^p(\mathbb{R}^d) \quad \text{for} \quad p > \frac{d}{\alpha}, \tag{4.22}$$

in particular, $g_2 \in L^2(\mathbb{R}^d)$. Hence we obtain that

$$g_3(z,z')1_{(1,\infty)}(|z+z'|) \in L^p(\mathbb{R}^{2d}) \quad \text{for} \quad p > \frac{d}{\alpha}, \tag{4.23}$$

since

$$\int_{\mathbb{R}^{2d}} g_3^p(z,z')1_{(1,\infty)}(|z+z'|)dzdz' = \int_{|z|>1} g_2^p * g_2^p(z)dz.$$

We have also $\varphi_1 \in L^p(\mathbb{R}^d)$ for $p \geq 1$, and moreover

$$\int_{\substack{|z|\leq 1\\ |z'|\leq 1}} \frac{1}{|z+z'|^d} dzdz' < \infty, \tag{4.24}$$

$$\int_{\substack{|z|\leq 1\\ |z+z'|\leq 1}} \frac{1}{|z+z'|^{2\alpha}} dzdz' < \infty, \tag{4.25}$$

$$\int_{\substack{|z|\leq 1\\ |z+z'|>1}} \frac{1}{|z+z'|^{2\alpha}} dzdz' < \infty. \tag{4.26}$$

Now, (4.16) is obvious by (4.24). The integral in (4.17) has the form (4.5) with

$$\Psi_2(z,z') = \frac{1}{|z+z'|^\alpha} 1_{[0,1]^2}(|z|,|z'|) \in L^1(\mathbb{R}^{2d})$$

and

$$\Psi_1 = \Psi_1^{(1)} + \Psi_1^{(2)},$$

where

$$\begin{aligned} \Psi_1^{(1)}(y,y') &= g_1(y+y')1_{(1,\infty)}(|y|)1_{[0,1]}(|y'|) \in L^1(\mathbb{R}^{2d}) \quad \text{by (4.25)},\\ \Psi_1^{(2)}(y,y') &= g_2(y+y')1_{(1,\infty)}(|y|)1_{[0,1]}(|y'|) \in L^2(\mathbb{R}^{2d}) \quad \text{by (4.26)}. \end{aligned}$$

Hence (4.17) holds by Lemma 4.3 applied to $\Psi_1^{(1)}, \Psi_2$ and to $\Psi_1^{(2)}, \Psi_2$.

(4.18) is obtained the same way because now the integral has the form (4.5) with Ψ_1 as before and $\Psi_2 = \Psi_2^{(1)} + \Psi_2^{(2)}$, where $\Psi_2^{(1)} = \Psi_1^{(1)}, \Psi_2^{(2)} = \Psi_1^{(2)}$.

The same argument applies to all integrals where exactly one of $|z|, |z'|$ is ≤ 1 and exactly one of $|y|, |y'|$ is ≤ 1.

Next, the integral in (4.19) is bounded above by

$$\begin{aligned} &\int_{\substack{|z|\leq 1\\ |z'|\leq 1}} \frac{1}{|z+z'|^\alpha} dzdz' \int_{\substack{|y|>1\\ |y'|>1}} \frac{\varphi_1(y)}{|y|^\alpha} \frac{1}{|y+y'|^\alpha} \varphi_1(y')dy'dy\\ &\leq C\int_{\mathbb{R}^d} \varphi_1(y)(\varphi_1^* * g_1)(y)dy + C\int_{\mathbb{R}^d} \varphi_1(y)(\varphi_1^* * g_2)(y)dy \end{aligned}$$

(recall that $\varphi_1^*(x) = \varphi_1(-x)$). We have $\varphi_1 \in L^2(\mathbb{R}^d), \varphi_1^* * g_1 \in L^2(\mathbb{R}^d)$, so the first integral is finite. Also $\varphi_1^* * g_2 \in L^2(\mathbb{R}^d)$ since $g_2 \in L^2(\mathbb{R}^d)$ (by (4.22)) and $\varphi_1^* \in L^1(\mathbb{R}^d)$, so the second integral is finite as well.

