# Some Properties of Solutions of Double Stochastic Differential Equations

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**Abstract**

The paper deals with double stochastic differential equations. Existence and uniqueness are obtained under a Hölder-type hypothesis.

The convergence in probability of the successive approximations in the Hölder norm and the existence of a weak solution for continuous coefficients are also proved.

Under an additional independence hypothesis, the mean square convergence of fractional step approximations is shown. This last result is used in order to deduce a comparison result and as a consequence the existence of a strong solution in the case when the "double "noise is additive.

Finally, a counterexample which shows that the comparison result does not work for double noise depending on the state is given.

# 1 Introduction

The multiple stochastic integral is introduced by Meyer [10] for martingales as differentials. The concept allows to consider the associated stochastic equations (see [18]). We refer to these equations as multiple stochastic equations. Such stochastic equations include the Itô equations, integro-differential and as well some classes of Volterra equations.

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In [18] we considered double stochastic equations under the Lipschitz condition and we addressed the problem of convergence of some approximation schemes.
In the present paper we consider double Itô equations with the Lipschitz condition replaced by a Hölder-type one.
We obtain the existence and uniqueness of the strong solution and, by using an exponential Doob type inequality for stochastic integrals that are not martingales, we deduce the convergence in probability (in the Hölder norm) of the successive approximations.
By means of the standard procedure (Skorohod representation of the weak convergence) the existence of a weak solution is shown for continuous coefficients.
Next, under an independence assumption, we show the convergence of the fractional step approximations to the unique solution of the equation.
Such approximations are used if other more traditional methods are not easy to apply, especially for numerical treatment, qualitative properties (as positivity, comparison, etc.).
The fractional step (or splitting up) method is used in Rascanu and Tudor [13] for finite dimensional Itô equations, in Belopol'skaya and Nagolkina [2], Bensoussan-Glowinsky and Rascanu [3], Goncharuk and Kotelenez [6], for different classes of stochastic partial equations.
We also use the splitting up method in order to obtain a comparison result for double stochastic equations with additive "double" noise and in particular we derive the existence of strong solution for continuous and bounded double drifts.
Finally, we give a counterexample which shows that the comparison result fails if the double diffusion coefficient is state dependent.

## 2 Existence and Uniqueness of Strong Solutions

Let $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0\le t\le T}\right)$ be a complete filtered probability space and $\{W_t\}_{0\le t\le T}$, $\left\{\tilde{W}_t\right\}_{0\le t\le T}$ be $\mathcal{F}_t$-Brownian motions.
Define

$$C_2(t) = \{(t_1, t_2) : 0 \le t_1 \le t_2 \le t\},$$
$$C_2(a, b) = \{(t_1, t_2) : a \le t_1 \le t_2 \le b\}.$$

Let $F(t,x), G(t,x) : [0,T]\times R \longrightarrow R$, $\tilde{F}(t_1,t_2,x), \tilde{G}(t_1,t_2,x) : C_2(T)\times R \longrightarrow R$ be measurable functions and $\xi$ be a $\mathcal{F}_0$-measurable random variable.
Define $LS = \Big\{\alpha : R_+ \longrightarrow R_+ :\ \alpha$ is strictly increasing, continuous, concave, and $\int_{0+}^{1} \frac{du}{\alpha(u)} = \infty\Big\}$.
Obvious $\alpha_1(u) = Lu,\ L > 0, \alpha_2(u) = x\left|\log x\right|^{1-\varepsilon}, \alpha_3(u) = x\left|\log x\right|\left|\log\left|\log x\right|\right|^{1-\varepsilon}$, $0 < \varepsilon < 1$, belong to $LS$ and $\alpha_2, \alpha_3$ are not Lipschitz. Also if $\alpha_1, \alpha_2 \in LS$, $c_1, c_2 \ge 0, c_1 + c_2 > 0$, then $c_1\alpha_1 + c_2\alpha_2 \in LS$.
We consider the following double stochastic differential equation

$$X_t = \xi + \int_0^t F(s, X_s)ds + \int_0^t G(s, X_s)dW_s +$$

$$\int_{C_2(t)} \tilde{F}(s_1, s_2, X_{s_1})ds_1 ds_2 + \int_{C_2(t)} \tilde{G}(s_1, s_2, X_{s_1})d\tilde{W}_{s_1}d\tilde{W}_{s_2}, 0 \le t \le T. \quad (2.1)$$

We introduce the following hypotheses on the initial data and the coefficients of (2.1).
*($H_1$) (Growth condition on $F, G$):* there exists a constant $K > 0$ such that

$$\left|F(t,x)\right|^2 + \left|G(t,x)\right|^2 \le K\left(1 + |x|^2\right),\ \forall t \in [0,T], x \in R.$$

*($\tilde{H}_1$) (Growth condition on $\tilde{F}, \tilde{G}$):* there exists a constant $\tilde{K} > 0$ such that for all $(s_1, s_2) \in C_2(T), x \in R$,

$$\left|\tilde{F}(s_1, s_2, , x)\right|^2 + \left|\tilde{G}(s_1, s_2, x)\right|^2 \le \tilde{K}\left(1 + |x|^2\right).$$

$(H_2)$ *(Hölder-type condition on $F, G$)* : there exist $\alpha \in LS$ such that for all $t \in [0,T], x, y \in R$,

$$|F(t,x) - F(t,y)|^2 + |G(t,x) - G(t,y)|^2 \leq \alpha\left(|x-y|^2\right).$$

$(\tilde{H}_2)$ *(Hölder-type condition on $\tilde{F}, \tilde{G}$)* : there exist $\tilde{\alpha} \in LS$ such that for all $(s_1, s_2) \in C_2(T), x, y \in R$,

$$\left|\tilde{F}(s_1,s_2,x) - \tilde{F}(s_1,s_2,y)\right|^2 + \left|\tilde{G}(s_1,s_2,x) - \tilde{G}(s_1,s_2,y)\right|^2 \leq \tilde{\alpha}\left(|x-y|^2\right).$$

$(H_3)$ $\xi \in L^2(\Omega, \mathcal{F}_0, P)$.
Next for a process $U$ and $\alpha > 0$ we define

$$\|U\|_\alpha = \sup_{s \neq t} \frac{|U_t - U_s|}{|t-s|^\alpha}, \ \|U\|_\alpha = \|U\|_\infty + \|U\|_\alpha.$$

