# Size-and-shape Cone, Shape Disk and Configuration Densities for the Elliptical Models.

**By José A. Díaz-García** *

*Universidad Autónoma Agraria Antonio Narro*
*25315 Buenavista, Saltillo*
*Coahuila, México*

**Ramón Gutiérrez-Jáimez**

*Universidad de Granada*
*Facultad de Ciencias*
*18071 Granada, España*

**and**

**Rogelio Ramos-Quiroga**

*Centro de Investigación en Matemáticas*
*Callejón de Jalisco s/n*
*36240 Guanajuato, Gto., México*

**Abstract**

The size-and-shape cone, shape disk and configuration densities are studied under elliptically contoured distributions in the central case. We prove that the shape disk and configuration densities are invariant on the family of elliptical laws.

## 1. INTRODUCTION

The elliptically contoured distribution of a random matrix has been studied by various authors, including Fang and Zhang [4] and Gupta and Varga [7]. From the definition of Gupta and Varga [7, p. 12], $X : N \times K$ has a matrix variate elliptically contoured distribution if its characteristic function takes the form $\phi_X(T) = \operatorname{etr}(iT\mu^t)\psi(\operatorname{tr}(T^t\Sigma T\Theta))$ with $T : N \times K$, $\mu : N \times K$, $\Sigma : N \times N$, $\Theta : K \times K$, $\Sigma \geq 0$, $\Theta \geq 0$ and $\psi : [0, \infty) \to I\!R$. If $X$ has a density, this is given by:

$$f_X(X) = |\Sigma|^{-K/2}|\Theta|^{-N/2}h(\operatorname{tr}((X-\mu)^t\Sigma^{-1}(X-\mu)\Theta^{-1})),$$

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where $h$ and $\psi$ determine each other for specified $N$ and $K$. Such a distribution is termed $X \sim \mathcal{E}_{N\times K}(\mu, \Sigma, \Theta, \psi)$ or $X \sim \mathcal{E}_{N\times K}(\mu, \Sigma, \Theta, h)$, respectively.

Assuming full rank $X$, we wish to study the distribution of the singular values $l_1 \geq l_2 \geq \cdots \geq l_n \geq 0$, $n = \min(N, K)$ and that of the normalized singular values $l_i^* = l_i/r$, $r = \left(\sum_i^n l_i^2\right)^{1/2}$ of matrix $X$. In the context of shape theory, the corresponding densities are known as size-and-shape cone density and shape disk density, respectively, Goodall and Mardia [6] and Díaz- García, Gutiérrez and Mardia [3].

In the case when $X$ has a Gaussian (Normal)distribution, the density of the singular values has been studied by various authors. Goodall and Mardia [6] have found the respective joint densities of the $l_i$ and $l_i^*$ for an arbitrary $N$ and $K = 2$ in the nonsingular central case. Note that the singular values of $X$ are the positive square root of the latent roots of $B = XX'$ (Wishart matrix). Thus, among many others, Srivastava and Khatri [9, p. 68] have found the joint density of the latent roots of $B$ when $\Sigma = I_N$ and $\Theta = I_K$ in the central case. A more general result has been obtained by Muirhead [8, p. 388] for an arbitrary $\Sigma > 0$ in the central case. In the nonsingular noncentral case, Davis [2] has studied the density of the latent roots of $B$ using a generalization of zonal polynomials called invariant polynomials. For the case of the noncentral Pseudo-Wishart matrix, Díaz, Gutirrez and Mardia [3] have found the shape disk density. When $X$ has an elliptically contoured distribution, Fang and Zhang [4] and Anderson and Fang [1] have found the joint density of the latent roots of $B$, when $\Sigma = I_N$ and $\Theta = I_K$ for the central case. Similarly, Teng et al. [10], by means of an expansion of Taylor series, have found the distribution of latent roots of $B$, when $\Sigma = I_n$ and $\Theta > 0$ in the noncentral case.