We write the integral in (4.20) as the sum $I_1 + I_2$, where

$$I_1 = \int_{\dots}\int_{|y+y'|>1} \dots, \qquad I_2 = \int_{\dots}\int_{|y+y'|\le 1} \dots .$$

I_1 has the form (4.5) with

$$\Psi_2(z,z') = \frac{1}{|z+z'|^\alpha} 1_{[0,1]}(|z|)1_{(1,\infty)}(|z'|), \quad \Psi_1(y,y) = g_3(y',y)1_{(1,\infty)}(|y+y'|).$$

We already know that Ψ_2 is the sum of two functions, one of them in $L^1(\mathbb{R}^{2d})$, the other one in $L^2(\mathbb{R}^{2d})$, and $\Psi_1 \in L^2(\mathbb{R}^{2d})$ by (4.23). Therefore $I_1 < \infty$ by Lemma 4.3. Now, $I_2 = I_2^{(1)} + I_2^{(2)}$, where

$$I_2^{(1)} = \int_{|z+z'|\le 1} \dots \int_{\dots}, \qquad I_2^{(2)} = \int_{|2+2'|>1} \dots \int_{\dots} .$$

We have

$$\begin{aligned} I_2^{(1)} &\le \sup_x |\varphi(x)| \int_{\substack{|z|\le 1\\ |z'|>1 \\ |z+z'|\le 1}} \frac{1}{|z+z'|^\alpha} dz dz' \int_{|y'|>1}\int_{|y|>1} \frac{\varphi_1(y)}{|y|^\alpha}\frac{1}{|y+y'|^\alpha} 1_{[0,1]}(|y+y'|) dy dy' \\ &\le C \int_{|y'|>1} \varphi_1^* * g_1(y') dy' \quad \text{(by (4.25))} \\ &< \infty, \end{aligned}$$

since $\varphi_1, g_1 \in L^1(\mathbb{R}^d)$.

$$\begin{aligned} I_2^{(2)} &= \int_{|z|\le 1}\int_{\mathbb{R}^d} \frac{1}{|z+z'|^\alpha} 1_{(1,\infty)}(|z+z'|)1_{(1,\infty)}(|z'|) \int_{|y'|>1} |\varphi(z'+y')| \\ &\quad \cdot \int_{|y|>1} \frac{|\varphi(z+y)|}{|y|^\alpha}\frac{1}{|y+y'|^\alpha} 1_{[0,1]}(|y+y'|) dy dy' dz dz' \\ &\le \int_{|z|\le 1} g_2 * (|\varphi| * (\varphi_1 * g_1))(z) dz. \end{aligned}$$

We have $|\varphi| * (\varphi_1 * g_1) \in L^2(\mathbb{R}^d)$ since $|\varphi| \in L^2(\mathbb{R}^d)$ and $\varphi_1 * g_1 \in L^1(\mathbb{R}^d)$; also $g_2 \in L^2(\mathbb{R}^d)$ by (4.22). Hence the function under the integral is bounded and the integral is finite. Thus (4.20) is proved. Note, moreover, that if in (4.20) we replace $\frac{1}{|y'|^\alpha}$ by $\frac{1}{|y|^\alpha}$, then the argument for I_1 remains unchanged, and in I_2 it suffices to observe that $\frac{1}{|y'|^\alpha} \le \frac{2^\alpha}{|y|^\alpha}$ if $|y'| > 1, |y+y'| \le 1$, so we are led to previous case.