**Theorem 2.1** *Suppose* $(H_1) - (H_3)$, $(\tilde{H}_1), (\tilde{H}_2)$ *are satisfied. Let* $\{X_t^n\}_{t\in[0,T]}$ *be the successive approximations defined by*

$$X_t^0 = \xi, \ X_t^{n+1} = \xi + \int_0^t F(s, X_s^n)ds + \int_0^t G(s, X_s^n)dW_s +$$

$$\int_{C_2(t)} \tilde{F}(s_1,s_2,X_{s_1}^n)ds_1 ds_2 + \int_{C_2(t)} \tilde{G}(s_1,s_2,X_{s_1}^n)d\tilde{W}_{s_1}d\tilde{W}_{s_2}, 0 \leq t \leq T.$$

*(a) Then the equation (2.1) has a pathwise unique solution* $\{X_t\}_{t\in[0,T]}$ *such that*

$$E\left(\sup_{t\leq T}|X_t|^2\right) < \infty. \tag{2.2}$$

*Moreover*

$$\lim_{n\to\infty} E\left(\sup_{t\leq T}|X_t^n - X_t|^2\right) = 0. \tag{2.3}$$

*(b) In addition assume that one of the following two conditions is satisfied:*
*$(b_1)$ For some $r > 1$, $E\left(|\xi|^{2r}\right) < \infty$, either,*
*$(b_2)$ There exists $\alpha \in (0,1)$ and a positive constant $K_1$ such that*

$$\left|\tilde{G}(s_1,t_2,x) - \tilde{G}(s_1,t_1,x)\right| \leq K|t_1 - t_2|^\alpha \ \text{for all } s_1 \leq t_i \leq T. \tag{2.4}$$

*Then for every $0 < \beta < \min(\frac{1}{2}, 2\alpha)$ we have* $\|X^n - X\|_\beta \xrightarrow[n\to\infty]{p} 0.$

**Proof.** (a) *Existence.* By standart computation it follows that

$$\sup_n E\left(\sup_{t\leq T}|X_t^n|^2\right) < \infty. \tag{2.5}$$

Next from $(H_2), (\tilde{H}_2)$ we deduce

$$g_{m,n}(t) := E\left(\sup_{s\leq t}|X_s^m - X_s^n|^2\right) \leq \int_0^t (\alpha+\tilde{\alpha})\left(g_{m,n}(s)\right)ds,$$

$$\overline{\lim}_{m,n\to\infty} g_{m,n}(t) \leq \int_0^t (\alpha+\tilde{\alpha})\,(\overline{\lim}_{m,n\to\infty} g_{m,n}(s))ds,$$

where from $\overline{\lim}_{m,n\to\infty} g_{m,n}(t) = 0$.
Therefore there exists a continuous process $\{X_t\}_{t\in[0,T]}$ such that

$$\lim_{n\to\infty} E\left(\sup_{t\leq T}|X_t^n - X_t|^2\right) = 0.$$

It is clear that $X$ is a solution of (2.1).
*Uniqueness.* If $\{X_t\}_{t\in[0,T]}, \{Y_t\}_{t\in[0,T]}$ are solutions of (2.1) and denote $\delta(t) = E\left(|X_t - Y_t|^2\right)$ then by $(H_2), (\tilde{H}_2)$ we have

$$\delta(t) \leq \int_0^t (\alpha+\tilde{\alpha})\,(\delta(s))ds,$$

and this implies $\delta(t) = 0$.
(b) Assume *($b_1$)* is satisfied. Then in a standard manner it follows

$$\sup_n E\left(\sup_{t\leq T}|X_t^n|^{2r}\right) < \infty,$$

where from we get

$$E\left(|X_t^n - X_s^n|^{2r}\right) \leq K_2\,(t-s)^r\,,\; E\left(|X_t - X_s|^{2r}\right) \leq K_2\,(t-s)^r\,,$$

and, if we define $Y^n = X^n - X$, the previuos inequalities imply

$$E\left(|Y_t^n - Y_s^n|^{2r}\right) \leq K_3\,(t-s)^r\,. \tag{2.6}$$

As in [1], (2.6) and the Garcia-Rodemich-Rumsey criterion imply the result (see also Lemma 8.27 of [15]).
Now assume that $(b_2)$ hold. Choose $\eta > 0$ such that $\beta + \eta < \frac{1}{2}$. Taking into account the previous result from (a) all we need to show is that $[Y^n]_\beta \xrightarrow{p} 0$, or since for all $\mu, \varepsilon > 0$,

$$P([Y^n]_\beta > \varepsilon) \le P([Y^n]_{\beta+\mu} > \varepsilon\mu^{-\eta}) + 2P\left(\|X^n - X\|_\infty > \varepsilon\mu^\beta\right),$$

to show that

$$\lim_{M\to\infty} \sup_n P([Y^n]_\theta > M) = 0,\ \forall 0 < \theta < \min(\frac{1}{2}, 2\alpha).$$

By using (2.2), (2.5) we obtain

$$P([Y^n]_\theta > M) \le P(\|X^n\|_\infty > M_1) + P(\|X\|_\infty > M_1) +$$
$$P([Y^n]_\theta > M, \|X^n\|_\infty \le M_1, \|X\|_\infty \le M_1) \le$$
$$\frac{K_3}{M_1^2} + P([Y^n]_\theta > M, \|X^n\|_\infty \le M_1, \|X\|_\infty \le M_1),$$

so we have to prove that

$$\lim_{M\to\infty} \sup_n P([Y^n]_\theta > M, \|X^n\|_\infty \le M_1, \|X\|_\infty \le M_1) = 0,\ \forall M_1 > 0. \quad (2.7)$$

Denote $A = \{\|X^n\|_\infty \le M_1, \|X\|_\infty \le M_1\}$. Then we have

$$P([Y^n]_\theta > M, A) \le P\left(\left[\int_0^\cdot (G(s, X_s^n) - G(s, X_s))\, dW_s\right]_\theta > \frac{M - K_4\sqrt{1 + M_1^2}}{2}, A\right) +$$

$$P\left(\left[\int_{C_2(.)} \left(\tilde{G}(s_1, s_2, X_{s_1}^n) - \tilde{G}(s_1, s_2, X_{s_1})\right) d\tilde{W}_{s_1} d\tilde{W}_{s_2}\right]_\theta > \frac{M - K_4\sqrt{1 + M_1^2}}{2}, A\right),$$

and hence (2.7) follows if we prove that

$$\lim_{M\to\infty} \sup_n P\left(\left[\int_0^\cdot (G(s, X_s^n) - G(s, X_s))\, dW_s\right]_\theta > M, A\right) = 0, \quad (2.8)$$

$$\lim_{M\to\infty} \sup_n P\left(\left[\int_{C_2(.)} \left(\tilde{G}(s_1, s_2, X_{s_1}^n) - \tilde{G}(s_1, s_2, X_{s_1})\right) d\tilde{W}_{s_1} d\tilde{W}_{s_2}\right]_\theta > M, A\right) = 0. \quad (2.9)$$

