# Time–Localization of Random Distributions on Wiener Space II: Convergence, Fractional Brownian Density Processes *

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**Abstract.** For a random element $\mathcal{X}$ of a nuclear space of distributions on Wiener space $C([0,1],\mathbb{R}^d)$, the localization problem consists in "projecting" $\mathcal{X}$ at each time $t \in [0,1]$ in order to define an $\mathcal{S}'(\mathbb{R}^d)$-valued process $X = \{X(t), t \in [0,1]\}$, called the time-localization of $\mathcal{X}$. The convergence problem consists in deriving weak convergence of time-localization processes (in $C([0,1],\mathcal{S}'(\mathbb{R}^d))$ in this paper) from weak convergence of the corresponding random distributions on $C([0,1],\mathbb{R}^d)$. Partial steps towards the solution of this problem were carried out in [BG1, BGN], the tightness having remained unsolved. In this paper we complete the solution of the convergence problem via an extension of the time-localization procedure. As an example, a fluctuation limit of a system of fractional Brownian motions yields a new class of $\mathcal{S}'(\mathbb{R}^d)$-valued Gaussian processes, the "fractional Brownian density processes".

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## 1. Introduction

We begin by describing the problem treated in this paper for a reader acquainted with the subject. Further down we give some background for the benefit of a non-acquainted reader and we introduce the fractional Brownian density processes.

A nuclear space of distributions on the space of continuous functions $C \equiv C([0,1], \mathbb{R}^d)$ was constructed in [GN] for the purpose of providing an appropriate state space for trajectorial fluctuation limits of particle systems (see the examples and the references in [GN], as well as the motivations and a detailed formulation of the problem). Let us denote this space of distributions by $\mathcal{S}(C)'$. The question of obtaining temporal results for the convergence of the fluctuations from the trajectorial results was studied in [BG1] and [BGN]. This involves the time-localization problem and the convergence problem. The time-localization of a random element $\mathcal{X}$ of $\mathcal{S}(C)'$ consists in "projecting" $\mathcal{X}$ at each $t \in [0,1]$ in order to define a process $X = \{X(t), t \in [0,1]\}$ with values in $\mathcal{S}' \equiv \mathcal{S}'(\mathbb{R}^d)$ (the space of tempered distributions on $\mathbb{R}^d$, which is the dual of the spaces $\mathcal{S} \equiv \mathcal{S}(\mathbb{R}^d)$ of $C^\infty$ rapidly decreasing functions). The process $X$ is called the *time-localization* of $\mathcal{X}$. In the convergence problem one has a sequence $(\mathcal{X}_n)_n$ of random elements of $\mathcal{S}(C)'$ which converge weakly and, assuming that they can be time-localized, one wishes to know if the corresponding time-localization processes $(X_n)_n$ also converge weakly. This is not a straightforward matter because, assuming that the processes $X_n$ have trajectories in the space of continuous functions $C([0,1], \mathcal{S}')$, there is no direct relationship between the topologies of $\mathcal{S}(C)'$ and $C([0,1], \mathcal{S}')$ which would allow to conclude by means of a continuous mapping theorem. The time-localization already poses technical problems because the projection procedure involves functions on $C$ that do not belong to the space of test functions $\mathcal{S}(C)$ which is paired with $\mathcal{S}(C)'$. Hence some additional technical conditions must be imposed in order to pass from the trajectorial analysis to the temporal analysis. This passage has been partially achieved in [BG1] and [BGN], the main question left open being the tightness of the time-localizations. In this paper we refine the time-localization method developed in [BGN] in a way which in some important cases solves the question of convergence of the time-localizations, including the tightness. The idea of the time-localization was introduced in [G2] in an intuitive way.

This scheme for proving convergence of temporal fluctuations, i.e., first "going up" to the trajectorial approach for the proof of convergence of fluctuations and then "coming back down" by means of time-localization, is apparently awkward. However, it is useful because it turns out to be easier to apply for some particle systems than proving convergence in $C([0,1], \mathcal{S}')$ directly. Moreover, the Markov property or associated martingales that the systems may have, play no role in our approach. Temporal fluctuation limits have been obtained for many particle systems by several authors, starting with the papers of Martin Löf [ML] and Holley and Stroock [HS]. The present method is useful for a class of systems which includes those in [ML, I1], and also for an analogous system of fractional Brownian motions, which yields a new $\mathcal{S}'(\mathbb{R}^d)$-valued Gaussian process described at the end of the Introduction.

In order to fix ideas let us consider a Poisson system of particles in $\mathbb{R}^d$. For each $x \in \mathbb{R}^d$, let $\xi^x = \{\xi^x(t), t \in [0,1]\}$ denote a continuous process starting from $x$. The processes $\xi^x$ can be for example trajectories of a given diffusion, or translations (by $x$) of a given process. Let $\Pi$ be a Poisson random measure on $\mathbb{R}^d$ with ($\sigma$-finite) intensity measure $\mu$. The system is defined by letting the process $\xi^x$ evolve from each point $x$ of $\Pi$, these processes being independent. We rescale by taking the intensity of the Poisson measure to be $n\mu, n > 0$, and we denote the Poisson measure by $\Pi_n$. Let

$$N_n(t) = \sum_{i=1}^{\infty} \delta_{\xi^{x_i}(t)}, \quad t \in [0,1], \tag{1.1}$$

where $\{x_i\}_i$ are the points of $\Pi_n$. Thus $N_n(t)$ is the empirical measure of the system at time $t$. Consider the fluctuation

$$X_n(t) = \frac{1}{\sqrt{n}}(N_n(t) - EN_n(t)), \quad t \in [0,1], \tag{1.2}$$

which defines a signed measure valued process. The temporal fluctuation limit result for the Poisson system is the following assertion, which holds under some additional assumptions on the processes $\xi^x$.

1.1. ASSERTION. *$X_n$ converges weakly in $C([0,1], \mathcal{S}')$ as $n \to \infty$ to the centered Gaussian process $X = \{X(t), t \in [0,1]\}$ with covariance functional*

$$Cov(\langle X(s), \varphi\rangle, \langle X(t), \psi\rangle) = \int_{\mathbb{R}^d} E\varphi(\xi^x(s))\psi(\xi^x(t))\mu(dx), \quad \varphi, \psi \in \mathcal{S}. \tag{1.3}$$

Our objective is to prove Assertion 1.1 by means of the trajectorial and time-localization approach.

For the Poisson system, let us now describe the approach based on trajectorial convergence of the fluctuations and time-localization. Let

$$\mathcal{N}_n = \sum_{i=1}^{\infty} \delta_{\xi^{x_i}}, \tag{1.4}$$

where $\{x^i\}_i$ are the points of $\Pi_n$ as above, but now $\delta_\omega$ is the Dirac measure at $\omega \in C$. Hence $\mathcal{N}_n$ is a random point measure on $C$, which is the trajectorial empirical measure of the system. Consider the fluctuation

$$\mathfrak{X}_n = \frac{1}{\sqrt{n}}(\mathcal{N}_n - E\mathcal{N}_n), \tag{1.5}$$

which is a random signed measure on $C$. The trajectorial fluctuation limit of the Poisson system is given in the following theorem.

1.2. THEOREM. *$\mathcal{X}_n$ converges weakly in $\mathcal{S}(C)'$ as $n \to \infty$ to a centered Gaussian random element $\mathcal{X}$ with covariance functional*

$$Cov(\langle \mathcal{X}, F\rangle, \langle \mathcal{X}, G\rangle) = \int_C FG d\mathbb{M}, \quad F, G \in \mathcal{S}(C), \tag{1.6}$$

*where $\mathbb{M}$ is the measure on $C$ defined by*

$$d\mathbb{M} = dM^x \mu(dx) \tag{1.7}$$

*and $M^x$ is the distribution measure on $C$ of the process $\xi^x$, provided that $\int_C F^2 d\mathbb{M} < \infty$ for all $F \in \mathcal{S}(C)$.*

($\mathcal{X}$ can be described as the "white noise" on $C$ based on $\mathbb{M}$).

Trajectorial fluctuation limits of particle systems (which are stochastic models of interest by themselves) have been obtained in [ML, G1, T, S], but they were not expressed as weak convergence of random elements of some space because an appropriate state space such as $\mathcal{S}(C)'$ was not available. The nuclearity of $\mathcal{S}(C)'$ is important in order to have weak convergence. Note that the temporal fluctuation limit in Assertion 1.1 depends only on the two-dimensional distributions of $\xi^x$, whereas the trajectorial fluctuation limit in Theorem 1.2 depends on the whole distribution of $\xi^x$. Hence the trajectorial result is more intimately connected with the motion process than the temporal one. A direct proof of Assertion 1.1 requires showing weak convergence of finite-dimensional distributions or some other approach that identifies the limit (which may be difficult if the processes $\xi^x$ do not have the Markov property) and tightness. On the other hand, the proof of Theorem 1.2 is quite simple, as it is just an ordinary central limit theorem for $\mathcal{S}(C)'$-valued random variables. Our aim is to derive Assertion 1.1 from Theorem 1.2. As we have said above, this is not automatic because Assertion 1.1 involves the topology of $C$, which has no direct relation with the space $\mathcal{S}(C)'$.

First of all, we need the "temporal" versions of the random elements $\mathcal{X}_n$ and $\mathcal{X}$ of $\mathcal{S}(C)'$. They are obtained by the time-localization procedure. The temporal version of the random element $\mathcal{X}_n$ defined by (1.5) is the process $X_n = \{X_n(t), t \in [0,1]\}$ given by (1.2), and the temporal version of $\mathcal{X}$ in Theorem 1.2 is the process $X = \{X(t), t \in [0,1]\}$ described in Assertion 1.1.

