PROOFS

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Natural quotients on split supercotangent bundles and their canonical supersymplectic structures*

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Abstract: The Batchelor model of the supercotangent bundle of a given base supermanifold is studied. Under the assumption that the supercotangent bundle splits, two different fibrations over the given base can be globally defined. The total spaces of these fibrations are in turn quotient supermanifolds of the supercotangent bundle, and each of them is equipped with a supersymplectic structure. Their corresponding supersymplectic 2-forms are actually exact, and homogeneous of different degrees. The homogeneous 1-forms from which they come from are natural with respect to Batchelor trivializations. Each of these 1-forms can be pulled back to the supercotangent bundle via the quotient maps, and can be added together in the supercotangent bundle to produce a nonhomogeneous 1-form there. Such a 1-form in the supercotangent bundle is canonical; it is characterized by the fact that the pullback of it under any 1-form on the base supermanifold yields the same 1-form on the base. The exterior derivative of this canonical 1-form is degenerate. Its radical produces an example of an involutive subsheaf, which is not integrable. This phenomenon is explained at the light of Frobenius for supermanifolds. The radicals of its homogeneous components, on the other hand, taken separately, do produce two globally defined foliations on the supercotangent bundle, and the corresponding spaces of leaves are precisely the two quotients of the supercotangent bundle we started with.

Keywords:.

MS classification:.

1. Introduction and description of results

The general framework of this paper is that of $\mathbb{Z}_2$-graded manifolds as defined in [13, 18, 20]. We refer the reader to [2, 7, 27, 28] for alternative generalizations and further examples. Following [20] a $\mathbb{Z}_2$-graded manifold is a pair $(M, \mathcal{A})$, where M is a smooth, finite-dimensional manifold, and $\mathcal{A}$ is a sheaf of $\mathbb{Z}_2$-graded algebras over M locally isomorphic to the sheaf of sections of the exterior algebra bundle $\Lambda E \to M$ of a vector bundle $E \to M$—called the *Batchelor*

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bundle—canonically attached to $\mathcal{A}$. More specifically, there is always defined a sheaf morphism $\mathcal{A} \to \Gamma(\Lambda E)$ which restricts to an isomorphism of the corresponding $\mathbb{Z}_2$-graded algebras of local sections over a suitable neighborhood of each point. When $\mathcal{A}$ is *globally* isomorphic to $\Gamma(\Lambda E)$—i.e., when $\mathcal{A} \to \Gamma(\Lambda E)$ can be inverted—the supermanifold is called *split* (or "Batchelor-trivial"). Split supermanifolds exhaust all the examples of smooth supermanifolds (see [3, 9, 15, 20]). In the holomorphic category, this is far from being the case (see [10, 29]). This does not mean, however, that the category of smooth supermanifolds gets subsummed into the category of smooth vector bundles: Supermanifold morphisms are a lot more general than the induced vector bundle morphisms at the level of the Batchelor trivializations involved. In fact, the class of split graded manifolds is interesting, and wide enough so as to deserve special consideration.

A fundamental relationship between splittings and connections has been given by Koszul in [15]: A supermanifold is split if and only if it admits a $\mathbb{Z}_2$-*graded connection.* When such a connection exists, it induces a connection on $E \to M$. Since, connections always exist on smooth vector bundles, but may not exist in the holomorphic category (see [1], and [22]), the Koszul approach sheds new light into the earlier results on non-split holomorphic supermanifolds given in [10] and [29]. For the purposes of our work, the relevant corollary of [15] is that if a connection fails to exist on either TM or E, the supertangent and the supercotangent bundles of a split supermanifold do not split (this was also pointed out by M. Rothstein in a beautiful paper [30] where the general theory for *even* symplectic structures on split supermanifolds was studied, and canonical forms for such structures were given).

A vector bundle over $(M, \mathcal{A})$ is a locally free sheaf $\mathcal{F}$ of $\mathcal{A}$-modules over M with a given $\mathbb{Z}_2$-grading, i.e., $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$. Given such an $\mathcal{F}$, a geometrical and functorial construction of a $\mathbb{Z}_2$-graded manifold $(\mathcal{V}_{\mathcal{F}}(M), \mathcal{V}_{\mathcal{F}}(\mathcal{A}))$ equipped with a $\mathbb{Z}_2$-graded submersion

$$\pi\colon \big(\mathcal{V}_{\mathcal{F}}(M), \mathcal{V}_{\mathcal{F}}(\mathcal{A})\big) \to (M, \mathcal{A}), \tag{1}$$

can be performed in such a way that the abstract sections of $\mathcal{F}$ correspond in a one-to-one fashion with the geometric sections of the submersion π:

$$\{\, \beta \mid \beta \in \mathcal{F}(M) \,\} \longleftrightarrow \{\, \beta\colon (M, \mathcal{A}) \to (\mathcal{V}_{\mathcal{F}}(M), \mathcal{V}_{\mathcal{F}}(\mathcal{A})) \mid \pi \circ \beta = \mathrm{id} \,\}. \tag{2}$$

It was one of the main issues of the early supermanifold theory to prove (or to guarantee by means of suitable axioms), that $\mathcal{F} = \mathrm{Der}\,\mathcal{A}$, and $\mathcal{F}^* = \mathrm{Hom}_{\mathcal{A}}(\mathrm{Der}\,\mathcal{A}, \mathcal{A})$ were actually locally free sheaves of $\mathcal{A}$-modules. In particular, the construction of (1) above can be applied to them, and respectively the tangent and cotangent $\mathbb{Z}_2$-graded manifolds associated to $(M, \mathcal{A})$ can be obtained. They shall be hereby denoted by $ST(M, \mathcal{A})$, and $ST^*(M, \mathcal{A})$, respectively, and we are specifically interested in the case when $\mathcal{A} = \Gamma(\Lambda E)$.

When a connection ∇ exists on the Batchelor bundle $E \to M$, it can be used to establish an isomorphism between the derivation sheaf $\mathrm{Der}\,\Gamma(\Lambda E)$ and $\mathcal{F} = \Gamma((\Lambda E) \otimes (TM \oplus E^*))$ which directly exhibits its $\Gamma(\Lambda E)$-module structure (cf. [21, 30]). Similarly, a connection on $E \to M$ makes $\mathrm{Hom}_{\Gamma(\Lambda E)}(\mathrm{Der}\,\Gamma(\Lambda E), \Gamma(\Lambda E))$ isomorphic to $\mathcal{F}^* = \Gamma(\Lambda E \otimes (T^*M \oplus E))$. If furthermore a connection ∇' exists on TM, the supermanifolds $ST(M, \Gamma(\Lambda E))$ and $ST^*(M, \Gamma(\Lambda E))$ are split themselves, and the functorial construction referred to above is particularly simple to describe globally. In particular, the supercotangent bundle $ST^*(M, \Gamma(\Lambda E))$ is the supermanifold $(T^*M \oplus E, \Gamma(\Lambda(TM \oplus E^* \oplus E)))$, and the fibration (1) becomes (see Section 3

below)

$$\pi : \big(T^*M \oplus E, \Gamma(\Lambda(TM \oplus E^* \oplus E))\big) \to (M, \Gamma(\Lambda E)), \tag{3}$$

which is induced by the inclusion $\Gamma(\Lambda E) \hookrightarrow \Gamma(\Lambda(TM \oplus E^* \oplus E))$. (Note: the bundle projection $T^*M \oplus E \to M$ is used to pull the sheaves of sections of the bundles E, TM, and E^* back to $T^*M \oplus E$. Nevertheless, we have written $\Gamma(\Lambda(E \oplus TM \oplus E^*))$ instead of showing explicitly the map to keep the notation simple. In what follows *the bundles whose sections we refer to, must be understood as pullbacks under the obvious submersion.*)

The main objective of this work is to address the question of whether or not a canonical 1-form can be defined on a split supercotangent bundle $ST^*(M, \Gamma(\Lambda E))$ as in nongraded differential geometry, and to see whether or not its exterior derivative can define a canonical supersymplectic structure on it. It will be thus assumed throughout this work that the following "splitting hypothesis" is satisfied: Connections exist on E and TM.

Now, a $\mathbb{Z}_2$-graded 1-form β in $(M, \Gamma(\Lambda E))$ is, according to the basic definitions in [13], a section of $\mathrm{Hom}_{\Gamma(\Lambda E)}(\mathrm{Der}\,\Gamma(\Lambda E), \Gamma(\Lambda E))$ which in turn gets uniquely identified with a $\mathbb{Z}_2$-graded manifold morphism

$$\beta\colon (M, \Gamma(\Lambda E)) \to ST^*(M, \Gamma(\Lambda E)) \tag{4}$$

such that $\pi \circ \beta = \mathrm{id}$; π being the supercotangent bundle projection (3). Since $\mathbb{Z}_2$-graded 1-forms can be pulled back under $\mathbb{Z}_2$-graded manifold morphisms, an equality like $\beta^*\Theta = \beta$ makes sense for a given 1-form Θ in $ST^*(M, \Gamma(\Lambda E))$. In fact, by requiring it to hold true *for any* β (regardless of whether it might be $\mathbb{Z}_2$-homogenous or not) one picks up the *canonical* 1-form of the graded cotangent bundle $ST^*(M, \Gamma(\Lambda E))$ (cf. [32] and Section 4 below).