On the set $A$ we have

$$|G(s,X_s^n)-G(s,X_s)| \le K_5\left(1+\|X^n\|_\infty+\|X\|_\infty\right) \le K_5\left(1+2M_1\right),$$

and similarly

$$\left|\tilde{G}(s_1,s_2,X_{s_1}^n)-\tilde{G}(s_1,s_2,X_{s_1})\right| \le K_5\left(1+2M_1\right).$$

¿From Lemma 10.2.2 of [8] it follows that on $A$,

$$\left[\int_0^\cdot \left(G(s,X_s^n)-G(s,X_s)\right)dW_s\right]_\theta \le K_6\zeta_n,$$

where $E\left(\zeta_n 1_A\right) \le K_7 T^2$. Therefore

$$P\left(\left[\int_0^\cdot \left(G(s,X_s^n)-G(s,X_s)\right)dW_s\right]_\theta > M, A\right) \le$$

$$P\left(A,\zeta_n>\frac{M}{K_6}\right) \le \frac{K_6 E\left(\zeta_n\right)}{M} \le \frac{K_8}{M} \underset{M\to\infty}{\longrightarrow} 0, \text{ uniformly in } n.$$

Next define

$$I_n(t)=\int_0^t \left(\tilde{G}(s_1,t,X_{s_1}^n)-\tilde{G}(s_1,t,X_{s_1})\right)d\tilde{W}_{s_1}.$$

Then

$$P\left(\left[\int_{C_2(.)} \left(\tilde{G}(s_1,s_2,X_{s_1}^n)-\tilde{G}(s_1,s_2,X_{s_1})\right)d\tilde{W}_{s_1}d\tilde{W}_{s_2}\right]_\theta > M, A\right) \le$$

$$P\left(A,\|I_n\|_\infty \le M_2, \left[\int_0^\cdot I_n(s)d\tilde{W}_s\right]_\theta > M\right) + P\left(\|I_n\|_\infty > M_2, A\right). \quad (2.10)$$

Now from Theorem 2 of [11] or Theorem 2.4 of [12] it follows

$$P\left(\|I_n\|_\infty > M_2, A\right) \le \exp\left\{-K_8(M_1)M_2^2\right\}, \quad (2.11)$$

and from Lemma 10.2.2 of [8]

$$P\left(A,\|I_n\|_\infty \le M_2, \left[\int_0^\cdot I_n(s)d\tilde{W}_s\right]_\theta > M\right) \le \frac{K_9(M_1,M_2)}{M}. \quad (2.12)$$

Finally from (2.10)-(2.12) we get (2.9).

**Theorem 2.2 (Existence of weak solutions)** *Assume that* $\left(F, G, \tilde{F}, \tilde{G}\right)$ *satisfy the hypotheses* $(H_1), (H_3)$, $(\tilde{H}_1)$. *Moreover suppose that:*
*(i) For some* $r > 1$, $E\left(|\xi|^{2r}\right) < \infty$.
*(ii)* $F, G, \tilde{F}, \tilde{G}$ *are continuous in all variables.*
*(iii) There exists* $\alpha \in (0,1)$ *and a positive constant* $\bar{K}$ *such that*

$$\left|\tilde{G}(s_1, t_1, x) - \tilde{G}(s_1, t_2, x)\right| \leq \bar{K} \left|t_1 - t_2\right|^{\alpha}, \ \forall s_1 \leq t_i \leq T, x \in R.$$

*Then (2.1) has a weak solution.*

We need the following convergence result for double stochastic integrals.
**Lemma 2.3.** *Let* $\{V_t^n\}_{t\in[0,T]}$, $\{V_t\}_{t\in[0,T]}$ *be Brownian motions and* $\{h_n(s_1, s_2)\}_{(s_1,s_2)\in C_2(T)}$, $\{h(s_1, s_2)\}_{(s_1,s_2)\in C_2(T)}$ *be predictable processes such that*

$$E\left(\int_{C_2(T)} |h_n(s_1, s_2)|^2 \, ds_1 ds_2\right) < \infty, E\left(\int_{C_2(T)} |h(s_1, s_2)|^2 \, ds_1 ds_2\right) < \infty.$$

*Assume that:*
*(1) For each* $t$, $V_t^n \xrightarrow{p} V_t$.
*(2) For each* $(s_1, s_2) \in C_2(T)$, $h_n(s_1, s_2) \xrightarrow{p} h(s_1, s_2)$ *and*

$$\lim_{C\to\infty} \sup_n P\left(\|h_n\|_\infty > C\right) = 0. \tag{2.13}$$

*(3) There exists* $\alpha \in (0, 2]$ *and for every* $n$ *there exists a nonnegative random variable* $\xi_n$ *such that*

$$\lim_{C\to\infty} \sup_n P\left(\xi_n > C\right) = 0, \tag{2.14}$$

$$\int_0^{t_1\wedge t_2} |h_n(s_1, t_1) - h_n(s_1, t_2)|^2 \, ds_1 \leq \xi_n \left|t_1 - t_2\right|^{\alpha}, \ \forall 0 \leq t_i \leq T. \tag{2.15}$$

*(4) For each* $\varepsilon > 0, 0 \leq t \leq T$,

$$\lim_{h\to 0} \lim_{n\to\infty} \sup_{\substack{|s_1 - s_2| \leq h \\ 0 \leq s_i \leq t}} P\left(|h_n(s_1, t) - h_n(s_2, t)| > \varepsilon\right) = 0. \tag{2.16}$$

*Then*

$$\int_{C_2(T)} h_n(s_1,s_2)dV^n_{s_1}dV^n_{s_2} \xrightarrow{p} \int_{C_2(T)} h(s_1,s_2)dV_{s_1}dV_{s_2}.$$