Finally, the integral in (4.21) can be written as $I_1 + I_2 + I_3 + I_4$, where

$$\begin{aligned} I_1 &= \int_{|z+z'|\le 1} \dots \int_{|y+y'|\le 1} \dots, \quad I_2 = \int_{|z+z'|>1} \dots \int_{|y+y'|\le 1} \dots, \quad I_3 = \int_{|z+z'|\le 1} \dots \int_{|y+y'|>1} \dots, \\ I_4 &= \int_{|z+z'|>1} \dots \int_{|y+y'|>1} \dots . \end{aligned}$$

We have

$$
\begin{aligned}
I_1 &\le \sup_x |\varphi(x)| \int_{\substack{|z|>1\\|y|>1}} \int_{\mathbb{R}^{2d}} \frac{1}{|z+z'|^\alpha} 1_{[0,1]}(|z+z'|) \frac{1}{|y+y'|^\alpha} 1_{[0,1]}(|y+y'|) \\
&\quad \cdot \frac{1}{|z'|^\alpha} 1_{(1,\infty)}(|z'|) \frac{1}{|y'|^\alpha} 1_{(1,\infty)}(|y'|) |\varphi(z'+y')| dz'dy'dzdy \\
&= \sup_x |\varphi(x)| \int_{\substack{|z|>1\\|y|>1}} (g_1 \otimes g_1) * F(z,y) dzdy,
\end{aligned}
$$

where

$$
F(z',y') = \frac{1}{|z'|^\alpha} 1_{(1,\infty)}(|z'|) \frac{1}{|y'|^\alpha} 1_{(1,\infty)}(|y'|) |\varphi(z'+y')|.
$$

As $g_1 \otimes g_1 \in L^1(\mathbb{R}^{2d})$, to obtain $I_1 < \infty$ it suffices to show that $F \in L^1(\mathbb{R}^{2d})$. But

$$
\int_{\mathbb{R}^{2d}} F(z',y') dz'dy' = \int_{\mathbb{R}^d} g_2(z')(g_2 * |\varphi|)|z'|dz' < \infty,
$$

since $g_2 \in L^2(\mathbb{R}^d)$ and $g_2 * |\varphi| \in L^2(\mathbb{R}^{2d})$.

Note that, by the remark above, $\frac{1}{|y'|^\alpha}$ can be replaced by $\frac{1}{|y|^\alpha}$ here, and the same is true for $|z|, |z'|$.

Next, in I_2 we put $y = x - y'$. Then

$$
\begin{aligned}
I_2 &= \int_{\substack{|z|>1\\|z'|>1\\|z+z'|>1}} \int_{\substack{|x-y'|>1\\|y'|>1\\|x|\le 1}} \frac{1}{|z'|^\alpha |z+z'|^\alpha} \frac{1}{|y'|^\alpha} \frac{1}{|x|^\alpha} |\varphi(z+x-y')||\varphi(z'+y')| dxdy'dzdz' \\
&\le \int_{\substack{|z|>1\\|z'|>1\\|z+z'|>1}} \int_{|y'|>1} \frac{1}{|z'|^\alpha |z+z'|^\alpha} \frac{1}{|y'|^\alpha} \int_{|x|\le 1} \frac{1}{|x|^\alpha} dx \varphi_1(z-y')|\varphi(z'+y')| dy'dzdz' \\
&= C \int_{\substack{|w+y'|>1\\|z'|>1\\|w+y'+z'|>1}} \int_{|y'|>1} \frac{1}{|z'|^\alpha |w+y'+z'|^\alpha} \frac{1}{|y'|^\alpha} \varphi_1(w)|\varphi(z'+y')| dy'dwdz' \\
&\le C \int_{|z'|>1} \int_{|y'|>1} \frac{1}{|z'|^\alpha |y'|^\alpha} \int_{|w+y'+z'|>1} \frac{1}{|w+y'+z'|^\alpha} \varphi_1(w) dw |\varphi(z'+y')| dy'dz' \\
&= C \int_{\mathbb{R}^d} g_2(z')(g_2 * ((g_2 * \varphi_1)|\varphi^*|))(z')dz'.
\end{aligned}
$$

Since $g_2 \in L^2(\mathbb{R}^d)$, to show that this integral is finite it suffices to observe that $(g_2 * \varphi_1)|\varphi^*| \in L^1(\mathbb{R}^d)$, as $\varphi_1, |\varphi^*| \in L^2(\mathbb{R}^d)$.

