

Topics in Infinitely Divisible Distributions and Lévy Processes

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Preface

Lévy processes, that is, stochastic processes with independent and stationary increments, and stochastically continuous, have been the objects of extensive research since Paul Lévy's work in 1930s. Brownian motion, Poisson process, Γ -process, and stable processes, some of the most important stochastic processes, are examples of Lévy processes. Their study is intimately connected with that of infinitely divisible distributions, which have n th roots in convolution sense for each n . The first three chapters of Sato [63] are referred to for basic facts on Lévy processes and infinitely divisible distributions. The book [63] is cited as [S] in this work.

The class of infinitely divisible distributions is too large for some analysis. Its subclass consisting of all selfdecomposable distributions is more tractable and still large enough to include most of important distributions. It contains the stable distributions and there is a chain of classes of distributions $L_m, m = 1, 2, \dots, \infty$, introduced by Urbanik [85], [86], between the class of selfdecomposable distributions and the class of stable distributions. In this work we present the study of these classes in relation to three important concepts in stochastic processes: Ornstein-Uhlenbeck type processes (OU type processes), selfsimilar additive processes and subordination.

The first chapter of this work is devoted to studying basic properties of the classes L_m . In particular, these classes are characterized as limit distributions of sums of certain independent random variables. Also, representations of the characteristic functions of distributions in L_m are given. The representation of those distributions in L_∞ is related to the representation of stable distributions. In Chapter 2, Ornstein-Uhlenbeck type processes are defined by means of Lévy processes. Necessary and sufficient conditions, in terms of the Lévy measures of the generating Lévy processes, for the generated OU type processes to have limit distributions of the class L_m are given. In Chapter 3, selfsimilar processes that are additive are studied. Specifically, it is proved that their distributions at fixed times are selfdecomposable, and conversely, given a selfdecomposable distribution there is a selfsimilar additive process whose distribution at time one coincides with that distribution. Furthermore, necessary and

sufficient conditions for joint distributions of such process to belong to the smaller classes L_m are shown.

Chapters 4 and 5 introduce K -parameter Lévy processes when K is a proper cone in \mathbb{R}^N , and study subordination of K -parameter Lévy processes by K -increasing Lévy processes. This concept is an extension of the multivariate subordination of \mathbb{R}_+^N -parameter Lévy processes as studied by Barndorff-Nielsen, Pedersen and Sato [4] (2001). Chapter 4 shows the relation between the generating triplets of processes involved in this generalized subordination when $K = \mathbb{R}_+^N$. Finally, Chapter 5 studies those properties which are inherited by the resulting process from the subordinator. In particular, it is proved that, when the subordinand process is strictly stable, the subordinate process inherits from the subordinator the properties of being in the class L_m and of being strictly stable.

This work is based on Ken-iti Sato's lectures on January 22–25, 2001, within the “Periodo de Concentración en Procesos de Lévy” held at CIMAT Guanajuato, México, from January to May. He gave 29 pages of notes to the audience at that time. They were collection of results with bibliographic references. Alfonso Rocha-Arteaga extended the material, supplying most of proofs and explanations. Sato polished them, giving proofs of Proposition 35, Theorems 55, 83, 124, and Remark 57 and adding Proposition 31, Theorem 49, Remark 58, Example 87 and some others.

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Chapter 1

Classes L_m and their characterization

Selfdecomposable distributions are extensions of stable distributions. In this chapter we will prove that between the class L_0 of selfdecomposable distributions and the class \mathfrak{S} of stable distributions there is a chain of subclasses called L_m , $m = 1, \dots, \infty$, with

$$L_0 \supset L_1 \supset L_2 \supset \dots \supset L_\infty \supset \mathfrak{S}.$$

In Section 1.1 basic properties are proved and these classes are characterized as limits of partial sums of independent random variables whose distributions are in certain classes closed under convolution and convergence.

A representation of characteristic functions of the classes above is presented in Section 1.2, showing, in particular, that L_∞ is the smallest class containing the class \mathfrak{S} , closed under convolution and convergence. The representation of distributions in L_∞ indicates a clear connection to the representation of stable distributions. (The notation described at the end of the notes will be freely used.)

1.1 Basic properties and characterization by limit theorems

Definition 1 *A distribution on \mathbb{R}^d ($\mu \in \mathfrak{P}$) is selfdecomposable if, for every $b > 1$ there is $\rho_b \in \mathfrak{P}$ such that*

$$\hat{\mu}(z) = \hat{\mu}(b^{-1}z) \hat{\rho}_b(z). \tag{1.1}$$

Sometimes it is called of class L .

Definition 2 $L_0 = L_0(\mathbb{R}^d)$ is the class of selfdecomposable distributions on \mathbb{R}^d .

For example, Gaussian distributions on \mathbb{R}^d and Γ -distributions on \mathbb{R} are selfdecomposable. It is known that all selfdecomposable distributions are infinitely divisible. Formally we have the following lemma (see [S] Proposition 15.5).

Lemma 3 Let $\mu \in L_0$. Then ρ_b is uniquely determined by μ and b , and both μ and ρ_b are in ID, that is, infinitely divisible.

Definition 4 For $m = 1, 2, 3, \dots$, $L_m = L_m(\mathbb{R}^d)$ is recursively defined as follows: $\mu \in L_m$ if and only if for every $b > 1$ there is $\rho_b \in L_{m-1}$ such that $\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\hat{\rho}_b(z)$.

It is immediate, by Lemma 3 and Definition 4, that $ID \supset L_0 \supset L_m$ for all $m \geq 1$. Next, we prove that these classes form a nested sequence. Thus, intersection over all L_m will give the limiting class.

Proposition 5 $ID \supset L_0 \supset L_1 \supset L_2 \supset \dots$

Proof. By induction. It is clear that $L_0 \supset L_1$ by former remark. Suppose that $L_m \supset L_{m+1}$, that is, if $\mu \in L_{m+1}$ then $\mu \in L_m$. Now, if $\mu \in L_{m+2}$ then for every $b > 1$ $\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\hat{\rho}_b(z)$ where $\rho_b \in L_{m+1}$. Induction hypothesis implies that $\rho_b \in L_m$, therefore $\mu \in L_{m+1}$. We have proved that $L_{m+1} \supset L_{m+2}$. This concludes the proof. ■

Definition 6 $L_\infty = L_\infty(\mathbb{R}^d) = \bigcap_{m=0}^{\infty} L_m(\mathbb{R}^d)$.

Remark 7 The class of trivial distributions is contained in L_∞ . Briefly, let δ_{x_0} with $x_0 \in \mathbb{R}^d$ be a probability measure concentrated at x_0 . Then $\hat{\delta}_{x_0}(z) = \hat{\delta}_{x_0}(b^{-1}z)\hat{\delta}_{x_0(1-b^{-1})}(z)$ for all $b > 1$. Hence $\delta_{x_0} \in L_m$ for all m .

Classes L_m are closed under convolution, convergence and type equivalence. These statements are proved in (i), (ii) and (iii) of the following lemma, respectively. These properties are important in characterizing the class L_m as limit distributions of independent random variables as it will be seen.

Lemma 8 Let $m \in \{0, 1, 2, 3, \dots, \infty\}$.

- (i) If μ_1 and μ_2 are in L_m , then $\mu_1 * \mu_2 \in L_m$.
- (ii) If $\mu_n \in L_m$ and $\mu_n \rightarrow \mu$, then $\mu \in L_m$.
- (iii) If $\mu_1 = \mathcal{L}(X) \in L_m$ and $\mu_2 = \mathcal{L}(aX + b)$ with $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$, then $\mu_2 \in L_m$.
- (iv) If $\mu_1 \in L_m$, $\mu_2 \in \mathfrak{P}$ and $\hat{\mu}_2(z) = \hat{\mu}_1(z)^a$ with some $a \geq 0$, then $\mu_2 \in L_m$.

Proof. (i) Induction. Let μ_1, μ_2 be in L_0 with $\rho_{1,b}, \rho_{2,b}$ given by (1.1), respectively. Then

$$\widehat{\mu_1 * \mu_2}(z) = \widehat{\mu_1 * \mu_2}(b^{-1}z) \rho_{1,b} * \rho_{2,b}(z),$$

which proves $\mu_1 * \mu_2$ is in L_0 .

Assume that the assertion is true for $m - 1$. Let μ_1, μ_2 be in L_m with $\rho_{1,b}, \rho_{2,b}$ in L_{m-1} . Then

$$\begin{aligned} \widehat{\mu_1 * \mu_2}(z) &= \widehat{\mu_1}(b^{-1}z) \widehat{\rho_{1,b}}(z) \widehat{\mu_2}(b^{-1}z) \widehat{\rho_{2,b}}(z) \\ &= \widehat{\mu_1 * \mu_2}(b^{-1}z) \rho_{1,b} * \rho_{2,b}(z). \end{aligned}$$

Since $\rho_{1,b} * \rho_{2,b} \in L_{m-1}$, we conclude $\mu_1 * \mu_2$ is in L_m .

(ii) Induction. Let $\mu_n \in L_0$ and $\mu_n \rightarrow \mu$. Recall that characteristic functions $\widehat{\mu}_n$ and $\widehat{\mu}$ have no zero, since they are infinitely divisible. Thus, for every $b > 1$

$$\widehat{\rho}_{n,b}(z) = \frac{\widehat{\mu}_n(z)}{\widehat{\mu}_n(b^{-1}z)}.$$

It follows that $\widehat{\rho}_{n,b}(z) \rightarrow \varphi_b(z) = \frac{\widehat{\mu}(z)}{\widehat{\mu}(b^{-1}z)}$, which is continuous at zero and therefore a characteristic function. This proves that $\mu \in L_0$.

Now assume that the assertion is true for $m - 1$. Let $\mu_n \in L_m$ and $\mu_n \rightarrow \mu$. The proof follows exactly as in the step above with $\widehat{\rho}_{n,b} \in L_{m-1}$ converging to a characteristic function $\varphi_b(z)$ in L_{m-1} , which shows that μ is in L_m .

(iii) Let $\mu_1 = \mathcal{L}(X) \in L_m$ and $\mu_2 = \mathcal{L}(aX + b)$ with $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$. For every $c > 1$

$$\widehat{\mu}_1(z) = \widehat{\mu}_1(c^{-1}z) \widehat{\rho}_{1,c}(z),$$

with $\rho_{1,c} \in L_{m-1}$. Now

$$\begin{aligned} \widehat{\mu}_2(z) &= \widehat{\mu}_1(az) \exp(i \langle b, z \rangle), \\ &= \widehat{\mu}_1(c^{-1}az) \widehat{\rho}_{1,c}(az) \exp(i \langle b, z \rangle), \\ &= \widehat{\mu}_1(c^{-1}az) \exp(i \langle b, c^{-1}z \rangle) \exp(i \langle b(1 - c^{-1}), z \rangle) \widehat{\rho}_{1,c}(az), \\ &= \widehat{\mu}_2(b^{-1}z) \widehat{\delta}_{b(1-c^{-1})}(z) \widehat{\rho}_{1,c}(az). \end{aligned}$$

Thus $\mu_2 \in L_m$.

(iv) Induction. Let $\mu_1 \in L_0$ and $b > 1$. By Lemma 3 μ_1 and $\rho_{1,b}$ are in ID , and hence μ_1^a and $\rho_{1,b}^a$ are well defined and in ID (see [S] Lemma 7.9). Then

$$\begin{aligned} \widehat{\mu}_2(z) &= \widehat{\mu}_1(z)^a = \widehat{\mu}_1(b^{-1}z)^a \widehat{\rho}_{1,b}(z)^a \\ &= \widehat{\mu}_2(b^{-1}z) \widehat{\rho}_{1,b}(z)^a. \end{aligned}$$

Hence $\mu_2 \in L_0$.

If the assertion is true for $m - 1$, then, similarly for μ_1 in L_m , $\widehat{\rho}_{1,b}(z)^a$ belongs to L_{m-1} in the former expression of $\widehat{\mu}_2$, therefore $\mu_2 \in L_m$. ■

Next we only prove that stable distributions are contained in the class L_∞ . It will be shown that L_∞ is the smallest class containing the stable distributions closed under convolution and convergence once the representation of characteristic functions of the class L_∞ is established. See Theorem 24.

Definition 9 For $0 < \alpha \leq 2$ we define the class $\mathfrak{S}_\alpha = \mathfrak{S}_\alpha(\mathbb{R}^d)$ as the class of $\mu \in \mathfrak{P}(\mathbb{R}^d)$ such that, for every $n \in \mathbb{N}$, there is $c \in \mathbb{R}^d$ satisfying

$$\widehat{\mu}(z)^n = \widehat{\mu}(n^{1/\alpha}z)e^{i\langle c, z \rangle}. \quad (1.2)$$

Then $\mathfrak{S} = \bigcup_{\alpha \in (0,2]} \mathfrak{S}_\alpha$ and, for any distinct α, α' in $(0, 2]$, $\mathfrak{S}_\alpha \cap \mathfrak{S}_{\alpha'}$ is exactly the class of trivial distributions. Sometimes we call $\mu \in \mathfrak{S}_\alpha$ α -stable, but this terminology is different from that of [S], p. 76, when μ is trivial. For $0 < \alpha \leq 2$, let $\mathfrak{S}_\alpha^0 = \mathfrak{S}_\alpha^0(\mathbb{R}^d)$ be the class of $\mu \in \mathfrak{P}(\mathbb{R}^d)$ such that, for every $n \in \mathbb{N}$,

$$\widehat{\mu}(z)^n = \widehat{\mu}(n^{1/\alpha}z). \quad (1.3)$$

Then $\mathfrak{S}^0 = \mathfrak{S}^0(\mathbb{R}^d) = \bigcup_{\alpha \in (0,2]} \mathfrak{S}_\alpha^0(\mathbb{R}^d)$ is the class of strictly stable distributions on \mathbb{R}^d . For any distinct α, α' in $(0, 2]$, $\mathfrak{S}_\alpha^0 \cap \mathfrak{S}_{\alpha'}^0$ consists of a single element δ_0 . Distributions δ_c with $c \neq 0$ belong only to \mathfrak{S}_1^0 . Sometimes we call $\mu \in \mathfrak{S}_\alpha^0$ strictly α -stable.

Proposition 10 A distribution μ is in $\mathfrak{S}_\alpha(\mathbb{R}^d)$ if and only if $\mu \in ID(\mathbb{R}^d)$ and, for every $a > 0$, there is $c \in \mathbb{R}^d$ satisfying

$$\widehat{\mu}(z)^a = \widehat{\mu}(a^{1/\alpha}z)e^{i\langle c, z \rangle}. \quad (1.4)$$

A distribution μ is in $\mathfrak{S}_\alpha^0(\mathbb{R}^d)$ if and only if $\mu \in ID(\mathbb{R}^d)$ and, for every $a > 0$,

$$\widehat{\mu}(z)^a = \widehat{\mu}(a^{1/\alpha}z)e^{i\langle c, z \rangle}. \quad (1.5)$$

See E18.4 of [S] for a proof.

Proposition 11 $L_\infty \supset \mathfrak{S}$.

Proof. Let $\mu \in \mathfrak{S}$. Then $\mu \in \mathfrak{S}_\alpha$ for some $\alpha \in (0, 2]$, that is, for every $a > 0$ there is c such that (1.4) holds. Given $b > 1$ let us define $a = b^{-\alpha} < 1$. Then $\widehat{\mu}(z)^a = \widehat{\mu}(b^{-1}z) \exp(i \langle c, z \rangle)$, and hence

$$\begin{aligned} \widehat{\mu}(z) &= \widehat{\mu}(b^{-1}z) \widehat{\mu}(z)^{1-a} \exp(i \langle c, z \rangle) \\ &= \widehat{\mu}(b^{-1}z) \widehat{\rho}_b(z), \end{aligned}$$

where $\widehat{\rho}_b(z) = \widehat{\mu}(z)^{1-a} \exp(i \langle c, z \rangle)$ is a characteristic function. Hence $\mu \in L_0$. Now, by Lemma 8 (iv) and Remark 7, $\rho_b \in L_0$. The last arguments are recursively applied to yield

$$\rho_b \in L_0 \Rightarrow \mu \in L_1 \Rightarrow \rho_b \in L_1 \Rightarrow \mu \in L_2 \Rightarrow \rho_b \in L_2 \Rightarrow \mu \in L_3 \Rightarrow \dots$$

Thus $\mu \in L_m$ for every $m \geq 0$. ■

Lemma 8 shows that the class of distributions L_m is a completely closed class, that is, closed under convolution, weak convergence and type equivalence. This property is essential in characterizing this class in terms of limits for sums of independent random variables.

Definition 12 Let \mathfrak{Q} be a subclass of \mathfrak{P} . Define $K(\mathfrak{Q}) \subset \mathfrak{P}$ as follows. $\mu \in K(\mathfrak{Q})$ if and only if there are independent \mathbb{R}^d -valued random variables Z_1, Z_2, Z_3, \dots , $b_n > 0$ and $c_n \in \mathbb{R}^d$ satisfying the following conditions.

- a) $\mathcal{L}(b_n \sum_{k=1}^n Z_k - c_n) \rightarrow \mu$ as $n \rightarrow \infty$.
- b) $\{b_n Z_k : k = 1, 2, \dots, n; n = 1, 2, \dots\}$ is a null array, that is, $\lim_{n \rightarrow \infty} \max_{1 \leq k \leq n} P[|b_n Z_k| > \varepsilon] = 0$ for all $\varepsilon > 0$.
- c) $\mathcal{L}(Z_k) \in \mathfrak{Q}$ for each k .

Using this operation K , the class of selfdecomposable distributions is comprehended as a class of limit distributions, as well as the class of distributions L_m .

Theorem 13 (i) $L_0 = K(\mathfrak{P}) = K(ID)$.

(ii) $L_m = K(L_{m-1})$ for $m = 1, 2, \dots$.

(iii) $L_\infty = K(L_\infty)$ and L_∞ is the greatest class \mathfrak{Q} that satisfies $\mathfrak{Q} = K(\mathfrak{Q})$.

Proof. (i) The first equality can be found in [S] Theorem 15.3. Thus $L_0 \supset K(ID)$. On the other hand, the same argument as in the proof of (ii) below combined with Lemma 3 yields $L_0 \subset K(ID)$.

(ii) Let $\mu \in L_m$. Since $\widehat{\mu}$ has no zero, we have, for every $b > 1$,

$$\widehat{\rho}_b(z) = \frac{\widehat{\mu}(z)}{\widehat{\mu}(b^{-1}z)}, \quad z \in \mathbb{R}^d,$$

with ρ_b in L_{m-1} . Let Z_1, Z_2, \dots independent random variables on \mathbb{R}^d with

$$\widehat{\mu}_{Z_k}(z) = \widehat{\rho}_{(k+1)/k}((k+1)z),$$

and define $S_n = n^{-1} \sum_{k=1}^n Z_k$. The notation $\mu_Z = \mathcal{L}(Z)$ for any random variable Z is used here. Then

$$\widehat{\mu}_{S_n}(z) = \prod_{k=1}^n \widehat{\mu}_{Z_k}(z/n) = \prod_{k=1}^n \frac{\widehat{\mu}(((k+1)/n)z)}{\widehat{\mu}((k/n)z)} = \frac{\widehat{\mu}(((n+1)/n)z)}{\widehat{\mu}((1/n)z)} \rightarrow \widehat{\mu}(z).$$

Hence $\mathcal{L}(S_n) \rightarrow \mu$.

By continuity of $\widehat{\mu}$, $\max_{1 \leq k \leq n} |\widehat{\mu}_{Z_k/n}(z) - 1| = \max_{1 \leq k \leq n} \left| \frac{\widehat{\mu}(((k+1)/n)z)}{\widehat{\mu}((k/n)z)} - 1 \right| \rightarrow 0$ as $n \rightarrow \infty$. This proves that $\{Z_k/n : k = 1, 2, \dots, n; n = 1, 2, \dots\}$ is a null array by Exercise 12.12 in [S]. Then a), b) and c) in the definition above hold for μ with $b_n = n^{-1}$ and $c_n = 0$. We conclude that $\mu \in K(L_{m-1})$.