**Proof.** Define

$$Z^n_t = \int_0^t h_n(s_1,t)dV^n_{s_1},\ Z_t = \int_0^t h(s_1,t)dV_{s_1}.$$

Then

$$\int_{C_2(T)} h_n(s_1,s_2)dV^n_{s_1}dV^n_{s_2} = \int_0^T Z^n_s dV^n_s,$$

$$\int_{C_2(T)} h(s_1,s_2)dV_{s_1}dV_{s_2} = \int_0^T Z_s dV_s.$$

By the convergence result for first order stochastic integrals due to Skorohod (see [14]-page 32) we have

$$Z^n_t \xrightarrow{p} Z_t,\ \forall 0 \le t \le T.$$

Define $A = \left\{\|h_n\|_\infty \le C_1, \xi_n \le C_1\right\}$. Then from (2.13)-(2.15) and Theorem 2 of [11] it follows

$$\lim_{C\to\infty} \sup_n P\left(\|Z^n\|_\infty > C\right) \le P\left(\|Z^n\|_\infty > C, A\right) +$$
$$P\left(\|h_n\|_\infty > C_1\right) + P\left(\xi_n > C_1\right) \le$$
$$\exp\left\{-\gamma(C_1)C^2\right\} + P\left(\|h_n\|_\infty > C_1\right) + P\left(\xi_n > C_1\right) \longrightarrow 0,$$

as $C \to \infty, C_1 \to \infty$.

Next Lemma 10.2.2 of [8] yields

$$\left|Z^n_t - Z^n_s\right| \le \eta_n \left|t-s\right|^\alpha \text{ on } A, \tag{2.17}$$

where

$$\sup_n E\left(1_A \eta_n\right) = C_2 < \infty. \tag{2.18}$$

Next by using (2.17), (2.18) we obtain

$P\left(\|Z^n_t - Z^n_s\| > \varepsilon\right) \le P\left(\|Z^n_t - Z^n_s\| > \varepsilon, A\right) + P\left(\|h_n\|_\infty > C_1\right) + P\left(\xi_n > C_1\right) \le$

$$P\left(1_A\eta_n > \varepsilon\,|t-s|^{-\alpha}\right) + P\left(\|h_n\|_\infty > C_1\right) + P\left(\xi_n > C_1\right) \leq$$
$$\frac{|t-s|^{\alpha}}{\varepsilon} E\left(1_A\eta_n\right) + P\left(\|h_n\|_\infty > C_1\right) + P\left(\xi_n > C_1\right) \leq$$
$$C_2\,|t-s|^{\alpha} + P\left(\|h_n\|_\infty > C_1\right) + P\left(\xi_n > C_1\right),$$

so that

$$\overline{\lim_{h\to 0}}\,\sup_n \sup_{\substack{|t-s|\leq h\\ 0\leq s,t\leq T}} P\left(|Z_t^n - Z_s^n| > \varepsilon\right) \leq$$
$$\sup_n P\left(\|h_n\|_\infty > C_1\right) + \sup_n P\left(\xi_n > C_1\right) \longrightarrow 0 \text{ as } C_1 \to \infty.$$

Applying once again the result of Skorohod ( [14]-page 32) we obtain that

$$\int_0^T Z_s^n dV_s^n \xrightarrow{p} \int_0^T Z_s dV_s.$$

**Proof of Theorem 2.2.** Choose a sequence $\left(F_n, G_n, \tilde{F}_n, \tilde{G}_n\right)$ of continuous functions such that
*($i_1$)* $\left(F_n, G_n, F_n, \tilde{G}_n\right)$ satisfy $(H_1)$,$(\tilde{H}_1)$ uniformly with respect to $n$.
*($i_2$)* $\left(F_n, G_n, \tilde{F}_n, \tilde{G}_n\right)$ are Lipschitz in the last variable.
*($i_3$)* For each $s$, $s_1$, $s_2$,

$$F_n(s,x) \longrightarrow F(s,x),\ G_n(s,x) \longrightarrow G(s,x),$$

$$\tilde{F}_n(s_1,s_2,x) \longrightarrow \tilde{F}(s_1,s_2,x), \tilde{G}_n(s_1,s_2,x) \longrightarrow \tilde{G}(s_1,s_2,x),$$

uniformly for $x$ in compact sets.
Let (by Theorem 2.1) $X_t^n$ be the strong solution of

$$X_t^n = \xi + \int_0^t F_n(s, X_s^n)ds + \int_0^t G_n(s, X_s^n)dW_s +$$

$$\int_{C_2(t)} \tilde{F}_n(s_1, s_2, X_{s_1}^n)ds_1 ds_2 + \int_{C_2(t)} \tilde{G}_n(s_1, s_2, X_{s_1}^n)d\tilde{W}_{s_1}d\tilde{W}_{s_2}, 0 \leq t \leq T.$$

¿From *(i)*, $(H_1)$,$(\tilde{H}_1)$ we deduce

$$\sup_n E\left(|X_t^n|^{2r}\right) < \infty,$$

and this implies

$$E\left(|X_t^n - X_s^n|^{2r}\right) \leq K_1 |t-s|^r , \ \forall n, s, t.$$

Therefore on some probability space $\left(\hat{\Omega}, \overset{\wedge}{\mathcal{F}}, \hat{P}\right)$ there exist $\left(\hat{\xi}_n, \hat{X}^n, \hat{W}^n, \tilde{V}^n\right)$, $\left(\hat{\xi}, \hat{X}, \hat{W}, \tilde{V}\right)$ such that:

(a) For every $n$, the laws of $\left(\hat{\xi}_n, \hat{X}^n, \hat{W}^n, \tilde{V}^n\right)$ and $\left(\xi, X^n, W, \hat{W}\right)$ coincide.

(b) For some subsequence $(n_k)$ (for simplicity we assume $n_k = k$ for all $k$ ),

$$\hat{\xi}_n \overset{a.s.}{\longrightarrow} \xi, \ \hat{X}^n \overset{a.s.}{\longrightarrow} \hat{X}, \ \hat{W}^n \overset{a.s.}{\longrightarrow} \hat{W}, \ \tilde{V}^n \overset{a.s.}{\longrightarrow} \tilde{V},$$

uniformly on every compact interval.

If we define

$$\overset{\wedge}{\mathcal{F}}{}_t^n = \mathcal{B}\left(\hat{\xi}_n, \hat{X}_s^n, \hat{W}_s^n, \tilde{V}_s^n : s \leq t\right), \ \overset{\wedge}{\mathcal{F}}_t = \mathcal{B}\left(\hat{\xi}, \hat{X}_s, \hat{W}_s, \tilde{V}_s : s \leq t\right),$$

then it is clear that $\hat{W}_t^n$, $\tilde{V}^n$(resp. $\hat{W}_t$, $\tilde{V}$) are $\overset{\wedge}{\mathcal{F}}{}_t^n$ (resp. $\overset{\wedge}{\mathcal{F}}_t$) Brownian motions.