Here, again we can replace $\frac{1}{|y'|^\alpha}$ by $\frac{1}{|y|^\alpha}$ and then the integrability for the pairs $\frac{1}{|z|^\alpha}, \frac{1}{|y|^\alpha}$ and $\frac{1}{|z|^\alpha}, \frac{1}{|y'|^\alpha}$ is obtained by a substitution $z \leftrightarrow z', y \leftrightarrow y'$.

$I_3 < \infty$ by symmetry.

Finally, $I_4 < \infty$ by Lemma 4.3 with $\Psi_1 = \Psi_2 = g_3 1_{(1,\infty)}(|z+z'|), p = q = 2$ (see (4.23). It is clear that nothing is changed here if we take other configurations of z, z' and y, y'.

So we have proved that if $\alpha < d < 2\alpha$, then the convolution integral process has SILT.

Continuity of the SILT can be proved again by repeating the argument of the proof of Theorem 3.14 of [BG3] and then carrying out carefully once more the estimations of the proof above. Some small amount of extra work is needed due to the fact that terms of the form $\frac{|z|^\alpha}{1+|z|^{\alpha(1-\varepsilon)}}$ will appear. We omit details.

We skip the proof of the converse, i.e., that if $d \geq 2\alpha$ then (3.10) does not hold for any measure ν, because this fact is not of central interest for us. We only point out that it suffices to prove that for a fixed $\varphi > 0$,

$$\int_{\substack{|z|>1\\ |z'|>1\\ |z+z'|\leq 1}} \int_{\substack{|y|>1\\ |y'|>1\\ |y+y'|\leq 1}} \frac{1}{|z'|^\alpha |z+z'|^\alpha} \frac{1}{|y'|^\alpha} \varphi(z+y)\varphi(z'+y')dydy'dzdz' = \infty \tag{4.27}$$

if $d \geq 2\alpha$. □

Proof of Corollary 3.15. Sufficiency is proved. To show neccessity we fix $\varphi \in \mathcal{S}_d$ such that $\widetilde{\varphi}_a > 0$ in the case $\nu = \delta_a$, or $\widetilde{\varphi} > 0$ in the case $\widetilde{\nu} > 0$. In both cases we can repeat the beginning of the proof of Proposition 3.9 (with $h(z,z) = \frac{c+|z|^\alpha+|z'|^\alpha}{|z+z'|^\alpha}$ this time), to find that existence of SILT for some $d \geq 2\alpha$ would be in contradiction to (4.27). □

Proof of Theorem 3.16. The sufficiency part can be deduced from the proofs of Proposition 3.10 and 3.14. To prove necessity we write the covariance of the convolution integral process associated with $q^{(2)}$ using formula (2.7). We have

$$\begin{aligned} K_{CI}^{(2)}(s,\varphi;t,\psi) &= \int_0^{s\wedge t} q^{(2)}(T_{s-\tau}\varphi, T_{t-\tau}\psi)d\tau \\ &= \int_0^{s\wedge t} \langle \gamma, c(T_{s-\tau}\varphi)(T_{t-\tau}\psi) - (T_{s-\tau}\varphi)(\Delta_\alpha T_{t-\tau}\psi) - (T_{t-\tau}\psi)(\Delta_\alpha T_{s-\tau}\varphi)\rangle d\tau. \end{aligned}$$