Now suppose $\mu \in K(L_{m-1})$. Then for every $b > 1$ conditions a) and b) and Lemma 15.4 of [S] imply that there are sequences of positive integers $\{m_j\}$ and $\{n_j\}$ with $m_j < n_j$ such that $b_{m_j}/b_{n_j} \rightarrow b$. Let us define

$$\begin{aligned} W_n &= b_n \sum_{k=1}^n Z_k + c_n, \\ U_j &= b_{n_j} \sum_{k=1}^{m_j} Z_k + b_{n_j} b_{m_j}^{-1} c_{m_j} \quad \text{and} \\ V_j &= b_{n_j} \sum_{k=m_j+1}^{n_j} Z_k + c_{n_j} - b_{n_j} b_{m_j}^{-1} c_{m_j}. \end{aligned}$$

Then $W_{n_j} = U_j + V_j$ and

$$\widehat{\mu}_{W_{n_j}}(z) = \widehat{\mu}_{U_j}(z) \widehat{\mu}_{V_j}(z) \tag{1.6}$$

by independence. Since $U_j = b_{n_j} b_{m_j}^{-1} W_{m_j}$,

$$\begin{aligned} \left| \widehat{\mu}_{U_j}(z) - \widehat{\mu}(b_{n_j} b_{m_j}^{-1} z) \right| &= \left| \widehat{\mu}_{W_{m_j}}(b_{n_j} b_{m_j}^{-1} z) - \widehat{\mu}(b_{n_j} b_{m_j}^{-1} z) \right| \\ &\leq \sup_{|w| \leq |z|} \left| \widehat{\mu}_{W_{m_j}}(w) - \widehat{\mu}(w) \right| \rightarrow 0 \end{aligned}$$

as $j \rightarrow \infty$ by condition a) and therefore $\widehat{\mu}_{U_j}(z) \rightarrow \widehat{\mu}(b^{-1}z)$. Since $\widehat{\mu}$ has no zeros, we can take $\widehat{\mu}_{V_j}(z) \rightarrow \widehat{\mu}(z)/\widehat{\mu}(b^{-1}z)$ from (1.6) as $j \rightarrow \infty$ and obtain a continuous limit which is the characteristic function of some probability measure ρ_b in L_{m-1} by condition c) and Lemma 8. Thus $\mu \in L_m$.

(iii) We have $L_m = K(L_{m-1}) \supset K(L_\infty)$ for all m , in consequence $L_\infty \supset K(L_\infty)$.

Now, if $\mu \in L_\infty$ then $\rho_b \in L_\infty$. Let us take Z_k as in the proof of (ii) and again $\mathcal{L}(\sum_{k=1}^n Z_k/n)$ tends to μ , where the distribution of Z_k is in L_∞ and $\{Z_k/n\}$ is a null array. This means that $\mu \in K(L_\infty)$.

Finally, let $\mathfrak{Q} = K(\mathfrak{Q})$. Then $\mathfrak{Q} = K(\mathfrak{Q}) \subset K(\mathfrak{P}) = L_0$. Now from $\mathfrak{Q} \subset L_0$ we obtain $\mathfrak{Q} = K(\mathfrak{Q}) \subset K(L_0) = L_1$. A cyclic application of the same arguments yields $\mathfrak{Q} \subset L_m$ for every m therefore $\mathfrak{Q} \subset L_\infty$. ■

1.2 Characteristic functions of distributions in L_m

Recall that $\mu \in ID(\mathbb{R}^d)$ is represented by its generating triplet (A, ν, γ) , that is,

$$\widehat{\mu}(z) = \exp \left[-\frac{1}{2} \langle z, Az \rangle + i \langle \gamma, z \rangle + \int_{\mathbb{R}^d} (e^{i \langle z, x \rangle} - 1 - i \langle z, x \rangle 1_{\{|x| \leq 1\}}(x)) \nu(dx) \right], \quad (1.7)$$

where A is a symmetric nonnegative-definite $d \times d$ matrix called the Gaussian covariance matrix, ν is a measure on \mathbb{R}^d satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty$ called the Lévy measure, and γ is a vector in \mathbb{R}^d . If $\{X_t\}$ is a Lévy process, then the generating triplet of $\mathcal{L}(X_1)$ is called the generating triplet of $\{X_t\}$. A Lévy process $\{X_t\}$ is called selfdecomposable if $\mathcal{L}(X_1) \in L_0$; it is called of class L_m if $\mathcal{L}(X_1) \in L_m$.

Let $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$, the unit sphere on \mathbb{R}^d . The following result is proved in [S] Theorem 15.10. Notice that selfdecomposability imposes no restriction on A and γ .

Theorem 14 *Let $\mu \in ID(\mathbb{R}^d)$. Then, $\mu \in L_0$ if and only if*

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{k_\xi(r)}{r} dr, \quad B \in \mathcal{B}(\mathbb{R}^d), \quad (1.8)$$

where λ is a finite measure on S and $k_\xi(r)$ is decreasing in $r \in (0, \infty)$ and measurable in $\xi \in S$.

Remark 15 The λ and $k_\xi(r)$ in Theorem 14 are not uniquely determined by $\mu \in L_0$. If $\nu \neq 0$, then we can choose λ to be a probability measure on S and $k_\xi(r)$ to be right continuous in r and to satisfy

$$\int_0^\infty (r^2 \wedge 1) \frac{k_\xi(r)}{r} dr = c > 0, \quad (1.9)$$

where c is a constant independent of ξ . If λ , $k_\xi(r)$ and λ^\sharp , $k_\xi^\sharp(r)$ both satisfy (1.8) and these conditions, then $\lambda = \lambda^\sharp$ and $k_\xi(\cdot) = k_\xi^\sharp(\cdot)$ for λ -a. e. ξ . Henceforth we assume that λ and $k_\xi(r)$ satisfy these conditions and call λ the spherical component of ν and $k_\xi(r)$ the k -function of ν (or μ). Define

$$h_\xi(u) = k_\xi(e^{-u}).$$

We call $h_\xi(u)$ the h -function of ν (or μ). We have

$$\begin{aligned} \widehat{\mu}(z) = & \exp \left[-\frac{1}{2} \langle z, Az \rangle + i \langle \gamma, z \rangle \right. \\ & \left. + \int_S \lambda(d\xi) \int_0^\infty (e^{i \langle z, r\xi \rangle} - 1 - i \langle z, r\xi \rangle 1_{(0,1]}(r)) \frac{k_\xi(r)}{r} dr \right]. \end{aligned} \quad (1.10)$$

In one dimension ($d = 1$), we have a unique expression

$$\widehat{\mu}(z) = \exp \left[-\frac{1}{2} Az^2 + i\gamma z + \int_{\mathbb{R}} (e^{izx} - 1 - izx 1_{[-1,1]}(x)) \frac{k(x)}{|x|} dx \right], \quad (1.11)$$

with $k(x)$ being decreasing and right continuous on $(0, \infty)$ and increasing and left-continuous on $(-\infty, 0)$, $k(x) \geq 0$, and $\int_{-\infty}^\infty (x^2 \wedge 1) \frac{k(x)}{|x|} dx < \infty$.

Remark 16 The representation (1.8) satisfying (1.9) of the Lévy measure is a special case of general polar decomposition. In general, if ν is the Lévy measure of $\mu \in ID(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) = c > 0$, then

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \rho_\xi(dr),$$

where λ is a probability measure on S and $\rho_\xi(\cdot)$ is a σ -finite measure on $(0, \infty)$ with

$$\int_0^\infty (r^2 \wedge 1) \rho_\xi(dr) = c$$

such that $\rho_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0, \infty))$. Moreover, if λ , $\rho_\xi(\cdot)$ and λ^\sharp , $\rho_\xi^\sharp(\cdot)$ both give this representation, then $\lambda = \lambda^\sharp$ and $\rho_\xi(\cdot) = \rho_\xi^\sharp(\cdot)$ for λ -a.e. ξ . Proof is given as an application of the existence theorem of conditional distributions.

Definition 17 For $\varepsilon > 0$, Δ_ε is the difference operator, $\Delta_\varepsilon f(u) = f(u + \varepsilon) - f(u)$. Δ_ε^n is the n th iteration of Δ_ε . Hence

$$\Delta_\varepsilon^n f(u) = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} f(u + j\varepsilon).$$

Define $\Delta_\varepsilon^0 f = f$. We say that $f(u)$, $u \in \mathbb{R}$, is monotone of order n if $\Delta_\varepsilon^j f \geq 0$ for $\varepsilon > 0$, $j = 0, 1, \dots, n$. We say that $f(u)$, $u \in \mathbb{R}$, is absolutely monotone if $\Delta_\varepsilon^j f \geq 0$ for $\varepsilon > 0$ and $j \in \mathbb{Z}_+$.

Lemma 18 (i) If $f(u)$ is monotone of order n , then, for all $\varepsilon > 0$ and $j = 0, 1, \dots, n-1$, $\Delta_\varepsilon^j f$ is increasing.

(ii) Let $n \geq 2$. A function $f(u)$ is monotone of order n if and only if $f \in C^{n-2}$, $f^{(j)} \geq 0$ for $j = 0, 1, \dots, n-2$, and $f^{(n-2)}$ is increasing and convex.

(iii) A function $f(u)$ is absolutely monotone if and only if $f \in C^\infty$ and $f^{(j)} \geq 0$ for $j = 0, 1, \dots$.

See Widder [93] pp. 144-151 for a proof. A consequence of (ii) is that, if $f \in C^n$ and $f^{(j)} \geq 0$ for $j = 0, 1, \dots, n$, then f is monotone of order n .

Remark 19 $f(-u)$ is absolutely monotone if and only if $f(u)$ is completely monotone, that is, $(-1)^j f^{(j)} \geq 0$ for $j = 0, 1, \dots$.

Now we will give characterization of the class $L_m(\mathbb{R}^d)$ in terms of its h -function.

Theorem 20 (i) Let $m \in \{0, 1, \dots\}$. Then $\mu \in L_m$ if and only if $\mu \in L_0$ and the h -function $h_\xi(u)$ is monotone of order $m+1$ for λ -a. e. ξ .

(ii) $\mu \in L_\infty$ if and only if $\mu \in L_0$ and $h_\xi(u)$ is absolutely monotone for λ -a. e. ξ .

Proof. (i) The assertion is trivial for $m = 0$. Let $m \geq 1$. We will show the validity of the assertion for m , assuming that it is valid for $m-1$. That is, we assume that $\mu \in L_{m-1}$ if and only if $\mu \in L_0$ and $h_\xi(u)$ is monotone of order m for λ -a. e. ξ .

Let $\mu \in L_0$ with $\nu \neq 0$. By definition of selfdecomposability $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z)$ for every $b > 1$. Notice that ρ_b is in ID with Lévy measure

$$\nu_b(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{\{k_\xi(r) - k_\xi(br)\}}{r} dr, \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where λ and $k_\xi(r)$ are the spherical component and the k -function of μ , respectively.

Let $a_b(\xi) = \int_0^\infty (r^2 \wedge 1) \frac{\{k_\xi(r) - k_\xi(br)\}}{r} dr$. We have that $0 < a_b(\xi) < c$ where c is as in Remark 15.

We will find the spherical component and the k -function for ρ_b . Let

$$\begin{aligned}\lambda_b(d\xi) &= c_b^{-1} a_b(\xi) \lambda(d\xi) \\ k_{b,\xi}(r) &= c_b a_b^{-1}(\xi) \{k_\xi(r) - k_\xi(br)\}\end{aligned}$$

where c_b is a constant such that λ_b is a probability measure. Then

$$h_{b,\xi}(u) = k_{b,\xi}(e^{-u}) = c_b a_b^{-1}(\xi) \{h_\xi(u) - h_\xi(u + \log b^{-1})\}.$$

Suppose that $\mu \in L_m$. Then $\rho_b \in L_{m-1}$ and its h -function $h_{b,\xi}$ is monotone of order m for λ -a.e. ξ by induction hypothesis. Hence

$$\Delta_\varepsilon^j h_\xi(u) - \Delta_\varepsilon^j h_\xi(u + \log b^{-1}) = c_b^{-1} a_b(\xi) \Delta_\varepsilon^j h_{b,\xi}(u) \geq 0, \quad j = 0, 1, 2, \dots, m.$$

It follows that $\Delta_\varepsilon^j h_\xi(u) \geq 0$ for $j = 1, 2, \dots, m+1$, and therefore h_ξ is monotone of order $m+1$ for λ -a.e. ξ .

Conversely, if h_ξ is monotone of order $m+1$ for λ a.e. ξ , then, by Lemma 18 (i), $\Delta_\varepsilon^j h_\xi(u)$ is increasing in u for $j = 1, 2, \dots, m$, and hence $h_{b,\xi}(u)$ is monotone of order m . Then, by induction hypothesis $\rho_b \in L_{m-1}$. Thus $\mu \in L_m$.

(ii) This assertion is an immediate consequence of (i) and the definition of L_∞ . ■

Lemma 21 *Let $0 < c < \infty$. A function $h_\xi(u)$ is absolutely monotone in $u \in \mathbb{R}$ and measurable in ξ and satisfies*

$$\int_{-\infty}^{\infty} (e^{-2u} \wedge 1) h_\xi(u) du = c \tag{1.12}$$

for all ξ if and only if

$$h_\xi(u) = \int_{(0,2)} e^{\alpha u} \Gamma_\xi(d\alpha), \tag{1.13}$$

where Γ_ξ is a measure on $(0, 2)$ for each ξ satisfying

$$\int_{(0,2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha} \right) \Gamma_\xi(d\alpha) = c \tag{1.14}$$

and Γ_ξ is measurable in ξ (that is, $\Gamma_\xi(B)$ is measurable in ξ for every $B \in \mathcal{B}((0, 2))$).

Proof. Suppose that $h_\xi(u)$ is absolutely monotone, measurable in ξ and (1.12) is valid. Then by Bernstein's theorem there is, for each ξ and u_o , a unique finite measure $\Gamma_\xi^{u_o}$ on $[0, \infty)$ such that

$$h_\xi(u_o + u) = \int_{[0, \infty)} e^{\alpha u} \Gamma_\xi^{u_o}(d\alpha), \quad \text{for } u \leq 0.$$

Letting $\Gamma_\xi(d\alpha) = e^{\alpha u} \Gamma_\xi^{u_o}(d\alpha)$ we eliminate independence of u_o and

$$h_\xi(u) = \int_{[0, \infty)} e^{\alpha u} \Gamma_\xi(d\alpha), \quad \text{for } u \in \mathbb{R}.$$

Now, $\Gamma_\xi(\{0\}) = 0$ since $h_\xi(u) \rightarrow 0$ as $u \rightarrow -\infty$. Moreover, from condition (1.12) we obtain

$$\int_{(0, \infty)} \Gamma_\xi(d\alpha) \int_0^\infty e^{u(\alpha-2)} du = \int_0^\infty e^{-2u} h_\xi(u) du < \infty,$$

which implies that Γ is concentrated in $(0, 2)$. Hence (1.13) follows.

We will prove (1.14). From (1.12)

$$\begin{aligned} c &= \int_{-\infty}^0 h_\xi(u) du + \int_0^\infty e^{-2u} h_\xi(u) du \\ &= \int_{(0, 2)} \Gamma_\xi(d\alpha) \int_{-\infty}^0 e^{\alpha u} du + \int_{(0, 2)} \Gamma_\xi(d\alpha) \int_0^\infty e^{u(\alpha-2)} du \\ &= \int_{(0, 2)} \Gamma_\xi(d\alpha) \left[\int_{-\infty}^0 e^{\alpha u} du + \int_0^\infty e^{-u(2-\alpha)} du \right] \\ &= \int_{(0, 2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha} \right) \Gamma_\xi(d\alpha). \end{aligned}$$

We claim that Γ_ξ is measurable in ξ . If Γ_ξ is a continuous measure for every ξ , then it is proved by the inversion formula for Laplace transforms (see Widder [78] pp. 295),

$$\int_0^\beta \Gamma_\xi(d\alpha) = \lim_{u \rightarrow -\infty} \sum_{m=0}^{[\beta|u]} \frac{u^m}{m!} h_\xi^{(m)}(u), \quad \text{for } \beta \geq 0.$$

If not, it is proved by approximating Γ_ξ with the convolutions with continuous measures. Conversely, from (1.14) we get (1.12) due to the one-to-one property in the expression above. Now from $h_\xi(-u) = \int_{(0, 2)} e^{-\alpha u} \Gamma_\xi(d\alpha)$, we obtain

$$(-1)^n \left(\frac{d}{du} \right)^n (h_\xi(-u)) = \int_{(0, 2)} e^{-\alpha u} \alpha^n \Gamma_\xi(d\alpha) \geq 0, \quad \text{for every } n.$$

Notice that the differentiation under the integral is permissible since, for each n , the new integrand is bounded and continuous as function of $u \in \mathbb{R}$. Thus $h_\xi(-u)$ is completely monotone and by Remark 19 $h_\xi(u)$ is absolutely monotone. Measurability of h_ξ in ξ is obtained from (1.13) as limit of ξ -measurable functions. ■

Theorem 22 (i) If $\mu \in L_\infty$, then

$$\begin{aligned} \widehat{\mu}(z) = & \exp \left[-\frac{1}{2} \langle z, Az \rangle + i \langle \gamma, z \rangle \right. \\ & \left. + \int_{(0,2)} \Gamma(d\alpha) \int_S \lambda_\alpha(d\xi) \int_0^\infty (e^{i \langle z, r\xi \rangle} - 1 - i \langle z, r\xi \rangle 1_{(0,1]}(r)) \frac{dr}{r^{1+\alpha}} \right], \end{aligned} \quad (1.15)$$

where A is a nonnegative-definite symmetric $d \times d$ matrix, $\gamma \in \mathbb{R}^d$, Γ is a measure on $(0, 2)$ satisfying

$$\int_{(0,2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha} \right) \Gamma(d\alpha) < \infty, \quad (1.16)$$

and λ_α is a probability measure on S for each α and is measurable in α . These A , γ , and Γ are uniquely determined by μ and λ_α is determined by μ up to α of Γ -measure 0.

(ii) Given A , γ , Γ , and λ_α satisfying the conditions above, we can find $\mu \in L_\infty$ satisfying (1.15).

Proof. Suppose that $\mu \in L_\infty$. Define Γ_ξ by Lemma 21 from the h -function h_ξ given by (1.13). We can find Γ and λ_α such that (1.16) holds, λ_α is measurable in α , and

$$\int_{(0,2)} \Gamma(d\alpha) \int_S \lambda_\alpha(d\xi) f(\alpha, \xi) = \int_S \lambda(d\xi) \int_{(0,2)} \Gamma_\xi(d\alpha) f(\alpha, \xi) \quad (1.17)$$

for every nonnegative measurable function $f(\alpha, \xi)$. In fact, if $\nu \neq 0$, then it suffices to apply the existence theorem of conditional distribution to the probability measure given by $c^{-1} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha} \right) \lambda(d\xi) \Gamma_\xi(d\alpha)$ on $(0, 2) \times S$. Then we have

$$\int_{\mathbb{R}^d} f(x) \nu(dx) = \int_{(0,2)} \Gamma(d\alpha) \int_S \lambda_\alpha(d\xi) f(r\xi) r^{-\alpha-1} dr \quad (1.18)$$

for every nonnegative measurable function $f(x)$, since, by (1.8), (1.13) and (1.17) this is true when $f(r\xi) = 1_B(r\xi)$, the indicator function of the set B in (1.8). It follows that

(1.18) is valid for every complex-valued ν -integrable function f . Letting $f(r\xi) = e^{i\langle z, r\xi \rangle} - 1 - i\langle z, r\xi \rangle 1_{(0,1]}(r)$, we get (1.15).

Conversely, given γ , Γ , A , and λ_α by (1.15) we can find λ and Γ_ξ such that (1.17) holds. Now define the increasing function h_ξ by $h_\xi(u) = \int_{(0,2)} e^{\alpha u} \Gamma_\xi(d\alpha)$ for every $u \in \mathbb{R}$ and $k_\xi(u) = h_\xi(-\log u)$. We claim that $\mu \in L_\infty$. By Theorem 20 (ii), it suffices to prove that h_ξ is absolutely monotone and $\mu \in L_0$. For the first part, condition (1.16) implies (1.14) and therefore we can apply Lemma 21. The second one follows from Theorem 14, since, by (1.15) and (1.17) the Lévy measure ν of μ has the form

$$\begin{aligned} \nu(B) &= \int_S \lambda(d\xi) \int_{(0,2)} \Gamma_\xi(d\alpha) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} dr \\ &= \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) dr \int_{(0,2)} \Gamma_\xi(d\alpha) r^{-\alpha-1} \\ &= \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{k_\xi(r)}{r} dr, \end{aligned}$$

and the required properties for λ and k_ξ , in this expression, are satisfied. Therefore $\mu \in L_0$. The correspondence between (Γ, λ_α) and (λ, Γ_ξ) is one-to-one up to α in a set of Γ -measure 0 and ξ in a set of λ -measure 0. Hence the reconstruction procedure of μ shows at the same time uniqueness of the representation. ■

Remark 23 We will clarify the relation of the representation of L_∞ with the class \mathfrak{S}_α . For $0 < \alpha < 2$, it is known (see [S] Theorem 14.3 and Remark 14.4), that $\mu \in \mathfrak{S}_\alpha$ if and only if $\mu \in ID$ with triplet (A, ν, γ) satisfying $A = 0$ and

$$\nu(B) = c \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} dr \quad \text{for every } B \in \mathcal{B}(\mathbb{R}^d), \quad (1.19)$$

with a probability measure λ and a nonnegative constant c . That is, $\mu \in \mathfrak{S}_\alpha$ with $0 < \alpha < 2$ if and only if $\mu \in L_\infty$ and, in its representation (1.15), $A = 0$ and Γ is concentrated at the point α . It is well known that $\mu \in \mathfrak{S}_2$ if and only if it is Gaussian, that is, $\nu = 0$.