Also we have (we omit the details)

$$\hat{X}_t^n = \hat{\xi}_n + \int_0^t F_n(s, \hat{X}_s^n) ds + \int_0^t G_n(s, \hat{X}_s^n) d\hat{W}_s^n +$$

$$\int_{C_2(t)} \tilde{F}_n(s_1, s_2, \hat{X}_{s_1}^n) ds_1 ds_2 + \int_{C_2(t)} \tilde{G}_n(s_1, s_2, \hat{X}_{s_1}^n) d\tilde{V}_{s_1}^n d\tilde{V}_{s_2}^n, 0 \leq t \leq T. \quad (2.19)$$

Next, it is easy to see that for each $t$,

$$\int_0^t F_n(s, \hat{X}_s^n) ds \overset{p}{\longrightarrow} \int_0^t F(s, \hat{X}_s) ds$$

$$\int_{C_2(t)} \tilde{F}_n(s_1, s_2, \hat{X}_{s_1}^n) ds_1 ds_2 \overset{p}{\longrightarrow} \int_{C_2(t)} \tilde{F}(s_1, s_2, \hat{X}_{s_1}) ds_1 ds_2,$$

and by [14], page 32,

$$\int_0^t G_n(s, \hat{X}_s^n) d\hat{W}_s^n \overset{p}{\longrightarrow} \int_0^t G(s, \hat{X}_s) d\hat{W}_s^n.$$

Finally by Lemma 2.3 we also have

$$\int_{C_2(t)} \tilde{G}_n(s_1, s_2, \hat{X}^n_{s_1}) d\tilde{V}^n_{s_1} d\tilde{V}^n_{s_2} \xrightarrow{p} \int_{C_2(t)} \tilde{G}(s_1, s_2, \hat{X}_{s_1}) d\tilde{V}_{s_1} d\tilde{V}_{s_2}.$$

It follows we can pass to the limit in (2.19) and obtain that

$$\hat{X}_t = \hat{\xi} + \int_0^t F(s, \hat{X}_s) ds + \int_0^t G(s, \hat{X}_s) d\hat{W}_s +$$

$$\int_{C_2(t)} \tilde{F}(s_1, s_2, \hat{X}_{s_1}) ds_1 ds_2 + \int_{C_2(t)} \tilde{G}(s_1, s_2, \hat{X}_{s_1}) d\tilde{V}_{s_1} d\tilde{V}_{s_2}, 0 \le t \le T,$$

i.e., $\hat{X}$ is a weak solution of (2.1).

# 3 Fractional Step Method. A Comparison Result

In this section we assume that:
*(i)* The filtration $(\mathcal{F}_t)_{0\le t\le T}$ is of the form $\mathcal{F}_t = F_t \vee \tilde{F}_t$ and $F_t$, $\tilde{F}_t$ are independent for all $t \ge 0$.
*(ii)* $W_t$ is $F_t$ -Brownian motion and $\tilde{W}_t$ is $\tilde{F}_t$-Brownian motion.
We introduce now the fractional step approximations associated to (2.1).
Let $\Delta : 0 = t_0 < t_1 < ....t_n = T$ be a partition of $[0, T]$ with the norm $\|\Delta\| = \max_{1\le i\le n} (t_i - t_{i-1})$.
We define *the fractional step approximations* as the processes $\{Z_\Delta (t)\}_{0\le t\le T}$, $\{Y_\Delta (t)\}_{0\le t\le T}$ defined by the following recursive procedure:
$Z_\Delta (0) = \xi$ and for $t \in [t_{i-1}, t_i)$, $i \ge 1$,

$$\begin{cases} Z_\Delta (t) &= Z_\Delta (t_{i-1}) + \int_{t_{i-1}}^t F(s, Z_\Delta (s)) ds + \int_{t_{i-1}}^t G(s, Z_\Delta (s)) dW_s \\ Y_\Delta (t_{i-1}) &= Z_\Delta (t_i -), \end{cases} \tag{3.20}$$

$$\begin{cases} Y_\Delta (t) &= Y_\Delta (t_{i-1}) + \int_{t_{i-1}}^t \int_0^{t_{i-1}} \tilde{F}(s_1, s_2, Y_\Delta(s_1)) ds_1 ds_2 \\ &+ \int_{C_2(t_{i-1},t)} \tilde{F}(s_1, s_2, Y_\Delta(s_1)) ds_1 ds_2 \\ &+ \int_{t_{i-1}}^t \int_0^{t_{i-1}} \tilde{G}(s_1, s_2, Y_\Delta(s_1)) d\tilde{W}_{s_1} d\tilde{W}_{s_2} \\ &+ \int_{C_2(t_{i-1},t)} \tilde{G}(s_1, s_2, Y_\Delta(s_1)) d\tilde{W}_{s_1} d\tilde{W}_{s_2} \\ Z_\Delta (t_i) &= Y_\Delta (t_i -). \end{cases} \tag{3.21}$$

**Lemma 3.1 (see [13])** *Let* $0 \leq \delta < T, b, c > 0$, *with* $4c\delta < 1$ *and* $f : [0,T] \longrightarrow R_+$ *be such that*

$$f(t+s) - f(t) \leq (s+\delta)\left(b + cf(t+s)\right), \ \ \forall s,t.$$

*Then*

$$f(t) \leq \left(f(0) + \frac{b}{c}\right) e^{4ct+1} - \frac{b}{c}.$$

**Proposition 3.2** *(a) For every* $t \in [t_{i-1}, t_i)$, $i \geq 1$,*the fractional step approximations* $(Z_\Delta(t), Y_\Delta(t))$ *is solution of the system of double stochastic equations*

$$\begin{cases} Z_\Delta(t) &= \xi + \int_0^t F(s, Z_\Delta(s))ds + \int_0^t G(s, Z_\Delta(s))dW_s \\ &+ \int_{C_2(t_{i-1})} \tilde{F}(s_1, s_2, Y_\Delta(s_1))ds_1 ds_2 + \int_{C_2(t_{i-1})} \tilde{G}(s_1, s_2, Y_\Delta(s_1))d\tilde{W}_{s_1} d\tilde{W}_{s_2} \\ Y_\Delta(t) &= \xi + \int_0^{t_i} F(s, Z_\Delta(s))ds + \int_0^{t_i} G(s, Z_\Delta(s))dW_s \\ &+ \int_{C_2(t)} \tilde{F}(s_1, s_2, Y_\Delta(s_1))ds_1 ds_2 + \int_{C_2(t)} \tilde{G}(s_1, s_2, Y_\Delta(s_1))d\tilde{W}_{s_1} d\tilde{W}_{s_2}, \end{cases} \tag{3.22}$$