In a general setting, the time-localization at time $t \in [0,1]$ of a random element $\mathcal{Y}$ of $\mathcal{S}(C)'$ is intuitively defined to be the random element $Y(t)$ of $\mathcal{S}'$ such that

$$\langle Y(t), \varphi\rangle = \langle \mathcal{Y}, F_{\varphi,t}\rangle, \quad \varphi \in \mathcal{S}, \tag{1.8}$$

where

$$F_{\varphi,t}(\omega) = \varphi(\omega(t)), \quad \omega \in C. \tag{1.9}$$

However, the function $F_{\varphi,t}$ defined by (1.9) does not belong to the space of test functions $\mathcal{S}(C)$, and therefore formula (1.8) does not make sense. The procedure of time-localization consists in formulating conditions on random elements $\mathcal{Y}$ of $\mathcal{S}(C)'$ under which it is possible to give a rigorous meaning to formula (1.8), and showing that $\mathcal{Y}$ satisfies such conditions. It turns out that this can be done in some cases by means of an "admissible measure" associated to $\mathcal{Y}$ (Definition 2.1). The time-localization result is given in Theorem 2.7. For the random element $\mathcal{X}$ of $\mathcal{S}(C)'$ in Theorem 1.2 an admissible measure is precisely $\mathbb{M}$ given by (1.7).

With a slightly stronger condition on an admissible measure associated to $\mathcal{Y}$ it is possible to show the time-localization process $Y = \{Y(t), t \in [0,1]\}$ has a version in $C([0,1], \mathcal{S}')$ (see Proposition 2.8).

For a sequence of random elements $\{\mathcal{Y}_n\}_n$ of $\mathcal{S}(C)'$ which convergence weakly to $\mathcal{Y}$, in order to obtain the convergence of the time-localization processes the idea is to have a common admissible measure associated to all the $\mathcal{Y}_n$ and $\mathcal{Y}$, which yields the time-localizations for all of them, and such that with some additional requirement provides also the weak-convergence of finite-dimensional distributions and the tightness of the time-localization processes. This is the subject of Proposition 3.1, Corollary 3.2 and Theorem 3.4.

To sum up, in this scheme proving weak convergence of random elements of $C([0,1], \mathcal{S}')$ consists in showing first that the trajectorial versions converge weakly in $\mathcal{S}(C)'$, and then showing that a common admissible measure for the approximations and the limit satisfies the required conditions. Note that the technical questions regarding to the topology of $C([0,1], \mathcal{S}')$, in particular those related with the tightness, have been trasformed into conditions on the associated admissible measure.

We now recall briefly the space of distributions on $C$ which we have denoted above by $\mathcal{S}(C)'$. Let $W^x$ denote the Wiener measure on $C$ supported on the functions that take the value $x \in \mathbb{R}^d$ at $t = 0$, and let $\lambda$ designate the Lebesgue measure on $\mathbb{R}^d$. Define the $\sigma$-finite measure $\mathbb{W}$ on $C$ by

$$d\mathbb{W} = dW^x \lambda(dx).$$

There exists a space of test functions $\mathcal{S}(\mathbb{R}^d, \mathcal{H})$ on $C$ such that

$$\mathcal{S}(\mathbb{R}^d, \mathcal{H}) \subset L^2(C, \mathbb{W}) \subset \mathcal{S}(\mathbb{R}^d, \mathcal{H})'$$

is a nuclear Gelfand triple, where $\mathcal{H}$ is a space of test functions on $C_0 = \{\omega \in C : \omega(0) = 0\}$ such that

$$\mathcal{H} \subset L^2(C_0, W^0) \subset \mathcal{H}'$$

is also a nuclear Gelfand triple. $\mathcal{S}(\mathbb{R}^d, \mathcal{H})$ is a space of smooth $\mathcal{H}$-valued functions on $\mathbb{R}^d$, and since $\mathcal{S}(\mathbb{R}^d, \mathcal{H}) \cong \mathcal{S} \widehat{\otimes} \mathcal{H}$, then $\mathcal{S}(\mathbb{R}^d, \mathcal{H})$ can indeed be regarded as a space of functions on $C$. The spaces

$\mathcal{S}(\mathbb{R}^d, \mathcal{H})$ and $\mathcal{S}(\mathbb{R}^d, \mathcal{H})'$ are what we have denoted by $\mathcal{S}(C)$ and $\mathcal{S}(C)'$, respectively. We will continue to use this shorter notation. See [GN] for the construction of $\mathcal{S}(C)'$.

Let us now go back to the Poisson systems discussed above. As examples of processes $\xi^x$ we will take the diffusions considered in [BG1], and we will also take fractional Brownian motions (fBm). FBm's were introduced by Kolmogorov [K]. The paper of Mandelbrot and Van Ness [MVN] generated considerable interest in fBm and its applications to various fields (e.g.[Ta, M], we omit a long list of recent references). It is worthwhile to note that fBm's appear also as fluctuation limits of occupation times of Poisson systems of Brownian motions [DW] (Theorem 0.4). A short proof of existence of fractional Brownian fields is given in [BG3].

We recall that the fBm (in $\mathbb{R}$) with Hurst parameter $h \in (0,1)$ is a continuous centered Gaussian process $\xi$ with covariance

$$Cov(\xi(s), \xi(t)) = \frac{1}{2}[t^{2h} + s^{2h} - |t-s|^{2h}], \quad t, s \geq 0.$$

The fBm in $\mathbb{R}^d$ is the Gaussian process whose $d$ components are independent one-dimensional fBm's. This process is self-similar, but it is Markovian only for $h = \frac{1}{2}$, which corresponds to the ordinary Brownian motion. We denote by $\xi^x$ the $d$-dimensional fBm started from $x \in \mathbb{R}^d$ (i.e. $\xi^x$ is $\xi^0$ translated by $x$). For this case the above mentioned convergence and time-localization procedure can be carried out to prove Assertion 1.1, and the covariance of the limit process $X$ is given by

$$\begin{aligned}
&Cov(\langle X(s), \varphi\rangle, \langle X(t), \psi\rangle) \\
&= (\pi[(t-s)^{2h}(2t^{2h} + 2s^{2h} - (t-s)^{2h}) - (t^{2h} - s^{2h})^2])^{-d} \\
&\quad \cdot \int_{\mathbb{R}^{3d}} \varphi(y_1, \ldots, y_d)\psi(z_1, \ldots, z_d) \\
&\quad \cdot exp\Big\{-2[(t-s)^{2h}(2t^{2h} + 2s^{2h} - (t-s)^{2h}) - (t^{2h} - s^{2h})^2]^{-1} \\
&\quad \cdot\Big(t^{2h}\sum_{i=1}^d (y_i - x_i)^2 + s^{2h}\sum_{i=1}^d (z_i - x_i)^2 - (t^{2h} + s^{2h} - (t-s)^{2h})\sum_{i=1}^d (y_i - x_i)(z_i - x_i)\Big)\Big\} \\
&\quad dy_1 \ldots dy_d dz_1 \ldots dz_d \mu(dx_1, \ldots, dx_d), \qquad s < t, \quad \varphi, \psi \in \mathcal{S}, \qquad (1.10)
\end{aligned}$$

$$\begin{aligned}
&Cov(\langle X(t), \varphi\rangle, \langle X(t), \psi\rangle) \\
&= \frac{1}{(2\pi t^{2h})^{d/2}} \int_{\mathbb{R}^{2d}} \varphi(y_1, \ldots, y_d)\psi(y_1, \ldots, y_d) exp\Big\{-\frac{1}{2t^{2h}}\sum_{i=1}^d (y_i - x_i)^2\Big\} \\
&\qquad dy_1 \ldots dy_d \mu(dx_1, \ldots, dx_d), \qquad \varphi, \psi \in \mathcal{S}. \qquad (1.11)
\end{aligned}$$

(see Proposition 4.6).

By analogy with the Brownian case, we call $X$ the *fractional Brownian density process with parameter*

$h$. Note that it is self-similar, i.e., the process $\{X_{ct}, t \in [0,1]\}$ has the same distribution as the process $\{c^h X_t, t \in [0,1]\}$ for each constant $c > 0$, but it is Markovian only for $h = \frac{1}{2}$ (for $h \neq \frac{1}{2}$ the covariance does not satisfy the condition which characterizes the Markov property for Gaussian $\mathcal{S}'$-processes [F]: for $r \leq s < t$, $Cov(\langle X_r, \varphi\rangle, \langle X_t, \psi\rangle) = Cov(\langle X_r, \varphi\rangle, \langle X_{s,t}, \psi\rangle)$, where $X_{s,t}$ is an $\mathcal{S}'$-valued random variable such that $\langle X_{s,t}, \psi\rangle$ belongs to the $L^2$-closure of $\{\langle X_s, \varphi\rangle, \varphi \in \mathcal{S}\}$). For $h = \frac{1}{2}$ the covariance (1.10), (1.11) reduces to

$$\begin{aligned} &Cov(\langle X(s), \varphi\rangle, \langle X(t), \psi\rangle) \\ = \quad & \frac{1}{(2\pi s)^{d/2}(2\pi(t-s))^{d/2}} \int_{\mathbb{R}^{3d}} \varphi(y_1, \ldots, y_d)\psi(z_1, \ldots, z_d) \\ & \cdot exp\Big\{-\frac{1}{2s}\sum_{i=1}^{d}(y_i - x_i)^2 - \frac{1}{2(t-s)}\sum_{i=1}^{d}(z_i - y_i)^2\Big\} dy \ldots dy_d dz_1 \ldots dz_d \mu(dx_1, \ldots, dx_d), \\ & \qquad\qquad s \leq t, \end{aligned}$$

and in the case $\mu = \lambda$, $X$ is the well-known Brownian density process [A, BG2].