Let us now assume that $X$ represents a geometric figure comprising $N$ points in $\mathbb{R}^K$ and that $Y : (N-1) \times K$ is invariant to translations of the figure $X$. We then have $Y = LX$, where $L$ is a matrix of dimension $(N-1) \times N$ with orthonormal rows, orthogonal to the vector $\mathbf{1} = (1, 1, \ldots, 1)'$, Goodall and Mardia[6]. Then, assuming that $Y \sim \mathcal{E}_{(N-1)\times K}(0, \sigma^2 I_{N-1}, I_K, h)$ we show that the size-and-shape cone density may be found from the singular value decomposition of matrix $Y$ (see Theorem 2.1). From this density, we are able to determine the shape disk density and show that this is invariant under the family of elliptically contoured distributions presenting the above parameters; such is the case of the Gaussian distribution (see Theorem 2.2), we also give the expression for the shape disk density in the Gaussian setting for $K = n = 2$ (see Corollary 2), this result was given in an incorrect form by Goodal and Mardia [5] and [6]; in these same references the configuration density is stated incorrectly too. Therefore, Section 3 provides a complete proof of the configuration density, with the corresponding corrections (see Theorem 3.1). Additionally we show that configuration density is invariant under the family of elliptical distributions (see Theorem 3.2).

## 2. SIZE-AND-SHAPE CONE DENSITY AND SHAPE DISK DENSITY.

When $N - 1 \leq K$, the size-and-shape cone density and the shape disk density, in the Gaussian case, are generally obtained from the the density of the Wishart $B$ matrix or from the density of $T$, the triangular matrix in the QR decomposition of $Y$, see Goodall and Mardia [5], [6]. When $N - 1 > K$, it is possible to find such densities as a function of that of $T$, the triangular matrix QR decomposition of the $Y^t$ matrix, see Goodall and Mardia [5]. Díaz, Gutirrez and Mardia [3] used the singular value decomposition of matrix $X$ and found a general expression for the density of $B$ for any case. From the latter density, they were able to identify the shape disk density. For the rank of $Y = n = \min(N-1, K)$, it is possible to apply the singular value decomposition of $Y$, thus obtaining:

THEOREM 2.1. *Let $Y \sim \mathcal{E}_{N-1\times K}(0, \sigma^2 I_{N-1}, I_K, h)$. Then the size-and-shape cone density is*

$$\frac{2^n \pi^{n((N-1)+K)/2} \prod_{j=1}^{n} l_j^{N-1+K-2n} \prod_{i<j}^{n} (l_i^2 - l_j^2)}{\sigma^{K(N-1)} \Gamma_n[\frac{1}{2}K] \Gamma_n[\frac{1}{2}(N-1)]} h\left(\frac{1}{\sigma^2}\sum_{j=1}^{n} l_j^2\right). \tag{1}$$

*Proof.* We know that

$$f_Y(Y) = \frac{1}{\sigma^{K(N-1)}} h\left(\frac{1}{\sigma^2} \operatorname{tr} YY^t\right)$$

Perform the factorization $Y = H_1 L P_1^t$, where $L = \operatorname{diag}(l_1, \cdots, l_n)$, $H_1 \in V_{n,N-1}$ the Stiefel manifold, and $P_1 \in V_{n,K}$. The Jacobian is (see Díaz, Gutirrez and Mardia [3])

$$(dY) = 2^{-n} \prod_{j=1}^{n} l_j^{N-1+K-2n} \prod_{i<j}^{n} (l_i^2 - l_j^2)(dL) \wedge (H_1 dH_1^t) \wedge (P_1 dP_1^t).$$

Therefore, the joint density of $H_1$, $L$ and $P_1$ is,

$$\frac{2^{-n} \prod_{j=1}^{n} l_j^{N-1+K-2n} \prod_{i<j}^{n} (l_i^2 - l_j^2)}{\sigma^{K(N-1)}} h\left(\frac{1}{\sigma^2}\sum_{j=1}^{n} l_j^2\right) (dL) \wedge (H_1 dH_1^t) \wedge (P dP^t).$$

Integrating with respect to $H_1$ and $P_1$ (see Muirhead [8]), we have

$$\int_{V_{n,K}} (P_1 dP_1^t) = \frac{2^n \pi^{nK/2}}{\Gamma_n[\frac{1}{2}K]} \quad \text{and} \quad \int_{V_{n,N-1}} (H_1 dH_1^t) = \frac{2^n \pi^{n(N-1)/2}}{\Gamma_n[\frac{1}{2}(N-1)]}$$

the result is obtained. ∎

We present below the particular cases of two elliptically distributed families, the matrix variate symmetric Kotz type and Pearson Type VII distributions, (see Gupta and Varga [pp. 75-76][7]).