*and* $Z_\Delta(t)$ *(resp.* $Y_\Delta(t)$*) is* $\mathcal{F}_{tt_{i-1}} = F_t \vee \tilde{F}_{t_{i-1}}$ *(resp.* $\mathcal{F}_{t_i t} = F_{t_i} \vee \tilde{F}_t$ *) measurable.*
*(b) There exist* $\delta_0 > 0$ *and constants* $\gamma_i = \gamma_i\left(E(|\xi|^2), T, K, \tilde{K}\right) < \infty, i = 1, 2, 3,$ *such that for every* $\|\Delta\| \leq \delta_0$,

$$E\left(\sup_{0 \leq t \leq T} |Z_\Delta(t)|^2\right) \leq \gamma_1, \tag{3.23}$$

$$E\left(\sup_{0 \leq t \leq T} |Y_\Delta(t)|^2\right) \leq \gamma_2, \tag{3.24}$$

$$\sup_{0 \leq t \leq T} E\left(|Z_\Delta(t) - Y_\Delta(t)|^2\right) \leq \gamma_3 \|\Delta\|. \tag{3.25}$$

**Proof.** Define for $t \in [t_{i-1}, t_i)$

$$f_\Delta(t) = C_1 \left\{ E\left(|\xi|^2\right) + \int_0^{t_i} E\left(|F(s, Z_\Delta(s))|^2 + |G(s, Z_\Delta(s))|^2\right) ds + \right.$$

$$\left. \int_{C_2(t_i)} E\left(\left|\tilde{F}(s_1, s_2, Y_\Delta(s_1))\right|^2 + \left|\tilde{G}(s_1, s_2, Y_\Delta(s_1))\right|^2\right) ds_1 ds_2 \right\},$$

where $C_1$ is a constant sufficiently large, which do not depent on $\Delta$.

Of course we have

$$E\left(\sup_{0\leq s\leq t}\left|Z_{\Delta}\left(s\right)\right|^{2}\right)\leq f_{\Delta}(t),\ E\left(\sup_{0\leq s\leq t}\left|Y_{\Delta}\left(s\right)\right|^{2}\right)\leq f_{\Delta}(t). \tag{3.26}$$

Now let $t\in\left[t_{i-1},t_{i}\right),\ t+s\in\left[t_{j-1},t_{j}\right),\ i\leq j.$
If $i=j$ then $f_{\Delta}(t+s)-f_{\Delta}(t)=0$. If $i<j$ then by $(H_1),(\tilde{H}_1)$ and (3.26) we have for some constants $a_1,a_2,a_3>0$, which are independent of $\Delta$,

$$f_{\Delta}(t+s)-f_{\Delta}(t)\leq\int_{t_i}^{t_j}\left(a_1+a_2f_{\Delta}(t_{j-1})\,ds\leq\right.$$

$$\int_{C_2(t_j)\backslash C_2(t_i)}\left(a_1+a_2f_{\Delta}(t_{j-1})\,ds_1ds_2\leq\right.$$

$$a_3\left(a_1+a_2f_{\Delta}(t+s)\left(t_j-t_i+\lambda\left(C_2(t_j)\backslash C_2(t_i)\right)\right)\leq\right.$$

$$a_3\left(a_1+a_2f_{\Delta}(t+s)\left(s+\|\Delta\|\right).\right.$$

Then by Lemma 3.1 we get (3.23), (3.24).
Next from (3.22), $\left(\tilde{H}_1\right),\left(\tilde{H}_2\right)$ and (3.23), (3.24) it follows easily (3.25).

**Theorem 3.3** *Assume that* $(H_1)-(H_3)\,,\left(\tilde{H}_1\right),\left(\tilde{H}_2\right)$ *are satisfied. If* $\{X_t\}_{0\leq t\leq T}$ *is the unique solution of (2.1) then*

$$\lim_{\|\Delta\|\longrightarrow 0}\sup_{0\leq t\leq T}E\left(\left|Z_{\Delta}\left(t\right)-X_t\right|^2\right)=0, \tag{3.27}$$

$$\lim_{\|\Delta\|\longrightarrow 0}\sup_{0\leq t\leq T}E\left(\left|Y_{\Delta}\left(t\right)-X_t\right|^2\right)=0. \tag{3.28}$$

**Proof.** Define

$$\tilde{F}(s_1,s_2,Z_{\Delta},Y_{\Delta})=\begin{cases}\tilde{F}(s_1,s_2,Y_{\Delta}(s_1)-\tilde{F}(s_1,s_2,Z_{\Delta}(s_1), & (s_1,s_2)\in C_2(t_{i-1})\\ -\tilde{F}(s_1,s_2,Z_{\Delta}(s_1), & (s_1,s_2)\in C_2(t_{i-1},t_i)\cup R_{\Delta},\end{cases}$$

and $\tilde{G}(s_1,s_2,Z_{\Delta},Y_{\Delta})$ similarly, where $R_{\Delta}=[0,t_{i-1})\times[t_{i-1},t_i)\,.$
Then we have the equality

$$Z_{\Delta}\left(t\right)=\int_0^t F(s,Z_{\Delta}\left(s\right))ds+\int_0^t G(s,Z_{\Delta}\left(s\right))dW_s+$$

$$\int_{C_2(t)} \tilde{F}(s_1, s_2, Z_\Delta(s_1))ds_1ds_2 + \int_{C_2(t)} \tilde{G}(s_1, s_2, Y_\Delta(s_1))d\tilde{W}_{s_1}d\tilde{W}_{s_2} + \gamma_\Delta(t), \tag{3.29}$$

where

$$\gamma_\Delta(t) = \int_{C_2(t)} \tilde{F}(s_1, s_2, Z_\Delta, Y_\Delta)ds_1ds_2 + \int_{C_2(t)} \tilde{G}(s_1, s_2, Z_\Delta, Y_\Delta)d\tilde{W}_{s_1}d\tilde{W}_{s_2}.$$

¿From (2.1) and (3.29) it follows

$$E\left(|Z_\Delta(t) - X_t|^2\right) \le C_2 \left\{ E\left(|\gamma_\Delta(t)|^2\right) + \int_0^t (\alpha + \tilde{\alpha})\left(E\left(|Z_\Delta(s) - X_s|^2\right)\right) ds \right\} \tag{3.30}$$

and $(H_1)$, $(\tilde{H}_1)$, (3.23)-(3.25) yield for $\|\Delta\| \le \delta_0$,

$$\sup_{t \le T} E\left(|\gamma_\Delta(t)|^2\right) \le C_3 \|\Delta\|.$$

¿From (3.30) and Gronwall's lemma we deduce

$$g(t) := \overline{\lim_{\|\Delta\| \to 0}} \sup_{s \le t} E\left(|Z_\Delta(s) - X_s|^2\right) \le C_4 \int_0^t (\alpha + \tilde{\alpha})(E(g(s)))\, ds,$$

and hence $g(t) \equiv 0$.