1.3. REMARK. (1.11) can be obtained from (1.10) by taking limit as $s \to t$.

In addition to the Poisson systems discussed above, we will also consider systems of the type studied by Itô [I1]. Although these systems are quite different from the Poisson model, it turns out that the same scheme works for them with very minor changes. Moreover, the admissible measure is the same one for both models.

The outline of the paper is as follows. In Section 2 we carry out the new time-localization. Section 3 deals with the convergence of the time-localization processes. Section 4 is devoted to the examples.

## 2. $L^p$-localization

In this section we show that the space $L^2$ employed in [BG1, BGN] for the time-localization plays in fact no special role and can be replaced by any $L^p, p \geq 1$. We will use this extension later on.

Generic constants will be denoted by $K, K_1, K_2, \ldots$, with dependencies on parameters in parenthesis.

First of all we recall the definition of an admissible measure.

2.1. DEFINITION. We say that a measure $\mathbb{M}$ on $C$ is admissible if it has a desintegration of the form

$$d\mathbb{M} = dM^x \mu(dx), \tag{2.1}$$

where

(i) $\mu$ is a $\sigma$-finite measure on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d}(1+|x|^2)^{-k}\mu(dx)<\infty \tag{2.2}$$

for some $k\geq 0$, and

(ii) $M^x$ is a probability measure on $\{\omega\in C:\omega(0)=x\}, x\in\mathbb{R}^d$, such that the measure $M_0^x(\cdot):=M^x(x+\cdot)$ satisfies

$$\int_{\mathbb{R}^d}\left(\int_{C_0}||\omega||_\infty^n M_0^x(d\omega)\right)^2(1+|x|^2)^{-k}\mu(dx)\leq J^n n!, \tag{2.3}$$

for all $n=1,2,\ldots$, and for some constant $J$, where $k$ is the same as in (2.2) and $||\cdot||_\infty$ stands for the sup norm.

We also define the measure

$$\widetilde{\mathbb{M}}(d\omega)=(1+|\omega(0)|^2)^k\mathbb{M}(d\omega), \tag{2.4}$$

where $k$ is as in (2.2).

Note that if $M_0^x$ does not depend on $x$, i.e., $M_0^x\equiv M_0$, then, given (2.1), condition (2.3) is equivalent to

$$\int_{C_0}||\omega||_\infty^n M_0(d\omega)\leq J_1\sqrt{n!} \tag{2.5}$$

for all $n=1,2,\ldots$ and for some constant $J_1$.

We shall need be the following immediate generalization of Lemma 2.4 of [BGN].

2.2. LEMMA. *Let* $\mathbb{M}$ *be an admissible measure,* $t\in[0,1]$ *and* $p\geq 1$. *Then* $F_{\varphi,t}\in L^p(C,\widetilde{\mathbb{M}})$ *for each* $\varphi\in\mathcal{S}$, *and the mapping* $\varphi\mapsto F_{\varphi,t}$ *is continuous from* $\mathcal{S}$ *into* $L^p(C,\widetilde{\mathbb{M}})$.

*Proof.* We argue as in the proof of Lemma 2.4 in [BGN] and we obtain

$$\int_C|F_{\varphi,t}|^p d\widetilde{\mathbb{M}}\leq K(k)\sup_{y\in\mathbb{R}^d}(1+|y|^{4k})|\varphi(y)|^p,$$

hence the assertions follow since $\sup_y(1+|y|^{4k})^{\frac{1}{p}}|\varphi(y)|$ is a continuous seminorm in $\mathcal{S}$. □

2.3. REMARK. By the definition of an admissible measure it is clear that the measure $\widetilde{\mathbb{M}}$ defined by (2.4) can be replaced here by any measure of the form $(1+|\omega(0)|^2)^q\mathbb{M}(d\omega)$, $q\geq 1$, and, moreover, we have

$$\sup_{t\in[0,1]}\int_C|F_{\varphi,t}|^p(1+|\omega(0)|^2)^q\mathbb{M}(d\omega)<\infty.$$

The following theorem is crucial for the time-localization procedure.

2.4. THEOREM. *Let* $\mathbb{M}$ *be an admissible measure and* $p\geq 1$. *Then* *(a)* $\mathcal{S}(C) \subset L^p(C, \widetilde{\mathbb{M}})$ *and the embedding is continuous.*

*(b) For each* $t \in [0,1]$ *and* $\varphi \in \mathcal{S}$*, the functional* $F_{\varphi,t}$ *belongs to* $\overline{\mathcal{S}}^p_{\mathbb{M}} :=$ *the closure of* $\mathcal{S}(C)$ *in the topology of* $L^p(C, \widetilde{\mathbb{M}})$.

*Proof.* (a) We know that $\mathbb{M} \in \mathcal{S}(C)'$ [BGN] (Theorem 2.5). It follows from the proof of this fact that the assertion is true for $p = 1$. Then it is also true for any natural $p$ since we can replace $F \in \mathcal{S}(C)$ by $F^p$ because we know that $F^p(x, \cdot) \in \mathcal{H}$. Now it is easily seen that it is true for any $p \geq 1$ as well.

(b) We present only a sketch of the proof since the idea is the same as in the proof of Theorem 2.5(c) in [BGN].

Fix any $\psi \in C_0^\infty(\mathbb{R}^d)$ (compact support) and $t \in (0,1]$. It suffices to prove that

$$F_{\psi,0} F_{\varphi,t} \in \overline{\mathcal{S}}^p_{\mathbb{M}}. \tag{2.6}$$

Let

$$\chi(B) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_B(x+y) M_0^x \circ \pi_t^{-1}(dy) |\psi(x)|^p (1+|x|^2)^k \mu(dx), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where $\pi_t(\omega) = \omega(t)$. For each $\gamma \in C^\infty(\mathbb{R}^d)$ we have

$$\int_C |F_{\psi,0} F_{\gamma,t}|^p d\widetilde{\mathbb{M}} = \int_{\mathbb{R}^d} |\gamma|^p \chi(dx). \tag{2.7}$$

Using the admissibility property (2.2) of $\mathbb{M}$ it can be proved that

$$m_n = \int_{\mathbb{R}^d} |x|^n \chi(dx) \leq K^n \sqrt{n!}, \quad n = 1, 2, \ldots, \tag{2.8}$$

hence $\gamma \in L^p(\mathbb{R}^d, \chi)$ for each polynomial $\gamma$.

Let $q = \frac{p}{p-1}$. For any $f \in L^q(\mathbb{R}^d, \chi)$ we define $d\chi^{(f)} = |f| d\chi$ and $m_n^{(f)} = \int |x|^n \chi^{(f)}(dx)$. The Hölder inequality, (2.8) and the Stirling formula imply that $\frac{1}{n}(m_n^{(f)})^{1/n} \to 0$ as $n \to \infty$, hence the measure $\chi^{(f)}$ is uniquely determined by its moments. From this we infer that if $f \in L^q(\mathbb{R}^d, \chi)$ is such that $\int \gamma(x) f(x) \chi(dx) = 0$ for each polynomial $\gamma$, then $f = 0$ $\chi$-a.e. As a consequence we obtain that the polynomials are dense in $L^p(\mathbb{R}^d, \chi)$ (see, e.g., [Tr], page 186, Corollary 1). By (2.7) it is now clear that it suffices to prove (2.6) assuming that $\varphi$ is a polynomial, and this fact can be shown in the same way as in Step 4 of the proof of Theorem 2.5 of [BGN]. One should only replace the squares by the $p$-th powers in the appropriate places. □

The class of random elements of $\mathcal{S}(C)'$ which can be time-localized is described in the next definition.

2.5. DEFINITION. A random element $\mathcal{X}$ of $\mathcal{S}(C)'$ is said to be admissible if there exist an admissible measure $\mathbb{M}$ and $p \geq 1$ such that the mapping $F \mapsto \langle \mathcal{X}, F \rangle$ is a continuous random linear functional (in the sense of [I2]) on $\mathcal{S}(C)$ with the topology induced by $L^p(C, \widetilde{\mathbb{M}})$. To stress the dependence on $\mathbb{M}$ and $p$ we will sometimes say that $\mathfrak{X}$ is $\mathbb{M}$-$p$-admissible.

This definition makes sense by Theorem 2.4(a). If $\mathfrak{X}$ is $\mathbb{M}$-$p$-admissible, its (unique) extension on $\overline{\mathcal{S}}^p_{\mathbb{M}}$ will be denoted by $\mathfrak{X}_{\mathbb{M}_p}(\cdot)$. The next proposition implies that this extension in fact does not depend on $\mathbb{M}$ and $p$ on the functions of the form $F_{\varphi,t}, \varphi \in \mathcal{S}, t \in [0,1]$ (which belong to $\overline{\mathcal{S}}^p_{\mathbb{M}}$ by Theorem 2.4(b)).

2.6. PROPOSITION. *Let $\mathbb{M}, \mathbb{M}'$ be admissible measures and $p, p'$ be arbitrary numbers $\geq 1$. Then for each $\varphi \in \mathcal{S}, t \in [0,1]$, there exists a sequence $(F_n)_n$ of elements of $\mathcal{S}(C)$ such that $F_n \to F_{\varphi,t}$ in $L^p(C, \widetilde{\mathbb{M}})$ and in $L^{p'}(C, \widetilde{\mathbb{M}}')$ simultaneously.*

*Proof.* The argument is a generalization of the proof of Corollary 2.6 of [BGN].