COROLLARY 1. *Let $Y \sim \mathcal{E}_{N-1\times K}(0, \sigma^2 I_{N-1}, I_K, h)$. Then*

1. *if $Y$ has a matrix variate symmetric Kotz type distribution with parameters $q, p, s \in \mathbb{R}$, with $p > 0$, $s > 0$ and $2q + K(N-1) > 0$, the size-and-shape cone density is,*

$$\frac{2^n s p^{2q+K(N-1)-2)/2s} \Gamma[\frac{1}{2}(K(N-1))] \pi^{[n((N-1)+K)-K(N-1)]/2} \prod_{j=1}^{n} l_j^{N-1+K-2n} \prod_{i<j}^{n} (l_i^2 - l_j^2)}{\sigma^{K(N-1)+2(q-1)} \Gamma_n[\frac{1}{2}K] \Gamma_n[\frac{1}{2}(N-1)] \Gamma[2q + K(N-1) - 2)/2s]}$$

$$\left(\sum_{j=1}^{n} l_j^2\right)^{q-1} \exp\left[-\frac{r}{\sigma^{2s}}\left(\sum_{j=1}^{n} l_j^2\right)^{s}\right].$$

*If we take* $q = s = 1$ *and* $p = \dfrac{1}{2}$*, we obtain the density in the Gaussian case.*

2. *if* $Y$ *has a matrix variate symmetric Pearson type VII distribution with parameters* $q, p \in \mathbb{R}$*,* $p > 0$ *and* $q > \dfrac{K(N-1)}{2}$*, the size-and-shape cone density is,*

$$\frac{2^n \pi^{[n((N-1)+K)-K(N-1)]/2}\Gamma[q]\prod_{j=1}^{n} l_j^{N-1+K-2n}\prod_{i<j}^{n}(l_i^2 - l_j^2)}{p^{K(N-1)/2}\sigma^{K(N-1)}\Gamma_n[\frac{1}{2}K]\Gamma_n[\frac{1}{2}(N-1)]\Gamma[\frac{1}{2}(2q-K(N-1))]}\left(1+\frac{1}{p\sigma^2}\sum_{j=1}^{n} l_j^2\right)^{-q}.$$

*In particular, when* $q = (K(N-1)+p)/2$ *we obtain the density in the matrix variate t-distribution case. And if* $p = 1$*, in the definition of matrix variate t-distribution, we have the density in matrix variate Cauchy distribution.*

Now for the shape disk density we have:

THEOREM 2.2. *Let* $Y \sim \mathcal{E}_{N-1\times K}(0, \sigma^2 I_{N-1}, I_K, h)$*. Then the shape disk density is*

$$\frac{2^{n-1}\Gamma[\frac{1}{2}(n(N+K-2n+2)-2)]\prod_{j=1}^{n} l_j^{*N+K-2n-1}\prod_{i<j}^{n}(l_i^{*2} - l_j^{*2})}{\pi^{-[n(2n-3)/2+1]}\sigma^{K(N-1)-n(N+K-2n+2)+2}\Gamma_n[\frac{1}{2}K]\Gamma_n[\frac{1}{2}(N-1)]}\left(\prod_{m=1}^{n-2}\sin^{n-m-1}\psi_m\right) \tag{2}$$

*where*

$$\begin{aligned}
l_j^* &= \left(\prod_{m=1}^{j-1}\sin\psi_m\right)\cos\psi_j \quad ; \quad 1 \le j \le n-2 \quad , \quad 0 \le \psi_t \le \pi \quad , \quad 1 \le t \le n-2 \\
l_{n-1}^* &= \left(\prod_{m=1}^{n-2}\sin\psi_m\right)\cos\theta \quad ; \quad 0 \le \theta \le 2\pi \\
l_n^* &= \left(\prod_{m=1}^{n-2}\sin\psi_m\right)\sin\theta \quad .
\end{aligned}$$

*Proof.* Define $l_j^* = l_j/r$ with $r = \left(\sum_{j=1}^{n} l_j^{1/2}\right)^2$ and observe that $\sum_{j=1}^{n}(l_j^*)^2 = 1$. Consider the following transformation,

$$\begin{aligned}
l_j &= r\left(\prod_{m=1}^{j-1}\sin\psi_m\right)\cos\psi_j \quad ; \quad 1 \le j \le n-2 \quad , \quad 0 \le \psi_t \le \pi \quad , \quad 1 \le t \le n-2 \\
l_{n-1} &= r\left(\prod_{m=1}^{n-2}\sin\psi_m\right)\cos\theta \quad ; \quad 0 \le \theta \le 2\pi \\
l_n &= r\left(\prod_{m=1}^{n-2}\sin\psi_m\right)\sin\theta \quad ; \quad 0 \le r < \infty \quad .
\end{aligned}$$