Theorem 24 The class L_∞ is the smallest class containing \mathfrak{S} and closed under convolution and convergence.

Proof. It has already been shown that the class L_∞ contains \mathfrak{S} and is closed under convolution and convergence. Let \mathfrak{Q} be a class containing \mathfrak{S} and closed under convolution and convergence. Let $\mu \in L_\infty$ and take the representation (1.15) for $\hat{\mu}$. We will prove that $\mu \in \mathfrak{Q}$.

Suppose $\gamma = 0$ and $A = 0$. First, assume that Γ is supported by $[\varepsilon, 2 - \varepsilon]$ for some positive ε . Let $M(d\alpha d\xi) = \Gamma(d\alpha) \lambda_\alpha(d\xi)$. It is a finite measure on $[\varepsilon, 2 - \varepsilon] \times S$. Choose M_n such that they converge to M and that each M_n is supported by $E_n \times S$ where E_n is a finite set in $[\varepsilon, 2 - \varepsilon]$. We have that $M_n(d\alpha d\xi) = \Gamma_n(d\alpha) \lambda_{n,\alpha}(d\xi)$, where Γ_n is supported by E_n . Let

$$f(\alpha, \xi) = \int_0^\infty (e^{i\langle z, r\xi \rangle} - 1 - i\langle z, r\xi \rangle 1_{(0,1]}(r)) \frac{dr}{r^{1+\alpha}}$$

and let μ_n be the probability measure with characteristic function given by the expression $\exp\left(\int_{(0,2)} \Gamma_n(d\alpha) \int_S \lambda_{n,\alpha}(d\xi) f(\alpha, \xi)\right)$. As μ_n is convolution of stable distributions, it belongs to \mathfrak{Q} . Since $f(\alpha, \xi)$ is continuous in (α, ξ) , $\widehat{\mu}_n(z)$ converges to $\widehat{\mu}(z)$. Hence $\mu \in \mathfrak{Q}$. Next consider a general μ . Restrict Γ to $[\frac{1}{n}, 2 - \frac{1}{n}]$ and let μ_n be the corresponding distribution. Since $\int_{(0,2)} \Gamma(d\alpha) \int_S \lambda_\alpha(d\xi) |f(\alpha, \xi)|$ is finite, $\widehat{\mu}_n(z)$ tends to $\widehat{\mu}(z)$. Hence $\mu \in \mathfrak{Q}$. This concludes the proof. ■

Example 25 If $\{X_t\}$ is a Γ -process with parameter $q > 0$, then $\mathcal{L}(X_t)$ is Γ -distribution; that is,

$$P[X_t \in B] = \frac{q^t}{\Gamma(t)} \int_{B \cap (0, \infty)} x^{t-1} e^{-qx} dx \quad (1.20)$$

for $t > 0$. We will prove that $\mathcal{L}(X_t)$ is in $L_0(\mathbb{R})$ but not in $L_1(\mathbb{R})$ for $t > 0$.

Let $\mu = \mathcal{L}(X_1)$. The Laplace transform of μ^t is

$$\begin{aligned} L_{\mu^t}(u) &= \frac{q^t}{\Gamma(t)} \int x^{t-1} e^{-(u+q)x} dx \\ &= \frac{q^t}{(u+q)^t} \int \frac{(u+q)^t}{\Gamma(t)} x^{t-1} e^{-(u+q)x} dx \\ &= \left(1 + \frac{u}{q}\right)^{-t}, \quad u \geq 0. \end{aligned}$$

On the other hand

$$\begin{aligned} \log\left(1 + \frac{u}{q}\right) &= \log(u+q) - \log(q) \\ &= \int_q^{u+q} \frac{dy}{y} = \int_0^u \frac{dy}{q+y} = \int_0^u \left(\int_0^\infty e^{-(q+y)x} dx\right) dy \\ &= \int_0^\infty \left(\int_0^u e^{-yx} dy\right) e^{-qx} dx = \int_0^\infty (1 - e^{-ux}) \frac{e^{-qx}}{x} dx. \end{aligned}$$

Then

$$L_{\mu^t}(u) = \exp \left[t \int_0^\infty (e^{-ux} - 1) \frac{e^{-qx}}{x} dx \right].$$

Extending this equality to the left half plane $\{\omega \in \mathbb{C} : \operatorname{Re}(\omega) \leq 0\}$ by analyticity and continuity to the boundary, we get, for $\omega = iz$ with $z \in \mathbb{R}$,

$$\widehat{\mu}^t(z) = \exp \left[t \int_0^\infty (e^{izx} - 1) \frac{e^{-qx}}{x} dx \right].$$

Hence the Lévy measure of $\{X_t\}$ is

$$\nu(dx) = x^{-1} e^{-qx} 1_{(0,\infty)}(x) dx, \quad (1.21)$$

with k -function $k(x) = e^{-qx} 1_{(0,\infty)}(x)$. This proves the first part.

We can assume that $q = 1$. Its h -function is $h(u) = k(e^{-u}) = e^{-e^{-u}}$ for $u \in \mathbb{R}$. Then

$$\begin{aligned} h'(u) &= e^{-u} e^{-e^{-u}}, \\ h''(u) &= (e^{-2u} - e^{-u}) e^{-e^{-u}}. \end{aligned}$$

Note that $h''(u) < 0$ for $u > 0$. The h -function is not a monotone function of order 2 by Lemma 18 (ii) because it is not convex in $(0, \infty)$. Finally, by Theorem 20 (i), $\mu \notin L_1(\mathbb{R})$.

Example 26 The distribution μ on \mathbb{R} in Linnik–Ostrovskii book ([S] E 18.19), that is,

$$\mu(dx) = c_0 \exp(bx - ce^{ax}) dx \quad (a, b, c, c_0 > 0), \quad (1.22)$$

is infinitely divisible with Lévy measure

$$\nu(dx) = |x|^{-1} e^{bx} (1 - e^{ax})^{-1} 1_{(-\infty, 0)}(x) dx. \quad (1.23)$$

Hence it is in L_0 . Akita and Maejima [1] (2001) show that, if $\{X_t\}$ is a Γ -process as in Example 25, then $\mathcal{L}(\log X_t)$ is in $L_2(\mathbb{R})$ for $t \geq 1$. Since $\mathcal{L}(\log X_t)$ has density equal to $\operatorname{const} \exp(tx - qe^x)$ for $t > 0$, this is a special case of (1.22). Therefore the μ in (1.22) is in $L_2(\mathbb{R})$ if $b/a \geq 1$.

Example 27 Lévy's distribution μ of stochastic area and its Lévy measure ν ([S] Example 15.15) are

$$\mu(dx) = (\pi \cosh x)^{-1} dx, \quad \nu(dx) = (2x \sinh x)^{-1} dx. \quad (1.24)$$

This distribution μ is selfdecomposable with k -function $k(x) = (2|\sinh x|)^{-1}$ in (1.11). Let us show that $\mu \in L_1(\mathbb{R})$.

Since μ is symmetric, it is enough to consider the h -function $h(u) = 2^{-1}(\sinh(e^{-u}))^{-1}$. Differentiating twice, we have

$$\begin{aligned} h'(u) &= 2^{-1}e^{-u} \cosh(e^{-u}) (\sinh(e^{-u}))^{-2} > 0, \\ h''(u) &= 2^{-1}e^{-u} (\sinh(e^{-u}))^{-3} [e^{-u} + 2^{-1} \sinh(2e^{-u}) \{e^{-u} \coth(e^{-u}) - 1\}]. \end{aligned}$$

Let $f(x) = x \coth x - 1$. If we can show that $f(x) > 0$ for $x > 0$, then $h''(u) > 0$ and $\mu \in L_1$ by Theorem 20 and Lemma 18. We have $f(x) = e^{-x} (e^x - e^{-x})^{-1} g(x)$ with $g(x) = x(e^{2x} + 1) - e^{2x} + 1$. Checking g' and g'' , we see that $g(x)$ is convex, increasing and positive for $x > 0$. Thus $f(x)$ is positive for $x > 0$.

It does not seem to be known whether μ is in $L_m(\mathbb{R})$ for $m \geq 2$.

Notes

This chapter is based on Sato [57] (1980). The classes L_m , $m = 1, 2, \dots$, and L_∞ were introduced by Urbanik [85] (1972b), [86] (1973). Then Kumar and Schreiber [31] (1978), [32] (1979) and Thu [82] (1979) followed. The classes were reformulated by Sato [57] in the form of Theorem 13.

Many properties of selfdecomposable distributions are known. Unimodality on \mathbb{R} (Yamazato [98] (1978)), singularity of densities and degree of smoothness on \mathbb{R} (Sato and Yamazato [72] (1978)), and absolute continuity on \mathbb{R}^d (Sato [58] (1982)) are among them. See [S] for more accounts.

Historically, selfdecomposable distributions were, without the name, introduced by Lévy [35] (1937) with characterizations similar to Theorem 13 (i) and Theorem 14. Stable distributions were discussed by Lévy [34] (1925) under the name "lois exceptionnelles".

Chapter 2

Classes L_m and Ornstein–Uhlenbeck type processes

It is well known that the Ornstein–Uhlenbeck process on \mathbb{R}^d induced by Brownian motion has a limit distribution as $t \rightarrow \infty$, which is Gaussian, while nonzero Lévy processes on \mathbb{R}^d do not have limit distributions. Processes of Ornstein–Uhlenbeck type are analogues of Ornstein–Uhlenbeck process where the Brownian motion part is replaced by a general Lévy process. In this chapter we shall give conditions under which processes of Ornstein–Uhlenbeck type on \mathbb{R}^d have limit distributions.

In Section 2.1, stochastic integrals of deterministic integrands by a Lévy process $\{Z_t\}$ are defined in order to construct the Ornstein–Uhlenbeck type process $\{X_t\}$. They are defined on a bounded time interval as limits in probability of stochastic integrals of step functions. The process $\{X_t\}$ is expressed as a stochastic integral by the Lévy process $\{Z_t\}$.

In Section 2.2 it is proved that the Ornstein–Uhlenbeck type process $\{X_t\}$ is a temporally homogeneous Markov process and that it has a limit distribution under an integrability condition on the Lévy measure ν_0 of $\{Z_t\}$. Specifically, if ν_0 satisfies

$$\int_{|x|>2} \log |x| \nu_0(dx) < \infty, \quad (2.1)$$

then, as $t \rightarrow \infty$, $\mathcal{L}(X_t)$ converges to a probability measure μ in the class L_0 . Conversely, every selfdecomposable distribution μ appears as limit distribution of some Ornstein–Uhlenbeck type process in this way. If condition (2.1) does not hold, $\mathcal{L}(X_t)$ does not tend to any distribution as t tends to ∞ .

Furthermore, in Section 2.3 it is shown that there is a one-to-one and onto correspondence between the distributions of the class L_{m-1} whose Lévy measures satisfy (2.1) and the distributions of the class L_m which appear as limit distributions of Ornstein-Uhlenbeck type processes. This correspondence preserves α -stability.

Section 2.4 considers stationary OU type processes and reformulates the main results in Section 2.2.

2.1 Stochastic integrals based on Lévy processes

In this section we define the stochastic integral of a bounded measurable function defined on a bounded closed interval on \mathbb{R} with respect to a given Lévy process and we obtain its characteristic function in terms of the characteristic function of this process.

Let $\{Z_t : t \geq 0\}$ be a Lévy process on \mathbb{R}^d with $\mathcal{L}(Z_1) = \mu_0$ and

$$Ee^{i\langle z, Z_t \rangle} = \widehat{\mu}_0(z)^t = e^{t\psi_0(z)}, \quad (2.2)$$

where $\psi_0(z)$ is the distinguished logarithm of $\widehat{\mu}_0(z)$. By a Lévy process we mean a stochastically continuous process with stationary independent increments, starting at 0 a. s., with sample functions a. s. being right continuous with left limits.

Definition 28 *Let $0 \leq t_0 < t_1 < \infty$. A function $f(s)$ on $[t_0, t_1]$ is called a step function if there are a finite number of points $t_0 = s_0 < s_1 < \dots < s_n = t_1$ such that*

$$f(s) = \sum_{j=1}^n a_j 1_{[s_{j-1}, s_j)}(s) \quad (2.3)$$

with some $a_1, \dots, a_n \in \mathbb{R}$. When $f(s)$ is a step function of this form, define

$$\int_{t_0}^{t_1} f(s) dZ_s = \sum_{j=1}^n a_j (Z_{s_j} - Z_{s_{j-1}}). \quad (2.4)$$

Note that the right-hand side of (2.4) is determined by $f(s)$, independently of the choice of the expression (2.3). Furthermore, note that, for any step function in (2.3), the distribution of (2.4) is infinitely divisible with characteristic function of the form

$$\begin{aligned}
E e^{i \langle z, \int_{t_0}^{t_1} f(s) dZ_s \rangle} &= \prod_{j=1}^n E \exp \left(i \langle a_j z, Z_{s_j} - Z_{s_{j-1}} \rangle \right) \\
&= \prod_{j=1}^n e^{(s_j - s_{j-1}) \psi_0(a_j z)} \\
&= \exp \left[\sum_{j=1}^n (s_j - s_{j-1}) \psi_0(a_j z) \right] \\
&= \exp \int_{t_0}^{t_1} \psi_0(f(s) z) ds.
\end{aligned} \tag{2.5}$$

Let (A_0, ν_0, γ_0) be the generating triplet of μ_0 . Then

$$\psi_0(z) = -\frac{1}{2} \langle z, A_0 z \rangle + i \langle \gamma_0, z \rangle + \int_{\mathbb{R}^d} (e^{i \langle z, x \rangle} - 1 - i \langle z, x \rangle 1_{\{|x| \leq 1\}}(x)) \nu_0(dx). \tag{2.6}$$

Proposition 29 *Let $f(s)$ be a real-valued bounded measurable function on $[t_0, t_1]$ such that there are uniformly bounded step functions $f_n(s)$, $n = 1, 2, \dots$, on $[t_0, t_1]$ satisfying $f_n \rightarrow f$ almost everywhere. Then $\int_{t_0}^{t_1} f_n(s) dZ_s$ converges to an \mathbb{R}^d -valued random variable X in probability. The limit X does not depend on the choice of f_n up to probability zero. The law of X is infinitely divisible and represented as*

$$E e^{i \langle z, X \rangle} = \exp \int_{t_0}^{t_1} \psi_0(f(s) z) ds. \tag{2.7}$$

Proof. Due to the continuity of the function ψ_0 in (2.6), $\psi_0((f_n(s) - f_m(s)) z) \rightarrow 0$ for almost every s , as $n, m \rightarrow \infty$. Then

$$\int_{t_0}^{t_1} \psi_0((f_n(s) - f_m(s)) z) ds \rightarrow 0$$

as $n, m \rightarrow \infty$. By (2.5)

$$\int_{t_0}^{t_1} f_n(s) dZ_s - \int_{t_0}^{t_1} f_m(s) dZ_s \rightarrow 0$$

in probability, therefore converges to 0 in the metric of convergence in probability. Thus there exists a random variable X which is limit in probability of the random variables $\int_{t_0}^{t_1} f_n(s) dZ_s$.

The law of X is infinitely divisible, since those of $\int_{t_0}^{t_1} f_n(s) dZ_s$ are. Moreover,

$$\int_{t_0}^{t_1} \psi_0(f_n(s) z) ds = \sum_{j=1}^n \psi_0(a_j z) (s_j - s_{j-1}) \rightarrow \int_{t_0}^{t_1} \psi_0(f(s) z) ds$$

by Lebesgue's bounded convergence theorem. Then, by (2.5)

$$E e^{i \langle z, \int_{t_0}^{t_1} f_n(s) dZ_s \rangle} \rightarrow \exp \int_{t_0}^{t_1} \psi_0(f(s)z) ds. \quad (2.8)$$

From this follows (2.7), applying the continuity theorem.

To see that the limit X does not depend on approximating sequences, let $f_n(s) \rightarrow f(s)$ and $g_n(s) \rightarrow f(s)$ a.e. both boundedly. Then

$$E e^{i \langle z, \int_{t_0}^{t_1} (f_n - g_n) dZ_s \rangle} = e^{\int_{t_0}^{t_1} \psi_0((f_n(s) - g_n(s))z) ds} \rightarrow 1$$

as $n \rightarrow \infty$, showing that $\int_{t_0}^{t_1} f_n dZ_s - \int_{t_0}^{t_1} g_n dZ_s \rightarrow 0$ in probability. ■

Definition 30 The \mathbb{R}^d -valued random variable X in the proposition above is the stochastic integral of f by $\{Z_t\}$, denoted by

$$X = \int_{t_0}^{t_1} f(s) dZ_s. \quad (2.9)$$

Proposition 31 If $f(s)$ is a real-valued bounded measurable function on $[t_0, t_1]$, then $X = \int_{t_0}^{t_1} f(s) dZ_s$ is definable and (2.7) holds.

Proof. By Proposition 29, it is enough to show the existence of uniformly bounded step functions $f_n(s)$ such that $f_n(s) \rightarrow f(s)$ a. e. Let $|f(s)| \leq C$. By Lusin's theorem (Halmos [19] p. 243), for each n , there is a closed set $F_n \subset [t_0, t_1]$ such that $[t_0, t_1] \setminus F_n$ has Lebesgue measure $< 2^{-n}$ and the restriction of f to F_n is continuous. Then, by Urysohn's theorem in general topology, there is a continuous function g_n on $[t_0, t_1]$ with $|g_n(s)| \leq C$ such that $g_n = f$ on F_n . We can choose step functions f_n on $[t_0, t_1]$ such that $|f_n(s) - g_n(s)| < 2^{-n}$ and $|f_n(s)| \leq C$. Let $G = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} ([t_0, t_1] \setminus F_n)$. Then G has Lebesgue measure 0. If $s \notin G$, then $s \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} F_n$ and $f_n(s) \rightarrow f(s)$, since

$$|f_n(s) - f(s)| = |f_n(s) - g_n(s)| < 2^{-n}$$

for large n . ■

Definition 32 A stochastic process $\{X_t: t \geq 0\}$ on \mathbb{R}^d is called an additive process if it has independent increments, is stochastically continuous, and starts at 0 a. s. and if, almost surely, $X_t(\omega)$ is right continuous with left limits in t . If the last condition is not assumed, we say that $\{X_t: t \geq 0\}$ is an additive process in law.

Proposition 33 *Let $f(s)$ be a locally bounded, measurable function on $[0, \infty)$. Then there is an additive process $\{X_t: t \geq 0\}$ on \mathbb{R}^d such that, for every $t > 0$,*

$$P \left[X_t = \int_0^t f(s) dZ_s \right] = 1. \quad (2.10)$$

Proof. Let $Y_0 = 0$ and $Y_t = \int_0^t f(s) dZ_s$ for $t > 0$, whose existence is guaranteed by Proposition 31. We claim that $\{Y_t: t \geq 0\}$ is an additive process in law on \mathbb{R}^d . Indeed, if $0 \leq t_0 < t_1 < t_2$, then

$$\int_{t_0}^{t_1} f(s) dZ_s + \int_{t_1}^{t_2} f(s) dZ_s = \int_{t_0}^{t_2} f(s) dZ_s \quad \text{a. s.},$$

as is proved from the case of step functions. This and the independent increment property of $\{Z_t\}$ prove that $\{Y_t\}$ has independent increments. If $t_n \downarrow t$, then

$$E e^{i\langle z, Y_{t_n} - Y_t \rangle} = E e^{i\langle z, \int_t^{t_n} f(s) dZ_s \rangle} = e^{\int_t^{t_n} \psi_0(f(s)z) ds} \rightarrow 1$$

and, similarly, if $t > 0$ and $t_n \uparrow t$, then $E e^{i\langle z, Y_t - Y_{t_n} \rangle} \rightarrow 1$. Hence $\{Y_t\}$ is stochastically continuous. This shows that $\{Y_t\}$ is an additive process in law. Now, $\{Y_t\}$ has a modification $\{X_t\}$ which is an additive process, by Theorem 11.5 of [S]. ■

Remark 34 *Henceforth, $\int_0^t f(s) dZ_s$ is understood to be the modification X_t in Proposition 33. Likewise, $\int_{t_0}^t f(s) dZ_s$ is understood to be $X_t - X_{t_0}$.*

We need a Fubini type theorem involving the stochastic integrals to prove the existence of the so-called Ornstein-Uhlenbeck type process. We establish this fact in the following proposition.