It is clear from the proof of Theorem 2.4 that it suffices to show that, given any $\psi \in C_0^\infty(\mathbb{R}^d)$, there exists a sequence $(v_n)_n$ of polynomials such that

$$F_{\psi,0}F_{v_n,t} \to F_{\psi,0}F_{\varphi,t} \text{ as } n \to \infty,$$

in $L^p(C, \widetilde{\mathbb{M}})$ and in $L^{p'}(C, \widetilde{\mathbb{M}}')$ simultaneously.

Let $\chi_p, \chi'_{p'}$ be the measures defined in the proof of Theorem 2.4, corresponding to $\mathbb{M}, p$ and $\mathbb{M}', p'$, respectively. (2.8) implies that $\int |x|^n (\chi_p + \chi'_{p'})(dx) \leq K_2^n \sqrt{n!}, n = 1, 2, \ldots$, hence we infer that the polynomials are dense in both $L^p(\mathbb{R}^d, \chi_p + \chi'_{p'})$ and $L^{p'}(\mathbb{R}^d, \chi_p + \chi'_{p'})$. Assume $p \leq p'$. Let $(v_n)_n$ be polynomials converging to $\varphi$ in $L^{p'}(\mathbb{R}^d, \chi_p + \chi'_{p'})$; then the convergence also holds in $L^p(\mathbb{R}^d, \chi_p + \chi'_{p'})$ since $\chi'_p + \chi'_{p'}$ is a finite measure. Hence we obtain $\lim_{n\to\infty} \int |\varphi - v_n|^p d\chi_p = \lim_{n\to\infty} \int |\varphi - v_n|^{p'} d\chi'_{p'} = 0$. To finish the proof it suffices to recall that

$$\int |F_{\psi,0}|^p |F_{\varphi,t} - F_{v_n,t}|^p d\widetilde{\mathbb{M}} = \int |\varphi - v_n|^p d\chi_p,$$

and the same is true for $\mathbb{M}', p'$. □

We are now ready to state the main result of this section.

2.7. THEOREM. *Let $\mathfrak{X}$ be an admissible random element of $\mathcal{S}(C)'$. Then for each $t \in [0,1]$ there exists a unique $\mathcal{S}'$-valued random variable $X(t)$ such that for each $\mathbb{M}, p$, if $\mathfrak{X}$ is $\mathbb{M}$-$p$-admissible, then*

$$\langle X(t), \varphi \rangle = \mathfrak{X}_{\mathbb{M}_p}(F_{\varphi,t}) \quad a.s.$$

*for all $\varphi \in \mathcal{S}$. The process $X = \{X(t), t \in [0,1]\}$ is called the time-localization process of $\mathfrak{X}$.*

*Proof.* If $\mathfrak{X}$ is $\mathbb{M}$-$p$-admissible, then Lemma 2.2 and Theorem 2.4 imply that the mapping $\varphi \mapsto \mathfrak{X}_{\mathbb{M}_p}(F_{\varphi,t})$ is a continuous linear random functional on $\mathcal{S}$. The existence of $X(t)$ now follows from the regularization theorem since $\mathcal{S}$ is nuclear [I2]. Finally, Proposition 2.6 implies that $X(t)$ does not depend on the particular $\mathbb{M}, p$. $\square$

We close this section with a generalization of Proposition 2.14 of [BGN]. This generalization is two-fold: firstly, it concerns the present, more general, setup, and secondly, no restriction is imposed on the Hölder exponent $\delta$.

2.8. PROPOSITION. *Let $\mathbb{M}$ be an admissible measure given by (2.1) such that*

$$\int_{\mathbb{R}^d}\int_{C_0} |\omega(t)-\omega(s)|^q M_0^x(d\omega)(1+|x|^2)^{-k}\mu(dx) \le L|t-s|^\delta \tag{2.9}$$

*for all $s,t \in [0,1]$ and for some constants $q \ge 1, \delta > 0, L > 0$. Then for each $p \ge 1$ and for each $\mathbb{M}$-$p$-admissible Gaussian random element $\mathcal{X}$ of $\mathcal{S}(C)'$, the time-localization process $X$ of $\mathcal{X}$ has a continuous version.*

*Proof.* As the measure $M_0^x(d\omega)(1+|x|^2)^{-k}\mu(dx)$ is finite, it suffices to prove the proposition for $q = 1$.

Fix an arbitrary $\varphi \in \mathcal{S}$ and denote $F_{s,t}(\omega) = \varphi(\omega(t)) - \varphi(\omega(s))$ for $\omega \in C$. We have

$$\begin{aligned}\int_C |F_{s,t}|^p d\widetilde{\mathbb{M}} &= \int_{\mathbb{R}^d}\int_{C_0} (|F_{s,t}(x+\omega)|^{\frac{1}{2}}(1+|x|^2)^{-\frac{k}{2}})(|F_{s,t}(x+\omega)|^{p-\frac{1}{2}}(1+|x|^2)^{\frac{3}{2}k} \\ &\qquad M_0^x(d\omega)\mu(dx) \\ &\le \left(\int_{\mathbb{R}^d}\int_{C_0} |F_{s,t}(x+\omega)(1+|x|^2)^{-k}M_0^x(d\omega)\mu(dx)\right)^{\frac{1}{2}} \\ &\quad \cdot\left(\int_{\mathbb{R}^d}\int_{C_0} |F_{s,t}(x+\omega)|^{2p-1}(1+|x|^2)^{3k}M_0^x(d\omega)\mu(dx)\right)^{\frac{1}{2}}.\end{aligned}$$

By Remark 2.3, the second factor is bounded by a constant $K(\varphi,p,k)$. On the other hand, the function $\varphi$ satisfies the Lipschitz condition, therefore by assumption (2.9) we obtain

$$\int_C |F_{s,t}|^p d\mathbb{M} \le K(\varphi,p,k,L)|t-s|^{\frac{\delta}{2}}.$$

As $\mathcal{X}$ is $\mathbb{M}$-$p$-admissible and Gaussian, we have

$$E\mathcal{X}_{\mathbb{M}_p}(F)^2 \le K\int_C |F|^p d\mathbb{M} \quad \text{for all } F \in \overline{\mathcal{S}}^p_{\mathbb{M}}.$$

Joining these two estimations we obtain

$$E(\langle X(t),\varphi\rangle - \langle X(s),\varphi\rangle)^2 = E\mathcal{X}_{\mathbb{M}_p}(F_{\varphi,t}-F_{\varphi,s})^2 \le K_1|t-s|^{\frac{\delta}{2}}.$$

We can assume $E\mathcal{X} = 0$ (see [BGN]). Then

$$E(\langle X(t),\varphi\rangle - \langle X(s),\varphi\rangle)^{2n} \le K_1K(n)|t-s|^{\frac{n\delta}{2}},$$

since $\langle X(t), \varphi\rangle - \langle X(s), \varphi\rangle$ is a centered Gaussian random variable.

It suffices to fix $n$ such that $\frac{n\delta}{2} > 1$, and then continuity of $X$ follows from Kolmogorov's criterion and Mitoma's theorem [Mi1]. □

## 3. Convergence of time-localization processes

In what follows $\underset{f}{\Rightarrow}$ denotes the weak convergence of finite-dimensional distributions and $\Rightarrow$ denotes weak (functional) convergence.

We start with the convergence of the finite-dimensional distributions. Some results on this subject had been obtained for a very special case in [BG1].

3.1. PROPOSITION. *Let $\mathcal{X}_n$ be admissible random elements of $\mathcal{S}(C)', n = 1, 2, \ldots,$ such that $\mathcal{X}_n \Rightarrow \mathcal{X}$, where $\mathcal{X}$ is an $\mathbb{M}$-$p$-admissible random element of $\mathcal{S}(C)'$ for some admissible measure $\mathbb{M}$ and some $p \geq 1$. Denote*

$$\mathcal{L} = span\{\mathcal{S}(C) \cup \{F_{\varphi,t} : \varphi \in \mathcal{S}, t \in [0, 1]\}\},$$

*and assume that $F \mapsto \limsup_{n\to\infty} E(|\mathcal{X}_n(F)| \wedge 1)$ as a function on $\mathcal{L}$ is continuous at $F = 0$ in the topology of $L^p(C, \widetilde{\mathbb{M}})$. Then $X_n \underset{f}{\Rightarrow} X$, where $X_n, X$ denote the time-localization processes of $\mathcal{X}_n, \mathcal{X}$, respectively.*

*Proof.* First observe that by Theorem 2.4 and Definition 2.5, $\mathcal{X}_n(F)$ is well (and uniquely) defined for any $F \in \mathcal{L}$. Fix arbitrary $t_1, \ldots, t_j \in [0, 1], \varphi_1, \ldots, \varphi_j \in \mathcal{S}$ and denote $F = F_{\varphi_1,t_1} + \ldots + F_{\varphi_j,t_j}$. We want to prove that

$$\mathcal{X}_n(F) \Rightarrow \mathcal{X}(F). \tag{3.1}$$

Fix a sequence $F_m \in \mathcal{S}(C), m = 1, 2, \ldots,$ such that $F_m \to F$ in $L^p(C, \widetilde{\mathbb{M}})$ as $m \to \infty$; its existence is guaranteed by Theorem 2.4. By assumption, $\langle \mathcal{X}_n, F_m\rangle \Rightarrow \langle \mathcal{X}, F_m\rangle$ as $n \to \infty, m = 1, 2, \ldots,$ and also $\langle \mathcal{X}, F_m\rangle \Rightarrow \mathcal{X}(F)$ as $m \to \infty$ (the convergence is even in probability) since $\mathcal{X}$ is $\mathbb{M}$-$p$-admissible. On the other hand, for each $\varepsilon \in (0, 1)$ we have

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(|\langle \mathcal{X}_n, F_m\rangle - \mathcal{X}_n(F)| > \varepsilon) \leq \frac{1}{\varepsilon} \lim_{m\to\infty} \limsup_{n\to\infty} E(|\mathcal{X}_n(F_m - F)| \wedge 1) = 0,$$

since $F_m - F \in \mathcal{L}$. Hence (3.1) follows by [B] (Theorem 4.2). □

Observe that in this proposition the $\mathcal{X}_n$'s are not necessarily associated with the same admissible measure. On the other hand, we are not able to get rid of the (fortunately rather weak) assumption on continuity of $\limsup_{n\to\infty} E(|\mathcal{X}_n(F)| \wedge 1)$ at zero.