Where,

$$J[(l_1,\cdots,l_n)\to(r,\psi_1,\cdots,\psi_{n-2},\theta)]=r^{n-1}\left(\prod_{m=1}^{n-2}\sin^{n-m-1}\psi_m\right).$$

Hence, the joint density of $l_j^*,\quad j=1,\cdots n$ is

$$\frac{2^n\pi^{n((N-1)+K)/2}\prod_{j=1}^{n}l_j^{*N-1+K-2n}\prod_{i<j}^{n}(l_i^{*2}-l_j^{*2})}{\sigma^{K(N-1)}\Gamma_n[\frac{1}{2}K]\Gamma_n[\frac{1}{2}(N-1)]}\left(\prod_{m=1}^{n-2}\sin^{n-m-1}\psi_m\right)$$
$$\int_0^\infty r^{n(N+K-2n+2)-3)}h\left(\frac{r^2}{\sigma^2}\right)dr.$$

Now observe that

$$\begin{aligned}\int_0^\infty r^{n(N+K-2n+2)-3}h\left(\frac{r^2}{\sigma^2}\right)dr &= \sigma^{n(N+K-2n+2)-2}\int_0^\infty s^{n(N+K-2n+2)-3}h(s^2)ds\\ &= \frac{\sigma^{n(N+K-2n+2)-2}\Gamma[\frac{1}{2}(n(N+K-2n+2)-2)]}{2\pi^{(n(N+K-2n+2)-2)/2}}\end{aligned}$$

and the result is as follows. ∎

Corollary 2. *Take $n=K=2$ in the density (2). Then the shape disk density is given by*

$$2(N-2)(\sin 2\theta)^{N-3}\cos 2\theta \tag{3}$$

*where $l_1^*=\cos\theta,\quad l_2^*=\sin\theta,\quad 0\le\theta\le\pi/4$.*

*Proof.* The proof is immediate from Theorem 2.2. Simply by observing that when $n=K=2$, we have

1. $\pi^{n(2n-3)/2+1}=\pi^2$.
2. $\Gamma_n[\frac{1}{2}K]=\pi$.
3. $\Gamma_n[\frac{1}{2}(N-1)]=(\pi(N-3)!)/2^{N-3}$.
4. $\Gamma[\frac{1}{2}(n(N+K-2n+2)-2)]=(N-2)(N-3)!$
5. $\sigma^{K(N-1)-n(N+K-2n+2)+2}=1$
6. $\left(\prod_{m=1}^{n-2}\sin^{n-m-1}\psi_m\right)=1$
7. $l_1^*=\cos\theta$ and $l_2^*=\sin\theta$
8. and $\cos 2\theta=\cos^2\theta-\sin^2\theta$ and $\sin\theta\cos\theta=\frac{1}{2}\sin 2\theta$. ∎

*Remark* 2.1. Note that the distribution of the shape disk is independent of $h$, that is, the distribution of normalized singular values of $X$ is invariant on $\mathcal{E}_{N-1\times K}(0, \sigma^2 I_{N-1}, I_K, h)$, which is a very interesting fact. Therefore, this distribution coincides with the Gaussian distribution case.

## 3. CONFIGURATION DENSITIES

Consider the following definition, Goodall and Mardia [6]:

DEFINITION 3.1. *Two figures $X : N \times K$ and $X' : N \times K$ have the same configuration, or affine shape, if for some (e,E)*

$$X' = XE + \mathbf{1}_N e^t$$

*where $e : N \times 1$ is the translation and $E : K \times K$ is nonsingular.*

The configuration coordinates are constructed in two steps summarized in the expression

$$LX = Y = UE$$

The matrix $U : (N-1) \times K$ contains configuration coordinates of $X$. Our interest now lies in finding the distribution of $U^t = (I|V^t)^t$, where if, $Y = (Y_1^t|Y_2^t)^t$ is partitioned into $Y_1 : K \times K$ (we assume that $Y_1$ is nonsingular), and $Y_2 : q \times K$, where $q = N - K - 1 \geq 1$, then $E = Y_1$ and $V = Y_2 Y_1^{-1}$.