Proposition 35 *Let $f(s)$ and $g(s)$ be bounded measurable functions on $[t_0, t_1]$. Then*

$$\int_{t_0}^{t_1} g(s) ds \int_{t_0}^s f(u) dZ_u = \int_{t_0}^{t_1} f(u) dZ_u \int_u^{t_1} g(s) ds \quad \text{a. s.} \quad (2.11)$$

Proof. Let

$$X = \int_{t_0}^{t_1} g(s) ds \int_{t_0}^s f(u) dZ_u, \quad Y = \int_{t_0}^{t_1} f(u) dZ_u \int_u^{t_1} g(s) ds. \quad (2.12)$$

Existence of these integrals follows from Propositions 31 and 33.

Step 1. We show that

$$Ee^{i\langle z, X \rangle} = Ee^{i\langle z, Y \rangle} = e^{\int_{t_0}^{t_1} \psi_0(f(u) \int_u^{t_1} g(s) ds) du} \quad (2.13)$$

for any bounded measurable functions f and g on $[t_0, t_1]$. Since Y is the stochastic integral of $f(u) \int_u^{t_1} g(s) ds$, the second equality in (2.13) is a consequence of (2.7). Let us calculate $Ee^{i\langle z, X \rangle}$. Let $t_{n,k} = t_0 + k2^{-n}(t_1 - t_0)$ for $n = 1, 2, \dots$ and $k = 0, 1, \dots, 2^n$. For $s \in [t_0, t_1)$, define $\lambda_n(s) = t_{n,k}$ if $t_{n,k-1} \leq s < t_{n,k}$. Let

$$X_n = \int_{t_0}^{t_1} g(s) ds \int_{t_0}^{\lambda_n(s)} f(u) dZ_u.$$

Since $\int_{t_0}^s f(u) dZ_u$ is right continuous and locally bounded in s a.s., X_n tends to X a.s. as $n \rightarrow \infty$. Hence $Ee^{i\langle z, X_n \rangle} \rightarrow Ee^{i\langle z, X \rangle}$. We have

$$X_n = \sum_{k=1}^{2^n} c_k \int_{t_0}^{t_{n,k}} f(u) dZ_u = \int_{t_0}^{t_1} \sum_{k=1}^{2^n} c_k 1_{[t_0, t_{n,k})}(u) f(u) dZ_u \quad \text{a. s.}$$

with $c_k = \int_{t_{n,k-1}}^{t_{n,k}} g(s) ds$. Thus, by (2.7),

$$\begin{aligned} Ee^{i\langle z, X_n \rangle} &= \exp \int_{t_0}^{t_1} \psi_0 \left(\sum_{k=1}^{2^n} c_k 1_{[t_0, t_{n,k})}(u) f(u) z \right) du \\ &= \exp \int_{t_0}^{t_1} \psi_0 \left(\sum_{k=1}^{2^n} 1_{[t_0, t_{n,k})}(u) \int_{t_{n,k-1}}^{t_{n,k}} g(s) ds f(u) z \right) du \\ &= \exp \int_{t_0}^{t_1} \psi_0 \left(\int_{\lambda_n(u) - 2^{-n}(t_1 - t_0)}^{t_1} g(s) ds f(u) z \right) du, \end{aligned}$$

which tends to the rightmost member of (2.13) as $n \rightarrow \infty$.

Step 2. Let us show that $X = Y$ a.s., assuming that f and g are step functions. Without loss of generality, we can assume that

$$f(s) = \sum_{j=1}^N a_j 1_{[s_{j-1}, s_j)}(s), \quad g(s) = \sum_{j=1}^N b_j 1_{[s_{j-1}, s_j)}(s)$$

with $t_0 = s_0 < s_1 < \dots < s_N = t_1$. First we prepare the identity

$$\int_{t_0}^{t_1} s dZ_s = t_1 Z_{t_1} - t_0 Z_{t_0} - \int_{t_0}^{t_1} Z_s ds \quad \text{a. s.} \quad (2.14)$$

Define $t_{n,k}$ and $\lambda_n(s)$ as in step 1. Since $\lambda_n(s)$, $n = 1, 2, \dots$, are step functions and $\lambda_n(s) \rightarrow s$, we have

$$\int_{t_0}^{t_1} \lambda_n(s) dZ_s \rightarrow \int_{t_0}^{t_1} s dZ_s \quad \text{in probability.}$$

Notice that

$$\begin{aligned} \int_{t_0}^{t_1} \lambda_n(s) dZ_s &= \sum_{k=1}^{2^n} t_{n,k} (Z_{t_{n,k}} - Z_{t_{n,k-1}}) \\ &= \sum_{k=1}^{2^n} t_{n,k} Z_{t_{n,k}} - \sum_{k=0}^{2^n-1} (t_{n,k} + 2^{-n} (t_1 - t_0)) Z_{t_{n,k}} \\ &= t_1 Z_{t_1} - t_0 Z_{t_0} - \sum_{k=0}^{2^n-1} 2^{-n} (t_1 - t_0) Z_{t_{n,k}} \\ &= t_1 Z_{t_1} - t_0 Z_{t_0} - \int_{t_0}^{t_1} Z_{\lambda_n(s)} ds - 2^{-n} (t_1 - t_0) Z_{t_0} + 2^{-n} (t_1 - t_0) Z_{t_1} \\ &\rightarrow t_1 Z_{t_1} - t_0 Z_{t_0} - \int_{t_0}^{t_1} Z_s ds \quad \text{a. s.} \end{aligned}$$

as $n \rightarrow \infty$, since $Z_{\lambda_n(s)} \rightarrow Z_s$ boundedly on $[t_0, t_1)$ a. s. This proves (2.14). Now

$$\begin{aligned} X &= \sum_{k=1}^N b_k \int_{s_{k-1}}^{s_k} ds \int_{t_0}^s f(u) dZ_u \\ &= \sum_{k=1}^N b_k \int_{s_{k-1}}^{s_k} \sum_{j=1}^N a_j (Z_{s \wedge s_j} - Z_{s \wedge s_{j-1}}) ds = I_1, \text{ say.} \end{aligned}$$

Since

$$\int_{s_{k-1}}^{s_k} (Z_{s \wedge s_j} - Z_{s \wedge s_{j-1}}) ds = \begin{cases} 0 & \text{for } k \leq j-1 \\ \int_{s_{k-1}}^{s_k} (Z_s - Z_{s_{k-1}}) ds & \text{for } k = j \\ (Z_{s_j} - Z_{s_{j-1}}) (s_k - s_{k-1}) & \text{for } k \geq j+1, \end{cases}$$

we have

$$I_1 = \sum_{j=1}^N a_j \left(b_j \int_{s_{j-1}}^{s_j} (Z_s - Z_{s_{j-1}}) ds + \sum_{k=j+1}^N b_k (Z_{s_j} - Z_{s_{j-1}}) (s_k - s_{k-1}) \right) = I_2, \text{ say.}$$

Use of (2.14) gives

$$I_2 = \sum_{j=1}^N a_j \left(-b_j \int_{s_{j-1}}^{s_j} s dZ_s + (Z_{s_j} - Z_{s_{j-1}}) \left(b_j s_j + \sum_{k=j+1}^N b_k (s_k - s_{k-1}) \right) \right) \quad \text{a. s.}$$

On the other hand,

$$\begin{aligned} Y &= \sum_{j=1}^N a_j \int_{s_{j-1}}^{s_j} dZ_u \int_u^{t_1} g(s) ds \\ &= \sum_{j=1}^N a_j \int_{s_{j-1}}^{s_j} \left(b_j (s_j - u) + \sum_{k=j+1}^N b_k (s_k - s_{k-1}) \right) dZ_u \\ &= \sum_{j=1}^N a_j \left(b_j \int_{s_{j-1}}^{s_j} (s_j - u) dZ_u + (Z_{s_j} - Z_{s_{j-1}}) \sum_{k=j+1}^N b_k (s_k - s_{k-1}) \right). \end{aligned}$$

Therefore $X = Y$ a. s. when f and g are step functions.

Step 3. We show $X = Y$ a. s. when f is bounded measurable and g is a step function. By Proposition 31 there are uniformly bounded step functions f_n such that $f_n \rightarrow f$ a. e. on $[t_0, t_1]$. Let

$$X_n = \int_{t_0}^{t_1} g(s) ds \int_{t_0}^s f_n(u) dZ_u, \quad Y_n = \int_{t_0}^{t_1} f_n(u) dZ_u \int_u^{t_1} g(s) ds.$$

We have $X_n = Y_n$ a. s. by step 2. Since

$$X - X_n = \int_{t_0}^{t_1} g(s) ds \int_{t_0}^s (f(u) - f_n(u)) dZ_u$$

and

$$Y - Y_n = \int_{t_0}^{t_1} (f(u) - f_n(u)) dZ_u \int_u^{t_1} g(s) ds,$$

step 1 gives

$$E e^{i\langle z, X - X_n \rangle} = E e^{i\langle z, Y - Y_n \rangle} = e^{\int_{t_0}^{t_1} \psi_0((f(u) - f_n(u)) \int_u^{t_1} g(s) ds) du},$$

which tends to 1 as $n \rightarrow \infty$. It follows that $X_n \rightarrow X$ and $Y_n \rightarrow Y$ in probability. Therefore $X = Y$ a. s.

Step 4. Now let us show $X = Y$ a.s. when f and g are bounded measurable functions. Choose uniformly bounded step functions g_n such that $g_n \rightarrow g$ a. e. on $[t_0, t_1]$. Let

$$\tilde{X}_n = \int_{t_0}^{t_1} g_n(s) ds \int_{t_0}^s f(u) dZ_u, \quad \tilde{Y}_n = \int_{t_0}^{t_1} f(u) dZ_u \int_u^{t_1} g_n(s) ds.$$

We have now $\tilde{X}_n = \tilde{Y}_n$ by step 3 and, by the same method, we can show that $\tilde{X}_n \rightarrow X$ and $\tilde{Y}_n \rightarrow Y$ in probability. Hence (2.11) is proved. ■

Example 36 Let $f(s)$ be of class C^1 on $[t_0, t_1]$. As an example of the use of Proposition 35, let us show the integration-by-parts formula

$$\int_{t_0}^{t_1} f(u) dZ_u = f(t_1)Z_{t_1} - f(t_0)Z_{t_0} - \int_{t_0}^{t_1} Z_s f'(s) ds \quad a. s. \quad (2.15)$$

Indeed, notice that

$$\int_{t_0}^{t_1} f(u) dZ_u = - \int_{t_0}^{t_1} dZ_u \int_u^{t_1} f'(s) ds + f(t_1) \int_{t_0}^{t_1} dZ_u = I, \quad \text{say.}$$

By Proposition 35,

$$\begin{aligned} I &= - \int_{t_0}^{t_1} f'(s) ds \int_{t_0}^s dZ_u + f(t_1) \int_{t_0}^{t_1} dZ_u \\ &= - \int_{t_0}^{t_1} f'(s) (Z_s - Z_{t_0}) ds + f(t_1) (Z_{t_1} - Z_{t_0}) \\ &= - \int_{t_0}^{t_1} f'(s) Z_s ds + (f(t_1) - f(t_0)) Z_{t_0} + f(t_1) (Z_{t_1} - Z_{t_0}), \end{aligned}$$

which is the right-hand side of (2.15).

2.2 Ornstein–Uhlenbeck type processes and limit distributions

Let $\{Z_t: t \geq 0\}$ be a Lévy process on \mathbb{R}^d with (2.2) and with generating triplet (A_0, ν_0, γ_0) satisfying (2.6). Let M be a random variable on \mathbb{R}^d such that M and $\{Z_t: t \geq 0\}$ are independent. Given $c \in \mathbb{R}$, consider the equation

$$X_t = M + Z_t - c \int_0^t X_s ds, \quad t \geq 0. \quad (2.16)$$

A stochastic process $\{X_t: t \geq 0\}$ is said to be a solution of (2.16) if X_t is right continuous with left limits in t and satisfies (2.16) a. s.

Proposition 37 *The equation (2.16) has an almost surely unique solution $\{X_t: t \geq 0\}$ and, almost surely,*

$$X_t = e^{-ct}M + e^{-ct} \int_0^t e^{cs} dZ_s, \quad t \geq 0. \quad (2.17)$$

Proof. Define X_t by (2.17). Then

$$\begin{aligned} c \int_0^t X_s ds &= Mc \int_0^t e^{-cs} ds + c \int_0^t e^{-cs} ds \int_0^s e^{cu} dZ_u \\ &= M(1 - e^{-ct}) + c \int_0^t e^{cu} dZ_u \int_u^t e^{-cs} ds \\ &= M(1 - e^{-ct}) + \int_0^t \left(c \int_u^t e^{-c(s-u)} ds \right) dZ_u \\ &= M(1 - e^{-ct}) + \int_0^t (1 - e^{-c(t-u)}) dZ_u \\ &= M - Me^{-ct} + Z_t - e^{-ct} \int_0^t e^{cu} dZ_u \\ &= M - X_t + Z_t, \end{aligned}$$

where we have applied the Proposition 35. Therefore (2.16) holds.

We will prove the uniqueness of the solution of (2.16). Suppose that $X_t^1(\omega)$ and $X_t^2(\omega)$ satisfy (2.16). For a fixed ω define a bounded function $f(t)$ on $[t_0, t_1]$ by $f(t) = X_t^1(\omega) - X_t^2(\omega)$. Recursive application gives

$$\begin{aligned} f(t) &= -c \int_0^t (X_s^1(\omega) - X_s^2(\omega)) ds \\ &= -c \int_0^t f(s) ds, \end{aligned}$$

and

$$\begin{aligned} f(t) &= -c \int_0^t (-c \int_0^s f(u) du) ds \\ &= (-c)^2 \int_0^t \left(\int_0^s f(u) du \right) ds \\ &= (-c)^2 \int_0^t f(u) \left(\int_u^t ds \right) du \\ &= (-c)^2 \int_0^t (t-s) f(s) ds. \end{aligned}$$

By induction we get

$$f(t) = \frac{(-c)^n}{(n-1)!} \int_0^t (t-s)^{n-1} f(s) ds, \quad \text{for } n = 1, 2, 3, \dots$$

Since $\sum_{n=1}^{\infty} \frac{|-c|^n}{(n-1)!} (t-s)^{n-1}$ is finite for every $0 < s < t$, the term $\frac{(-c)^n}{(n-1)!} (t-s)^{n-1}$ tends to 0 as $n \rightarrow \infty$, uniformly for $0 < s < t$. Hence $f(t) = 0$. This concludes the proof. ■

Proposition 38 *The process $\{X_t\}$ of Proposition 37 is a temporally homogeneous Markov process starting from $X_0 = M$ with transition probability $P_t(x, dy)$ infinitely divisible and satisfying*

$$\int_{\mathbb{R}^d} e^{i\langle z, y \rangle} P_t(x, dy) = \exp \left[i e^{-ct} \langle x, z \rangle + \int_0^t \psi_0(e^{-cs} z) ds \right]. \quad (2.18)$$

Proof. For every $s \in [0, t]$ we have

$$X_t = e^{-c(t-s)} X_s + e^{-ct} \int_s^t e^{cu} dZ_u \quad (2.19)$$

from (2.17) and from $X_s = e^{-cs} M + e^{-cs} \int_0^s e^{cu} dZ_u$. Because $e^{-ct} \int_s^t e^{cu} dZ_u$ and $\{X_u : u \leq s\}$ are independent, the identity (2.19) shows that $\{X_t : t \geq 0\}$ is a Markov process with transition probability

$$P[X_t \in B \mid X_s = x] = P \left[e^{-c(t-s)} x + e^{-ct} \int_s^t e^{cu} dZ_u \in B \right], \quad x \in \mathbb{R}^d, B \in \mathcal{B}(\mathbb{R}^d)$$

for $0 \leq s \leq t$ (use Proposition 1.16 of [S] for a proof). Denote the right-hand side of the last equality by $\rho(B)$. Then ρ is infinitely divisible and, by (2.7),

$$\begin{aligned} \widehat{\rho}(z) &= E e^{i\langle z, e^{-c(t-s)} x + e^{-ct} \int_s^t e^{cu} dZ_u \rangle} \\ &= e^{i\langle z, e^{-c(t-s)} x \rangle + \int_s^t \psi_0(e^{cu} e^{-ct} z) du} \\ &= e^{i\langle z, e^{-c(t-s)} x \rangle + \int_0^{t-s} \psi_0(e^{-cv} z) dv}. \end{aligned}$$

Hence ρ depends only on $t-s$ and x . Therefore $\{X_t : t \geq 0\}$ is a temporally homogeneous Markov process with transition probability $P_t(x, dy)$ satisfying (2.18). ■

Definition 39 *If $c > 0$, then the process $\{X_t\}$ of Proposition 37 is called Ornstein–Uhlenbeck type process (or OU type process) generated by $\{Z_t\}$ and c , or generated by μ_0 and c , or generated by $(A_0, \nu_0, \gamma_0, c)$, starting from $X_0 = M$. Sometimes the process $\{Z_t\}$ is called the background driving Lévy process.*

Remark 40 *Proposition 38 shows that the definition of Ornstein–Uhlenbeck type processes coincides with that of Chapter 3 of [S].*

Lévy processes do not have limit distributions as $t \rightarrow \infty$ except in the case of the zero process. But, in the case of OU type processes, drift force toward the origin of the magnitude proportional to the distance from the origin works, so that they are likely to have limit distributions. This is true only if they do not have too many big jumps.

Theorem 41 *Let $c > 0$ be fixed.*

(i) *Let $\{Z_t\}$ be a Lévy process on \mathbb{R}^d with $\mathcal{L}(Z_1) = \mu_0$ and generating triplet (A_0, ν_0, γ_0) . Let $\{X_t\}$ be the OU type process generated by $(A_0, \nu_0, \gamma_0, c)$, starting from $X_0 = M$. Assume that*

$$\int_{|x|>2} \log |x| \nu_0(dx) < \infty. \quad (2.20)$$

Then

$$\mathcal{L}(X_t) \rightarrow \mu \quad \text{as } t \rightarrow \infty \quad (2.21)$$

for some $\mu \in \mathfrak{P}$, and this μ does not depend on M . Moreover,

$$\int_0^\infty |\psi_0(e^{-cs}z)| ds < \infty, \quad (2.22)$$

$$\widehat{\mu}(z) = \exp \int_0^\infty \psi_0(e^{-cs}z) ds, \quad (2.23)$$

and μ belongs to $L_0(\mathbb{R}^d)$. The generating triplet (A, ν, γ) of μ is as follows:

$$A = \frac{1}{2c} A_0 \quad (2.24)$$

$$\nu(B) = \frac{1}{c} \int_{\mathbb{R}^d} \nu_0(dx) \int_0^\infty 1_B(e^{-s}x) ds, \quad B \in \mathcal{B}(\mathbb{R}^d), \quad (2.25)$$

$$\gamma = \frac{1}{c} \gamma_0 + \frac{1}{c} \int_{|x|>1} \frac{x}{|x|} \nu_0(dx). \quad (2.26)$$

(ii) For any $\mu \in L_0(\mathbb{R}^d)$, there exists a unique triplet (A_0, ν_0, γ_0) satisfying (2.20) such that μ satisfies (2.21) for the OU type process generated by $(A_0, \nu_0, \gamma_0, c)$ starting from an arbitrary M . Using λ and $k_\xi(r)$ in Theorem 14 for the Lévy measure ν of μ , we have

$$\nu_0(B) = -c \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) dk_\xi(r). \quad (2.27)$$

(iii) In the set-up of (i), assume

$$\int_{|x|>2} \log |x| \nu_0(dx) = \infty \quad (2.28)$$

instead of (2.20). Then $\mathcal{L}(X_t)$ does not tend to any distribution as $t \rightarrow \infty$ and, moreover, for any $a > 0$,

$$\sup_{x,y} P_t(x, D_a(y)) \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (2.29)$$

where $D_a(y) = \{z: |z - y| \leq a\}$.