3.2. COROLLARY. *Let $\mathbb{M}$ be an admissible measure and assume that random elements $\mathcal{X}_n$ of $\mathcal{S}(C)', n =$*

$1, 2, \ldots$, *satisfy the following property: there exist* $r > 0, p \geq 1, \beta > 0$ *and* $K > 0$ *such that*

$$\sup_n E|\langle \mathcal{X}_n, F\rangle|^r \leq K\left(\int_C |F|^p d\widetilde{\mathbb{M}}\right)^\beta \tag{3.2}$$

*for all* $F \in \mathcal{S}(C)$*. Assume moreover that* $\mathcal{X}_n \Rightarrow \mathcal{X}$*. Then* $\mathcal{X}$ *is* $\mathbb{M}$*-*$p$*-admissible and* $X_n \underset{f}{\Rightarrow} X$*, where* $X_n, X$ *are the time-localization processes of* $\mathcal{X}_n, \mathcal{X}$*, respectively.*

*Proof.* (3.2) obviously implies that each $\mathcal{X}_n$ is $\mathbb{M}$-$p$-admissible. For each $F \in \mathcal{S}(C)$ we have $\langle \mathcal{X}_n, F\rangle \Rightarrow \langle \mathcal{X}, F\rangle$, hence

$$E|\langle \mathcal{X}, F\rangle|^r \leq \liminf_{n\to\infty} E|\langle \mathcal{X}_n, F\rangle|^r \leq K\left(\int_C |F|^p d\mathbb{M}\right)^\beta$$

by [B], Theorem 5.3. Hence $\mathcal{X}$ is $\mathbb{M}$-$p$-admissible as well. It is now clear that the convergence $X_n \underset{f}{\Rightarrow} X$ follows from Proposition 3.1, since (3.2) extends to $F \in \overline{\mathcal{S}}^p_{\mathbb{M}} \supset \mathcal{L}$. □

We now pass to the problem of tightness of time-localization processes. First of all we show that in general weak convergence in $\mathcal{S}(C)'$ does not imply tightness in $C([0,1], \mathcal{S}')$ of the time-localization processes, even if these processes do have continuous paths.

3.3. EXAMPLE. Fix $\omega_n \in C_0([0,1], \mathbb{R})$ such that

$$supp(\omega_n) \subset \left(0, \frac{2}{n}\right), \quad \omega_n\left(\frac{1}{n}\right) = 1, \quad ||\omega_n||_\infty \leq 1, \quad n = 1, 2, \ldots,$$

and let $\omega = 0$. Put $\mathcal{X}_n = \delta_{\omega_n}, \mathcal{X} = \delta_\omega$. It is clear that $\mathcal{X}_n, \mathcal{X}$ are admissible (deterministic) elements of $\mathcal{S}(C)'$, and their time-localization (deterministic) processes are $X_n = \omega_n, X = 0$, respectively.

It is immediately seen that $X_n \not\Rightarrow X$ in $C([0,1], \mathcal{S}'(\mathbb{R}))$, since otherwise we would have $\varphi(\omega_n) = \langle X_n, \varphi\rangle \Rightarrow \langle X, \varphi\rangle = \varphi(0)$ in $C([0,1], \mathbb{R})$ for each $\varphi \in \mathcal{S}(\mathbb{R})$, i.e., $\sup_{t\in[0,1]} |\varphi(\omega_n(t)) - \varphi(0)| \to 0$, but this obviously does not hold, e.g., if $\varphi(1) = 1, \varphi(0) = 0$.

On the other hand, $\mathcal{X}_n \Rightarrow \mathcal{X}$ in $\mathcal{S}(C)'$. Indeed, this convergence is equivalent to

$$F(\omega_n) \to F(\omega) \quad \text{for each} \quad F \in \mathcal{S}(C). \tag{3.3}$$

Any such $F$ has a chaos expansion $F = \sum_{j=0}^\infty I_j(f_j)$, where the kernels $f_j$ are infinitely differentiable and vanish on the boundary of $[0,1]^j$ together with all their derivatives; in particular all their derivatives are bounded. Moreover, we know from [NZ, GN] that the following explicit formula holds:

$$\begin{aligned} &I_j(f_j)(\omega_n) \\ &= \sum_{i=0}^{[j/2]} (-1)^i \frac{j! 2^{-i}}{(j-2i)!i!} \int_{[0,1]^{j-2i}} \int_{[0,1]^i} \frac{\partial^{j-2i}}{\partial t_1 \ldots \partial t_{j-2i}} f_j(t_1, \ldots, t_{j-2i}, u_1, u_1, \ldots, u_i, u_i) \\ &\quad \cdot \omega_n(t_1) \ldots \omega_n(t_{j-2i}) du_1 \ldots du_i dt_1 \ldots dt_{j-2i}. \end{aligned}$$

Hence $I_j(f_j)(\omega_n) \to 0$ for all $j > 0$. As a consequence (3.3) holds since, obviously, $F(\omega) = I_0(f_0)$ and the series $\sum_j I_j(f_j)$ converges uniformly on bounded subsets of $C_0$ (see the proof of Theorem A of [GN]). □

The main result of this section is the following theorem.

3.4. THEOREM. *Let $\mathbb{M}$ be an admissible measure satisfying (2.9) with $\delta > 1$. Then for each sequence $(\mathcal{X}_n)_n$ of random elements of $\mathcal{S}(C)'$ such that*

*(i) for any $\varphi \in \mathcal{S}$ the real random variables $\{\langle \mathcal{X}_n, F_{\varphi,0}\rangle, n = 1, 2, \ldots\}$ are tight;*

*(ii) condition (3.2) is satisfied with some $p \geq q$, $\beta\delta > 1$;*

*the family of time-localization processes $\{X_n\}_n$ of $\{\mathcal{X}_n\}_n$ is tight in $C([0,1], \mathcal{S}')$.*

*Proof.* First observe that $F_{\varphi,0} \in \mathcal{S}(C)$ if $\varphi \in \mathcal{S}$, hence condition (i) makes sense (independently of (ii)).

Condition (ii) implies that each $\mathcal{X}_n$ is $\mathbb{M}$-$p$-admissible, so the localization process $X_n$ exists.

Fix an arbitrary $\varphi \in \mathcal{S}$. By Mitoma's theorem [Mi2] it suffices to prove that the family of real valued processes $\{\langle X_n, \varphi\rangle\}_n$ is tight in $C$.

The argument is similar to that in the proof of Proposition 2.8. Again, for $s, t \in [0,1]$ we denote $F_{s,t}(\omega) = \varphi(\omega(t)) - \varphi(\omega(s))$. We have, for $0 < \varepsilon < q$,

$$\begin{aligned}
&\int_C |F_{s,t}|^p d\widetilde{\mathbb{M}} \\
&= \int_{\mathbb{R}^d}\int_{C_0} (|F_{s,t}|^{q-\varepsilon}(1+|x|^2)^{-k\frac{q-\varepsilon}{q}})(|F_{s,t}|^{p-q+\varepsilon}(1+|x|^2)^{k+k\frac{q-\varepsilon}{q}})dM_0^x\mu(dx) \\
&\leq \left(\int_{\mathbb{R}^d}\int_{C_0} |F_{s,t}|^q(1+|x|^2)^{-k}dM_0^x\mu(dx)\right)^{\frac{q-\varepsilon}{q}} \\
&\quad\cdot\left(\int_{\mathbb{R}^d}\int_{C_0} |F_{s,t}|^{\frac{(p-q+\varepsilon)q}{\varepsilon}}(1+k|^2)^{\frac{k}{\varepsilon}(2q-\varepsilon)}dM_0^x\mu(dx)\right)^{\frac{\varepsilon}{q}},
\end{aligned}$$

by the Hölder inequality. Now, by Remark 2.3 the second factor is bounded by a constant independent of $s, t$. Hence, using the Lipschitz property of $\varphi$ and assumption (2.9), we obtain

$$\int_C |F_{s,t}|^p d\mathbb{M} \leq K(\varepsilon, k, p, q, L, \varphi)|t-s|^{\delta(1-\frac{\varepsilon}{q})}.$$

Inequality (3.2) extends to $\overline{\mathcal{S}}^p_{\mathbb{M}}$, so

$$\begin{aligned}
&E|\langle X_n(t), \varphi\rangle - \langle X_n, (s), \varphi\rangle|^r = E|\mathcal{X}_n(F_{\varphi,t} - F_{\varphi,s})|^r \\
&\leq K_1\left(\int_C |F_{\varphi,t} - F_{\varphi,s}|^p d\widetilde{\mathbb{M}}\right)^\beta \leq K(\varepsilon, \beta, k, p, q, L, K_1, \varphi)|t-s|^{\beta\delta(1-\frac{\varepsilon}{q})}.
\end{aligned}$$

It suffices now to take $\varepsilon$ such that $\beta\delta(1-\frac{\varepsilon}{q})>1$ and apply the classical tightness criterion in $C$ [B] (Theorem 2.3). □

3.5. REMARKS. (a) This theorem, applied to just one random element $\mathcal{X}$, yields a continuity criterion for the time-localization process in the non-Gaussian case.