It turns out that Θ is nonhomogeneous in the $\mathbb{Z}_2$-grading of $\mathrm{Hom}_{\Gamma(\Lambda E)}(\mathrm{Der}\,\Gamma(\Lambda E), \Gamma(\Lambda E))$, i.e., $\Theta = \Theta_0 + \Theta_1$, and $\Theta_\mu \neq 0$ ($\mu = 0, 1$). Whence, $\omega = -\,\mathrm{d}\,\Theta = \omega_0 + \omega_1$ (with d as in [13]) is nonhomogeneous, too. The first difference observed with respect to nongraded differential geometry is that the 2-form ω degenerates in the graded setting (cf. Proposition 5.1 below). We then look at the distribution defined by its radical, $\mathrm{Rad}\,\omega = \{D \mid i_D\omega = 0\}$, where D stands for any graded derivation of the supercotangent bundle structure sheaf $\Gamma(\Lambda(TM \oplus E^* \oplus E))$. It is immediate to verify that $\mathrm{Rad}\,\omega$ is an involutive subsheaf of the locally free module of graded derivations. But, even though $\mathrm{Rad}\,\omega$ is finitely generated and locally free, its behavior as an involutive distribution is rather pathological. The reason is that $\mathrm{Rad}\,\omega$ *is not* generated by a set of *homogeneous* derivations; in other words, *it is not a direct subsheaf* in the sense of [26]. At this point the expert reader may take a look at Example 5.2 below where we explicitly exhibit the pathology referred to above; plainly, there is no way to define a reasonable subsupermanifold structure on what would be the leaves of the foliation generated by $\mathrm{Rad}\,\omega$. In fact, if such a structure were possible, there would also be—locally, at least—a supermanifold structure on the quotient modulo the foliation, but the fibers of any quotient projection always have a homogeneously generated tangent sheaf structure (see [26]). Put in different words: $\mathrm{Rad}\,\omega$ gives an example of an involutive distribution which is not integrable.

One may look, however, at the homogeneous components of ω, and try to determine separately whether or not $\mathrm{Rad}\,\omega_0$, and $\mathrm{Rad}\,\omega_1$ are homogeneously generated involutive distributions on the supercotangent bundle. The result is (cf. Section 6 below) that both are involutive direct subsheaves; but more remarkably: Their corresponding foliations $\mathcal{F}(\mathrm{Rad}\,\omega_\mu)$ are regular in the

sense of [26]. That is, there are supermanifold structures which are globally defined on the space of leaves of each distribution, equipped with graded submersions

$$q_\mu : ST^*(M, \Gamma(\Lambda E)) \to ST^*(M, \Gamma(\Lambda E))/\mathcal{F}(\operatorname{Rad}\omega_\mu), \quad \mu = 0, 1. \tag{5}$$

Furthermore, each quotient $ST^*(M, \Gamma(\Lambda E))/\mathcal{F}(\operatorname{Rad}\omega_\mu)$, $\mu = 0, 1$ has a globally defined $\mathbb{Z}_2$-graded symplectic structure $\hat{\omega}_\mu$ which is homogeneous of degree μ, and

$$q_\mu{}^*\hat{\omega}_\mu = \omega_\mu = -\,\mathrm{d}\Theta_\mu, \quad \mu = 0, 1. \tag{6}$$

More specifically, the quotients $ST^*(M, \Gamma(\Lambda E))/\mathcal{F}(\operatorname{Rad}\omega_\mu)$, are

$$q_0\colon \big(T^*M \oplus E, \Gamma(\Lambda(E \oplus TM \oplus E^*))\big) \to \big(T^*M, \Gamma(\Lambda(E \oplus E^*))\big) \tag{6a}$$

and

$$q_1\colon \big(T^*M \oplus E, \Gamma(\Lambda(E \oplus TM \oplus E^*))\big) \to \big(E, \Gamma(\Lambda(E \oplus TM))\big) \tag{6b}$$

which are defined by the canonical inclusions $\Gamma(\Lambda(E \oplus E^*)) \hookrightarrow \Gamma(\Lambda(E \oplus TM \oplus E^*))$ and $\Gamma(\Lambda(E \oplus TM)) \hookrightarrow \Gamma(\Lambda(E \oplus TM \oplus E^*))$, respectively.

We furthermore find in Section 6 coordinate-free expressions for Θ, ω and the symplectic forms $\hat{\omega}_0$ on $(T^*M, \Gamma(\Lambda(E \oplus E^*)))$, and $\hat{\omega}_1$ on $(E, \Gamma(\Lambda(E \oplus TM)))$, respectively: First of all, using the fact (Lemma 3.3) that $\operatorname{Der}\Gamma(\Lambda(E \oplus E^*))$ is generated as a $\Gamma(\Lambda(E \oplus E^*))$-module over T^*M by the derivations ∇_X (with $X \in \operatorname{Der} C^\infty(T^*M)$), i_φ (with $\varphi \in \Gamma(E)$), and i_ξ (with $\xi \in \Gamma(E^*)$), the *canonical* 1-*form* Θ_0 of $(T^*M, \Gamma(\Lambda(E \oplus E^*)))$ is defined as the $\Gamma(\Lambda(E \oplus E^*))$-linear extension of the mapping given on generators by

$$\langle \nabla_X + i_\varphi + i_\xi \mid \Theta_0 \rangle = \langle X \mid \theta_0 \rangle + (\varphi \mid \xi) \tag{7a}$$

where θ_0 is the canonical 1-form on the cotangent bundle, $(X, \theta_0) \mapsto \langle X \mid \theta_0 \rangle$ is just the evaluation of the 1-form θ_0 on the tangent vector X, and $(\varphi, \xi) \mapsto (\varphi \,|\, \xi)$ stands for the duality bilinear pairing of sections from E and E^*. Its $\mathbb{Z}_2$-graded exterior derivative $\mathrm{d}\Theta_0 = -\omega_0$ is computed in Proposition 6.2.

Similarly, since $\operatorname{Der}\Gamma(\Lambda \tilde{p}_1^*(E \oplus TM))$ is generated as a $\Gamma(\Lambda(E \oplus TM))$-module over the manifold E by ∇_X, ∇_ξ, i_η, and i_φ (with $X \in \operatorname{Der} C^\infty(M)$, $\xi \in \Gamma(E)$, $\eta \in \Omega^1(M)$, and $\varphi \in \Gamma(E^*)$—see Lemma 3.3 below), the *canonical* 1-*form* Θ_1 of the graded manifold $(E, \Gamma(\Lambda(E \oplus TM)))$ is defined as the $\Gamma(\Lambda(E \oplus TM))$-linear extension of the mapping given on generators by

$$\langle \nabla_X + \nabla_\xi + i_\eta + i_\varphi \mid \Theta_1 \rangle = \langle X \mid \eta \rangle + (\varphi \mid \xi) \tag{7b}$$

and its exterior derivative $\mathrm{d}\,\Theta_1 = -\omega_1$ is computed in Proposition 6.3.

2. Geometric supervector bundles

Let $(M, \Gamma(\Lambda E))$ be a split graded manifold. We shall adhere ourselves to the following notation: Write $\mathcal{E}$ and $\Lambda\mathcal{E}$ instead of $\Gamma(E)$ and $\Gamma(\Lambda E)$, respectively, and write $\mathfrak{X}_M$ and Ω^1_M instead of $\Gamma(TM)$ and $\Gamma(T^*M)$, respectively. We recall in this section two basic results (Theorem 2.1, and Proposition 2.2, resp.), and provide a sketch of their proofs for the sake of self-containedness.

2.1. Theorem. *Let $(M, \mathcal{A}_M)$ be a supermanifold. There exists a functorial correspondence between the isomorphism classes of locally free sheaves of $\mathcal{A}$-modules over M, and the isomorphism classes of fibrations $\pi\colon (H, \mathcal{A}_H) \to (M, \mathcal{A}_M)$ over $(M, \mathcal{A}_M)$ given in such a way that the abstract sections of a given locally free sheaf of $\mathcal{A}_M$-modules correspond in a one-to-one fashion with the geometric sections $(M, \mathcal{A}_M) \to (H, \mathcal{A}_H)$ of its corresponding fibration ($\mathbb{Z}_2$-graded vector bundle).*

Sketch of proof. Basically, one must perform local constructions over a given open cover $\{U_i\}$ of M, in such a way that on the overlaps the sheaf maps involved satisfy the appropriate compatibility conditions in terms of transition functions. One then invokes the sheaf axioms to conclude the existence of the globally defined structures. We shall only indicate here how the local construction works, and will leave the details to the reader; therefore, we may assume that $(M, \Lambda\mathcal{E})$ is a split graded manifold of dimension (m, n) and that $\mathcal{F}$ is a locally free sheaf of $\Lambda\mathcal{E}$-modules of rank (p, q) having the following structure:

$$\mathcal{F} = (\Lambda\mathcal{E}) \otimes (\mathcal{F}_0 \oplus \mathcal{F}_1),$$

where $\mathcal{F}_0$ (resp., $\mathcal{F}_1$) is the sheaf of sections of some vector bundle $F_0 \to M$ (resp., $F_1 \to M$), of rank p (resp. q). In other words, the $\mathbb{Z}_2$-grading is such that the sections from $\mathcal{F}_0$ are even and those from $\mathcal{F}_1$ are odd. Morphisms of the fibrations obtained are defined locally by the condition of being $\Lambda\mathcal{E}$-linear on sections in order to produce the desired correspondence on isomorphism classes. But then, the central point is to prove the following (see [24, 32]):

2.2. Proposition. *The total space $F = F_0 \oplus F_1$ of the Whitney sum bundle admits the structure of an $(m + p + q, n + p + q)$-dimensional graded manifold: Its structure sheaf $\mathcal{A}_F$ is $\Lambda\tilde{\pi}^*(\mathcal{E} \oplus \mathcal{F}_0^* \oplus \mathcal{F}_1^*)$, and the bundle projection $\tilde{\pi}\colon F_0 \oplus F_1 \to M$ is used to pull the bundle $E \oplus F_0^* \oplus F_1^*$ back to $F_0 \oplus F_1$. Moreover, there is a graded-manifold submersion*

$$\pi\colon (F_0 \oplus F_1, \mathcal{A}_F) \to (M, \Lambda\mathcal{E})$$

uniquely defined by the following conditions:

(1) *M can be covered by open subsets U such that there is a graded diffeomorphism* (*defined through an isomorphism of $\Lambda\mathcal{E}|_U$-modules*)

$$\varphi_U\colon \left(\tilde{\pi}^{-1}(U), \mathcal{A}_F|_{\tilde{\pi}^{-1}(U)}\right) \to (U, \Lambda\mathcal{E}|_U) \times (V_0 \oplus V_1, \mathcal{A}_{V_0 \oplus V_1})$$

satisfying $p_2 \circ \varphi_U = \pi$, where p_2 is the projection of the product onto the fixed graded manifold $(V_0 \oplus V_1, \mathcal{A}_{V_0 \oplus V_1})$ whose underlying manifold is a $\mathbb{Z}_2$-graded vector space of dimension $(p + q, p + q)$.