Proof. (i) From (2.18), the characteristic function of X_t is

$$E e^{i\langle z, X_t \rangle} = \left(E e^{ie^{-ct}\langle z, M \rangle} \right) \exp \int_0^t \psi_0(e^{-cs}z) ds. \quad (2.30)$$

Now use $\psi_0(z)$ in terms of its triplet (A_0, ν_0, γ_0) , given in (2.6), to obtain

$$\int_0^t \psi_0(e^{-cs}z) ds = -\frac{1}{2}\langle z, \tilde{A}_t z \rangle + i\langle \tilde{\gamma}_t, z \rangle + \int_{\mathbb{R}^d} g(z, x) \tilde{\nu}_t(dx) \quad (2.31)$$

where $g(z, x) = e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle 1_{\{|x| \leq 1\}}(x)$ and

$$\begin{aligned} \tilde{A}_t &= \int_0^t e^{-2cs} ds A_0, \\ \tilde{\nu}_t(B) &= \int_{\mathbb{R}^d} \nu_0(dx) \int_0^t 1_B(e^{-cs}x) ds \quad \text{for every } B \in \mathcal{B}(\mathbb{R}^d), \\ \tilde{\gamma}_t &= \int_0^t e^{-cs} ds \gamma_0 + \int_{\mathbb{R}^d} \nu_0(dx) \int_0^t e^{-cs} x [1_D(e^{-cs}x) - 1_D(x)] ds \end{aligned} \quad (2.32)$$

with $D = \{x: |x| \leq 1\}$. Observe that, as $t \rightarrow \infty$,

$$\tilde{A}_t \rightarrow \frac{1}{2c} A_0,$$

$$\begin{aligned} \int_{|x| \leq 1} |x|^2 \tilde{\nu}_t(dx) &= \int_{\mathbb{R}^d} \nu_0(dx) \int_0^t |e^{-cs} x|^2 1_D(e^{-cs} x) ds \\ &\rightarrow \int_{\mathbb{R}^d} |x|^2 \nu_0(dx) \int_0^\infty e^{-2cs} 1_{\{|x| \leq e^{cs}\}}(x) ds \\ &= \frac{1}{2c} \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu_0(dx), \end{aligned}$$

$$\begin{aligned} \int_{|x| > 1} \tilde{\nu}_t(dx) &= \int_{\mathbb{R}^d} \nu_0(dx) \int_0^t 1_{D^c}(e^{-cs} x) ds \\ &\rightarrow \int_{\mathbb{R}^d} \nu_0(dx) \int_0^\infty 1_{\{|x| > e^{cs}\}}(x) ds \\ &= \frac{1}{c} \int_{|x| > 1} \log |x| \nu_0(dx), \end{aligned}$$

$$\begin{aligned} \tilde{\gamma}_t &\rightarrow \frac{1}{c} \gamma_0 + \int_{\mathbb{R}^d} \nu_0(dx) \int_0^\infty e^{-cs} x 1_{\{1 < |x| \leq e^{cs}\}}(x) ds \\ &= \frac{1}{c} \gamma_0 + \frac{1}{c} \int_{|x| > 1} \frac{x}{|x|} \nu_0(dx). \end{aligned}$$

These limits are finite by condition (2.20). We have

$$|g(z, x)| \leq \frac{1}{2} |z|^2 |x|^2 1_{\{|x| \leq 1\}}(x) + 2 \cdot 1_{\{|x| > 1\}}(x).$$

It follows from (2.6) that

$$\begin{aligned} \psi_0(e^{-cs} z) &= -\frac{1}{2} e^{-2cs} \langle z, A_0 z \rangle + i e^{-cs} \langle \gamma_0, z \rangle \\ &\quad + \int_{\mathbb{R}^d} g(z, e^{-cs} x) \nu_0(dx) + i \int_{\mathbb{R}^d} \langle z, e^{-cs} x \rangle 1_{\{1 < |x| \leq e^{cs}\}}(x) \nu_0(dx). \end{aligned}$$

Hence

$$\begin{aligned} |\psi_0(e^{-cs} z)| &\leq \frac{1}{2} e^{-2cs} \langle z, A_0 z \rangle + e^{-cs} |\gamma_0| |z| + \frac{1}{2} |z|^2 \int |e^{-cs} x|^2 1_D(e^{-cs} x) \nu_0(dx) \\ &\quad + 2 \int 1_{D^c}(e^{-cs} x) \nu_0(dx) + |z| \int |e^{-cs} x| 1_{\{1 < |x| \leq e^{cs}\}}(x) \nu_0(dx). \end{aligned}$$

Therefore, the convergences above show (2.22). In fact, it is shown that, as $t \rightarrow \infty$, \tilde{A}_t and $\tilde{\gamma}_t$ tends to some \tilde{A}_∞ and $\tilde{\gamma}_\infty$, respectively, and $\tilde{\nu}_t$ increases to some measure $\tilde{\nu}_\infty$ satisfying

$\int (|x|^2 \wedge 1) \tilde{\nu}_\infty(dx) < \infty$. Hence, using the density $d\tilde{\nu}_t/d\tilde{\nu}_\infty$ and the dominated convergence theorem, we see that the right-hand side of (2.31) tends to

$$-\frac{1}{2}\langle z, \tilde{A}_\infty z \rangle + i\langle \tilde{\gamma}_\infty, z \rangle + \int_{\mathbb{R}^d} g(z, x) \tilde{\nu}_\infty(dx).$$

Thus the distribution with characteristic function $\exp \int_0^t \psi_0(e^{-cs}z) ds$ tends to a $\mu \in ID$ with triplet $(\tilde{A}_\infty, \tilde{\nu}_\infty, \tilde{\gamma}_\infty)$. This μ satisfies (2.23) by (2.31). Now convergence (2.21) of $\mathcal{L}(X_t)$ to μ follows from (2.23) and (2.30). Observe that μ does not depend on M . The triplet $(\tilde{A}_\infty, \tilde{\nu}_\infty, \tilde{\gamma}_\infty)$ of μ is identical with the triplet (A, ν, γ) described by (2.24), (2.25) and (2.26). Now from (2.23)

$$\begin{aligned} \widehat{\mu}(b^{-1}z) &= \exp \int_0^\infty \psi_0(e^{-cs}b^{-1}z) ds \\ &= \exp \int_{(\log b)/c}^\infty \psi_0(e^{-cs}z) ds. \end{aligned}$$

We can write

$$\frac{\widehat{\mu}(z)}{\widehat{\mu}(b^{-1}z)} = \exp \int_0^t \psi_0(e^{-cs}z) ds$$

for $t = \frac{1}{c} \log b$, which is the characteristic function of $P_t(0, \cdot)$. Here, $P_t(x, dy)$ denotes the transition probability of the OU type process X_t given by Proposition 38. Thus μ is selfdecomposable.

(ii) By Theorem 14, the Lévy measure of the selfdecomposable distribution μ on \mathbb{R}^d has the form

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{k_\xi(r)}{r} dr, \quad B \in \mathcal{B}(\mathbb{R}^d), \quad (2.33)$$

where λ is a probability measure on S and $k_\xi(r)$ is nonnegative, decreasing in $r \in (0, \infty)$, measurable in $\xi \in S$ and

$$\int_S \lambda(d\xi) \int_0^\infty (r^2 \wedge 1) \frac{k_\xi(r)}{r} dr < \infty.$$

Define a measure $\nu_0(B)$ by the right hand of (2.27). To prove that ν_0 is a Lévy measure satisfying (2.20), we will show that

$$\int_{|x| \leq 2} |x|^2 \nu_0(dx) + \int_{|x| > 2} \log |x| \nu_0(dx) < \infty. \quad (2.34)$$

Let

$$l(u) = \int_0^u (r^2 \wedge 1) \frac{dr}{r} = \int_0^\infty (e^{-2t} u^2 \wedge 1) dt = \begin{cases} \frac{1}{2} u^2 & 0 \leq u \leq 1, \\ \frac{1}{2} + \log u & u > 1. \end{cases}$$

Below we use the following fact (see [S] Lemma 17.6). For every $l(r)$ and $k_\xi(r)$ nonnegative and right continuous functions on $(0, \infty)$ such that $k_\xi(r)$ is decreasing and $k_\xi(\infty) = 0$ and $l(r)$ is increasing and $l(0+) = 0$, we have

$$\int_{0+}^\infty l(r) dk_\xi(r) = - \int_{0+}^\infty k_\xi(r) dl(r). \quad (2.35)$$

Then, by definition of l and ν_0 ,

$$\begin{aligned} \int_{\mathbb{R}^d} l(|x|) \nu_0(dx) &= -c \int_S \lambda(d\xi) \int_0^\infty l(r) dk_\xi(r) \\ &= c \int_S \lambda(d\xi) \int_0^\infty k_\xi(r) dl(r) \\ &= c \int_S \lambda(d\xi) \int_0^\infty (r^2 \wedge 1) \frac{k_\xi(r)}{r} dr < \infty. \end{aligned}$$

But,

$$\begin{aligned} \int_{\mathbb{R}^d} l(|x|) \nu_0(dx) &= \int_{|x| \leq 1} |x|^2 \nu_0(dx) + \int_{|x| > 1} \left(\frac{1}{2} + \log |x|\right) \nu_0(dx) \\ &\geq \int_{|x| \leq 2} \frac{1}{8} |x|^2 \nu_0(dx) + \int_{|x| > 2} \log |x| \nu_0(dx). \end{aligned}$$

Therefore (2.34) follows.

Now, if $B \in \mathcal{B}(\mathbb{R}^d)$ satisfies $B \subset \{x : |x| > \varepsilon\}$ for some $\varepsilon > 0$ then, applying again (2.35) now to (2.33)

$$\begin{aligned} \nu(B) &= - \int_S \lambda(d\xi) \int_0^\infty dk_\xi(r) \int_0^r 1_B(u\xi) \frac{du}{u} \\ &= - \int_S \lambda(d\xi) \int_0^\infty dk_\xi(r) \int_0^\infty 1_B(e^{-s} r \xi) ds \\ &= \frac{1}{c} \int_{\mathbb{R}^d} \nu_0(dy) \int_0^\infty 1_B(e^{-s} y) ds. \end{aligned}$$

That is, (2.25) holds.

Next, define A_0 and γ_0 by (2.24) and (2.26) respectively, then, the OU type process generated by $(A_0, \nu_0, \gamma_0, c)$ has μ as limit distribution. This proves (2.21).

Uniqueness. Suppose that two processes of OU type with common c have the limit distribution μ . Let $\{Z_t^a\}$ and $\{Z_t^b\}$ the associated Lévy processes with respective distinguished logarithms $\psi_a(z) = \log Ee^{i\langle z, Z_1^a \rangle}$ and $\psi_b(z) = \log Ee^{i\langle z, Z_1^b \rangle}$. By (2.23)

$$\exp \int_0^\infty \psi_a(e^{-cs}z) ds = \exp \int_0^\infty \psi_b(e^{-cs}z) ds.$$

Now we use the same argument at the end of the proof (i) to get

$$\exp \int_0^t \psi_a(e^{-cs}z) ds = \exp \int_0^t \psi_b(e^{-cs}z) ds$$

for every $t > 0$ and $z \in \mathbb{R}^d$. Differentiating at $t = 0$ we get $\psi_a(z) = \psi_b(z)$.

(iii) We assume (2.28). Suppose that, for some $x_0 \in \mathbb{R}^d$, $P_t(x_0, \cdot)$ tends to a probability measure μ as $t \rightarrow \infty$. Since $P_t(x_0, \cdot)$ is infinitely divisible, μ is infinitely divisible ([S] Lemma 7.8). Let ν be the Lévy measure of μ . Then, by Theorem 8.7 of [S],

$$\int f(x) \tilde{\nu}_t(dx) \rightarrow \int f(x) \nu(dx), \quad t \rightarrow \infty,$$

for any bounded continuous function vanishing on a neighborhood of 0. Here $\tilde{\nu}_t$ is the Lévy measure of $P_t(x, \cdot)$. It follows from (2.32) that

$$\int_{|x|>1} \tilde{\nu}_t(dx) = \int_{|y|>1} \left(t \wedge \left(\frac{1}{c} \log |y| \right) \right) \nu_0(dy),$$

which tends to ∞ by assumption (2.28). This is absurd. Hence, for any $x \in \mathbb{R}^d$, $P_t(x, \cdot)$ does not tend to a probability measure as $t \rightarrow \infty$.

We use Lemma 42 below and condition (2.28) to obtain

$$P_t(x, D_a(y)) \leq K_d \left(\int_{|u|>a/\pi} \tilde{\nu}_t(du) \right)^{-1/2} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

for any a , x , and y , with K_d a constant depending only on d . Thus we get (2.29). For any M , the OU type process $\{X_t\}$ generated by $(A_0, \nu_0, \gamma_0, c)$ starting from M satisfies

$$P[X_t \in D_a(y)] = E[P_t(M, D_a(y))] \rightarrow 0$$

for any a and y . Thus $\mathcal{L}(X_t)$ does not tend to any distribution as $t \rightarrow \infty$. ■

Lemma 42 Let $C(x, a) = [x_1 - a, x_1 + a] \times \cdots \times [x_d - a, x_d + a]$, a cube in \mathbb{R}^d with center $x = (x_j)_{1 \leq j \leq d}$. Let $\mu \in ID(\mathbb{R}^d)$ with Lévy measure ν . Then

$$\mu(C(x, a)) \leq K_d \left(\int_{|y| > a/\pi} \nu(dy) \right)^{-1/2}, \quad (2.36)$$

where K_d is a constant which depends only on d .

Proof. First we show that, for any $\mu \in \mathfrak{P}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $a > 0$, and $b > 0$ with $b \leq \pi/a$,

$$\mu(C(x, a)) \leq \left(\frac{\pi}{2}\right)^{2d} b^{-d} \int_{C(0, b)} |\widehat{\mu}(z)| dz. \quad (2.37)$$

Let $f(u) = \left(\frac{\sin(u/2)}{u/2}\right)^2$ and $h(v) = (1 - |v|)1_{\{|v| \leq 1\}}(v)$. Then

$$f(u) = \int_{-\infty}^{\infty} e^{iuv} h(v) dv, \quad h(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuv} f(u) du.$$

For $x, z \in \mathbb{R}^d$, let $\tilde{f}(x) = \prod_{j=1}^d f(x_j)$ and $\tilde{h}(z) = \prod_{j=1}^d h(z_j)$. Then, for every $x \in \mathbb{R}^d$ and $b > 0$,

$$\int \tilde{f}(b(y-x)) \mu(dy) = \int \mu(dy) \int e^{i\langle b(y-x), z \rangle} \prod_{j=1}^d h(z_j) dz = b^{-d} \int e^{-i\langle x, z \rangle} \widehat{\mu}(z) \tilde{h}(b^{-1}z) dz.$$

Since $f(u) \geq (2/\pi)^2$ for $|u| \leq \pi$, it follows that

$$b^{-d} \int_{C(0, b)} |\widehat{\mu}(z)| dz \geq \int_{C(x, a)} \tilde{f}(b(y-x)) \mu(dy) \geq \left(\frac{\pi}{2}\right)^{2d} \mu(C(x, a))$$

if $ab \leq \pi$, that is, (2.37).

Now let $\mu \in ID$ with Lévy measure ν . We claim that, for any $b > 0$,

$$b^{-d} \int_{C(0, b)} |\widehat{\mu}(z)| dz \leq K'_d \left(\int_{|y| > 1/b} \nu(dy) \right)^{-1/2}, \quad (2.38)$$

where K'_d is a constant which depends only on d . We have

$$|\widehat{\mu}(z)| \leq \exp \left[\operatorname{Re} \int g(z, x) \nu(dx) \right] \leq \exp \left[- \int_{|y| > 1/b} (1 - \cos \langle z, x \rangle) \nu(dy) \right].$$

Let $V = \int_{|y|>1/b} \nu(dy)$. If $V = 0$, then (2.38) is trivial. Suppose that $V > 0$, and let $\tilde{\nu}(dy) = V^{-1}1_{\{|y|>1/b\}}(y)\nu(dy)$. Use Jensen's inequality to get

$$|\hat{\mu}(z)| \leq \int e^{-V(1-\cos\langle z,y \rangle)} \tilde{\nu}(dy).$$

Hence

$$\int_{C(0,b)} |\hat{\mu}(z)| dz \leq \int_{|y|>1/b} F(y) \tilde{\nu}(dy) \text{ with } F(y) = \int_{|z|\leq\sqrt{db}} e^{-V(1-\cos\langle z,y \rangle)} dz.$$

We fix $y \neq 0$ and consider an orthogonal transformation that carries $y/|y|$ to $e_1 = (\delta_{1j})_{1\leq j\leq d}$. Then

$$F(y) = \int_{|z|\leq\sqrt{db}} e^{-V(1-\cos(z_1|y|))} dz.$$

Let $E_k = \{z \in \mathbb{R}^d: |z| \leq \sqrt{db} \text{ and } 2\pi k/|y| < z_1 \leq 2\pi(k+1)/|y|\}$ and $n = \lceil \sqrt{db}|y|/2\pi \rceil$ with brackets denoting integer part. Then

$$\begin{aligned} F(y) &= 2 \sum_{k=0}^n \int_{E_k} e^{-V(1-\cos(z_1|y|))} dz \\ &\leq 2(n+1) \int \cdots \int_E dz_2 \cdots dz_d \int_0^{2\pi/|y|} e^{-V(1-\cos(z_1|y|))} dz_1 \\ &\leq 4K_d'' b^{d-1} (n+1) |y|^{-1} \int_0^\pi e^{-V(1-\cos u)} du, \end{aligned}$$

where $E = \{z' \in \mathbb{R}^{d-1}: |z'| \leq \sqrt{db}\}$ and K_d'' is the volume of the ball with radius \sqrt{d} in \mathbb{R}^{d-1} . Using $1 - \cos u \geq 2\pi^{-2}u^2$ for $0 \leq u \leq \pi$, we have

$$\int_0^\pi e^{-V(1-\cos u)} du \leq \int_0^\infty e^{-2V\pi^{-2}u^2} du = KV^{-1/2}$$

with an absolute constant K . Noting that

$$\sup_{|y|>1/b} (n+1)|y|^{-1} = \sup_{|y|>1/b} \left(\left\lceil \frac{\sqrt{db}|y|}{2\pi} \right\rceil + 1 \right) |y|^{-1} = bK_d'''$$

with a constant K_d''' depending only on d , we obtain (2.38).

Taking $b = \pi/a$ and combining (2.37) and (2.38), we get (2.36). ■

Remark 43 In Theorem 41 (i), $\mathcal{L}(X_t)$ converges as $t \rightarrow \infty$. But, if $\{Z_t\}$ is non-trivial, then X_t does not converge in probability; see Remark 57.

Remark 44 If $\mu_0 \in ID$ with Lévy measure ν_0 , then the condition (2.20) for ν_0 is equivalent to

$$\int_{|x|>2} \log |x| \mu_0(dx) < \infty. \quad (2.39)$$

See [S] Theorem 25.3 and Proposition 25.4.

2.3 Relations to classes L_m and \mathfrak{S}_α

Theorem 41 shows the importance of the following class.

Definition 45 $ID_{\log} = ID_{\log}(\mathbb{R}^d)$ is the class of $\mu_0 \in ID(\mathbb{R}^d)$ such that its Lévy measure ν_0 satisfies (2.20).

We fix $c > 0$. For a Lévy process $\{Z_t : t \geq 0\}$ on \mathbb{R}^d with $\mathcal{L}(Z_1) = \mu_0 \in ID_{\log}$, the OU type process $\{X_t : t \geq 0\}$ generated by $\{Z_t\}$ and c , starting from an arbitrary M , has a limit distribution $\mu \in L_0$. This is contained in Theorem 41. Define the mapping $\Phi : ID_{\log} \rightarrow L_0$ by $\Phi(\mu_0) = \mu$.

Now we clarify the relation of this mapping Φ with the classes L_m and \mathfrak{S}_α .

Theorem 46 (i) Let $m \in \{0, 1, \dots, \infty\}$. Then $\mu_0 \in L_{m-1} \cap ID_{\log}$ if and only if $\Phi(\mu_0) \in L_m$, where $L_{-1} = ID$. The correspondence of $L_{m-1} \cap ID_{\log}$ and L_m by Φ is one-to-one and onto.

(ii) Let $0 < \alpha \leq 2$. Then, $\mu_0 \in \mathfrak{S}_\alpha$ if and only if $\mu_0 \in ID_{\log}$ and $\Phi(\mu_0) \in \mathfrak{S}_\alpha$; $\mu_0 \in \mathfrak{S}_\alpha^0$ (that is, strictly α -stable) if and only if $\mu_0 \in ID_{\log}$ and $\Phi(\mu_0) \in \mathfrak{S}_\alpha^0$. There are $a > 0$ and $\gamma \in \mathbb{R}^d$ satisfying

$$\widehat{\Phi(\mu_0)}(z) = \widehat{\mu_0}(z)^a e^{i\langle \gamma, z \rangle} \quad (2.40)$$

if and only if $\mu_0 \in \mathfrak{S}$. There is $a > 0$ satisfying

$$\widehat{\Phi(\mu_0)}(z) = \widehat{\mu_0}(z)^a \quad (2.41)$$

if and only if $\mu_0 \in \mathfrak{S}^0$.