(b) If assumption (ii) holds with $r>1$, then assumption (i) is automatic.

Putting together Corollary 3.2 and Theorem 3.4 we obtain the following corollary:

3.6. COROLLARY. *If $\mathbb{M}$ and $(\mathcal{X}_n)_n$ satisfy the assumptions if Theorem 3.4 and $\mathcal{X}_n \Rightarrow \mathcal{X}$, then $\mathcal{X}$ is $\mathbb{M}$-$p$-admissible (with $p$ of Corollary 3.2) and $X_n \Rightarrow X$ in $C([0,1],S')$, where $X$ is the time-localization process of $X$.*

## 4. Examples.

We consider first the Poisson system of motions described in the Introduction. We assume throughout that the measure $\mu$ satisfies condition (2.2).

It follows from the structure of Poisson systems that the mean and the second moments of $\mathcal{N}_n$ defined by (1.4) are given by

$$E\langle\mathcal{N}_n,F\rangle=\int_{\mathbb{R}^d}EF(\xi^x)n\mu(dx)=\int_{\mathbb{R}^d}\left(\int_C FdM^x\right)n\mu(dx)=n\int_C Fd\mathbb{M} \qquad F\in\mathcal{S}(C), \tag{4.1}$$

$$\begin{aligned}E\langle\mathcal{N}_n,F\rangle\langle\mathcal{N}_n,G\rangle&=\int_{\mathbb{R}^d}EF(\xi^x)G(\xi^x)n\mu(dx)\\&=\int_{\mathbb{R}^d}\left(\int_C FGdM^x\right)n\mu(dx)=n\int_C FGd\mathbb{M},\quad F,G\in\mathcal{S}(C),\end{aligned} \tag{4.2}$$

provided that the integrals on the right-hand sides are well defined, and the characteristic functional of $\mathcal{N}_n$ is given by

$$E\exp\{i\langle\mathcal{N}_n,F\rangle\}=\exp\left\{n\int_{\mathbb{R}^d}(E\exp\{iF(\xi^x)\}-1)\mu(dx)\right\},\quad F\in\mathcal{S}(C). \tag{4.3}$$

To show this, the basic idea is that for $A\in\mathcal{B}(\mathbb{R}^d)$ such that $\mu(A)<\infty$, conditioned on $\Pi_n(A)=k$, the $k$ points are distributed independently, each with distribution $d\mu/\mu(A)$ (see also [ML]).

Hence, by (4.1) and (4.3) the characteristic functional of $\mathcal{X}_n$ defined by (1.5) is

$$E\exp\{i\langle\mathcal{X}^n,F\rangle\}=\exp\left\{n\int_{\mathbb{R}^d}E\left(\exp\left\{i\frac{1}{\sqrt{n}}F(\xi^x)\right\}-1-\frac{1}{\sqrt{n}}F(\xi^x)\right)\mu(dx)\right\},\quad F\in\mathcal{S}(C). \tag{4.4}$$

The proof of Theorem 1.2 is straightforward but we include if for completeness.

*Proof of Theorem 1.2.* By expanding the exponential inside the integral on the right-hand side of (4.4) and using (4.1) and (4.2) we obtain

$$\lim_{n\to\infty} E\exp\{i\langle \mathcal{X}_n, F\rangle\} = \exp\left\{-\frac{1}{2}\int_{\mathbb{R}^d} EF^2(\xi^x)\mu(dx)\right\} = \exp\left\{-\frac{1}{2}\int_C F^2 d\mathbb{M}\right\}$$

for all $F \in \mathcal{S}(C)$. This implies that $\mathcal{X}_n \Rightarrow \mathcal{X}$ in $\mathcal{S}(C)'$, by Lévy's continuity theorem on nuclear spaces [Me, Bo], where $\mathcal{X}$ is the centered Gaussian random element of $\mathcal{S}(C)'$ determined by (1.6), (1.7). □

An analogous proof gives the convergence of the finite-dimensional distributions in Assertion 1.1 under the condition that the covariance functionals are well defined. (Note that formulas analogous to (4.1), (4.2), (4.3) hold for $N_n(t)$ defined by (1.1)). However, such a proof involves more cumbersome notation, and moreover it is not needed for the examples since convergence of finite-dimensional distributions is a consequence in our approach.

We will need the fourth moments of $\mathcal{X}_n$ (provided that they exist), which can be computed from the characteristic functional (4.4) in a standard way:

$$E\langle \mathcal{X}_n, F\rangle^4 = \frac{1}{n}\int_C F^4 d\mathbb{M} + 3\left(\int_C F^2 d\mathbb{M}\right)^2, \quad F \in \mathcal{S}(C). \tag{4.5}$$

We now proceed to the time-localization and convergence problems.

4.1. THEOREM. *Assertion 1.1 holds if the measure* $\mathbb{M}$ *defined by (1.7) is admissible and satisfies condition (2.9) with some* $q \le 4$ *and* $\delta > 1$.

*Proof.* Note that finiteness of $\int_C |F|^p d\mathbb{M}$ for any $p \ge 1$ is guaranteed by Theorem 2.4 (a). Therefore $\mathcal{X}_n \Rightarrow \mathcal{X}$ in $\mathcal{S}(C)'$ by Theorem 1.2.

We have, using the Schwarz inequality,

$$\begin{aligned}
\left(\int_C F^2 d\widetilde{\mathbb{M}}\right)^2 &= \left(\int_{\mathbb{R}^d}\int_{C_0} F^2(x+\omega) M_0^x(d\omega)\mu(dx)\right)^2 \\
&= \left(\int_{\mathbb{R}^d}\int_{C_0} F^2(x+\omega)(1+|x|^2)^k M_0^x(d\omega)(1+|x|^2)^{-k}\mu(dx)\right)^2 \\
&\le K\int_{\mathbb{R}^d}\int_{C_0} F^4(x+\omega)(1+|x|^2)^{2k} M_0^x(d\omega)(1+|x|^2)^{-k}\mu(dx) \\
&= K\int_C F^4 d\widetilde{\mathbb{M}},
\end{aligned}$$

so, from (4.5),

$$\sup_n E\langle \mathcal{X}_n, F\rangle^4 \le K_1 \int_C F^4 d\widetilde{\mathbb{M}} \quad \text{for all} \quad F \in \mathcal{S}(C). \tag{4.6}$$

Hence condition (3.2) is satisfied with $r = p = 4$ and $\beta = 1$. Therefore, by Theorem 2.7 and Corollary 3.2, $\mathcal{X}_n$ and $\mathcal{X}$ have time-localization processes $X_n$ and $X$ (and $X_n \underset{f}{\Rightarrow} X$). It is now clear that the localizations $X_n$ are the fluctuation processes defined by (1.2).

Now, by (4.6), Corollary 3.6 and Remark 3.5(b) we have $X_n \Rightarrow X$ in $C(0,1], \mathcal{S}')$.

It remains to prove that the covariance of $X$ is given by (1.3).

For $s, t \in [0,1]$, $\varphi, \psi \in \mathcal{S}$, fix sequences $(F_n)_n$, $(G_n)_n$, $F_n, G_n \in \mathcal{S}(C)$, such that $F_n \to F_{\varphi,s}$, $G_n \to F_{\psi,t}$ in $L^4(C, \widetilde{\mathbb{M}})$ and in $L^2(C, \widetilde{\mathbb{M}})$ simultanelously. It is possible to find such sequences by Proposition 2.6. $\mathcal{X}$ is $\mathbb{M}$-4-admissible, so $\langle \mathcal{X}, F_n \rangle \to \langle \mathcal{X}, F_{\varphi,s} \rangle$ in probability and hence in $L^2(\Omega)$ as well, since $\mathcal{X}$ is Gaussian. The same holds for $\langle \mathcal{X}, G_n \rangle$. Then,

$$
\begin{aligned}
& Cov(\langle X(s), \varphi \rangle, \langle X(t), \psi \rangle) = E\mathcal{X}(F_{\varphi,s})\mathcal{X}(F_{\psi,t}) \\
= \ & \lim_{n\to\infty} E\langle \mathcal{X}, F_n \rangle \langle \mathcal{X}, G_n \rangle = \lim_{n\to\infty} \int_C F_n G_n d\mathbb{M} = \int_C F_{\varphi,s} F_{\psi,t} d\mathbb{M} \\
= \ & \int_{\mathbb{R}^n} E\varphi(\xi^x(s))\psi(\xi^x(t))\mu(dx),
\end{aligned}
$$

by (1.6), (1.7), since $\mathbb{M} \le \widetilde{\mathbb{M}}$. □

The condition $q \le 4$ in Theorem 4.1 is suited to the examples below. In general it would be possible to have any $q \ge 1$ and to work with $p$-th moments of $\langle \mathcal{X}_n, F \rangle$ for some $p \ge q$.

Now we consider the specific examples of motions $\xi^x$ mentioned in the Introduction.

*Diffusions.*

Suppose that $\xi^x$ satisfies the Itô stochastic differential equation

$$
\xi^x(t) = x + \int_0^t b(s, \xi^x(s))ds + \int_0^t \sigma(s, \xi^x(s))dW(s), \quad x \in \mathbb{R}^d,
$$

where $W$ is a standard Wiener process in $\mathbb{R}^d$.

We consider two cases:

(i) $\sigma$ and $b$ are bounded.

(ii) $\sigma$ is bounded, $b$ has (at most) linear growth, and $\mu$ satisfies

$$
\int_{\mathbb{R}^d} |x|^{2n} \mu(dx) \le L^n n!
$$

for all $n$ and some constant $L$.