(2) *The abstract sections from $\mathcal{F}(U)$ correspond in a one-to-one fashion with graded manifold morphisms*

$$\sigma_U\colon (U, \Lambda\mathcal{E}|_U) \to (V_0 \oplus V_1, \mathcal{A}_{V_0 \oplus V_1})$$

over a trivializing neighborhood U of the form described in (1).

Sketch of proof. Actually, one first establishes property (2) of the statement, and then proceeds to (1). Now, (2) is in turn a consequence of the following fact (see [6, 26, 34] for its proof):

(*) *There is a universal object in the category of* (*say, real*) *supermanifolds*, $\mathbb{R}^{1|1}$, *such that*

$$\Lambda\mathcal{E} \leftrightarrow \mathrm{Maps}\big((M, \Lambda\mathcal{E}), \mathbb{R}^{1|1}\big).$$

That is, the abstract sections from $\Lambda\mathcal{E}$ *correspond in a one-to-one fashion with actual supermanifold morphisms. In fact,* $\mathbb{R}^{1|1} = (\mathbb{R}, \Gamma(\Lambda T))$, *where* $T = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is the rank* 1 *trivial vector bundle.*

Taking this result into account, the correspondence in (2) can be made evident by first thinking of it algebraically: the sections from

$$\mathcal{F} = \Lambda\mathcal{E} \otimes (\mathcal{F}_0 \oplus \mathcal{F}_1) \simeq \mathrm{Hom}\left((\mathcal{F}_0 \oplus \mathcal{F}_1)^*, \Lambda\mathcal{E}\right)$$

can be extended to maps of $\mathbb{Z}$-graded algebras

$$\mathrm{Hom}\left(\Lambda(\mathcal{F}_0 \oplus \mathcal{F}_1)^*, \Lambda\mathcal{E}\right) \simeq \Lambda(\mathcal{E} \oplus \mathcal{F}_0 \oplus \mathcal{F}_1).$$

Thus, if $\mathcal{A}_F$ is to have property (2), then it must be the sheaf of sections of the vector bundle $\Lambda\tilde{\pi}^*(E \oplus F_0^* \oplus F_1^*) \to F_0 \oplus F_1$. One then proves that this works analytically (with its $\mathbb{Z}_2$-grading) as well. Finally, the graded manifold morphism π is given by the composition $\Lambda\mathcal{E} \to \tilde{\pi}_*\tilde{\pi}^*\Lambda\mathcal{E} \hookrightarrow \tilde{\pi}_*\tilde{\pi}^*\Lambda(\mathcal{E} \oplus \mathcal{F}_0^* \oplus \mathcal{F}_1^*)$ (cf. [36]). □

3. Supertangent and supercotangent bundles

For example, the structure of the graded tangent manifold associated to the locally free sheaf $\mathrm{Der}\,\Lambda\mathcal{E}$, is completely determined by the fact that, locally $\mathcal{F}_0 \simeq \mathfrak{X}_M$ and $\mathcal{F}_1 \simeq \mathcal{E}^*$. If a connection exists, the statement becomes global:

3.1. Proposition. *Let* $(M, \Lambda\mathcal{E})$ *be a split supermanifold, and let* ∇ *be a connection on* $\mathcal{E}$. *Then*,

$$\mathrm{Der}\,\Lambda\mathcal{E} = \Lambda\mathcal{E} \otimes (\mathfrak{X}_M \oplus \mathcal{E}^*).$$

The proof of this proposition can be found in several references: cf. [13, 21, 24, 30]; it is based on the proofs given in [8] and [12] for derivations on the space of differential forms. Now, Proposition 3.1 implies that $\mathrm{Hom}(\mathrm{Der}\,\Lambda\mathcal{E}, \Lambda\mathcal{E}) \simeq \Lambda\mathcal{E} \otimes (\mathfrak{X}_M{}^* \oplus \mathcal{E})$, and one may immediately apply the construction indicated in the previous section to obtain the following:

3.2. Proposition. *The* $\mathbb{Z}_2$*-graded fibrations corresponding to the rank* (m, n) *locally free sheaves of* $\Lambda\mathcal{E}$*-modules over* M, $\mathrm{Der}\,\Lambda\mathcal{E}$, *and* $\mathrm{Hom}(\mathrm{Der}\,\Lambda\mathcal{E}, \Lambda\mathcal{E})$, *are respectively given under the bijection of Theorem* 2.1 *by*

$$\mathrm{Der}\,\Lambda\mathcal{E} \longleftrightarrow \pi\colon \left(TM \oplus E^*, \Lambda\tilde{\pi}^*(\mathcal{E} \oplus \mathfrak{X}_M{}^* \oplus \mathcal{E})\right) \to (M, \Lambda\mathcal{E}),$$

and

$$\mathrm{Hom}(\mathrm{Der}\,\Lambda\mathcal{E}, \Lambda\mathcal{E}) \longleftrightarrow \pi\colon \left(T^*M \oplus E, \Lambda\tilde{\pi}^*(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)\right) \to (M, \Lambda\mathcal{E}).$$

Moreover, $(T^*M \oplus E, \Lambda\tilde{\pi}^*(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*))$ *has two natural quotients induced by the inclusions* $\Lambda(\mathcal{E} \oplus \mathcal{E}^*) \hookrightarrow \Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)$, *and* $\Lambda(\mathcal{E} \oplus \mathfrak{X}_M) \hookrightarrow \Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)$:

$$q_0\colon \left(T^*M \oplus E, \Lambda\tilde{\pi}^*(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)\right) \to \left(T^*M, \Lambda\tilde{p}_0^*(\mathcal{E} \oplus \mathcal{E}^*)\right),$$

and

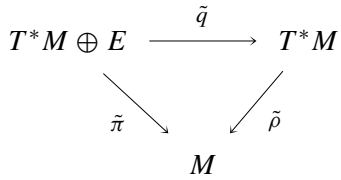
$$q_1\colon \big(T^*M \oplus E,\, \Lambda\tilde{\pi}^*(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)\big) \to \big(E,\, \Lambda\tilde{p}_1^*(\mathcal{E} \oplus \mathfrak{X}_M)\big),$$

respectively. In particular, $(T^*M, \Lambda\tilde{p}_0^*(\mathcal{E} \oplus \mathcal{E}^*))$ *is a* $(2m, 2n)$*-dimensional supermanifold, while* $(T^*M, \Lambda\tilde{p}_0^*(\mathcal{E} \oplus \mathcal{E}^*))$ *is* $(m+n, m+n)$*-dimensional.*

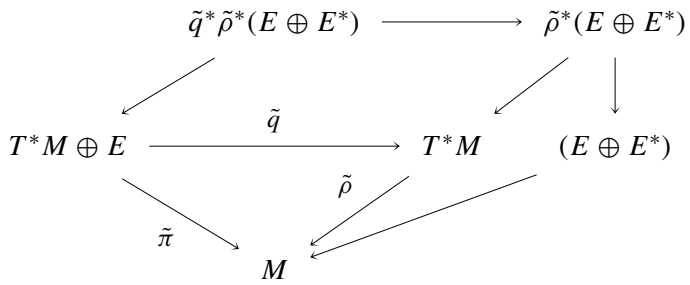
Proof. The first statement is a direct consequence of (the proof of) Theorem 2.1 and Proposition 2.2. Thus, all that really needs to be elucidated are the quotient projections of the second statement. These are better understood in terms of the appropriate diagrams. Indeed, let

$$0 \longrightarrow E \longrightarrow T^*M \oplus E \xrightarrow{\tilde{q}} T^*M \longrightarrow 0$$

be the vector bundle sequence given by the Whitney sum of E and T^*M. We also have the commutative diagram



which can be used to pull back the bundle $E \oplus E^*$ over M to produce the bundle $\tilde{\rho}^*(E \oplus E^*)$ over T^*M, and the bundle $\tilde{q}^*\tilde{\rho}^*(E \oplus E^*)$ over $T^*M \oplus E$ as indicated in the following diagram



Note that there is a natural graded manifold epimorphism

$$q\colon \big(T^*M \oplus E, \tilde{\pi}^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)\big) \to \big(T^*M, \tilde{\rho}^*\Lambda(\mathcal{E} \oplus \mathcal{E}^*)\big)$$

if and only if there is a natural sheaf monomorphism,

$$\tilde{\rho}^*\Lambda(\mathcal{E} \oplus \mathcal{E}^*) \hookrightarrow \tilde{q}_*\tilde{\pi}^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*).$$