Proof. (i) Let $\mu \in L_m$. Then, the probability measure ρ_b corresponding to μ in (1.1) is in L_{m-1} and its characteristic function has the form (see end of the proof of Theorem 41 (i))

$$\widehat{\rho}_b(z) = \exp \int_0^{\frac{1}{c} \log b} \psi_0(e^{-cs}z) ds. \quad (2.42)$$

Then

$$\widehat{\rho}_b(z)^{\frac{c}{\log b}} = \exp \left[\frac{c}{\log b} \int_0^{\frac{1}{c} \log b} \psi_0(e^{-cs}z) ds \right] \rightarrow e^{\psi_0(z)} = \widehat{\mu}_0(z)$$

as $b \downarrow 1$. Thus $\mu_0 \in L_{m-1}$ since the class L_{m-1} is closed under convergence.

Let $\mu_0 \in L_{m-1} \cap ID_{\log}$. Then, the OU type process given by Theorem 41 (i) has limit distribution $\mu \in L_0(\mathbb{R}^d)$ where $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z) \widehat{\rho}_b(z)$ with $\widehat{\rho}_b(z)$ as in (2.42). Recall that (2.42) is the characteristic function of the random variable $X = \int_0^{\frac{1}{c} \log b} e^{-cs} dZ_s$ given by (2.7), where $\{Z_t: t \geq 0\}$ is the Lévy process with $\mathcal{L}(Z_1) = \mu_0 \in L_{m-1}$. Hence, the stochastic integrals of step functions by $\{Z_t\}$ belong to the class L_{m-1} , see (2.4). Then, by Propositions 29 and 31 and the closedness of the class L_{m-1} described in Lemma 8 it follows that the distribution of X is in L_{m-1} . Therefore $\mu \in L_m$.

The fact that Φ is a one-to-one and onto mapping between $L_{m-1} \cap ID_{\log}$ and L_m follows immediately from the above argument and from the uniqueness of (A_0, ν_0, γ_0) corresponding to $\mu \in L_0(\mathbb{R}^d)$ in Theorem 41 (ii).

(ii) By (1.4), a distribution μ_0 in ID is α -stable if and only if, for any $a > 0$, there is $\gamma_{0,a} \in \mathbb{R}^d$ such that

$$a\psi_0(z) = \psi_0(a^{1/\alpha}z) + i\langle \gamma_{0,a}, z \rangle. \quad (2.43)$$

We note that any stable distribution is in ID_{\log} (see Example 25.10 of [S]). If $\mu_0 \in \mathfrak{S}_\alpha$ and $\Phi(\mu_0) = \mu$, then, by (2.23)

$$\begin{aligned} \widehat{\mu}(z)^a &= \exp \left[a \int_0^\infty \psi_0(e^{-cs}z) ds \right] \\ &= \exp \int_0^\infty (\psi_0(a^{1/\alpha}e^{-cs}z) + i\langle \gamma_{0,a}, e^{-cs}z \rangle) ds \\ &= \widehat{\mu}(a^{1/\alpha}z) e^{i\langle \gamma_a, z \rangle} \end{aligned}$$

with $\gamma_a = \frac{1}{c}\gamma_{0,a}$, which shows that $\mu \in \mathfrak{S}_\alpha$.

Conversely, assume that $\mu \in \mathfrak{S}_\alpha$ and $\mu = \Phi(\mu_0)$. Then $\widehat{\mu}(z)^a = \widehat{\mu}(a^{1/\alpha}z)e^{i\langle\gamma_a, z\rangle}$ with some γ_a , and hence, by (2.23),

$$a \int_0^\infty \psi_0(e^{-cs}z)ds = \int_0^\infty \psi_0(e^{-cs}a^{1/\alpha}z)ds + i\langle\gamma_a, z\rangle$$

for all $z \in \mathbb{R}^d$. Replacing z by $e^{-ct}z$ and making change of variables, we get

$$a \int_t^\infty \psi_0(e^{-cs}z)ds = \int_t^\infty \psi_0(e^{-cs}a^{1/\alpha}z)ds + i\langle e^{-ct}\gamma_a, z\rangle.$$

Differentiation in t gives

$$a\psi_0(e^{-ct}z) = \psi_0(e^{-ct}a^{1/\alpha}z) + i\langle ce^{-ct}\gamma_a, z\rangle.$$

Letting $t \downarrow 0$, we have $a\psi_0(z) = \psi_0(a^{1/\alpha}z) + i\langle c\gamma_a, z\rangle$, that is, $\mu_0 \in \mathfrak{S}_\alpha$.

The argument above simultaneously shows that $\mu_0 \in \mathfrak{S}_\alpha^0$ if and only if $\mu \in \mathfrak{S}_\alpha^0$.

Next, let $\mu_0 \in \mathfrak{S}_\alpha$. We will show (2.40) for some $a > 0$ and $\gamma \in \mathbb{R}^d$. Since (2.43) holds for all $a > 0$, we have

$$\int_0^\infty \psi_0(e^{-cs}z)ds = \int_0^\infty (e^{-\alpha cs}\psi_0(z) - i\langle\gamma_{0,\exp(-\alpha cs)}, z\rangle) ds.$$

It follows that

$$\int_0^\infty \psi_0(e^{-cs}z)ds = \frac{1}{\alpha c}\psi_0(z) + i\langle\gamma, z\rangle$$

with $\gamma = -\lim_{t \rightarrow \infty} \int_0^t \gamma_{0,\exp(-\alpha cs)} ds$, where the existence of the limit comes from the finiteness of $\int_0^\infty \psi_0(e^{-cs}z)ds$ and $\int_0^\infty e^{-\alpha cs}\psi_0(z)ds$. By (2.23) this gives (2.40) with $a = \frac{1}{\alpha c}$.

Conversely, suppose that $\mu_0 \in ID_{\log}$ satisfies (2.40) with some $a > 0$ and γ . This means that

$$\int_0^\infty \psi_0(e^{-cs}z)ds = a\psi_0(z) + i\langle\gamma, z\rangle.$$

Replace z by $e^{-ct}z$ to obtain

$$\int_t^\infty \psi_0(e^{-cs}z)ds = a\psi_0(e^{-ct}z) + i\langle e^{-ct}\gamma, z\rangle.$$

Fix z for a while and denote $f(t) = i\langle ce^{-ct}\gamma, z \rangle$ and $g(t) = \psi_0(e^{-ct}z)$. Then we see that g is differentiable and $g(t) = -ag'(t) + f(t)$. Hence

$$\frac{d}{dt} (e^{t/a}g(t)) = e^{t/a}g'(t) + \frac{1}{a}e^{t/a}g(t) = \frac{1}{a}e^{t/a}f(t),$$

that is,

$$e^{t/a}g(t) = \int_0^t \frac{1}{a}e^{s/a}f(s)ds + g(0).$$

Now we have

$$\begin{aligned} \psi_0(e^{-ct}z) &= e^{-t/a} \int_0^t \frac{1}{a}e^{s/a}ic\langle e^{-cs}\gamma, z \rangle ds + e^{-t/a}\psi_0(z) \\ &= i\langle \eta_t, z \rangle + e^{-t/a}\psi_0(z) \end{aligned}$$

with some $\eta_t \in \mathbb{R}^d$. Thus, for every $b \in (0, 1)$, there is $\gamma_{0,b}$ such that

$$b\psi_0(z) = \psi_0(b^{ac}z) + i\langle \gamma_{0,b}, z \rangle \quad \text{for } z \in \mathbb{R}^d.$$

Changing z , we also get

$$\frac{1}{b}\psi_0(z) = \psi_0(b^{-ac}z) - i\langle \frac{1}{b}b^{-ac}\gamma_{0,b}, z \rangle \quad \text{for } z \in \mathbb{R}^d.$$

These show that $ac \geq 1/2$ and μ_0 is $\frac{1}{ac}$ -stable by Theorem 13.15 of [S].

The argument above also shows the last assertion in (ii) that (2.41) holds if and only if $\mu_0 \in \mathfrak{S}^0$. ■

Remark 47 If μ_0 is stable, then there are $b > 0$ and $\gamma \in \mathbb{R}^d$ satisfying

$$\widehat{\Phi(\mu_0)}(z) = \widehat{\mu_0}(bz)e^{i\langle \gamma, z \rangle}.$$

But the converse is not true. See Wolfe [94].

Remark 48 By introducing a stronger convergence concept in ID_{\log} and using the usual weak convergence in L_0 , the mapping Φ and its inverse are continuous. See Sato and Yamazato [74].

Let us give another formulation of the relation of Φ with the classes L_m . For $m \in \mathbb{N}$, let Φ^m be the m th iteration of Φ . That is, $\Phi^1 = \Phi$ and, for $m \geq 2$, $\Phi^m(\mu)$ is defined with $\Phi^m(\mu) = \Phi(\Phi^{m-1}(\mu))$ if and only if $\Phi^{m-1}(\mu)$ is defined and in ID_{\log} .

Theorem 49 *Let $m \geq 0$.*

(i) *Let $\mu_m \in ID(\mathbb{R}^d)$ with triplet (A_m, ν_m, γ_m) and $\log \hat{\mu}_m = \psi_m$. Assume that*

$$\int_{|x|>2} (\log |x|)^{m+1} \nu_m(dx) < \infty. \quad (2.44)$$

Then $\Phi^{m+1}(\mu_m)$ is definable. Let $\mu = \Phi^{m+1}(\mu_m)$. Then $\mu \in L_m(\mathbb{R}^d)$ and

$$\int_0^\infty s^m |\psi_m(e^{-cs}z)| ds < \infty, \quad (2.45)$$

$$\hat{\mu}(z) = \exp \int_0^\infty \frac{s^m}{m!} \psi_m(e^{-cs}z) ds. \quad (2.46)$$

The triplet (A, ν, γ) of μ is expressed as

$$A = \frac{1}{(2c)^{m+1}} A_m, \quad (2.47)$$

$$\nu(B) = \frac{1}{m! c^{m+1}} \int_{\mathbb{R}^d} \nu_m(dx) \int_0^\infty s^m 1_B(e^{-s}x) ds, \quad B \in \mathcal{B}(\mathbb{R}^d), \quad (2.48)$$

$$\gamma = \frac{1}{c^{m+1}} \gamma_m + \frac{1}{m! c^{m+1}} \int_0^\infty s^m e^{-s} ds \int_{1 < |x| \leq e^s} x \nu_m(dx) \quad (2.49)$$

(ii) *For any $\mu \in L_m(\mathbb{R}^d)$ there exists a unique $\mu_m \in ID$ with triplet (A_m, ν_m, γ_m) satisfying (2.44) such that $\Phi^{m+1}(\mu_m) = \mu$.*

(iii) *If $\mu_m \in ID$ with triplet (A_m, ν_m, γ_m) does not satisfy (2.44), then $\Phi^{m+1}(\mu_m)$ is not definable.*

Proof. Induction. When $m = 0$, the statements reduce to Theorem 41. Let $m \geq 1$. Assume that the assertions are true for $m - 1$ in place of m . Let us show the assertions for m .

(i) We assume (2.44). Noting that $\mu_m \in ID_{\log}$, write $\Phi(\mu_m) = \mu_{m-1}$, its triplet $(A_{m-1}, \nu_{m-1}, \gamma_{m-1})$, and $\log \hat{\mu}_{m-1} = \psi_{m-1}$. Then, using (2.25), we get

$$\begin{aligned} \int_{|x|>2} (\log |x|)^m \nu_{m-1}(dx) &= \frac{1}{c} \int_{\mathbb{R}^d} \nu_m(dx) \int_0^\infty (\log |e^{-s}x|)^m 1_{\{|e^{-s}x|>2\}} ds \\ &= \frac{1}{c} \int_{|x|>2} \nu_m(dx) \int_0^{\log(|x|/2)} (\log |x| - s)^m ds \\ &= \frac{1}{c(m+1)} \int_{|x|>2} [(\log |x|)^{m+1} - (\log 2)^{m+1}] \nu_m(dx) < \infty. \end{aligned}$$

It follows that $\Phi^m(\mu_{m-1})$ is definable and in L_{m-1} . Thus $\mu = \Phi^{m+1}(\mu_m)$ is definable. Repeated application of Theorem 46 (i) shows that $\mu \in L_m$.

Let us show (2.45). Define ρ by

$$\rho(B) = \int_{\mathbb{R}^d} \nu_m(dx) \int_0^\infty s^m 1_B(e^{-s}x) ds.$$

Then

$$\begin{aligned} \int_{|x| \leq 1} |x|^2 \rho(dx) &= \int \nu_m(dx) \int_0^\infty s^m |e^{-s}x|^2 1_{\{|e^{-s}x| \leq 1\}} ds \\ &= \int_{|x| \leq 1} |x|^2 \nu_m(dx) \int_0^\infty s^m e^{-2s} ds + \int_{|x| > 1} |x|^2 \nu_m(dx) \int_{\log|x|}^\infty s^m e^{-2s} ds, \end{aligned}$$

which is finite since $\int_{\log|x|}^\infty s^m e^{-2s} ds \sim \text{const}|x|^{-2}(\log|x|)^m$ as $|x| \rightarrow \infty$. Moreover

$$\int_{|x| > 1} \rho(dx) = \int_{\mathbb{R}^d} \nu_m(dx) \int_0^\infty s^m 1_{\{|e^{-s}x| > 1\}} ds = \int_{|x| > 1} \frac{1}{m+1} (\log|x|)^{m+1} \nu_m(dx),$$

which is finite by (2.44). Hence, writing

$$\begin{aligned} \psi_m(e^{-cs}z) &= -\frac{1}{2} e^{-2cs} \langle z, A_m z \rangle + i e^{-cs} \langle \gamma_m, z \rangle \\ &+ \int_{\mathbb{R}^d} g(z, e^{-cs}x) \nu_m(dx) + i \int_{\mathbb{R}^d} \langle z, e^{-cs}x \rangle 1_{\{1 < |x| \leq e^{cs}\}} \nu_m(dx), \end{aligned} \quad (2.50)$$

and estimating in the same way as in the proof of Theorem 41 (i), we obtain (2.45). Here we have also used the estimate

$$\int_0^\infty s^m e^{-s} ds \int_{1 < |x| \leq e^s} |x| \nu_m(dx) = \int_{|x| > 1} |x| \nu_m(dx) \int_{\log|x|}^\infty s^m e^{-s} ds < \infty.$$

since $\psi_{m-1}(z) = \int_0^\infty \psi_m(e^{-cs}z) ds$, we have

$$\begin{aligned} \widehat{\mu}(z) &= \exp \int_0^\infty \frac{s^{m-1}}{(m-1)!} \psi_{m-1}(e^{-cs}z) ds = \exp \int_0^\infty \frac{s^{m-1}}{(m-1)!} ds \int_s^\infty \psi_m(e^{-cu}z) du \\ &= \exp \int_0^\infty \psi_m(e^{-cu}z) du \int_0^u \frac{s^{m-1}}{(m-1)!} ds, \end{aligned}$$

which gives (2.46). The use of Fubini theorem in the above is permitted by (2.45). Now, calculating $\int_0^\infty \frac{s^m}{m!} \psi_m(e^{-cs}z) ds$ from (2.50), we see that μ has triplet (A, ν, γ) described by (2.47), (2.48), (2.49).

We prove (iii) before (ii).

(iii) Suppose that $\mu_m \in ID$ satisfies $\int_{|x|>2} (\log |x|)^{m+1} \nu_m(dx) = \infty$. Choose $n \in \{0, 1, \dots, m\}$ such that $\int_{|x|>2} (\log |x|)^n \nu_m(dx) < \infty$ and $\int_{|x|>2} (\log |x|)^{n+1} \nu_m(dx) = \infty$. Then $\Phi^n(\mu_m)$ is definable (if $n = 0$, then we let $\Phi^0(\mu_m) = \mu_m$). Let $\tilde{\mu} = \Phi^n(\mu_m)$ and let $\tilde{\nu}$ be the Lévy measure of $\tilde{\mu}$. We claim that

$$\int_{|x|>2} \log |x| \tilde{\nu}(dx) = \infty.$$

By (i) we have

$$\tilde{\nu}(B) = \frac{1}{(n-1)! c^n} \int_{\mathbb{R}^d} \nu_m(dx) \int_0^\infty s^{n-1} 1_B(e^{-s}x) ds.$$

Hence

$$\begin{aligned} \int_{|x|>2} \log |x| \tilde{\nu}(dx) &= \text{const} \int \nu_m(dx) \int_0^\infty s^{n-1} (\log |e^{-s}x|) 1_{\{|e^{-s}x|>2\}} ds \\ &= \text{const} \int_{|x|>2} \nu_m(dx) \int_0^{\log(|x|/2)} s^{n-1} (\log |x| - s) ds = \infty, \end{aligned}$$

because $\int_0^{\log(|x|/2)} s^{n-1} (\log |x| - s) ds \sim \frac{1}{n(n+1)} (\log |x|)^{n+1}$ as $|x| \rightarrow \infty$. This shows that $\Phi(\tilde{\mu})$ is not definable. Thus $\Phi^{m+1}(\mu_m)$ is not definable.

(ii) Let $\mu \in L_m$. Apply Theorem 46 (i). There is $\mu_0 \in L_{m-1} \cap ID_{\log}$ such that $\Phi(\mu_0) = \mu$. Then, there is $\mu_1 \in L_{m-2} \cap ID_{\log}$ such that $\Phi(\mu_1) = \mu_0$. Continuing this, we get finally $\mu_m \in L_{-1} \cap ID_{\log} = ID_{\log}$ such that $\Phi(\mu_m) = \mu_{m-1}$. Hence μ_m is in the domain of definition of Φ^{m+1} and $\Phi^{m+1}(\mu_m) = \mu$. It follows from (iii) that μ_m satisfies (2.44). ■

Remark 50 *In order that $\mu_m \in ID$ has Lévy measure ν_m satisfying (2.44), it is necessary and sufficient that*

$$\int_{|x|>2} (\log |x|)^{m+1} \mu_m(dx) < \infty. \quad (2.51)$$

See [S] Theorem 25.3 and Proposition 25.4 for a proof.

Remark 51 *The expression (2.48) of the Lévy measure of $\mu \in L_m(\mathbb{R}^d)$ is rewritten in the following way. Let λ_m and $k_{m,\xi}(r)$ be the spherical component and the k -function of ν_m ,*

respectively. Then $\int_S \lambda_m(d\xi) \int_2^\infty (\log r)^{m+1} k_{m,\xi}(r) \frac{dr}{r} < \infty$ and

$$\begin{aligned} \nu(B) &= \frac{1}{m! c^{m+1}} \int_S \lambda_m(d\xi) \int_0^\infty k_{m,\xi}(r) \frac{dr}{r} \int_0^\infty s^m 1_B(e^{-s} r \xi) ds \\ &= \frac{1}{m! c^{m+1}} \int_S \lambda_m(d\xi) \int_0^\infty k_{m,\xi}(r) \frac{dr}{r} \int_0^r \left(\log \frac{r}{u}\right)^m 1_B(u\xi) \frac{du}{u} \\ &= \frac{1}{m! c^{m+1}} \int_S \lambda_m(d\xi) \int_0^\infty 1_B(u\xi) \frac{du}{u} \int_u^\infty k_{m,\xi}(r) \left(\log \frac{r}{u}\right)^m \frac{dr}{r}. \end{aligned}$$

2.4 Stationary Ornstein–Uhlenbeck type processes

Let us define stochastic integrals over infinite time parameter set. For this purpose, let

$$\{Z_t: -\infty < t < \infty\}$$

be a stochastically continuous process on \mathbb{R}^d with stationary independent increments such that, almost surely, $Z_t(\omega)$ is right continuous with left limits. Let $\mu_0 = \mathcal{L}(Z_1 - Z_0)$ and $\psi_0(z) = \log \widehat{\mu}_0(z)$. Then

$$E[e^{i\langle z, Z_t - Z_s \rangle}] = e^{(t-s)\psi_0(z)} \quad \text{for } s \leq t. \quad (2.52)$$

The distribution $\mathcal{L}(Z_t)$ is not determined by μ_0 . Indeed, if Y and $\{Z_t\}$ are independent, then $\{Y + Z_t\}$ also satisfies these requirements. For $t_0 \in \mathbb{R}$, $\{Z_{t_0+t}: -\infty < t < \infty\}$ fulfills the requirements, too.