THEOREM 3.1. *With the Gaussian model and the above notation, the configuration density is*

$$\frac{2^K \Gamma_K[(N-1)/2]}{\pi^{Kq/2}|I + V^t V|^{(N-1)/2}\Gamma_K[K/2]} \operatorname{etr}\left\{\frac{\Omega}{2\sigma^2} - \frac{\mu^t\mu}{2\sigma^2}\right\} {}_1F_1\left(-\frac{q}{2}; \frac{k}{2}; -\frac{\Omega}{2\sigma^2}\right)$$

*where $\Omega = \mu^t U(U^t U)^{-1} U^t \mu = \mu^t Y(Y^t Y)^{-1} Y^t \mu$ and $M = (N-1)K$, for all $Y$ for which rank $Y_1 = K$. When $q$ is a positive integer, the expression ${}_1F_1$ is a polynomial with degree $Kq/2$ in the latent roots of $\Omega$.*

*Proof.* We have

$$\begin{aligned} f_Y(Y) &= \frac{1}{(2\pi)^{M/2}\sigma^M} \operatorname{etr}\left(-\frac{1}{2\sigma^2}(Y-\mu)(Y-\mu)^t\right) \\ &= \frac{1}{(2\pi)^{M/2}\sigma^M} \operatorname{etr}\left(-\frac{1}{2\sigma^2}YY^t - \frac{1}{2\sigma^2}\mu\mu^t + \frac{1}{2\sigma^2}\mu^t Y\right). \end{aligned}$$

Perform the substitution $Y = UF^{\frac{1}{2}}H$ where $H \in \mathcal{O}(K)$ and $(F^{\frac{1}{2}})^2 > 0$. The Jacobian is $(dY) = |F|^{(q-1)/2}(dV)(dF)(HdH^t)$, from which the joint density of $F$, $V$ and $H$ is,

$$\frac{\operatorname{etr}\left(-\frac{1}{2\sigma^2}\mu\mu^t\right)}{(2\pi)^{M/2}\sigma^M}|F|^{(q-1)/2} \operatorname{etr}\left(-\frac{1}{2\sigma^2}UFU^t\right) \operatorname{etr}\left(\frac{1}{2\sigma^2}\mu^t UF^{\frac{1}{2}}H\right)(dF)(dV)(HdH^t)$$

we now find that (see Muirhead [8, p. 262])

$$\int_{\mathcal{O}(K)} \operatorname{etr}\left(\frac{1}{2\sigma^2}\mu^t U F^{\frac{1}{2}} H\right)(H dH^t) = \frac{2^K \pi^{K^2/2}}{\Gamma_K[K/2]} {}_0F_1\left(\tfrac{1}{2}K; \frac{1}{4\sigma^4}\mu^t U F U^t \mu\right).$$

Then the joint density of $V$ and $F$ is

$$\frac{\sigma^{-M} \operatorname{etr}\left(-\dfrac{1}{2\sigma^2}\mu\mu^t\right)}{2^{(M-2K)/2}\pi^{(M-K^2)/2}\Gamma_K[K/2]} |F|^{(q-1)/2} \operatorname{etr}\left(-\frac{1}{2\sigma^2} U F U^t\right) {}_0F_1\left(\tfrac{1}{2}K; \frac{1}{4\sigma^4}\mu^t U F U^t \mu\right)(dF)(dV).$$

Consider the following integral

$$I = \int_{F>0} |F|^{(q-1)/2} \operatorname{etr}\left(-\frac{1}{2\sigma^2} U F U^t\right) {}_0F_1\left(\tfrac{1}{2}K; \frac{1}{4\sigma^4}\mu^t U F U^t \mu\right)(dF).$$

Taking $Z = \left(\dfrac{1}{2\sigma^2}U^t U\right)$, $Y = \left(\dfrac{1}{4\sigma^4}U^t \mu\mu^t U\right)$, $a = (N-1)/2$ and $m = K$ in Theorems 7.2.7 and 7.3.4 (see Muirhead [8, pp. 248 and 260]) we have,

$$I = \Gamma_K[(N-1)/2] \left|\frac{1}{2\sigma^2}U^t U\right|^{-(N-1)/2} {}_1F_1\left(\tfrac{1}{2}(N-1); \tfrac{1}{2}K; \left(\frac{1}{4\sigma^4}U^t \mu\mu^t U\right)\left(\frac{1}{2\sigma^2}U^t U\right)^{-1}\right)$$

and observe that

$$\left|\frac{1}{2\sigma^2}U^t U\right|^{-(N-1)/2} = 2^{K(N-1)/2}\sigma^{K(N-1)}|U^t U|^{-(N-1)/2} = 2^{M/2}\sigma^M |I + V^t V|^{-(N-1)/2}$$

and

$$\operatorname{tr}\left(\frac{1}{4\sigma^4}U^t \mu\mu^t U\right)\left(\frac{1}{2\sigma^2}U^t U\right)^{-1} = \frac{1}{2\sigma^2}\operatorname{tr}\mu^t U (U^t U)^{-1} U^t \mu.$$