But since $\tilde{\pi} = \tilde{\rho} \circ \tilde{q}$, we have, $\tilde{\pi}^* = \tilde{q}^*\tilde{\rho}^*$. Thus, the natural map $\mathcal{T} \to \tilde{q}_*\tilde{q}^*\mathcal{T}$ defined for any sheaf $\mathcal{T}$ (cf. [36]), yields part of the natural map above as,

$$\tilde{\rho}^*\Lambda(\mathcal{E} \oplus \mathcal{E}^*) \to \tilde{q}_*\tilde{q}^*\big(\tilde{\rho}^*\Lambda(\mathcal{E} \oplus \mathcal{E}^*)\big).$$

Since $\Lambda(\mathcal{E} \oplus \mathcal{E}^*) \hookrightarrow \Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)$, the rest is clear. The proof for the other quotient structure is completely analogous. □

Note. We mention in passing that other quotients are possible. In particular, the following two have been recently used in [38] with the aim of producing a $(\mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z})$-graded complex of differential forms:

$$\big(T^*M \oplus E, \Lambda\tilde{\pi}^*(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)\big) \to \big(T^*M, \Lambda\tilde{p}_0^*(\mathcal{E} \oplus \mathfrak{X}_M)\big),$$

and

$$\big(T^*M \oplus E, \Lambda\tilde{\pi}^*(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)\big) \to \big(E, \Lambda\tilde{p}_1^*(\mathcal{E} \oplus \mathcal{E}^*)\big).$$

We shall not deal with these quotients here, however.

3.3. Lemma. *Let $(M, \Lambda\mathcal{E})$ be a split supermanifold. Let ∇' be a connection on TM, and let ∇'' be a connection on E. Denote by ∇ either, the direct sum connection $\nabla' \oplus \nabla''$ on $T^*M \oplus E$, or the induced connection on $E \oplus TM \oplus E^*$. The derivation sheaf* $\operatorname{Der} p^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)$ *of the supercotangent bundle $(T^*M \oplus E, p^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*))$ of $(M, \Lambda\mathcal{E})$ may be decomposed as*

$$\operatorname{Der} p^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*) = p^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*) \otimes \big(\mathfrak{X}_{T^*M\oplus E} \oplus (\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)^*\big).$$

Furthermore,

$$\mathfrak{X}_{T^*M\oplus E} \simeq \mathfrak{X}_M \oplus \mathfrak{X}_M^* \oplus \mathcal{E}$$

as the sections of the vertical subbundles $\operatorname{Vert}(\mathfrak{X}_M^*) = \operatorname{Ker}(\mathrm{d}\pi_0 \colon T(T^*M) \to TM)$, *and* $\operatorname{Vert}(\mathcal{E}) = \operatorname{Ker}(\mathrm{d}\pi_1 \colon TE \to TM)$ *are naturally identified to the sections of $\mathfrak{X}_M^*$ and $\mathcal{E}$, respectively.*

Proof. All that has to be verified is the last statement, but that is well known. We shall, nevertheless, recall how the tangent sheaf $\mathfrak{X}_E$ of an arbitrary vector bundle $E \to M$, with connection ∇, decomposes as the $C^\infty(E)$-module $\mathfrak{X}_M \oplus \mathcal{E}$. Our argument follows [39]. First note that a set of local coordinates $\{q^i\}$ on some open subset $U \subset M$, and a local frame $\{\varphi^\mu\}$ of $\mathcal{E}^*|_U$, determines a set of local coordinates on E; namely $\{q^i \circ \pi\ ,\ \varphi^\mu\}$, where $\pi \colon E \to$ is the bundle projection, and $\varphi^\mu \in \mathcal{E}^*|_U$ is thought of as a smooth map $E|_U \to \mathbb{R}$. In particular, a local vector field $W \in \mathfrak{X}_E$ is completely determined by $W(q^i \circ \pi)$, and $W(\varphi^\mu)$, as

$$W = \sum_i W(q^i \circ \pi)\frac{\partial}{\partial(q^i \circ \pi)} + \sum_\mu W(\varphi^\mu)\frac{\partial}{\partial(\varphi^\mu)}.$$

A global vector field $W \in \mathfrak{X}_E$ is then completely determined by specifying

$$W(f \circ \pi) \ \text{ for any } \ f \in C^\infty(M) \quad \text{ and } \quad W(\varphi) \ \text{ for any } \ \varphi \in \Gamma(E^*).$$

In particular, W is vertical if and only if $W(f \circ \pi) = 0$ for any $f \in C^\infty(M)$. Now, each section $\sigma \in \mathcal{E}$ defines a *vertical* field—denoted by $W_\sigma \in \mathfrak{X}_E$—upon specifying that

$$\begin{aligned} &W_\sigma(f \circ \pi) = 0, && f \in C^\infty(M), \\ &W_\sigma(\varphi) = (\sigma \mid \varphi) \circ \pi, && \varphi \in \mathcal{E}^*, \end{aligned}$$

where $(\cdot \mid \cdot) : \mathcal{E} \times \mathcal{E}^* \to C_M^\infty$ denotes the duality bilinear pairing between E and E^*. If a connection ∇ is given on $E \to M$, then any vector field $X \in \mathfrak{X}_M$ from the base can be lifted

to define a *horizontal* vector field $X^{\mathrm{h}} \in \mathfrak{X}_E$. Thus, the $C^\infty(E)$-module $\mathfrak{X}_E$ is generated by $\mathfrak{X}_M$ and $\mathcal{E}$. $\square$

3.4. Notation. We shall introduce the following conventions in order to express the even and odd generators of $\operatorname{Der} \Lambda \tilde{p}^*(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)$ in a convenient manner. First of all, the even generators (i.e., the elements from $\mathfrak{X}_{T^*M \oplus E} \simeq \mathfrak{X}_M \oplus \mathfrak{X}_M^* \oplus \mathcal{E}$), will be written as follows:

$$\nabla_X \leftrightarrow X \in \mathfrak{X}_M, \qquad \nabla_\theta \leftrightarrow \theta \in \mathfrak{X}_M^*, \qquad \nabla_{\bar{\sigma}} \leftrightarrow \sigma \in \mathcal{E},$$

where X really means the horizontal lift X^{h} of X to $T^*M \oplus E$ via ∇. Similarly, θ (resp. $\bar{\sigma}$) really means the vertical field W_θ in T^*M (resp. E) defined by the section $\theta \in \Gamma(T^*M) = \mathfrak{X}_M^*$ (resp. $\sigma \in \Gamma(E) = \mathcal{E}$).

On the other hand, we know that the odd generators of $\operatorname{Der} p^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)$ (i.e., the elements from $(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)^*$), act algebraically by contraction (cf. [21, 30]); we thus write them as

$$i_\varphi \leftrightarrow \varphi \in \mathcal{E}^*, \qquad i_{\bar{\theta}} \leftrightarrow \theta \in \mathfrak{X}_M^*, \qquad i_\sigma \leftrightarrow \sigma \in \mathcal{E}.$$

Convention. The bar helps in distinguishing when is it that a given section θ from $\mathfrak{X}_M^*$ (resp. σ from $\mathcal{E}$) appears as an even generator, and when is it that it appears as an odd generator.

Remark. In order to compute with the generators just given, we simply have to take into account the following two facts: First, for any A and B in $\{X, \bar{\eta}, \xi\}$, and any ρ and τ in $\{\varphi, \eta, \bar{\xi}\}$,

$$[\nabla_A, \nabla_B] = \nabla_{[A,B]} - i_{R(A,B)}, \qquad [\nabla_A, i_\rho] = i_{\nabla_A \rho}, \qquad [i_\rho, i_\tau] = 0.$$

Second, for $\nabla = \nabla' \oplus \nabla''$,

$$[\nabla'_A, \nabla''_B] = 0.$$

4. The canonical one-form on the supercotangent bundle

The purpose of this section is to prove, and elucidate the nature of the following (see [32]):

4.1. Theorem. *There exists a* 1*-form* Θ *on* $ST^*(M, \Gamma(\Lambda E))$ *which is uniquely characterized by the property that, for any* 1*-form* $\beta \in \Omega^1(M, \Gamma(\Lambda E))$

$$\beta^*\Theta = \beta.$$

For the proof, use will be made of the well-known properties of $\mathbb{Z}_2$-graded differential forms as given in [13]. In particular, we shall write $(D_1, \ldots, D_k) \mapsto \langle D_1, \ldots, D_k \mid \beta \rangle$ for the value of the $\mathbb{Z}_2$-graded k-form β on the k-tuple of $\mathbb{Z}_2$-graded derivations $(D_1, \ldots, D_k)$, and we shall denote by $\mathbf{d}\colon \Omega^k(M, \mathcal{A}_M) \to \Omega^{k+1}(M, \mathcal{A}_M)$ the exterior derivative of $\mathbb{Z}_2$-graded differential forms.

4.2. Coordinatization of the supercotangent bundle. Theorem 4.1 will be proved by means of a local coordinate computation as in the nongraded case, and we shall then give a transition function argument to show that the local characterization can be globalized. In Section 6, however, we shall give an intrinsic (global) formula for Θ.