Construction of $\{Z_t\}$ with $Z_0 = 0$ is as follows. Let $\{Z_t^{(1)}: t \geq 0\}$ and $\{Z_t^{(2)}: t \geq 0\}$ be independent Lévy processes on \mathbb{R}^d such that $E[e^{i\langle z, Z_t^{(1)} \rangle}] = e^{t\psi_0(z)}$ and $E[e^{i\langle z, Z_t^{(2)} \rangle}] = e^{t\psi_0(-z)}$. Define $Z_t = Z_t^{(1)}$ for $t \geq 0$ and $Z_t = Z_{(-t)_-}^{(2)}$ for $t < 0$. Let us check the condition (2.52), while the other conditions are evidently satisfied. If $s \leq t < 0$, then

$$\begin{aligned} Ee^{i\langle z, Z_t - Z_s \rangle} &= Ee^{i\langle z, Z_{(-t)_-}^{(2)} - Z_{(-s)_-}^{(2)} \rangle} = \lim_{\varepsilon \downarrow 0} Ee^{i\langle z, Z_{-t-\varepsilon}^{(2)} - Z_{-s-\varepsilon}^{(2)} \rangle} \\ &= \lim_{\varepsilon \downarrow 0} Ee^{i\langle -z, Z_{-s-\varepsilon}^{(2)} - Z_{-t-\varepsilon}^{(2)} \rangle} = e^{(t-s)\psi_0(z)}. \end{aligned}$$

If $s < 0 \leq t$, then

$$\begin{aligned} Ee^{i\langle z, Z_t - Z_s \rangle} &= Ee^{i\langle z, Z_t^{(1)} - Z_{(-s)_-}^{(2)} \rangle} = \left(Ee^{i\langle z, Z_t^{(1)} \rangle} \right) \lim_{\varepsilon \downarrow 0} Ee^{i\langle -z, Z_{-s-\varepsilon}^{(2)} \rangle} \\ &= e^{t\psi_0(z)} \lim_{\varepsilon \downarrow 0} e^{(-s-\varepsilon)\psi_0(z)} = e^{(t-s)\psi_0(z)}. \end{aligned}$$

Let $f(s)$, $-\infty < s < \infty$, be a real-valued, locally bounded, measurable function. Define, for $-\infty < t_0 < t_1 < \infty$, the integral $\int_{t_0}^{t_1} f(s)dZ_s$ in the same way as before. We have

$$E \exp \left(i \left\langle z, \int_{t_0}^{t_1} f(s)dZ_s \right\rangle \right) = \exp \int_{t_0}^{t_1} \psi_0(f(s)z)ds. \quad (2.53)$$

Definition 52 *If the limit in probability of $\int_{t_0}^t f(s)dZ_s$ as $t \rightarrow \infty$ exists, then the limit is denoted by $\int_{t_0}^{\infty} f(s)dZ_s$. Likewise, if the limit in probability of $\int_t^{t_1} f(s)dZ_s$ as $t \rightarrow -\infty$ exists, then the limit is denoted by $\int_{-\infty}^{t_1} f(s)dZ_s$.*

Fix $c > 0$. Consider the equation

$$X_t = X_{t_0} + Z_t - Z_{t_0} - c \int_{t_0}^t X_s ds \quad \text{for } -\infty < t_0 \leq t < \infty. \quad (2.54)$$

Definition 53 *A stochastic process $\{X_t: -\infty < t < \infty\}$ is said to be a stationary solution of (2.54) if it is right continuous with left limits a. s. and satisfies (2.54) and, for every $t_0 \in \mathbb{R}$,*

$$X_{t_0} \text{ and } \{Z_t - Z_{t_0}: t \geq t_0\} \text{ are independent} \quad (2.55)$$

and if it is stationary in the sense that, for every $s \in \mathbb{R}$,

$$\{X_{t+s}: -\infty < t < \infty\} \stackrel{d}{=} \{X_t: -\infty < t < \infty\}. \quad (2.56)$$

Definition 54 *A stationary solution $\{X_t: -\infty < t < \infty\}$ of (2.54) is called the stationary Ornstein–Uhlenbeck type process (or stationary OU type process) generated by $\{Z_t\}$ and c , or generated by μ_0 and c , or generated by $(A_0, \nu_0, \gamma_0, c)$.*

Theorem 55 (i) *Suppose that (2.20) holds. Then, the stochastic integral $\int_{-\infty}^t e^{cs}dZ_s$ is definable for each $t \in \mathbb{R}$. Further we can define a process $\{X_t: -\infty < t < \infty\}$ right continuous with left limits in t a. s. such that*

$$P \left[X_t = e^{-ct} \int_{-\infty}^t e^{cs}dZ_s \right] = 1 \quad \text{for } t \in \mathbb{R}. \quad (2.57)$$

This process $\{X_t: -\infty < t < \infty\}$ is a stationary solution of the equation (2.54). For each $t \in \mathbb{R}$, $\mathcal{L}(X_t) = \Phi(\mu_0)$, where Φ is the mapping defined in the preceding section. A stationary solution of (2.54) is unique in the sense of law.

(ii) *Suppose that (2.28) holds. Then, for any $t \in \mathbb{R}$, the stochastic integral $\int_{-\infty}^t e^{cs}dZ_s$ does not exist. The equation (2.54) does not have a stationary solution.*

Proof. Let $Z_u^{(s)} = Z_{s+u} - Z_s$. Then, for any $s \in \mathbb{R}$, $\{Z_u^{(s)} : u \geq 0\}$ is a Lévy process with $\mathcal{L}(Z_1^{(s)}) = \mu_0$. For any bounded measurable function f on $[t_0, t_0 + s]$, we have

$$\int_{t_0}^{t_0+s} f(u) dZ_u = \int_0^s f(t_0 + u) dZ_u^{(t_0)} \quad \text{a. s.}$$

from the definition of stochastic integrals. For $t_0 \in \mathbb{R}$, let

$$X_t^{(t_0)} = e^{-ct} \int_{t_0}^t e^{cs} dZ_s \quad \text{for } t \geq t_0. \quad (2.58)$$

Then $\{X_{t_0+s}^{(t_0)} : s \geq 0\}$ is the OU type process generated by μ_0 and c , starting from 0, and

$$\mathcal{L}\left(X_t^{(t_0)}\right) = \mathcal{L}\left(X_{t-t_0}^{(0)}\right), \quad (2.59)$$

because

$$X_{t_0+s}^{(t_0)} = e^{-c(t_0+s)} \int_{t_0}^{t_0+s} e^{cu} dZ_u = e^{-cs} \int_0^s e^{cu} dZ_u^{(t_0)} \quad \text{a. s.}$$

(i) We assume (2.20). The distribution $\mu = \Phi(\mu_0)$ satisfies (2.23). Let $t_0 < t_1 < t$. Then

$$\begin{aligned} E \exp\left(i \left\langle z, \int_{t_0}^t e^{cs} dZ_s - \int_{t_1}^t e^{cs} dZ_s \right\rangle\right) &= E \exp\left(i \left\langle z, \int_{t_0}^{t_1} e^{cs} dZ_s \right\rangle\right) \\ &= \exp \int_{t_0}^{t_1} \psi_0(e^{cs} z) ds = \exp \int_{-t_1}^{-t_0} \psi_0(e^{-cs} z) ds, \end{aligned}$$

which tends to 1 as $t_0, t_1 \rightarrow -\infty$ by (2.23). Thus, for any $t_n \rightarrow -\infty$ and fixed t , $\left\{\int_{t_n}^t e^{cs} dZ_s\right\}$ is a Cauchy sequence in the metric of convergence in probability. Hence it is convergent in probability. The limit does not depend on the choice of $\{t_n\}$. It follows that the stochastic integral $\int_{-\infty}^t e^{cs} dZ_s$ exists. Denote

$$Y_t = e^{-ct} \int_{-\infty}^t e^{cs} dZ_s.$$

Then $\mathcal{L}(Y_t) = \lim_{u \rightarrow \infty} \mathcal{L}(X_u^{(0)}) = \mu$ by (2.59). There is a modification $\{X_t\}$ of $\{Y_t\}$ with sample functions right continuous with left limits, which follows from [S] Theorem 11.5, as

$\left\{ \int_{-\infty}^{t_0+u} e^{cs} dZ_s - \int_{-\infty}^{t_0} e^{cs} dZ_s : u \geq 0 \right\}$ is an additive process in law for any $t_0 \in \mathbb{R}$. Notice that, for any $t_0 \in \mathbb{R}$, (2.55) is satisfied. We have, for $t_0 \in \mathbb{R}$ and $s \geq 0$,

$$\begin{aligned} X_{t_0+s} &= e^{-c(t_0+s)} \int_{-\infty}^{t_0+s} e^{cu} dZ_u = e^{-c(t_0+s)} \int_{-\infty}^{t_0} e^{cu} dZ_u + e^{-c(t_0+s)} \int_{t_0}^{t_0+s} e^{cu} dZ_u \\ &= e^{-cs} X_{t_0} + e^{-cs} \int_0^s e^{cu} dZ_u^{(t_0)} \quad \text{a. s.} \end{aligned}$$

It follows from Proposition 37 that

$$X_{t_0+s} = X_{t_0} + Z_s^{(t_0)} - c \int_0^s X_{t_0+u} du \quad \text{for } s \geq 0, \quad (2.60)$$

that is, (2.54) is satisfied. Let us check the stationarity (2.56). We have, for $\tilde{Z}_u^{(s)} = Z_{s+u}$,

$$\begin{aligned} X_{t+s} &= e^{-c(t+s)} \int_{-\infty}^{s+t} e^{cu} dZ_u = e^{-ct} \int_{-\infty}^t e^{cu} d\tilde{Z}_u^{(s)} \\ &\stackrel{d}{=} e^{-ct} \int_{-\infty}^t e^{cu} dZ_u = X_t. \end{aligned}$$

Similarly,

$$(X_{t_1+s}, \dots, X_{t_n+s}) \stackrel{d}{=} (X_{t_1}, \dots, X_{t_n}),$$

that is, (2.56).

Let us show the uniqueness in law of a stationary solution $\{X_t : -\infty < t < \infty\}$ of (2.54). From stationarity, $\mathcal{L}(X_t) = \rho$ does not depend on t . It follows from (2.54) that (2.60) is satisfied. Hence, using (2.55), we see

$$X_{t_0+t} = e^{-ct} X_{t_0} + e^{-ct} \int_0^t e^{cs} dZ_s^{(t_0)}, \quad t \geq 0. \quad (2.61)$$

by Proposition 37. Thus

$$\begin{aligned} Ee^{i\langle z, X_{t+t_0} \rangle} &= \left(Ee^{i\langle z, e^{-ct} X_{t_0} \rangle} \right) \exp \int_0^t \psi_0(e^{-ct+cs} z) ds \\ &= \left(Ee^{i\langle z, e^{-ct} X_{t_0} \rangle} \right) \exp \int_0^t \psi_0(e^{-cs} z) ds. \end{aligned}$$

Since $Ee^{i\langle z, X_{t+t_0} \rangle} = \widehat{\rho}(z)$, we get

$$\widehat{\rho}(z) = \exp \int_0^\infty \psi_0(e^{-cs}z) ds = \widehat{\Phi(\mu_0)}(z),$$

letting $t \rightarrow \infty$. Hence $\rho = \Phi(\mu_0)$. (2.55) and (2.61) show that the distribution of (X_{t_0}, X_{t_0+t}) is determined by μ_0 . Similarly, for any $-\infty < t_1 < \dots < t_n < \infty$, the distribution of $(X_{t_1}, \dots, X_{t_n})$ is determined by μ_0 .

(ii) We assume (2.28). Then, by Theorem 41 (iii) and by (2.59), $\mathcal{L}(X_t^{(t_0)})$ is not convergent as $t_0 \rightarrow -\infty$. That is, $\mathcal{L}\left(\int_{t_0}^t e^{cs} dZ_s\right)$ is not convergent as $t_0 \rightarrow -\infty$. Hence $\int_{-\infty}^t e^{cs} dZ_s$ does not exist.

Suppose that a stationary solution $\{X_t: -\infty < t < \infty\}$ of (2.54) exists. Let $\mathcal{L}(X_t) = \mu$, which is independent of t . As before, we have

$$\widehat{\mu}(z) = \left(Ee^{i\langle z, e^{-ct} X_{t_0} \rangle} \right) \exp \int_0^t \psi_0(e^{-cs}z) ds.$$

Hence $e^{\int_0^t \psi_0(e^{-cs}z) ds} \rightarrow \widehat{\mu}(z)$ as $t \rightarrow \infty$. This contradicts the assertion (iii) of Theorem 41 on non-existence of limit distribution. ■

Theorem 55, combined with Theorem 41, gives the following result.

Corollary 56 *Any stationary OU type process $\{X_t: -\infty < t < \infty\}$ has distribution $\mathcal{L}(X_t)$ in L_0 . Conversely, any distribution in L_0 is the distribution of a stationary OU type process.*

Remark 57 *The same proof as that of Theorem 55 (i) gives the result that, under the condition (2.20), $\int_0^\infty e^{-cs} dZ_s$ exists and*

$$\mathcal{L}\left(\int_0^\infty e^{-cs} dZ_s\right) = \Phi(\mu_0). \quad (2.62)$$

Note that

$$\mathcal{L}\left(\int_0^t e^{-cs} dZ_s\right) = \mathcal{L}\left(e^{-ct} \int_0^t e^{cs} dZ_s\right) = \mathcal{L}(X_t), \quad (2.63)$$

where $\{X_t\}$ is the OU type process generated by μ_0 and c , starting from 0. Thus (2.62) is equivalent to (2.21). But, if $\{Z_t\}$ is nontrivial, then X_t does not converge in probability as $t \rightarrow \infty$. The process $\{\int_0^t e^{-cs} dZ_s: t \geq 0\}$ is an additive process; it is not identical in law with $\{X_t\}$ except in the trivial case.

Proof. Note that $X_t = e^{-ct} \int_0^t e^{cs} dZ_s$. The first equality in (2.63) is because both have characteristic function $\exp \int_0^t \psi_0(e^{-cs} z) ds$. Let us prove that, if $\{Z_t\}$ is nontrivial, then limit in probability of X_t as $t \rightarrow \infty$ does not exist. Suppose, on the contrary, that $X_t \rightarrow Y$ in probability as $t \rightarrow \infty$. Then $X_t - X_{t-1} \rightarrow 0$ in probability, since

$$P[|X_t - X_{t-1}| > \varepsilon] \leq P[|X_t - Y| > \varepsilon/2] + P[|X_{t-1} - Y| > \varepsilon/2] \rightarrow 0$$

for any $\varepsilon > 0$. Hence $Ee^{i\langle z, X_t - X_{t-1} \rangle} \rightarrow 1$ as $t \rightarrow \infty$. From the nontriviality there is $z_0 \in \mathbb{R}^d$ such that $|\widehat{\mu}_0(z_0)| < 1$ (see [S] Lemma 13.9). It follows that $\operatorname{Re} \psi_0(z_0) < 0$ since $|\widehat{\mu}_0(z_0)| = e^{\operatorname{Re} \psi_0(z_0)}$. As $\operatorname{Re} \psi_0 \leq 0$ and ψ_0 is continuous, it follows that $\int_0^1 \operatorname{Re} \psi_0(e^{-cs} z_0) ds < 0$. We have

$$X_t - X_{t-1} = e^{-ct} \int_{t-1}^t e^{cs} dZ_s + (e^{-ct} - e^{-c(t-1)}) \int_0^{t-1} e^{cs} dZ_s$$

and the two terms on the right are independent. Hence

$$\begin{aligned} |Ee^{i\langle z, X_t - X_{t-1} \rangle}| &\leq \left| Ee^{i\langle z, e^{-ct} \int_{t-1}^t e^{cs} dZ_s \rangle} \right| = \left| e^{\int_{t-1}^t \psi_0(e^{-ct+cs} z) ds} \right| \\ &= \left| e^{\int_0^1 \psi_0(e^{-cs} z) ds} \right| = e^{\int_0^1 \operatorname{Re} \psi_0(e^{-cs} z) ds} < 1 \quad \text{if } z = z_0. \end{aligned}$$

This is a contradiction. ■

Remark 58 Characterization of L_m in Theorem 49 is given a form of stochastic integrals over $[0, \infty)$. Let

$$p_m(t) = ((m+1)!t)^{1/(m+1)}.$$

Suppose that $\mu_m \in L_m(\mathbb{R}^d)$. Then the representation of $\widehat{\mu}(z)$ in (2.46) is rewritten to

$$\widehat{\mu}(z) = \exp \int_0^\infty \psi_m(e^{-cp_m(t)} z) dt.$$

This gives

$$\mu = \mathcal{L} \left(\int_0^\infty e^{-cp_m(t)} dZ_t \right),$$

where the existence of the stochastic integral in the right-hand side is proved from (2.45). Here $\{Z_t : t \geq 0\}$ is a Lévy process with $\mathcal{L}(Z_1) = \mu_m$ satisfying (2.44) or, equivalently, (2.51). If μ_m does not satisfy (2.44), then the stochastic integral $\int_0^\infty e^{-cp_m(t)} dZ_t$ does not exist.

Notes

Most of this chapter follows Wolfe [94] (1982a) and Sato and Yamazato [73] (1983), [74] (1984), but some of detailed treatment of stochastic integrals are new. A study of stochastic integrals based on additive processes is found in Rajput and Rosinski [54] (1989). In the case of bounded p -variation, a treatment of the integral equation (2.16) by pathwise integrals is done by Mikosch and Norvaiša [48] (2000). Early discussion of OU type processes is a paper of Doob [14] (1942). Construction of OU type processes without using stochastic integrals is given in Section 17 of [S].

The representation (2.25) of the Lévy measures of selfdecomposable distributions was discovered by Urbanik [83] (1969), although he did not recognize the connection to limit distributions of OU type processes. When $d = 1$, the expression of the Lévy measures of distributions of L_m in Remark 51 is essentially the same as those given by Urbanik [85] (1972b), [86] (1973) and Sato [57] (1980).

The relation of the class of selfdecomposable distributions to OU type processes and stochastic integrals over an infinite time interval was recognized by Wolfe, Jurek–Vervaat, Sato–Yamazato, and Gravereaux almost at the same time. Sato and Yamazato started from the integro-differential equation for densities of selfdecomposable distributions on \mathbb{R} proved in [72] (1978), found its meaning related to OU type processes, and made extension to higher dimensions. Sato reported those results in a symposium at Research Institute for Mathematical Sciences, Kyoto University, in July 1981 on the occasion of H. Kesten’s visit to Kyoto, and also in his invited talk at the Tenth Conference on Stochastic Processes and Their Applications, held at Montreal in August 1981. There Sato met Wolfe and Vervaat. Wolfe [94] (1982a) seems to have been the earliest in finding the connection of L_0 to limit distributions of OU type processes; it was submitted in October 1979. The representation of a selfdecomposable distribution as the distribution of a stochastic integral on $[0, \infty)$ in (2.62) of Remark 57 is by Wolfe [95] (1982b) and Jurek and Vervaat [27] (1983). For more accounts see [74] (1984). Gravereaux [15] (1982) also got similar results. Most of these results were obtained in the form of operator generalization, which will be touched upon in Section 5.2.

In the case where $\{Z_t\}$ is an increasing Lévy process on \mathbb{R} , the limit theorem in Theorem 41 was discovered by Çinlar and Pinsky [13] (1971) in storage theory.

Theorem 46 was given by [27] (1983) and [73] (1983) for (i) on L_m and by [94] (1982a) and [27] (1983) for (ii) on \mathfrak{S} and \mathfrak{S}^0 . Theorem 49 on L_m was due to Jurek [25] (1983b) in a different formulation. The integral representation of L_m in Remark 58 is given in [25] (1983b). Theorem 55 on stationary OU type processes was given by [73] (1983).

Recurrence and transience of OU type processes

OU type processes on \mathbb{R}^d satisfying (2.20) are recurrent. But there are recurrent OU type processes which do not satisfy (2.20). Also there are transient OU type processes. These have been shown in Sato and Yamazato [74] (1984) and a criterion of recurrence/transience is given by Shiga [78] (1990) for $d = 1$ and by Sato, Watanabe, and Yamazato [69] (1994) for $d \geq 2$.

Chapter 3

Classes L_m and selfsimilar additive processes

Selfsimilar processes on \mathbb{R}^d are those stochastic processes whose finite dimensional distributions are invariant under change of time scale, in the sense that any change of time scale has the same effect as some change of spatial scale. This property is called selfsimilarity and it depends on a positive number H called exponent. These processes are called H -selfsimilar processes. Lévy processes which are selfsimilar constitute an important class, called strictly stable processes. In this case the exponent H is restricted to $H = 1/\alpha$ where $0 < \alpha \leq 2$.

In this chapter selfsimilar processes which are additive are studied. Section 3.1 gives their characterization in relation to the class L_0 . Section 3.2 discusses connections to the classes L_m .

3.1 Characterization by class L_0

It will be shown that, for any selfsimilar additive process $\{X_t\}$ on \mathbb{R}^d , the distribution of X_t is selfdecomposable for every t , that is, $\mathcal{L}(X_t) \in L_0(\mathbb{R}^d)$. Conversely, if μ belongs to the class $L_0(\mathbb{R}^d)$, then for every $H > 0$, there is a unique, in law, H -selfsimilar additive process $\{X_t\}$ on \mathbb{R}^d with $\mathcal{L}(X_1) = \mu$. As a consequence, there are many additive processes which are selfsimilar. In this set-up, $\{X_t\}$ is a Lévy process if and only if μ is strictly α -stable.