But

$$-(T_{s-\tau}\varphi)(\Delta_\alpha T_{t-\tau}\psi) - (T_{t-\tau}\psi)(\Delta_\alpha T_{s-\tau}\varphi) = \frac{d}{d\tau}(T_{s-\tau}\varphi)(T_{t-\tau}\psi)$$

by the semigroup property, hence

$$\begin{aligned} K_{CI}^{(2)}(s,\varphi;t,\psi) &= c\int_0^{s\wedge t} \langle \gamma, (T_{s-t}\varphi)(T_{t-\tau}\psi)\rangle d\tau \\ &\quad + \langle \gamma, (T_{s-s\wedge t}\varphi)(T_{t-s\wedge t}\psi)\rangle - \langle \gamma, (T_s\varphi)(T_t\psi)\rangle \\ &= cK_{CI}^{(1)}(s,\varphi;t,\psi) + \langle \gamma, (T_{s-s\wedge t}\varphi)(T_{t-s\wedge t}\psi)\rangle - K_F^{(1)}(s,\varphi;t,\psi). \end{aligned}$$

In consequence the covariance functional of the OU process is, by (3.8),

$$\begin{aligned} K_X^{(2)}(s,\varphi;t,\psi) &= K_{X_0}^{(2)}(T_s\varphi, T_t\psi) + K_{CI}^{(2)}(s,\varphi;t,\psi) \\ &= \langle \gamma, (T_{s-s\wedge t}\varphi)(T_{t-s\wedge t}\psi)\rangle + cK_X^{(1)}(s,\varphi;t,\psi). \end{aligned}$$

As $c \geq 0$ and $K_X^{(1)}(s,\varphi;t,\psi) \geq 0$ for $\varphi,\psi \geq 0$, we then have for $s<u, r<v, \Phi^{(1)},\Phi^{(2)} \in \mathcal{S}(\mathbb{R}^{2d}), \Phi^{(1)},\Phi^{(2)} \geq 0$,

$$J_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)}) \geq \int_{\mathbb{R}^{4d}} \Phi^{(1)}(x,x')\Phi^{(2)}(z,z')p_{u-s}(x,z)p_{v-r}(x',z')\gamma_1(x)\gamma_1(x')dzdz'dxdx',$$

where $\gamma_1(x) := 1 \wedge \gamma(x)$ (see Section 2 of [BG3]). Hence, for $\varphi > 0$, by Fatou's lemma,

$$\begin{aligned}(4.28)\qquad &\lim_{\varepsilon\to 0}\int_{[0,1]^4} J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi},\Phi^f_{\varepsilon,\varphi})dsdrdudv \\ &\geq \int_{\substack{s<u\\ r<v}} \liminf_{\varepsilon\to 0}\int_{\mathbb{R}^{4d}} f_\varepsilon(x-x')\varphi(x)f_\varepsilon(z-z')\varphi(z)p_{u-s}(x,z)p_{v-r}(x',z') \\ &\quad \cdot\gamma_1(x)\gamma_1(x')dxdzdx'dz'dsdrdudv.\end{aligned}$$

The function $(x,z) \mapsto \varphi(x)\varphi(z)p_{u-s}(x,z)\gamma_1(x)$ is bounded continuous, hence

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^{2d}} f_\varepsilon(x-x')f_\varepsilon(z-z')\varphi(x)\varphi(z)p_{u-s}(x,z)\gamma_1(x)dxdz = \varphi(x')\varphi(z')p_{u-s}(x',z')\gamma_1(x'),$$

and the right-hand side of (4.28) is bounded from below by

$$\int_{\substack{s<u\\ r<v}}\int_{\mathbb{R}^{2d}} \varphi(x')\varphi(z')p_{u-s}(x',z')p_{v-r}(x',z')\gamma_1^2(x')dx'dz'dsdrdudv.$$

Now we can apply Lemma 6.2.3 of [BG1] to obtain that this integral is bounded from below by

$$C(\varphi)\int_{\substack{s<u\\ r<v}} (u-s+v-r)^{-\frac{d}{\alpha}}dsdrdudv,$$

which is infinite if $d \geq 2\alpha$. Therefore SILT does not exist in this case (see Theorem 2.1.2 of [BG3]). □

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References.