First, choose a local frame $\{\nabla_{X_i}; i_{\varphi^\mu}\}$ for $\operatorname{Der}\Lambda\mathcal{E}$ defined by a local frame $\{X_i\}$ on $U \subset M$, and a local frame $\{\varphi^\mu\}$ for the sections of $E^*|_U$. Now consider the corresponding dual frames: Say $\{\theta^i\}$ for 1-forms on U (i.e., $\theta^i \in \Omega^1(U)$, with $\langle X_j \mid \theta^i\rangle = \delta_{ij}$), and $\{\sigma_\mu\}$ for the sections of $E|_U$ (with, $(\sigma_\mu \mid \varphi^\nu) = \delta_{\mu\nu}$). Let Y_i be the vertical vector field on $T^*M|_U$ defined by the section θ^i. That is

$$\begin{aligned} &Y_i(f\circ\pi) = 0 && \forall\ f \in C^\infty_M(U),\\ &Y_i(i_X) = \langle X \mid \theta^i\rangle\circ\pi && \forall\ X \in \mathfrak{X}_M(U). \end{aligned}$$

Similarly, let Z_μ be the vertical vector field on $E|_U$ defined by the section σ_μ. That is,

$$\begin{aligned} &Z_\mu(f\circ\pi) = 0 && \forall\ f \in C^\infty_M(U),\\ &Z_\mu(\varphi) = (\sigma_\mu \mid \varphi)\circ\pi && \forall\ \varphi \in \mathcal{E}^*(U). \end{aligned}$$

The connection ∇ allows us to define the local even generators for $\operatorname{Der}\Lambda(\mathcal{E}\oplus\mathfrak{X}_M\oplus\mathcal{E}^*)$ as:

$$\text{Even generators:}\quad \nabla_{X_i},\quad \nabla_{Y_i},\quad \nabla_{Z_\mu},$$

where now, X_i really means X_i^{h}; that is, the horizontal lift of X to the cotangent bundle T^*M via the connection ∇'.

The odd generators are defined by contraction against the elements of the chosen frames. We thus have:

$$\text{Odd generators:}\quad i_{\varphi^\mu},\quad i_{\theta^i},\quad i_{\sigma_\mu}.$$

Local coordinates are defined in such a way that

$$\begin{aligned} &\nabla_{X_i}x^j = \langle\nabla_{X_i} \mid \mathbf{d}x^j\rangle = \delta_{ij}, && i_{\varphi^\mu}\xi^\nu = \langle i_{\varphi^\mu} \mid \mathbf{d}\xi^\nu\rangle = \delta_{\mu\nu},\\ &\nabla_{Y_i}y^j = \langle\nabla_{Y_i} \mid \mathbf{d}y^j\rangle = \delta_{ij}, && i_{\theta^i}\bar{y}^j = \langle i_{\theta^i} \mid \mathbf{d}\bar{y}^j\rangle = \delta_{ij},\\ &\nabla_{Z_\mu}\bar{\xi}^\nu = \langle\nabla_{Z_\mu} \mid \mathbf{d}\bar{\xi}^\nu\rangle = \delta_{\mu\nu}, && i_{\sigma_\mu}\zeta^\nu = \langle i_{\sigma_\mu} \mid \mathbf{d}\zeta^\nu\rangle = \delta_{\mu\nu}. \end{aligned}$$

Thus, a 1-superform defined on the supercotangent bundle can be uniquely written in these coordinates as

$$\begin{aligned} \Theta = &\sum_i \mathbf{d}x^i\cdot A_i + \sum_i \mathbf{d}y^i\cdot B_i + \sum_\mu \mathbf{d}\bar{\xi}^\mu\cdot C_\mu\\ &+ \sum_\mu \mathbf{d}\xi^\mu\cdot D_\mu + \sum_i \mathbf{d}\bar{y}^i\cdot E_i + \sum_\mu \mathbf{d}\zeta^\mu\cdot F_\mu. \end{aligned}$$

Now, let $\beta \in \Omega^1(M,\mathcal{A}_M) = \operatorname{Hom}(\operatorname{Der}\mathcal{A}_M, \mathcal{A}_M)$ be a 1-superform on $(M,\mathcal{A}_M)$. Write it locally as

$$\beta = \sum_i \mathbf{d}q^i\cdot\beta_i + \sum_\mu \mathbf{d}\varphi^\mu\cdot\beta_\mu.$$

When this $\beta \in \Omega^1(M,\mathcal{A}_M)$ is viewed as a morphism $\beta\colon (M,\mathcal{A}_M) \to ST^*(M,\mathcal{A}_M)$, its effect on the given local coordinates of $ST^*(M,\mathcal{A}_M)$ is the following:

$$\begin{aligned} &\beta^*x^i = q^i \quad (x^i = \pi^*q^i), && \beta^*\xi^\mu = \varphi^\mu \quad (\xi^\mu = \pi^*\varphi^\mu),\\ &\beta^*y^i = (\beta_i)_0, && \beta^*\bar{y}^i = (\beta_i)_1,\\ &\beta^*\bar{\zeta}^\mu = (\beta_\mu)_0, && \beta^*\zeta^\mu = (\beta_\mu)_1, \end{aligned}$$

where $\beta_i = (\beta_i)_0 + (\beta_i)_1$ in its $\mathbb{Z}_2$-decomposition. Thus, one may directly compute the pullback

$$\begin{aligned}\beta^*\Theta &= \sum_i \mathbf{d}\beta^* x^i \cdot \beta^* A_i + \sum_i \mathbf{d}\beta^* y^i \cdot \beta^* B_i + \sum_\mu \mathbf{d}\beta^* \bar{\xi}^\mu \cdot \beta^* C_\mu \\ &\quad + \sum_\mu \mathbf{d}\beta^* \xi^\mu \cdot \beta^* D_\mu + \sum_i \mathbf{d}\beta^* \bar{y}^i \cdot \beta^* E_i + \sum_\mu \mathbf{d}\beta^* \zeta^\mu \cdot \beta^* F_\mu \\ &= \sum_i \mathbf{d}q^i \cdot \beta^* A_i + \sum_i \mathbf{d}(\beta_i)_0 \cdot \beta^* B_i + \sum_\mu \mathbf{d}(\beta_\mu)_0 \cdot \beta^* C_\mu \\ &\quad + \sum_\mu \mathbf{d}\varphi^\mu \cdot \beta^* D_\mu + \sum_i \mathbf{d}(\beta_i)_1 \cdot \beta^* E_i + \sum_\mu \mathbf{d}(\beta_\mu)_1 \cdot \beta^* F_\mu.\end{aligned}$$

We now need the following lemma, which is an immediate consequence of the fact $(*)$ mentioned in the proof of Proposition 2.2 (see also [26, 34]):

4.3. Lemma. *Let $\beta\colon (M, \mathcal{A}_M) \to (N, \mathcal{A}_N)$ be a supermanifold morphism. Suppose $A \in \mathcal{A}_N(U)$ is a local section, and also denote by $A\colon (U, \mathcal{A}_N|_U) \to \mathbb{R}^{1|1}$ its corresponding morphism into $\mathbb{R}^{1|1}$. Then the section $\beta^* A \in \mathcal{A}_M(\tilde{\beta}^{-1}(U))$ corresponds to the morphism given by the composition $A \circ \beta$:*

$$\beta^* A = A \circ \beta : \big(\tilde{\beta}^{-1}(U), \mathcal{A}_M|_{\tilde{\beta}^{-1}(U)}\big) \to \mathbb{R}^{1|1}.$$

Proof. In fact, the morphism $A \circ \beta\colon (M, \mathcal{A}_M) \to \mathbb{R}^{1|1}$ has the following effect on the standard local coordinates (t, τ) of $\mathbb{R}^{1|1}$ (t is a basis for the dual space of the vector space $\mathbb{R}$, and τ is the global section $t \mapsto (t, 1)$ of the trivial bundle $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$):

$$(A \circ \beta)^* t = \beta^* A^* t = \beta^*(A_0) \qquad \text{and} \qquad (A \circ \beta)^* \tau = \beta^* A^* \tau = \beta^*(A_1)$$

where $A = A_0 + A_1$ is the decomposition of the section A from $\mathcal{A}_N(U)$. □

Taking Lemma 4.3 into account, the condition $\beta^*\Theta = \beta$ yields,

$$\begin{aligned}&\sum_i \mathbf{d}q^i \cdot \beta_i + \sum_\mu \mathbf{d}\varphi^\mu \cdot \beta_\mu \\ &\qquad = \sum_i \mathbf{d}q^i \cdot A_i \circ \beta + \sum_i \mathbf{d}(\beta_i)_0 \cdot B_i \circ \beta + \sum_\mu \mathbf{d}(\beta_\mu)_0 \cdot C_\mu \circ \beta \\ &\qquad\quad + \sum_\mu \mathbf{d}\varphi^\mu \cdot D_\mu \circ \beta + \sum_i \mathbf{d}(\beta_i)_1 \cdot E_i \circ \beta + \sum_\mu \mathbf{d}(\beta_\mu)_1 \cdot F_\mu \circ \beta.\end{aligned}$$