Then

$$I = \frac{2^{M/2}\Gamma_K[(N-1)/2]\sigma^M}{|I + V^t V|^{(N-1)/2}} {}_1F_1\left(\frac{N-1}{2}; \frac{K}{2}; \frac{1}{2\sigma^2}\mu^t U (U^t U)^{-1} U^t \mu\right).$$

Applying the Kummer relation (see Muirhead [8, p. 265]), we obtain

$$I = \frac{2^{M/2}\Gamma_K[(N-1)/2]\sigma^M}{|I + V^t V|^{(N-1)/2}} \operatorname{etr}\left(\frac{1}{2\sigma^2}\mu^t U (U^t U)^{-1} U^t \mu\right) {}_1F_1\left(-\frac{q}{2}; \frac{K}{2}; -\frac{1}{2\sigma^2}\mu^t U (U^t U)^{-1} U^t \mu\right),$$

given that

$$\frac{K}{2} - \frac{N-1}{2} = \frac{K-N+1}{2} = -\frac{N-K-1}{2} = -\frac{q}{2}.$$

Hence

$$f_U(U) = \frac{2^K \Gamma_K[(N-1)/2]}{\pi^{(M-K^2)/2}|I + V^t V|^{(N-1)/2}\Gamma_K[K/2]} \operatorname{etr}\left\{\frac{\mu^t U (U^t U)^{-1} U^t \mu}{2\sigma^2} - \frac{\mu^t \mu}{2\sigma^2}\right\}$$
$$ {}_1F_1\left(-\frac{q}{2}; \frac{k}{2}; -\frac{1}{2\sigma^2}\mu^t U (U^t U)^{-1} U^t \mu\right)$$

Finally, with $Kq = (M - K^2)$ and $\Omega = \mu^t U (U^t U)^{-1} U^t \mu$, the result follows. ∎

In the following theorem, the above result is extended to the case of the central elliptical model. As in the case of the shape disk density, the configuration density is also invariant under the elliptical model.

THEOREM 3.2. *Let* $Y \sim \mathcal{E}_{N-1\times K}(0, \sigma^2 I_{N-1}, I_K, h)$. *Then the configuration density is invariant under the elliptical family, and is given by*

$$\frac{2^K \Gamma_K[(N-1)/2]}{\pi^{Kq/2}\Gamma_K[K/2]} |I + V^t V|^{-(N-1)/2}$$

*Proof.* Using the same procedure as in Theorem 3, the joint density of $F, U, H$ is given by:

$$\frac{1}{\sigma^M}|F|^{(q-1)/2} h\left(\frac{1}{\sigma^2}\operatorname{tr} U^t U F\right)(dV)(dF)(HdH^t)$$

and the density of $V$ is obtained by integrating over $H \in \mathcal{O}(K)$ and over $F >$. Thus:

$$\frac{2^K \pi^{K^2/2}}{\Gamma_K[\frac{1}{2}K]\sigma^M} \int\limits_{F>0} |F|^{(q-1)/2} h\left(\frac{1}{\sigma^2}\operatorname{tr} U^t U F\right)(dV)(dF).$$

The result follows by taking $n = N-1$, $p = k$ and $\Sigma = (\frac{1}{\sigma^2}U^t U)^{-1}$ in the following expression (see Gupta and Varga[p. 169][7]),

$$\int\limits_{A>0} |A|^{(n-p-1)/2} h(\operatorname{tr}\Sigma^{-1}A)(dA) = \frac{\Gamma_p[\frac{1}{2}n]|\Sigma|^{n/2}}{\pi^{np/2}}.$$

■

*Remark* 3.1. Observe that, according to the notation used in section 3.2.4 in Fang and Zhang [p.102][4], $Y$ has a Vector-Elliptical distribution, while $V$ has a Spherical distribution. Furthermore, $V$ has a matrix variate type $t$-distribution, see Fang and Zhang [p. 112][4].

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