4.2. LEMMA. $\mathbb{M}$ *defined by (1.7) is admissible and satisfies condition (2.9) with* $q = 4$ *and* $\delta = 2$.

*Proof.* Condition (2.3) for admissiblity is verifed in [BGN] (proof of Lemma 3.1.1). Now,

$$\int_{\mathbb{R}^d} \int_{C_0} |\omega(t) - w(s)|^4 M_0^x(d\omega)(1 + |x|^2)^{-k} \mu(dx)$$
$$= \int_{\mathbb{R}^d} E|\xi^x(t) - \xi^x(s)|^4 (1 + |x|^2)^{-k} \mu(dx),$$

and

$$E|\xi^x(t) - \xi^x(s)|^4 \leq 8E\left|\int_s^t b(u, \xi^x(u))du\right|^4 + 8E\left|\int_s^t \sigma(u, \xi^x(u)dW(u)\right|^4.$$

By Itô's rule and Burkholder's inequality,

$$E\left|\int_s^t \sigma(u, \xi^x(u))dW(u)\right|^4 \leq K(t - s)^2,$$

since $\sigma$ is bounded. Therefore condition (2.9) holds in case (i).

For case (ii) we have, using Hölder's inequality,

$$\left|\int_s^t b(u, \xi^x(u)du\right|^4 \leq K_1\left(\int_s^t (1 + |\xi^x(u)|)du\right)^4$$
$$\leq K_1(t - s)^3 \int_s^t (1 + |\xi^x(u)|)^4 du$$
$$\leq K_2(t - s)^3 \int_s^t (1 + |x|^4 + |\xi^x(u) - x|^4)du$$
$$\leq K_2(t - s)^4 (1 + |x|^4 + ||\xi^x - x||_\infty^4),$$

and the argument in the final part of the proof of Lemma 3.1.1 of [BGN] shows that

$$\int_{\mathbb{R}^d} (1 + |x|^4 + E||\xi^x - x||_\infty^4)(1 + |x|^2)^{-k} \mu(dx) < \infty,$$

so condition (2.9) holds. □

We have inmmediately from Theorem 4.1 and Lemma 4.2 :

4.3. PROPOSITION. *Assertion 1.1 holds for the Poisson system of diffusions.*

*Fractional Brownian motions.*

Let $\xi^x$ be the fBm with Hurst parameter $h \in (0, 1)$ started from $x \in \mathbb{R}^d$.

First we have to show that $\xi$ ($\equiv \xi^0$) satisfies the required maximal inequality (2.5).

4.4. LEMMA. *For all* $n \geq 1$,

$$E \sup_{t\in[0,1]} |\xi(t)|^n \leq K^n \sqrt{n!},$$

*where $K$ is a constant depending on $h$ and $d$.*

*Proof.* It suffices to do the proof for $d = 1$ and $n$ even. We have

$$E|\xi(t) - \xi(s)|^{2n} = a_n|t - s|^{2nh},$$

where $a_n = (2n - 1)(2n - 3)\ldots 1$. Let $n$ be large enough so that $nh > 1$.

Define the random variable

$$\begin{aligned}\eta &= \int_0^1 \int_0^1 [(\xi(t) - \xi(s))|t - s|^{-\frac{3}{2n}}]^{2n} dsdt \\ &= \int_0^1 \int_0^1 |\xi(t) - \xi(s)|^{2n}|t - s|^{-3} dsdt.\end{aligned}$$

The first integral expression is written with a view towards the application of the lemma of Garsia, Rodemich and Rumsey.

We have

$$\begin{aligned}E\eta &= a_n \int_0^1 \int_0^1 |t - s|^{2nh-3} dsdt = 2a_n \int_0^1 \int_0^t u^{2nh-3} dudt \\ &= \frac{a_n}{(nh - 1)(2nh - 1)},\end{aligned}$$

hence $\eta < \infty$ a.s. Applying the lemma of Garsia, Rodemich and Rumsey [GRR] (the version in [SV], page 60) we obtain

$$|\xi(t) - \xi(0)| \le 8 \int_0^{2t} \left(\frac{16\eta}{\gamma u^2}\right)^{\frac{1}{2n}} \frac{3}{2n} u^{\frac{3}{2n}-1} du,$$

where $\gamma$ is a universal constant, so

$$\begin{aligned}|\xi(t)| &\le K_1 \eta^{\frac{1}{2n}} \frac{1}{n} \int_0^{2t} u^{\frac{1}{2n}} du \\ &= 2K_1 \eta^{\frac{1}{2n}} (2t)^{\frac{1}{2n}}.\end{aligned}$$

Therefore

$$\sup_{t\in[0,1]} |\xi(t)|^{2n} \le K_2^{2n}\eta,$$

and

$$\begin{aligned}E \sup_{t\in[0,1]} |\xi(t)|^{2n} &\le K_2^{2n} E\eta = K_2^{2n} \frac{a_n}{(nh - 1)(2nh - 1)} \\ &\le K_3^{2n} \frac{(2n - 1)!}{2^{n-1}(n - 1)!} \le K^{2n}\sqrt{(2n)!},\end{aligned}$$

where we have used the Stirling formula in the last step. □

It is also possible to obtain the maximal inequality of Lemma 4.4 using recent martingale results from [NV], but only for $h > \frac{1}{2}$.

4.5. COROLLARY. $\mathbb{M}$ *defined by (1.7) is admissible and satisfies condition (2.9).*

*Proof.* Condition (2.5) follows from Lemma 4.4.

Now,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{C_0} |\omega(t)-\omega(s)|^q M_0(d\omega)(1+|x|^2)^{-k}\mu(dx) \\
&= KE|\xi(t)-\xi(s)|^q \le K(q)E(|\xi(t)-\xi(s)|^2)^{q/2} \\
&= K_1(q)|t-s|^{hq},
\end{aligned}$$

hence condition (2.9) is satified with $q$ such that $hq > 1$. □

4.6. PROPOSITION. *Assertion 1.1 holds for the Poisson systems of fBm's and the covariance (1.3) is given by (1.10), (1.11).*

*Proof.* The convergence follows from Theorem 4.1 and Corollary 4.5.

It remains only to obtain the explicit expressions (1.10), (1.11) for the covariance. This follows from formula (1.3) and the lemma below. □

4.7. LEMMA. *For* $\varphi, \psi \in \mathcal{S}$,

$$E\varphi(\xi^x(t)) = \frac{1}{(2\pi t^{2h})^{d/2}}\int_{\mathbb{R}^d}\varphi(y_1,\dots,y_d)\exp\left\{-\frac{1}{2t^{2h}}\sum_{i=1}^d (y_i-x_i)^2\right\}dy_1,\dots,dy_d, \tag{4.7}$$

$$\begin{aligned}
&E\varphi(\xi^x(s))\psi(\xi^x(t)) \\
&= (\pi[(t-s)^{2h}(2t^{2h}+2s^{2h}-(t-s)^{2h})-(t^{2h}-s^{2h})^2])^{-d} \\
&\cdot \int_{\mathbb{R}^{2d}} \varphi(y_1,\dots,y_d)\psi(z_1,\dots,z_d) \\
&\cdot \exp\Bigg\{-2[(t-s)^{2h}(2t^{2h}+2s^{2h}-(t-s)^{2h})-(t^{2h}-s^{2h})^2]^{-1} \\
&\cdot\Bigg(t^{2h}\sum_{i=1}^d (y_i-x_i)^2 + s^{2h}\sum_{i=1}^d (z_i-x_i)^2 - (t^{2h}+s^{2h}-(t-s)^{2h})\sum_{i=1}^d (y_i-x_i)(z_i-x_i)\Bigg)\Bigg\} \\
&\quad dy_1\dots dy_d dz_1\dots dz_d, \qquad\qquad s<t.
\end{aligned} \tag{4.8}$$

*Proof.* (4.7) is clear. We prove (4.8). Denote for brevity

$$c_h(s,t) = \frac{1}{2}[t^{2h} + s^{2h} - (t-s)^{2h}], \quad s < t.$$

$(\xi^x(s), \xi^x(t))$ is a $2d$-dimensional centered Gaussian random vector with covariance matrix

$$V = \begin{pmatrix} s^{2h}I & c_h(s,t)I \\ c_h(s,t)I & t^{2h}I \end{pmatrix},$$

where $I$ is the $d \times d$ identity matrix.

It is easy to see that the inverse of $V$ is

$$V^{-1} = \frac{1}{(st)^{2h} - c_h(s,t)^2} \begin{pmatrix} t^{2h}I & -c_h(s,t)I \\ -c_h(s,t)I & s^{2h}I \end{pmatrix}$$

The determinant of $V$ is given by

$$|V| = \frac{1}{4^d}[(t-s)^{2h}(2t^{2h} + 2s^{2h} - (t-s)^{2h}) - (t^{2h} - s^{2h})^2]^d.$$

This is proved as follows. $V$ can be written as $V = BB^T$ (where $B^T$ is the transpose of $B$), and therefore $|V| = |B|^2$, where

$$B = \begin{pmatrix} (s^{2h} - c_h(s,t)^2 t^{-2h})^{1/2} I & c_h(s,t)t^{-h}I \\ 0 & t^h I \end{pmatrix}.$$

The result then follows from

$$\begin{aligned} &E\varphi(\xi^x(s))\psi(\xi^x(t)) \\ &= \int_{\mathbb{R}^{2d}} \varphi(x_1 + y_1, \ldots, x_d + y_d)\psi(x_1 + z_1, \ldots, x_d + z_d) \\ &\quad \cdot \frac{1}{(2\pi)^d |V|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(y_1, \ldots, y_d, z_1, \ldots, z_d)V^{-1}(y_1, \ldots, y_d, z_1, \ldots, z_d)^T\right\} \\ &\quad dy_1 \ldots dy_d dz_1 \ldots dz_d. \end{aligned}$$

□

We now consider a model which is a generalization of the one investigated by Itô [I1]. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent, identically distributed processes in $\mathbb{R}^d$ with initial distribution $\mu$. Again we assume that the processes are diffusions or fBm's, as in the case of the Poisson systems above.