We now compute $\mathbf{d}(\beta_i)_\nu$ and $\mathbf{d}(\beta_\mu)_\nu$ for $\nu = 0$, and 1, according to (cf. [13])

$$\mathbf{d}(\beta_a)_\nu = \sum_j \mathbf{d}q^j \cdot \frac{\partial(\beta_a)_\nu}{\partial q^j} + \sum_\lambda \mathbf{d}\varphi^\lambda \cdot \frac{\partial(\beta_a)_\nu}{\partial \varphi^\lambda},$$

where $a = i$ or μ. Therefore, $\beta^*\Theta = \beta$ if and only if

$$\begin{aligned}\beta_i &= A_i \circ \beta + \sum_j \frac{\partial(\beta_j)_0}{\partial q^i} \cdot B_j \circ \beta + \sum_\nu \frac{\partial(\beta_\nu)_0}{\partial q^i} \cdot C_\nu \circ \beta \\ &\quad + \sum_j \frac{\partial(\beta_j)_1}{\partial q^i} \cdot E_j \circ \beta + \sum_\nu \frac{\partial(\beta_\nu)_1}{\partial q^i} \cdot F_\nu \circ \beta\end{aligned}$$

and

$$\beta_\mu = D_\mu \circ \beta + \sum_j \frac{\partial(\beta_j)_0}{\partial\varphi^\mu} \cdot B_j \circ \beta + \sum_\nu \frac{\partial(\beta_\nu)_0}{\partial\varphi^\mu} \cdot C_\nu \circ \beta + \sum_j \frac{\partial(\beta_j)_1}{\partial\varphi^\mu} \cdot E_j \circ \beta + \sum_\nu \frac{\partial(\beta_\nu)_1}{\partial\varphi^\mu} \cdot F_\nu \circ \beta.$$

Now, $\beta_i = (y^i + \bar{y}^i) \circ \beta$ and $\beta_\mu = (\bar{\zeta}^\mu + \zeta^\mu) \circ \beta$. Therefore, the condition $\beta^*\Theta = \beta$ *for any* β, implies

$$(y^i + \bar{y}^i - A_i) \circ \beta = \sum_j \frac{\partial(\beta_j)_0}{\partial q^i} \cdot B_j \circ \beta + \sum_\nu \frac{\partial(\beta_\nu)_0}{\partial q^i} \cdot C_\nu \circ \beta + \sum_j \frac{\partial(\beta_j)_1}{\partial q^i} \cdot E_j \circ \beta + \sum_\nu \frac{\partial(\beta_\nu)_1}{\partial q^i} \cdot F_\nu \circ \beta$$

and

$$(\bar{\zeta}^\mu + \zeta^\mu - D_\mu) \circ \beta = \sum_j \frac{\partial(\beta_j)_0}{\partial\varphi^\mu} \cdot B_j \circ \beta + \sum_\nu \frac{\partial(\beta_\nu)_0}{\partial\varphi^\mu} \cdot C_\nu \circ \beta + \sum_j \frac{\partial(\beta_j)_1}{\partial\varphi^\mu} \cdot E_j \circ \beta + \sum_\nu \frac{\partial(\beta_\nu)_1}{\partial\varphi^\mu} \cdot F_\nu \circ \beta,$$

for all possible values of the sections β_j and β_ν. In particular, for those having all the β_j's and β_ν's constant, in which case $\beta^*(y^i + \bar{y}^i - A_i) = y^i + \bar{y}^i - A_i = 0$, and $\beta^*(\bar{\zeta}^\mu + \zeta^\mu - D_\mu) = \bar{\zeta}^\mu + \zeta^\mu - D_\mu = 0$, for all i, and all μ. But then we still have

$$0 = \sum_j \frac{\partial(\beta_j)_0}{\partial q^i} \cdot B_j \circ \beta + \sum_\nu \frac{\partial(\beta_\nu)_0}{\partial q^i} \cdot C_\nu \circ \beta + \sum_j \frac{\partial(\beta_j)_1}{\partial q^i} \cdot E_j \circ \beta + \sum_\nu \frac{\partial(\beta_\nu)_1}{\partial q^i} \cdot F_\nu \circ \beta$$

and

$$0 = \sum_j \frac{\partial(\beta_j)_0}{\partial\varphi^\mu} \cdot B_j \circ \beta + \sum_\nu \frac{\partial(\beta_\nu)_0}{\partial\varphi^\mu} \cdot C_\nu \circ \beta + \sum_j \frac{\partial(\beta_j)_1}{\partial\varphi^\mu} \cdot E_j \circ \beta + \sum_\nu \frac{\partial(\beta_\nu)_1}{\partial\varphi^\mu} \cdot F_\nu \circ \beta$$

for all β. In particular, for those β's having $\partial(\beta_a)_\lambda/\partial\varphi^\mu = \delta_{a\mu}$ and all others equal to zero (resp. $\partial(\beta_a)_\lambda/\partial q^j = \delta_{aj}$ and all others equal to zero). Whence, $B_j = C_\nu = E_j = F_\nu = 0$ for all j and ν, and therefore

$$\Theta = \sum_i \mathbf{d}x^i \cdot (y^i + \bar{y}^i) + \sum_\mu \mathbf{d}\xi^\mu \cdot (\bar{\zeta}^\mu + \zeta^\mu).$$

It is now a straightforward matter to see that this is actually globally defined (i.e., independently of its local coordinate description): Indeed, the coefficients of this form are linear in the local coordinates; whence, the corresponding jacobian matrices for the change of coordinates simply replace the coordinates of one coordinate patch by the corresponding coordinates on the other. We shall not bother much with the local computations in terms of transition functions, since in

Section 6 below we shall give a global description of Θ in terms of its homogeneous components. At any rate, what remains to complete the proof of Theorem 4.1 can now be safely left to the reader. □

5. An involutive distribution in the supercotangent bundle which is not integrable

This section illustrates the pathology exhibited by the $\mathbb{Z}_2$-graded exterior derivative $\mathbf{d}\Theta$ of the canonical 1-form just obtained, in trying to define a symplectic graded manifold structure. From the local expression of Θ one obtains

$$\omega := -\mathbf{d}\Theta = \sum_i \mathbf{d}x^i \cdot \mathbf{d}(y^i + \bar{y}^i) + \sum_\mu \mathbf{d}\xi^\mu \cdot \mathbf{d}(\bar{\zeta}^\mu + \zeta^\mu).$$

Note that ω is not $\mathbb{Z}_2$-homogeneous. This property is inherited from the fact that Θ is *nonhomogeneous* either. Also note that ω is a closed 2-form on $ST^*(M, \Gamma(\Lambda E))$ which degenerates:

5.1. Proposition. *Let the setting, and the notation be as in* 4.2 *above. The space of sections,*

$$\operatorname{Rad} \mathrm{d}\Theta = \{\, D \in \operatorname{Der} p^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*) \mid \mathbf{i}_D\omega = 0 \,\}$$

is closed under the Lie bracket of $\mathbb{Z}_2$-graded derivations, and it is locally spanned as a free $\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)$-module, by

$$\partial_j := \nabla_{Y_j} - i_{\theta^j} =: \frac{\partial}{\partial y^i} - \frac{\partial}{\partial \bar{y}^i} \quad \textit{and} \quad \partial_\nu := \nabla_{Z_\nu} + i_{\sigma^\nu} =: \frac{\partial}{\partial \zeta^\nu} + \frac{\partial}{\partial \bar{\zeta}^\nu}.$$

Proof. The generators ∂_j and ∂_ν are obtained via a straightforward computation from the definitions and the local coordinate expressions. We leave the tedious but quite straightforward details to the reader. □

Note. One observes that $\operatorname{Rad}\Theta$ is not $\mathbb{Z}_2$-homogeneously generated (e.g., $|\nabla_{Y_j}| = 0$, while $|i_{\theta^j}| = 1$). In fact, *there is no way of giving to it a set of homogeneous generators*, as we shall now show through the following:

5.2. Example (The supercotangent bundle of $\mathbb{R}^{1|1}$). Let $\{q; \varphi\}$ be some choice of global coordinates on the base $\mathbb{R}^{1|1}$. The coordinates on $ST^*\mathbb{R}^{1|1}$ are given by $\{x, y, \bar{\zeta}; \xi, \zeta, \bar{y}\}$, where the first three are even, the last three are odd, and $x = \pi^*q, \xi = \pi^*\varphi$, with $\pi : ST^*\mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ the bundle projection. The canonical 1-form Θ in this set of coordinates is: $\Theta = \mathrm{d}x\,(y + \bar{y}) + \mathrm{d}\xi\,(\zeta + \bar{\zeta})$. It is easy to see (or can be concluded directly from Proposition 5.1) that the most general $X \in \operatorname{Rad}\mathrm{d}\Theta$ can be expressed in the form

$$X = (Xy)\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial \bar{y}}\right) + (X\zeta)\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}}\right),$$

where the values of $f = Xy$ and $g = X\zeta$ are nonhomogeneous, arbitrary, and independent of each other. Thus, we have a *rank 2 module with two nonhomogeneous free generators*. If Y_0, and Y_1 were homogeneous generators for the same module, sections f, g, h, and k of the structure

sheaf should exists, such that

$$Y_0 = f\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial \bar{y}}\right) + g\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}}\right),$$
$$Y_1 = h\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial \bar{y}}\right) + k\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}}\right).$$

Using the fact that $|Y_0| = 0$ and $|Y_1| = 1$ one easily reaches the following conclusion: *There is no way to pass to a set of homogeneous generators*, unless $f = g = h = k = 0$. Evidently, the same is true for the general case, as one only has to reproduce this argument separately on each of the generators ∂_j and ∂_ν of Proposition 5.1 above. Thus, the involutive subsheaf $\operatorname{Rad} \mathrm{d}\,\Theta$ is not a *direct subsheaf* of $\operatorname{Der} p^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)$ in the sense of [26].