[AFL] R.J. Adler, R.E. Feldman and M. Lewin, *Intersection local times for infinite systems of Brownian motions and for the Brownian density process*, Ann. Prob. 19 (1991), 192–220.

[AR] R.J. Adler and J.S. Rosen, *Intersection local times of all orders for Brownian and stable density processes–construction, renormalization and limit laws*, Ann. Prob. 21 (1993), 1073–1123.

[BRS] V.I. Bogachev, M. Röckner and B. Schmuland, *Generalized Mehler semigroups and applications*, Probab. Theory Rel. Fields 105 (1996), 193–225.

[BG1] T. Bojdecki and L. G. Gorostiza, *Self-intersection local time for Gaussian $\mathcal{S}'(\mathbb{R}^d)$-processes: existence, path continuity and examples*, Stoch. Proc. Appl. 60 (1995), 191–226.

[BG2] T. Bojdecki and L. G. Gorostiza, *Self-intersection local time for $\mathcal{S}'(\mathbb{R}^d)$-Wiener processes and related Ornstein-Uhlenbeck processes*, Infinite Dimensional Analysis, Quantum Probability and Related Topics 2 (1999), 569–615.

[BG3] T. Bojdecki and L. G. Gorostiza, *Self-intersection local time for some $\mathcal{S}'(\mathbb{R}^d)$-Ornstein-Unlenbeck processes related to inhomogeneous fields*, Math. Nachr. (in press).

[BJ] T. Bojdecki and J. Jakubowski, *Invariant measures for generalized Langevin equations in conuclear space*, Stoch. Proc. Appl. 84 (1999), 1–24.

[D] D.A. Dawson, *Measure–valued Markov processes*, in Lect. Notes Math. 1541 (1993), 1–260, Springer–Verlag, Berlin.

[Dy1] E. B. Dynkin, *Minimal excessive measures and functions*, Trans. Amer. Math. Soc. 258 (1980), 217–244.

[Dy2] E.B. Dynkin, *Branching particle systems and superprocesses*, Ann. Probab. 19 (1991), 1157–1194.

[GLM] L.G. Gorostiza and J.A. López–Mimbela, *The multitype measure branching process*, Adv. Appl. Prob. 22 (1990), 49–67.

[GL1] L. G. Gorostiza and Z.-H. Li, *Fluctuation limits of measure-valued immigration processes with small branching*, in "Modelos Estocásticos", Investigación 14 (1998), 261–268, Aportaciones Matemáticas, Soc. Mat. Mexicana.

[GL2] L. G. Gorostiza and Z.-H. Li, *High-density fluctuations of immigration branching particle systems*, in "Stochastic Models", Canadian Math. Soc. Conference Proceedings Series (2000), Amer. Math. Soc. (in press).

[KS] N. Konno and T. Shiga, *Stochastic partial differential equations for some measure–valued diffusions*, Probab. Theory Rel. Fields 79 (1988), 34–51.

[L1] Z.-H. Li, *Immigration structures associated with Dawson-Watanabe superprocesses*, Stoch. Proc. Appl. 62 (1996), 73–86.

[L2] Z.-H. Li, *Immigration processes associated with branching particle systems*, Adv. Appl. Probab. 30 (1998), 657–675.

[L3] Z.-H. Li, *Measure-valued immigration diffusion and generalized Ornstein-Uhlenbeck diffusions*, Acta Mathematicae Applicatae Sinica 15 (1999), 310–320.

[LS] Z.-H. Li and T. Shiga, *Measure–valued branching diffusions: immigrations, excursions and limit theorems*, J. Math. Kyoto Univ. 35 (1995), 233–274.

[T] A. Talarczyk, *Self-intersection local time of order k for Gaussian processes in* $\mathcal{S}'(\mathbb{R}^d)$, (in preparation).

[Ta] S. J. Taylor, *Multiple points for the sample paths of the symmetric stable process*, Z. Wahrsch. Verw. Geb. 5 (1966), 247–264.

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