In the trajectorial approach we define the empirical measure

$$\mathcal{N}_n = \sum_{i=1}^{n} \delta_{\xi_i}$$

and the fluctuation

$$\mathcal{X}_n = \frac{1}{\sqrt{n}}(\mathcal{N}_n - E\mathcal{N}_n),$$

where

$$E\mathcal{N}_n = n\mathbb{M}, \quad d\mathbb{M} = dM^x\mu(dx).$$

Everything works in a similar way to the Poisson system. The only difference is that instead of (4.5) we use the following estimation for the fourth moment:

$$E\langle \mathcal{X}_n, F\rangle^4 \leq K_1(\langle \mathbb{M}, F^2\rangle^2 + \langle \mathbb{M}, F\rangle^4) + K_2\frac{1}{n}(\langle \mathbb{M}, F^4\rangle + \langle \mathbb{M}, F\rangle^4), \qquad F \in \mathcal{S}(C),$$

which is easily obtained.

The trajectorial fluctuation limit is given the following theorem.

4.8. THEOREM. $\mathcal{X}_n$ *converges weakly in* $\mathcal{S}(C)'$ *as* $n \to \infty$ *to a centered Gaussian random element* $\mathcal{X}$ *with covariance functional*

$$Cov(\langle \mathcal{X}, F\rangle, \langle \mathcal{X}, G\rangle) = \int_C FGd\mathbb{M} - \int_C Fd\mathbb{M}\int_C Gd\mathbb{M}, \qquad F, G \in \mathcal{S}(C).$$

The time-localization of $\mathcal{X}_n$ is clearly the process $X_n(t) = \frac{1}{\sqrt{n}}(N_n(t) - EN_n(t)), t \in [0,1]$, where $N_n(t) = \sum_{i=1}^n \delta_{\xi_i(t)}$. The temporal fluctuation limit is the following result.

4.9. THEOREM. $X_n$ *converges weakly in* $C([0,1], \mathcal{S}')$ *as* $n \to \infty$ *to the centered Gaussian process* $X = \{X(t), t \in [0,1]\}$ *with covariance functional*

$$\begin{aligned} &Cov(\langle X(s), \varphi\rangle, \langle X(t), \psi\rangle) \\ &= \int_{\mathbb{R}^d} E\varphi(\xi^x(s))\psi(\xi^x(t))\mu(dx) - \int_{\mathbb{R}^d} E\varphi(\xi^x(s))\mu(dx)\int_{\mathbb{R}^d} E\psi(\xi^x(t))\mu(dx), \qquad \varphi, \psi \in \mathcal{S}. \end{aligned}$$

The process $X$ in Theorem 4.9 is the time-localization of $\mathcal{X}$ in Theorem 4.8. For the case of diffusions the time-localization was obtained in [BGN]. For the model with fBm's an explicit expression for the covariance of $X$ is obtained from formulas (4.7) and (4.8).

Note that the same admissible measure $d\mathbb{M} = dM^x\mu(dx)$ is useful for both models, although they are quite distinct. The only difference is the meaning of the measure $\mu$ in each case.

4.10. REMARK. The previous results are also valid for other processes $\xi$, provided that $\int F^2 d\mathbb{M} < \infty$ for all $F \in \mathcal{S}(C)$ in the case of Theorem 4.8, and the measure $\mathbb{M}$ satisfies the required conditions in the case of Theorem 4.9.

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**References**

[AFL] Adler, R. J., Feldman, R. E. and Lewin, M., Intersection local times for infinite systems of Brownian motions and for the Brownian density process, *Ann. Prob.* **19** (1991), 192–220

[B] Billingsley, P., *Convergence of Probability Measures*, Wiley, 1968.

[BG1] Bojdecki, T. and Gorostiza, L. G., A nuclear space of distributions on Wiener space and application to weak convergence, in *U.S.–Japan Bilateral Seminar on Stochastic Analysis in Infinite Dimensions*, K. Kunita and H. H. Kuo (eds.), Pitman Research Notes in Math., Vol. **310** (1994), pp. 12–25, Longman.

[BG2] Bojdecki, T. and Gorostiza, L. G., Self–intersection local time for Gaussian $\mathcal{S}'(\mathbb{R}^d)$–processes: existence, path continuity and examples, *Stoch. Proc. Appl.* **60** (1995), 1991–226.

[BG3] Bojdecki, T. and Gorostiza, L. G., Fractional Brownian motion via fractional Laplacian, *Stat. Prob. Lett.* **44** (1999), 107–108.

[BGN] Bojdecki, T., Gorostiza, L. G., and Nualart, D., Time-localization of random distributions on Wiener space, *Potential Analysis* **6** (1997), 183–205.

[Bo] Boulicaut, P., Convergence cylindrique et convergence étroite d'une suite de probabilités de Radon, *Z. Wahrsch. Verw. Geb.* **28** (1973), 43–52.

[DW] Deuschel. J.-D. and Wang, K., Large deviations of the occupation time functional of a Poisson system of independent Brownian particles, *Stoch. Proc. Appl.* **52** (1994), 183–209.

[F] Fernández, B., Markov properties of the fluctuation limit of a particle system, *J. Math. Anal. Appl.* **149** (1990), 160–179.

[GRR] Garsia, A. M., Rodemich, E. and Rumsey, M., A real variable lemma and the continuity of paths of some Gaussian processes, *Indiana Univ. Math. J.* **20** (1970), 565–578.

[G1] Gorostiza, L. G., Limites gaussiennes pour les champs aléatoires ramifiés supercritiques, in *Aspects Statistiques et Aspects Physiques des Processus Gaussiens*, p. 385–398. Editions du CNRS, Paris, (1981).

[G2] Gorostiza, L. G., Distributions on Wiener space and fluctuation limits of Poisson systems of Brownian bridges, in *Chaos Expansions, Multiples Wiener–Itô Integrals and Their Applications*, C. Houdré and V. Pérez–Abreu (eds.) Prob. and Stat. Series (1994), pp. 337–347, CRC Press.

[GN] Gorostiza, L. G., and Nualart, D., Nuclear Gelfand triples on Wiener space and applications to trajectorial fluctuations of particle systems, *J. Funct. Anal.* **125** (1994), 37–66.

[HS] Holley, R. and Stroock, D. W., Generalized Ornstein–Uhlenbeck processes and infinite particle branching Brownian motions, *Publ. Res. Inst. Math. Sci.* **14** (1981), 741–788.

[I1] Itô, K., Distribution valued processes arising from independent Brownian motions, *Math. Zeit* **182** (1983), 17–33.

[I2] Itô, K., *Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces.* SIAM, 1984.

[K] Kolmogorov, A. N., Wiener's spiral and some other interesting curves in Hilbert space, *Dokl. Akad. Nauk SSSR* **26** (1940), 115–118.

[M] Mandelbrot, B. B., *The Fractal Geometry of Nature*, Freeman, San Francisco, 1982.

[MVN] Mandelbrot, B. B. and Van Ness, J. W., Fractional Brownian motions, fractional noises and applications, *SIAM Review* **10** (1968), 422–437.

[ML] Martin–Löf, A., Limits theorems for the motion of a Poisson systems of independent Markovian particles with high density, Z. Wahrsch. Verw. Geb. **34** (1976), 205–223.

[Me] Meyer, P.–A., Le théorème de continuité de P. Lévy sur les espaces nucléaires, Sem. Bourbaki, 18e année, No. 311, 1965/66.

[Mi1] Mitoma, I., On the sample continuity of $\mathcal{S}'$-processes, *J. Math. Soc. Japan* **35** (1983), 629–636.

[Mi2] Mitoma, I., Tightness of probabilities on $C([0,1],\mathcal{S}')$ and $D([0,1],\mathcal{S}')$, *Ann. Prob.* **11** (1983), 989–999.

[NV] Novikov, A. and Valkeila, E., On some maximal inequalities for fractional Brownian motions, *Stat. Prob. Lett.* **44** (1999), 47–54.

[NZ] Nualart, D., and Zakai, M., Multiple Wiener–Itô integrals possessing a continuous extension, *Probab. Th. Rel. Fields* **85** (1990), 131–145.

[SV] Stroock, D. W. and Varadhan, S. R. S., *Multidimensional Diffusion Processes*, Springer–Verlag, Berlin, 1979.

[S] Sznitmann, A.–S., A fluctuation result for nonlinear diffusions, in *Infinite–dimensional analysis and stochastic processes, Research Notes in Math.* **124** (1985), 145–160, Pitman

[T] Tanaka, H., Limit theorems for certain diffusion processes with interaction, in *Proceedings, Taniguchi International Symposium on Stochastic Analysis, Katata and Kyoto*, Noth–Holland, 1984.

[Ta] Taqqu, M. S., A bibliographical guide to self-similar processes and long range dependence, in *Dependence in Probability and Statistics, A survey of Recent Results*, E. Eberlein and M. S. Taqqu (eds.), Progress in Probability and Statistics, Vol. **11** (1986), pp. 137–161, Birkhäuser.

[Tr] Treves, F., *Topological Vector Spaces, Distributions and Kernels*, Academic Press, 1967.