**Definition 3.4.** Let $(F_i, G_i)_{i=1,2}$ satisfy $(H_1), (H_2), \left(\tilde{H}_1\right), \left(\tilde{H}_2\right)$.We say that the comparison principle hold for $(F_i, G_i)_{i=1,2}$ if given two initial conditions $\xi_i$, $i = 1, 2$, ( $\xi_i$ are $\mathcal{F}_0$-measurable) with $\xi_1 \le \xi_2$ a.s. and strong solutions $\{X_i(t)\}$ of the Itô equations

$$X_i(t) = \xi_i + \int_0^t F_i(s, X_i(s))ds + \int_0^t G_i(s, X_i(s))dW_s,\ 0 \le t \le T,$$

then $X_1(t) \le X_2(t)$ a.s. for every $t \ge 0$.

**Theorem 3.5 (comparison).** *Let $(\xi_i, F_i, G_i)_{i=1,2}$ satisfy the hypotheses $(H_1)-(H_3)$, $\left(\tilde{H}_1\right), \left(\tilde{H}_2\right)$ and let $\tilde{G}(s_1, s_2) : C_2(T) \longrightarrow R$ be a measurable and bounded function.*
*Let $\{X_i(t)\}$ be the unique solution of the double stochastic equation with additive double noise*

$$X_i(t) = \xi_i + \int_0^t F_i(s, X_i(s))ds + \int_0^t G_i(s, X_i(s))dW_s +$$

$$\int_{C_2(t)} \tilde{F}_i(s_1, s_2, X_i(s_1))ds_1 ds_2 + \int_{C_2(t)} \tilde{G}(s_1, s_2)d\tilde{W}_{s_1} d\tilde{W}_{s_2}, 0 \le t \le T. \quad (3.31)$$

*Moreover suppose that:*
*(1)* $\xi_1 \le \xi_2$ *a.s..*
*(2) The comparison principle hold for* $(F_i, G_i)_{i=1,2}$ *.*
*(3)* $\tilde{F}_1(s_1, s_2, x) \le \tilde{F}_2(s_1, s_2, y)$ *if* $x \le y$*, for all* $s_i$*.*
*Then* $X_1(t) \le X_2(t)$ *a.s. for every* $t \ge 0$*.*

**Proof.** By induction over the intervals $[t_{i-1}, t_i), i = 1, n$, it is easily seen that the fractional step approximations satisfy

$$Z_{1,\Delta}(t) \le Z_{2,\Delta}(t), \ Y_{1,\Delta}(t) \le Y_{2,\Delta}(t), \quad P\text{-as for all } t,$$

and passing to the limit (by Theorem 3.2) we obtain the conclusion.

**Remark 3.6** The comparison result given by the previous Theorem is more useful if for the first order Itô equation the comparison principle holds only for markovian coefficients and not for functional ones.
Comparison results for Itô equations can be found in [5] (and the references therein), [7], [9].
It should be interesting to extend the Theorem 3.5 to double stochastic Volterra equations, by using comparison results for first order stochastic Volterra equations as proven in [**?**], [17].

**Theorem 3.7** *Let* $(\xi, F, G)$ *satisfy the hypotheses* $(H_1) - (H_3)$, $\left(\tilde{H}_1\right), \left(\tilde{H}_2\right)$ *and let* $\tilde{F}(s_1, s_2, x) : C_2(T) \times R \longrightarrow R$ *be bounded and continuous and* $\tilde{G}(s_1, s_2) : C_2(T) \longrightarrow R$ *be a measurable and bounded function.*
*Then the double stochastic equation*

$$X_t = \xi + \int_0^t F(s, X_s)ds + \int_0^t G(s, X_s)dW_s +$$

$$\int_{C_2(t)} \tilde{F}(s_1, s_2, X_{s_1})ds_1 ds_2 + \int_{C_2(t)} \tilde{G}(s_1, s_2)d\tilde{W}_{s_1} d\tilde{W}_{s_2}, 0 \le t \le T, \quad (3.32)$$

*has a strong solution.*

**Proof.** Choose a sequence $\tilde{F}_n(s_1, s_2, x)$ of continuous, bounded and Lipschitz in $x$, which converges decreasing to $\tilde{F}(s_1, s_2, x)$.
Let $X_t^n$ be the strong solutions of

$$X_t^n = \xi + \int_0^t F(s, X_s^n)ds + \int_0^t G(s, X_s^n)dW_s +$$

$$\int_{C_2(t)} \tilde{F}_n(s_1, s_2, X_{s_1}^n)ds_1 ds_2 + \int_{C_2(t)} \tilde{G}(s_1, s_2)d\tilde{W}_{s_1} d\tilde{W}_{s_2}, 0 \leq t \leq T.$$

It is clear that the comparison principle hold for $(F, G)$ (see for example [7]). Thenb by Theorem 3.5 we have

$$X_t^1 \geq X_t^2 \geq ... \geq X_t^n \geq ..... \quad P\text{-a.s for all } t \geq 0.$$

It is easily seen that $\lim_{n\to\infty} X_t^n = X_t$ is a solution of (3.32).