Remark. Despite of the fact that there is no way to obtain homogeneous free generators for $\operatorname{Rad} \mathrm{d}\,\Theta$, one might try to take advantage of the fact that the generators are *free*. In this setting, there is another thing to do: Try to find the $\mathbb{R}^{1|1}$-integral flows of the nonhomogeneous generators as in [23]. Since the homogeneous components of the two inhomogeneous generators commute among themselves, their corresponding integral flows will also commute in the sense of [23]. Hence, one may compose these integral flows, keeping their integration parameters (t_1, τ_1) and (t_2, τ_2) fixed. Then, proceed as in the proof of Frobenius Theorem in [26], and use these parameters to change the coordinates. In this way one obtains the smallest direct (i.e., homogeneous) distribution that contains the two inhomogeneous generators above; this turns out to be of rank (2, 2). The problem one faces then is that with this homogeneous distribution, there is no way to define a reasonable $\mathbb{R}^{1|1} \hookrightarrow \mathbb{R}^{1|1}$ action (in the sense of [6]) in order to obtain a (2, 2)-dimensional quotient.

One way out of these difficulties is obtained by considering separately the radicals defined by the homogeneous components $\omega_0 = -\mathrm{d}\Theta_0$ and $\omega_1 = -\mathrm{d}\Theta_1$, respectively:

5.3. Proposition. *Let* $\mathcal{D}_\mu = \{D \in \operatorname{Der} p^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*) \mid \mathbf{i}_D\omega_\mu = 0\}$. *Then,* $\mathcal{D}_\mu$ *is a direct subsheaf of* $\operatorname{Der} p^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)$ *of rank* (n, m) *when* $\mu = 0$ *and of rank* (m, n) *when* $\mu = 1$. *In fact*

$$\mathcal{D}_0 \simeq p^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*) \otimes (\mathcal{E} \oplus \mathfrak{X}_M^*),$$

where the sections from the $\mathcal{E}$*-summand at the right of the* $\otimes$*-sign enter as even generators, and those from* $\mathfrak{X}_M^*$ *as odd generators. Similarly*

$$\mathcal{D}_1 \simeq p^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*) \otimes (\mathfrak{X}_M^* \oplus \mathcal{E}),$$

where the sections from the $\mathfrak{X}_M^*$*-summand at the right of the* $\otimes$*-sign enter as even generators, and those from* $\mathcal{E}$ *as odd generators.*

Proof. This follows in a straightforward manner from the local generators of $\operatorname{Der} p^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*)$ given in Lemma 4.3 above. □

5.4. Theorem. *The distributions* $\mathcal{D}_\mu$ ($\mu = 0, 1$) *produce regular foliations* $\mathcal{F}(\mathcal{D}_\mu)$, *and the corresponding quotients have the following supermanifold structure*:

$$ST^*(M, \Gamma(\Lambda E))/\mathcal{F}(\mathcal{D}_\mu) = \begin{cases} (T^*M, \Lambda(\mathcal{E} \oplus \mathcal{E}^*)) & \mu = 0, \\ (E, \Lambda(\mathcal{E} \oplus \mathfrak{X} & \mu = 1. \end{cases}$$

Proof. Note, first of all, that the dimensions match, according to the general theorem in [26], and the corresponding ranks of $\mathcal{D}_0$ and $\mathcal{D}_1$ found in Proposition 5.3 above:

$$\dim ST^*(M, \Gamma(\Lambda E))/\mathcal{F}(\mathcal{D}_\mu) = \begin{cases} \dim ST^*(M, \Gamma(\Lambda E)) - (n, m) = (2m, 2n) & \mu = 0, \\ \dim ST^*(M, \Gamma(\Lambda E)) - (m, n) = (m + n, m + n) & \mu = 1. \end{cases}$$

We have to show that the supermanifold structures over T^*M and E given in the statement are precisely those obtained by looking at the $\mathbb{Z}_2$-graded algebra sheaves of first integrals of the distributions $\mathcal{D}_0$ and $\mathcal{D}_1$. We shall do it for $\mathcal{D}_0$ only, as the verification for $\mathcal{D}_1$ is completely analogous. Consider

$$\mathcal{B}_0 = \{f \in C^\infty(T^*M \oplus E) \otimes p^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M \oplus \mathcal{E}^*) \mid \text{ for all } Z \in \mathcal{D}_0 \; Zf = 0\}.$$

Now, $\mathcal{D}_0$ is generated by $\mathcal{E}$ in the even submodule, and $\mathfrak{X}_M^*$ in the odd submodule. Note that $(\mathcal{E})f = 0$ means that f does not depend on the coordinates of the fibers of E in $T^*M \oplus E$, and that $(\mathfrak{X}_M^*)f = 0$ means that f does not depend on the coordinates coming from the $\mathfrak{X}_M^*$-summand of the Batchelor bundle of $ST^*(M, \Gamma(\Lambda E))$. Therefore, $\mathcal{B}_0 \simeq C^\infty(T^*M) \otimes p_0^*\Lambda(\mathcal{E} \oplus \mathcal{E}^*)$, where $p_0 \colon T^*M \to M$ is the cotangent bundle projection. □

6. Homogeneous symplectic structures on the quotients

It will now be shown in a coordinate-free manner that the quotients $(T^*M, \Lambda\tilde{\rho}^*(\mathcal{E} \oplus \mathcal{E}^*))$ and $(E, \Lambda\tilde{p}_1^*(\mathcal{E} \oplus \mathfrak{X}_M))$ carry canonical nondegenerate closed graded 2-forms (see [21, 30]). These are, respectively, the graded exterior derivatives of the $\mathbb{Z}_2$-homogeneous componentes of the canonical 1-form Θ.

Consider first the case of $\mathcal{F} = \tilde{\rho}^*(\mathcal{E} \oplus \mathcal{E}^*)$. Then

$$\operatorname{Der}\Lambda\mathcal{F} = \Lambda\mathcal{F} \otimes (\mathfrak{X}_{T^*M} \oplus \mathcal{F}^*) \simeq \tilde{\rho}^*\Lambda(\mathcal{E} \oplus \mathcal{E}^*) \otimes (\mathfrak{X}_{T^*M} \oplus \tilde{\rho}^*(\mathcal{E} \oplus \mathcal{E}^*)^*),$$

whereas

$$\begin{aligned} \operatorname{Hom}(\operatorname{Der}\Lambda\mathcal{F}, \Lambda\mathcal{F}) &\simeq \Lambda\mathcal{F} \otimes (\{\mathfrak{X}_{T^*M}\}^* \oplus \mathcal{F}) \\ &\simeq \tilde{\rho}^*\Lambda(\mathcal{E} \oplus \mathcal{E}^*) \otimes (\{\mathfrak{X}_{T^*M}\}^* \oplus \tilde{\rho}^*(\mathcal{E} \oplus \mathcal{E}^*)). \end{aligned}$$

Now, at the light of the natural pairing $\mathfrak{X}_{T^*M} \times \{\mathfrak{X}_{T^*M}\}^* \to C^\infty_{T^*M}$, the canonical 1-form θ_0 in the cotangent bundle defines a $C^\infty_{T^*M}$-linear map

$$\mathfrak{X}_{T^*M} \to C^\infty_{T^*M} \qquad X \mapsto \langle X \mid \theta_0 \rangle.$$

Set, on the even generators ∇_X of the derivation sheaf,

$$\langle \nabla_X \mid \Theta_0 \rangle = \langle X \mid \theta_0 \rangle$$

and extend it in a $\Lambda\tilde{\rho}^*(\mathcal{E}\oplus\mathcal{E}^*)$-linear fashion to the whole submodule generated by the ∇_X's. Similarly, at the light of the induced $C^\infty_{T^*M}$-linear pairing $(\cdot\,,\cdot)\colon \tilde{\rho}^*\mathcal{E}\times\tilde{\rho}^*\mathcal{E}^*\to C^\infty_{T^*M}$, set

$$\langle i_\varphi + i_\xi \mid \Theta_0\rangle = (\varphi \mid \xi)$$

and extend it $\Lambda\tilde{\rho}^*(\mathcal{E}\oplus\mathcal{E}^*)$-linearly.

6.1. Definition. *The canonical* 1*-form* Θ_0 in the graded manifold $(T^*M, \tilde{\rho}^*\Lambda(\mathcal{E}\oplus\mathcal{E}^*))$ is defined as the $\tilde{\rho}^*\Lambda(\mathcal{E}\oplus\mathcal{E}^*)$-linear extension of the mapping given on generators by

$$\langle \nabla_X + i_\varphi + i_\xi \mid \Theta_0\rangle = \langle X \mid \theta_0\rangle + (\varphi \mid \xi),$$

where θ_0 is the canonical 1-form on the cotangent bundle.

Local coordinate expression. Note that with respect to the local coordinates introduced in Lemma 4.3 above

$$\Theta_0 = \sum_i \mathbf{d}x^i\cdot y^i + \sum_\mu \mathbf{d}\xi^\mu\cdot\zeta^\mu$$

as expected.

We now need to compute the graded exterior derivative of Θ_0. This can be conveniently expressed in terms of the natural fibered metric g defined on $\tilde{\rho}^*(\mathcal{E}\oplus\mathcal{E}^*)$ as follows:

$$g\colon \tilde{\rho}^*(\mathcal{E}\oplus\mathcal{E}^*)\times\tilde{\rho}^*(\mathcal{E}\oplus\mathcal{E}^*)\to C^\infty_{T^*M},$$
$$\big((\varphi,\xi),(\psi,\zeta)\big)\mapsto (\varphi\mid\zeta)+(\psi\mid\xi).$$

Note that this is *a fortiori* compatible with any given connection ∇.