Next we give an example (Lemma 3.9) which shows that the comparison principle of Theorem 3.5 is not satisfied if the diffusion coefficient of the double stochastic integral depends on the state variable.
For $f \in L^2_{loc}(R_+)$ and $0 \leq t \leq T$ we define

$$Z_t^f = \exp\left\{\int_0^t f(s)dW_s - \frac{1}{2}\int_0^t |f(s)|^2 ds\right\},$$

$$\Phi_t^f = \exp\left\{-2\int_0^t f(s)dW_s\right\}.$$

**Lemma 3.8** *(a) The unique solution of the linear double stochastic equation*

$$X_t^f = 1 + \int_{C_2(t)} f(s_1)f(s_2)X_{s_1}^f d\tilde{W}_{s_1} d\tilde{W}_{s_2}, 0 \leq t \leq T, \tag{3.33}$$

*is given by*

$$X_t^f = \frac{1}{2}Z_t^f\left(1 + \Phi_t^f\right) = \frac{1}{2}\left(Z_t^f + Z_t^{(-f)}\right). \tag{3.34}$$

*(b) If $(H_t)$ is a continuous semimartingale, then the unique solution of the double stochastic equation*

$$X_t = H_t + \int_{C_2(t)} f(s_1)f(s_2)X_{s_1} d\tilde{W}_{s_1} d\tilde{W}_{s_2}, 0 \leq t \leq T, \tag{3.35}$$

*is given by*

$$X_t = X_t^f H_0 + X_t^f \int_0^t \frac{\left(X_s^f dH_s - f(s)d\left[H, \tilde{W}\right]_s\right)}{\left(\left|X_s^f\right|^2 - \left|Y_s^f\right|^2\right)^2}$$

$$-Y_t^f \int_0^t \frac{Y_s^f dH_s}{\left(\left|X_s^f\right|^2 - \left|Y_s^f\right|^2\right)^2}, \tag{3.36}$$

$$Y_t^f = X_t^f - 1, \tag{3.37}$$

*where* $[.,.]$ *denotes the quadratic variation.*

**Proof.** *(a)* We have

$$\int_{C_2(t)} f(s_1)f(s_2)X_{s_1}^f d\tilde{W}_{s_1} d\tilde{W}_{s_2} = \int_0^t f(s_2)\left(\int_0^{s_2} f(s_1)X_{s_1}^f d\tilde{W}_{s_1}\right) d\tilde{W}_{s_2} =$$

$$\frac{1}{2}\int_0^t f(s_2)\left[\int_0^{s_2} f(s_1)\left(Z_{s_1}^f + Z_{s_1}^{(-f)}\right) d\tilde{W}_{s_1}\right] d\tilde{W}_{s_2} =$$

$$\frac{1}{2}\int_0^t f(s_2)\left(Z_{s_2}^f - 1 - Z_{s_1}^{(-f)} + 1\right) d\tilde{W}_{s_2} =$$

$$\frac{1}{2}\left(Z_t^f - 1 + Z_{s_1}^{(-f)} - 1\right) = X_t^f - 1.$$

*(b)* The previous computation shows that

$$Y_t^f = \int_0^t f(s_1)X_{s_1}^f d\tilde{W}_{s_1} = X_t^f - 1.$$

Define

$$\tilde{f}(t) = \begin{pmatrix} f(t) & 0 \\ 0 & f(t) \end{pmatrix}, V_t = \begin{pmatrix} X_t^f & Y_t^f \\ Y_t^f & X_t^f \end{pmatrix},$$

$$U_t^T = \left(X_t^f, Y_t^f\right),\ \tilde{X}_t^T = (X_t, Y_t),\ \tilde{H}_t^T = (H_t, 0).$$

Then

$$U_t = e_1 + \int_0^t \tilde{f}(s) U_s d\tilde{W}_s,$$

$$\tilde{X}_t = \tilde{H}_t + \int_0^t \tilde{f}(s) \tilde{X}_s d\tilde{W}_s.$$

By Theorem 4 of [19] it follows that

$$\tilde{X}_t = V_t \tilde{H}_0 + V_t \int_0^t V_s^{-1} \left( X_s^f dH_s - f(s) d \left[ H, \tilde{W} \right]_s \right)$$

$$-Y_t^f \int_0^t \frac{Y_s^f dH_s}{\left( \left| X_s^f \right|^2 - \left| Y_s^f \right|^2 \right)^2},$$

which implies (3.36).
We consider the following "pure" double stochastic equations

$$X_1(t) = 1 + \int_{C_2(t)} X_1(s_1) d\tilde{W}_{s_1} d\tilde{W}_{s_2},$$

$$X_2(t) = 1 + \frac{t^2}{2} + \int_{C_2(t)} X_2(s_1) d\tilde{W}_{s_1} d\tilde{W}_{s_2}.$$

By Lemma 3.8 with $f \equiv 1, H_t = \frac{t^2}{2}$ we have

$$X_1(t) = \frac{1}{2} \left( Z_t^1 + Z_t^{(-1)} \right), \; X_2(t) = X_1(t) + R(t),$$

$$R(t) = \int_0^t \frac{2X^1(s) ds}{\left(2X^1(s) - 1\right)^2} +$$

$$X_1(t) \int_0^t \frac{s ds}{\left(2X^1(s) - 1\right)^2} - \int_0^t \frac{s ds}{\left(2X^1(s) - 1\right)^2}.$$

**Lemma 3.9** *We have that*

$$\lim_{t \longrightarrow \infty} P(R(t) < 0) = 1,$$

*Therefore the comparison principle is not satisfied for the solutions* $X_1, X_2$*(Note that in the present situation* $\xi_1 = \xi_2 = 1, \tilde{F}_1 = 0 < \tilde{F}_2 = 1$*).*

**Proof.** It is known that

$$\lim_{t\longrightarrow\infty} Z_t^1 = \lim_{t\longrightarrow\infty} Z_t^{(-1)} = 0,$$

and hence $\lim_{t\longrightarrow\infty} X_1(t) = 0$.
Define

$$\delta(t) = \left[\int_0^t \frac{sX_1(s)ds}{(2X^1(s)-1)^2}\right]\left[\int_0^t \frac{sds}{(2X^1(s)-1)^2}\right]^{-1}.$$

Then

$$P(R(t)<0) = P\left(1-X^1(t) > \delta(t)\right) \text{ a.s.},$$

and all we need to show is that $\lim_{t\longrightarrow\infty}\delta(t) = 0$ a.s.
Obviously

$$\lim_{t\longrightarrow\infty}\int_0^t \frac{sds}{(2X^1(s)-1)^2} = \infty \; a.s.$$

Assume that

$$\lim_{t\longrightarrow\infty}\int_0^t \frac{sds}{(2X^1(s)-1)^2} = \sigma.$$

If $\sigma < \infty$ then $\lim_{t\longrightarrow\infty}\delta(t) = 0$ and if $\sigma = 0$ tha by l'Hôspital rule

$$\lim_{t\longrightarrow\infty}\delta(t) = \lim_{t\longrightarrow\infty} X_1(t) = 0.$$

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