6.2. Proposition. ([30]) *The graded* 2*-form* $\omega_0 = -\,\mathrm{d}\,\Theta_0$ *is a graded symplectic* 2*-form on the graded manifold* $(T^*M, \tilde{\rho}^*\Lambda(\mathcal{E}\oplus\mathcal{E}^*))$. *In fact*,

$$\langle \nabla_X, \nabla_Y \mid \mathrm{d}\,\Theta_0\rangle = \langle X, Y \mid \mathrm{d}\,\theta_0\rangle - g(R(X,Y)_,\cdot\,),$$
$$\langle \nabla_X, i_\varphi + i_\xi \mid \mathrm{d}\,\Theta_0\rangle = 0,$$
$$\langle i_\psi + i_\zeta, i_\varphi + i_\xi \mid \mathrm{d}\,\Theta_0\rangle = 2(\psi\mid\xi) + 2(\varphi\mid\zeta)$$

for any $X, Y\in\mathfrak{X}_{T^*M}$ *and any* $(\psi,\zeta),(\varphi,\xi)\in\tilde{\rho}^*(\mathcal{E}\oplus\mathcal{E}^*)^*$.

Proof. The form $\mathrm{d}\,\Theta_0$ is obviously closed. Its nondegeneracy is clear from the formulae above on generators. To obtain the formulae, let $\mathcal{F} = \tilde{\rho}^*(\mathcal{E}\oplus\mathcal{E}^*)$, so that $\mathrm{Der}\,\Lambda\mathcal{F}\simeq\Lambda\mathcal{F}\otimes(\mathfrak{X}_{T^*M}\oplus\mathcal{F}^*)$. We recall that the generators are given by ∇_X (for each $X\in\mathfrak{X}(T^*M)$) and i_χ (for each $\chi\in\mathcal{F}^*$), so that $i_\chi\eta = (\chi\mid\eta)\in C^\infty(T^*M)$ for any $\eta\in\mathcal{F}$, and then i_χ is extended to $\Lambda\mathcal{F}\to\Lambda\mathcal{F}$, as a degree -1 derivation. The fibered metric g on $\mathcal{F}$ allows an identification of $\mathcal{F}^*$ and $\mathcal{F}$; say, for each $\eta\in\mathcal{F}$ we write $i_\eta\colon\Lambda\mathcal{F}\to\Lambda\mathcal{F}$ for the degree -1 derivation which on $\mathcal{F}$ is given by $i_{\eta_2}\eta_1 = g(\eta_2,\eta_1)$. This is the form in which we will use it, and together with the identifications $\eta\leftrightarrow(\varphi,\xi)$, and $i_\eta\leftrightarrow i_\varphi + i_\xi$.

Now, since $Xg(\eta_1,\eta_2) = g(\nabla_X\eta_1,\eta_2) + g(\eta_1,\nabla_X\eta_2)$, it follows that, for any pair of vector fields $X, Y\in\mathfrak{X}(T^*M)$, and any pair of sections $\eta_1,\eta_2\in\mathcal{F}$,

$$g([\nabla_X,\nabla_Y]\eta_1,\eta_2) = g(\nabla_{[X,Y]}\eta_1,\eta_2) - g(R(X,Y)\eta_2,\eta_1).$$

That is,

$$[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} - i_{R(X,Y)_},$$

where, by definition, $(i_{R(X,Y)_}\eta_1)(\eta_2) = g(R(X,Y)\,\underline{\eta_2}, \eta_1)$. Thus $i_{R(X,Y)_} = g(R(X,Y)\,_, \cdot)$. Also note that $g(R(X,Y)\underline{\eta_2}, \eta_1) + g(R(X,Y)\underline{\eta_1}, \eta_2) = 0$. Therefore

$$\begin{aligned}\langle \nabla_X, \nabla_Y \mid \mathrm{d}\,\Theta_0\rangle &= \nabla_X\langle \nabla_Y \mid \Theta_0\rangle - \nabla_Y\langle \nabla_X \mid \Theta_0\rangle - \langle [\nabla_X, \nabla_Y] \mid \Theta_0\rangle \\ &= X\langle Y \mid \theta_0\rangle - Y\langle X \mid \theta_0\rangle - \langle \nabla_{[X,Y]} \mid \Theta_0\rangle + \langle i_{R(X,Y)_} \mid \Theta_0\rangle \\ &= \langle X, Y \mid \mathrm{d}\,\theta_0\rangle + g(R(X,Y)\,_, \cdot).\end{aligned}$$

Let us now consider the bracket $[\nabla_X, i_\eta]$. It is easily verified that $[\nabla_X, i_\eta] = i_{\nabla_X\eta}$. Therefore

$$\langle \nabla_X, i_\eta \mid \mathrm{d}\,\Theta_0\rangle = \nabla_X\langle i_\eta \mid \Theta_0\rangle - i_\eta\langle \nabla_X \mid \Theta_0\rangle - \langle i_{\nabla_X\eta} \mid \Theta_0\rangle,$$

but the second term is zero, since $\langle \nabla_X \mid \Theta_0\rangle = \langle X \mid \theta_0\rangle \in C^\infty(T^*M)$. Hence,

$$\langle \nabla_X, i_\eta \mid \mathrm{d}\,\Theta_0\rangle = \nabla_X\langle i_\eta \mid \Theta_0\rangle - \langle i_{\nabla_X\eta} \mid \Theta_0\rangle.$$

The definition of Θ_0 together with the compatibility of ∇ with g immediately implies that the right hand side is zero.

Finally, the fact that $[i_{\eta_1}, i_{\eta_2}] = 0$ yields

$$\langle i_{\eta_1}, i_{\eta_2} \mid \mathrm{d}\,\Theta_0\rangle = i_{\eta_1}\langle i_{\eta_2} \mid \Theta_0\rangle + i_{\eta_2}\langle i_{\eta_1} \mid \Theta_0\rangle$$

from which the last formula in the statement follows after making the identifications $\eta_1 \leftrightarrow (\psi, \zeta) \leftrightarrow i_\psi + i_\zeta$, and $\eta_2 \leftrightarrow (\varphi, \xi) \leftrightarrow i_\varphi + i_\xi$, and applying the definition of Θ_0. □

We shall now turn our attention to the *canonical odd* 1-form on the "odd quotient" $(E, \Lambda\tilde{p}_1^*(\mathcal{E} \oplus \mathfrak{X}_M))$ of $ST^*(M, \Gamma(\Lambda E))$. As before, we assume that a connection ∇ exists on $E \oplus TM$ in terms of which we may write the most general derivation $D \in \mathrm{Der}\,\Lambda\tilde{p}_1^*(\mathcal{E} \oplus \mathfrak{X}_M)$ as a linear combination of the generators (see 3.3 and 3.4 above)

$$\nabla_X, \quad \nabla_{\bar\xi}, \quad i_{\bar\eta}, \quad i_\varphi,$$

where $X \in \mathfrak{X}_M$, $\xi \in \mathcal{E}$, $\eta \in \mathfrak{X}_M^*$, and $\varphi \in \mathcal{E}^*$.

6.3. Definition. The *canonical* 1*-form* Θ_1 in the graded manifold $(E, p_1^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M))$ is defined as the $p_1^*\Lambda(\mathcal{E} \oplus \mathfrak{X}_M)$-linear extension of the mapping given on generators by

$$\langle \nabla_X + \nabla_{\bar\xi} + i_{\bar\eta} + i_\varphi \mid \Theta_1\rangle = \langle X \mid \bar\eta\rangle + (\varphi \mid \bar\xi).$$

Local coordinate expression. Note that with respect to the local coordinates introduced in Lemma 4.3 above,

$$\Theta_1 = \sum_i \mathbf{d}x^i \cdot \bar{y}^i + \sum_\mu \mathbf{d}\xi^\mu \cdot \bar\zeta^\mu.$$

6.4. Proposition. *The graded* 2*-form* $\omega_1 = -\,\mathrm{d}\,\Theta_1$ *is a graded symplectic* 2*-form on the graded manifold* $(E,\ p_1^*\Lambda(\mathcal{E}\oplus\mathfrak{X}_M))$. *In fact,*

$$\langle \nabla_X,\ i_{\tilde{\eta}} \mid \mathrm{d}\,\Theta_1\rangle = \langle X \mid \eta\rangle,$$
$$\langle \nabla_{\tilde{\xi}},\ i_{\varphi} \mid \mathrm{d}\,\Theta_1\rangle = -(\varphi \mid \xi)$$

and all others are equal to zero.

Proof. The proof is straightforward and considerably easier than that of Proposition 6.2 above, since the Lie brackets have already been computed there. □

6.5. A few words about naturality. It is a consequence of the results in [30] that any *even* supersymplectic structure on the split supermanifold $(T^*M,\ \Gamma(\Lambda(E\oplus E^*)))$, where the fibered metric on the bundle $E\oplus E^*\to M$ is given by g above, is (super) diffeomorphic to the canonical one. In other words, that there is always a (super) symplectic Batchelor trivialization map onto the canonical *even quotient.* For the odd quotient $(E,\ \Gamma(\Lambda(E\oplus TM)))$, the argument given in [30] to prove the statement we have just refered to, gets simplified because of the fact that the de Rham cohomology of the supermanifold defined by **d** is isomorphic to the de Rham cohomology on the base defined by d. In particular, any closed *odd* 2-form is exact. As a matter of fact, this type of argument has been given in [4] in order to produce a (super) symplectic Batchelor trivialization for the split supermanifold $(M,\ \Gamma(\Lambda(T^*M)))$ so as to map any *odd* (super) symplectic structure, onto a given one defined *a priori.* But after looking at the analysis given in [4], the reader will be easily convinced that the same argument carried on for the split supermanifold $(E,\ \Gamma(\Lambda(E\oplus TM)))$ shows that there is always a (super) symplectic Batchelor trivialization under which a given odd (super) symplectic structure is mapped onto the canonical form $\mathbf{d}\Theta_1$.

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