Definition 59 *A stochastic process $\{X_t: t \geq 0\}$ on \mathbb{R}^d is selfsimilar if, for any $a > 0$, there is $b > 0$ such that $\{X_{at}: t \geq 0\} \stackrel{d}{=} \{bX_t: t \geq 0\}$.*

Theorem 60 *Let $\{X_t: t \geq 0\}$ be a selfsimilar, stochastically continuous, nonzero process on \mathbb{R}^d with $X_0 = 0$ a. s. Then b in the definition above is uniquely determined by a and there is $H > 0$ such that, for any $a > 0$, $b = a^H$.*

See Theorem 13.11 and Remark 13.13 of [S].

Definition 61 *The number H in Theorem 60 is called the exponent of the selfsimilar process $\{X_t\}$. A nonzero selfsimilar process with exponent H is called H -selfsimilar.*

We study selfsimilar additive processes on \mathbb{R}^d . Recall that, if $\{X_t: t \geq 0\}$ is an additive process, then $\mathcal{L}(X_t) \in ID$ for any $t \geq 0$. This was proved by Lévy and Khintchine. See Theorem 9.1 of [S].

First, let us consider time change by powers of t .

Proposition 62 *If $\{X_t\}$ is an H -selfsimilar additive process on \mathbb{R}^d , then, for any $\eta > 0$, $\{X_{t^\eta}\}$ is an ηH -selfsimilar additive process.*

Proof. For any $a > 0$, $\{X_{(at)^\eta}\} = \{X_{a^\eta t^\eta}\} \stackrel{d}{=} \{a^{\eta H} X_{t^\eta}\}$. Thus $\{X_{t^\eta}\}$ is an ηH -selfsimilar process. The additivity of $\{X_{t^\eta}\}$ follows from that of $\{X_t\}$. ■

This shows that exponents are not important for selfsimilar additive processes, because we can freely change the exponent H .

Definition 63 *A process $\{X_t: t \geq 0\}$ is a strictly stable process on \mathbb{R}^d if it is a selfsimilar Lévy process. A process $\{X_t: t \geq 0\}$ is a stable process on \mathbb{R}^d if it is a Lévy process and, for any $a > 0$, there are $b > 0$ and $c \in \mathbb{R}^d$ such that $\{X_{at}: t \geq 0\} \stackrel{d}{=} \{bX_t + tc: t \geq 0\}$.*

Proposition 64 *If $\{X_t\}$ is a nontrivial stable process, then b and c are uniquely determined by a , and there is $0 < \alpha \leq 2$ such that $b = a^{1/\alpha}$ for all $a > 0$. If $\{X_t\}$ is a nonzero strictly stable process, then b is uniquely determined by a , and there is $0 < \alpha \leq 2$ such that $b = a^{1/\alpha}$ for all $a > 0$ (hence it is $(1/\alpha)$ -selfsimilar).*

See Theorem 13.15 and Definition 13.16 of [S].

Definition 65 *The α in Proposition 64 is called the index of the (strictly) stable process. The process is called (strictly) α -stable.*

Remark 66 If $\{X_t\}$ is an α -stable process, then, for any $t \geq 0$, $\mathcal{L}(X_t)$ is in \mathfrak{S}_α , that is, an α -stable distribution in Definition 9. Conversely, if $\mu \in \mathfrak{S}_\alpha$ and if μ is nontrivial, then for any $t_0 > 0$ there exists an α -stable process $\{X_t\}$ such that $\mathcal{L}(X_{t_0}) = \mu$, since the Lévy process $\{X_t\}$ with $\mathcal{L}(X_1) = \mu^{1/t_0}$ satisfies $X_{at} \stackrel{d}{=} a^{1/\alpha} X_t + \frac{t}{t_0} c$, where c is the vector in (1.4). If $\{X_t\}$ is a strictly α -stable process, then, for any $t \geq 0$, $\mathcal{L}(X_t)$ is in \mathfrak{S}_α^0 , that is, a strictly α -stable distribution in Definition 9. If $\mu \in \mathfrak{S}_\alpha^0$ and if $\mu \neq \delta_0$, then for any $t_0 > 0$ there is a strictly α -stable process $\{X_t\}$ such that $\mathcal{L}(X_{t_0}) = \mu$.

In the case of strictly α -stable processes, the index α or the exponent of selfsimilarity $H = 1/\alpha$ is very important, as a Lévy process turns into a non-Lévy process by nonlinear time change from t to t^η .

The following theorem establishes the relation between selfsimilar additive processes and selfdecomposable distributions.

Theorem 67 Fix $H > 0$.

(i) If $\{X_t: t \geq 0\}$ is an H -selfsimilar additive process on \mathbb{R}^d , then $\mathcal{L}(X_t) \in L_0(\mathbb{R}^d)$ for all $t \geq 0$.

(ii) For any $\mu \in L_0(\mathbb{R}^d)$ satisfying $\mu \neq \delta_0$, there is a unique (in law) H -selfsimilar additive process $\{X_t\}$ on \mathbb{R}^d such that $\mathcal{L}(X_1) = \mu$.

Proof. (i) Let μ_t and $\mu_{s,t}$ be the distributions of X_t and $X_t - X_s$, respectively. We have

$$\begin{aligned} \widehat{\mu}_t(z) &= \widehat{\mu}_s(z) \widehat{\mu}_{s,t}(z) \\ &= \widehat{\mu}_t\left(\left(\frac{s}{t}\right)^H z\right) \widehat{\mu}_{s,t}(z), \end{aligned} \quad (3.1)$$

by the independent increments and by $X_s = X_{(s/t)t} \stackrel{d}{=} (s/t)^H X_t$. Given $b > 1$ choose $0 < s < t$ such that $b = (s/t)^{-H}$. Then the identity above shows that $\mu_t \in L_0$ for $t > 0$. Note that $X_0 = 0$ a.s. since $X_0 \stackrel{d}{=} a^H X_0$ for all $a > 0$. Thus $\mu_0 \in L_0$ is evident.

(ii) By definition of selfdecomposability,

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z) \widehat{\rho}_b(z)$$

for every $b > 1$. Recall that $\widehat{\mu}(z) \neq 0$ and that the probability measure ρ_b is uniquely determined. See Lemma 3. Next define μ_t and $\mu_{s,t}$ by $\mu_0 = \delta_0$,

$$\begin{aligned} \widehat{\mu}_t(z) &= \widehat{\mu}(t^H z) && \text{for every } t > 0, \\ \widehat{\mu}_{s,t}(z) &= \widehat{\rho}_{(t/s)^H}(t^H z) && \text{for every } 0 < s < t, \end{aligned}$$

and $\mu_{0,t} = \mu_t$ for $t > 0$. We have $\mu_t = \mu_s * \mu_{s,t}$ for every $0 \leq s < t$, since

$$\widehat{\mu}(t^H z) = \widehat{\mu}(s^H z) \widehat{\rho}_{(t/s)^H}(t^H z)$$

for every $0 < s < t$. Therefore $\widehat{\mu}_{s,t}(z) = \widehat{\mu}_t(z)/\widehat{\mu}_s(z)$. It follows that, for $0 \leq r < s < t$,

$$\widehat{\mu}_{r,s}(z) \widehat{\mu}_{s,t}(z) = \frac{\widehat{\mu}_s(z) \widehat{\mu}_t(z)}{\widehat{\mu}_r(z) \widehat{\mu}_s(z)} = \frac{\widehat{\mu}_t(z)}{\widehat{\mu}_r(z)} = \widehat{\mu}_{r,t}(z).$$

Thus $\mu_{r,t} = \mu_{r,s} * \mu_{s,t}$ for every $0 \leq r < s < t$. Now Kolmogorov's extension theorem applies and we can construct a process $\{X_t : t \geq 0\}$ such that, for $0 \leq t_0 < t_1 < \dots < t_n$ and $B_0, \dots, B_n \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned} P[X_{t_0} \in B_0, \dots, X_{t_n} \in B_n] &= \int \mu_{t_0}(dx_0) 1_{B_0}(x_0) \int \mu_{t_0, t_1}(dx_1) 1_{B_1}(x_0 + x_1) \int \dots \\ &\quad \dots \int \mu_{t_{n-1}, t_n}(dx_n) 1_{B_n}(x_0 + \dots + x_{n-1} + x_n). \end{aligned}$$

It starts at 0 *a.s.*, it has independent increments, and it is stochastically continuous because $\mu_{s,t} \rightarrow \delta_0$ as $s \uparrow t$ or $t \downarrow s$. Therefore it is an additive process in law. By choosing a modification, it is an additive process ([S] Theorem 11.5).

The distribution of X_1 is μ . We have from the definition of μ_t that

$$X_{at} \stackrel{d}{=} a^H X_t \quad \text{for every } t \geq 0.$$

This implies that $\{X_{at}\}$ and $\{a^H X_t\}$ have a common system of finite-dimensional distributions, since both are additive processes. That is, $\{X_t\}$ is selfsimilar with exponent H . Since $X_t \stackrel{d}{=} t^H X_1$, the distribution of X_t is determined by μ and H . Hence the process $\{X_t\}$ with the properties required is unique up to equivalence in law. ■

Remark 68 *The process $\{X_t\}$ in Theorem 67 (ii) is a Lévy process if and only if μ is strictly α -stable and $H = 1/\alpha$. This follows from Definition 63 and Remark 66.*

Remark 69 *Fix $H > 0$. Given $\mu \in L_0(\mathbb{R}^d)$, $\mu \neq \delta_0$, let $\{X_t\}$ be the H -selfsimilar additive process with $\mathcal{L}(X_1) = \mu$ in Theorem 67 (ii). Let $\{Y_t\}$ be the Lévy process with $\mathcal{L}(Y_1) = \mu$. Let $\widetilde{X}_t = X_{t^{1/(\alpha H)}}$. Then $\{\widetilde{X}_t\}$ and $\{Y_t\}$ have the following relation.*

(i) *If μ is strictly α -stable, then $\{\widetilde{X}_t\} \stackrel{d}{=} \{Y_t\}$. In particular, $\{X_t\} \stackrel{d}{=} \{Y_t\}$ if $\alpha = 1/H$.*

(ii) Let μ be α -stable, and not strictly α -stable. Then $\{X_t\}$ and $\{Y_t\}$ are not identical in law. If $\alpha \neq 1$, then

$$\{\tilde{X}_t\} \stackrel{d}{=} \{Y_t + (t^{1/\alpha} - t)\tau\}, \quad (3.2)$$

where $\tau \neq 0$, and τ is the drift if $0 < \alpha < 1$ and the center if $1 < \alpha \leq 2$. If $\alpha = 1$, then

$$\{\tilde{X}_t\} \stackrel{d}{=} \left\{ Y_t + (t \log t) c \int_S \xi \lambda(d\xi) \right\}, \quad (3.3)$$

where c and λ are those in the expression (1.19) of the Lévy measure ν of μ , and $\int_S \xi \lambda(d\xi) \neq 0$.

Proof. By Proposition 62, $\{\tilde{X}_t\}$ is $(1/\alpha)$ -selfsimilar. Let $\mu_t = \mathcal{L}(\tilde{X}_t)$. Then $\mu_1 = \mu$ and $\hat{\mu}_t(z) = \hat{\mu}(t^{1/\alpha}z)$.

(i) This is Remark 68.

(ii) Let $\alpha \neq 1$. Then, by [S] Theorems 14.1, 14.2, and 14.7, and Remark 14.6, $\hat{\mu}(z) = \hat{\rho}(z)e^{i\langle \tau, z \rangle}$, $\tau \neq 0$, and ρ is strictly α -stable. Hence

$$\hat{\mu}_t(z) = \hat{\rho}(t^{1/\alpha}z)e^{it^{1/\alpha}\langle \tau, z \rangle} = \hat{\rho}(z)^t e^{it^{1/\alpha}\langle \tau, z \rangle} = \hat{\mu}(z)^t e^{i(t^{1/\alpha}-t)\langle \tau, z \rangle},$$

that is, $\tilde{X}_t \stackrel{d}{=} Y_t + (t^{1/\alpha} - t)\tau$.

Let $\alpha = 1$. Then,

$$\hat{\mu}(z) = \exp \left[c \int_S \lambda(d\xi) \int_0^\infty (e^{ir\langle \xi, z \rangle} - 1 - ir\langle \xi, z \rangle 1_{(0,1]}(r)) \frac{dr}{r^2} + i\langle \gamma, z \rangle \right]$$

with $\int_S \xi \lambda(d\xi) \neq 0$ ([S] Theorem 14.7). Hence

$$\hat{\mu}_t(z) = \hat{\mu}(tz) = \exp \left[c \int_S \lambda(d\xi) \int_0^\infty \left(e^{iu\langle \xi, z \rangle} - 1 - iu\langle \xi, z \rangle 1_{(0,1]} \left(\frac{u}{t} \right) \right) \frac{tdu}{u^2} + it\langle \gamma, z \rangle \right].$$

Since

$$1_{(0,1]} \left(\frac{u}{t} \right) = \begin{cases} 1_{(0,1]}(u) - 1_{(t,1]}(u) & \text{for } t < 1 \\ 1_{(0,1]}(u) + 1_{(1,t]}(u) & \text{for } t > 1, \end{cases}$$

we obtain

$$\hat{\mu}_t(z) = \hat{\mu}(z)^t \exp \left[i(t \log t) \left\langle c \int_S \xi \lambda(d\xi), z \right\rangle \right],$$

that is, $\tilde{X}_t \stackrel{d}{=} Y_t + (t \log t) c \int_S \xi \lambda(d\xi)$. Now, we get (3.2) for $\alpha \neq 1$ and (3.3) for $\alpha = 1$, since both sides are additive processes. ■

3.2 Joint distributions and classes L_m

When $\{X_t\}$ is a selfsimilar additive process, the distribution of X_t is selfdecomposable. But its joint distributions (finite-dimensional distributions) are not always selfdecomposable. Let us give conditions for joint distributions of $\{X_t\}$ to be selfdecomposable and, furthermore, conditions for them to belong to classes L_m .

Theorem 70 *Given $\mu \in L_0(\mathbb{R}^d)$, $\mu \neq \delta_0$, and $H > 0$, let $\{X_t\}$ be the H -selfsimilar additive process with $\mathcal{L}(X_1) = \mu$. Let $m \in \{1, 2, \dots, \infty\}$. Then the following six conditions are equivalent.*

- (a) $\mu \in L_m(\mathbb{R}^d)$.
- (b) $\mathcal{L}(X_t) \in L_m(\mathbb{R}^d)$ for all $t \geq 0$.
- (c) $\mathcal{L}((X_{t_k})_{k=1,2}) \in L_{m-1}(\mathbb{R}^{2d})$ for all $t_1, t_2 \geq 0$.
- (d) $\mathcal{L}(c_1 X_{t_1} + c_2 X_{t_2}) \in L_{m-1}(\mathbb{R}^d)$ for all $t_1, t_2 \geq 0$ and $c_1, c_2 \in \mathbb{R}$.
- (e) $\mathcal{L}((X_{t_k})_{1 \leq k \leq n}) \in L_{m-1}(\mathbb{R}^{nd})$ for all $n \in \mathbb{N}$ and $t_1, \dots, t_n \geq 0$.
- (f) $\mathcal{L}(c_1 X_{t_1} + \dots + c_n X_{t_n}) \in L_{m-1}(\mathbb{R}^d)$ for all $n \in \mathbb{N}$, $t_1, \dots, t_n \geq 0$, and $c_1, \dots, c_n \in \mathbb{R}$.

We understand $m - 1 = \infty$ if $m = \infty$.

Proof. (a) and (b) are equivalent, because $\mathcal{L}(X_t) = \mathcal{L}(t^H X_1)$ for $t > 0$ and $\mathcal{L}(X_0) = \delta_0$. Let $0 \leq s < t$ and let $\mu_t = \mathcal{L}(X_t)$ and $\mu_{s,t} = \mathcal{L}(X_t - X_s)$. Then (3.1) shows that $\mu_t \in L_m$ if and only if $\mu_{s,t} \in L_{m-1}$ for $0 < s < t$.

Now let us prove that (b) \Rightarrow (e) \Rightarrow (f) \Rightarrow (d) \Rightarrow (b) and that (e) \Rightarrow (c) \Rightarrow (d).

(b) \Rightarrow (e): $X_t - X_s \in L_{m-1}$ for all $0 \leq s \leq t$. Given $0 = t_0 \leq t_1 \leq \dots \leq t_n$, we see that $X_{t_k} - X_{t_{k-1}}$, $k = 1, \dots, n$, are independent and hence, by Lemma 71 (ii) below, $\mathcal{L}((X_{t_k} - X_{t_{k-1}})_{1 \leq k \leq n}) \in L_{m-1}$. Since $(X_{t_k})_{1 \leq k \leq n}$ is a linear image of $(X_{t_k} - X_{t_{k-1}})_{1 \leq k \leq n}$, $\mathcal{L}((X_{t_k})_{1 \leq k \leq n})$ is in L_{m-1} by Lemma 71 (i).

(e) \Rightarrow (f): Use Lemma 71 (i), since $c_1 X_{t_1} + \dots + c_n X_{t_n}$ is a linear image of $(X_{t_k})_{1 \leq k \leq n}$.

(f) \Rightarrow (d): A special case with $n = 2$.

(d) \Rightarrow (b): For $0 < s < t$, $\mu_{s,t}$ is in L_{m-1} , since $X_t - X_s$ is a special case of $c_1 X_{t_1} + c_2 X_{t_2}$.

(e) \Rightarrow (c): A special case.

(c) \Rightarrow (d): Use Lemma 71 (i) as in the proof that (e) \Rightarrow (f). ■

Lemma 71 *Let $m \in \{0, 1, \dots, \infty\}$.*

(i) *Let X be a random variable on \mathbb{R}^{d_1} and T a linear transformation from \mathbb{R}^{d_1} into \mathbb{R}^{d_2} . If $\mathcal{L}(X) \in L_m(\mathbb{R}^{d_1})$, then $\mathcal{L}(TX) \in L_m(\mathbb{R}^{d_2})$.*

(ii) *Let $d_1, \dots, d_n \in \mathbb{N}$ and $d = d_1 + \dots + d_n$. Let X_j be a random variable on \mathbb{R}^{d_j} for each j . If X_1, \dots, X_n are independent and if $\mathcal{L}(X_j) \in L_m(\mathbb{R}^{d_j})$ for each j , then $\mathcal{L}((X_j)_{1 \leq j \leq n}) \in L_m(\mathbb{R}^d)$.*

Proof. (i) Let $Y = TX$, $\mu = \mathcal{L}(X)$, and $\mu_Y = \mathcal{L}(Y)$. Suppose that $\mu \in L_0$. For any $b > 1$, $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z)$ with some ρ_b . Let T' be the transpose of T . Since $\widehat{\mu}_Y(z) = \widehat{\mu}(T'z)$, we have $\widehat{\mu}_Y(z) = \widehat{\mu}_Y(b^{-1}z)\widehat{\rho}_b(T'z)$. Hence $\mu_Y \in L_0$. This proves (i) for $m = 0$. By induction we can prove it for $m = 1, 2, \dots$. The validity for $m = \infty$ follows from this.

(ii) Let $X = (X_j)_{1 \leq j \leq n}$, $\mu_j = \mathcal{L}(X_j)$, and $\mu = \mathcal{L}(X)$. Assume that X_1, \dots, X_n are independent and $\mu_j \in L_0$ for each j . Then $\widehat{\mu}_j(z) = \widehat{\mu}_j(b^{-1}z)\widehat{\rho}_{j,b}(z)$, $z \in \mathbb{R}^{d_j}$, with some $\rho_{j,b}$. For $z = (z_j)_{1 \leq j \leq d}$ with $z_j \in \mathbb{R}^{d_j}$,

$$\widehat{\mu}(z) = \prod_{j=1}^n \widehat{\mu}_j(z_j) = \prod_{j=1}^n \widehat{\mu}_j(b^{-1}z_j)\widehat{\rho}_{j,b}(z_j) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z),$$

where $\widehat{\rho}_b(z) = \prod_{j=1}^n \widehat{\rho}_{j,b}(z_j)$. Hence (ii) is true for $m = 0$. It is true for $m = 1, 2, \dots$ by induction. Thus (ii) is true also for $m = \infty$. ■

Remark 72 *The situation is quite different for Lévy processes. Let $m \in \{0, 1, \dots, \infty\}$. If $\{Z_t\}$ is a Lévy process on \mathbb{R}^d with $\mathcal{L}(Z_1) \in L_m(\mathbb{R}^d)$, then, for every $n \in \mathbb{N}$ and $t_1, \dots, t_n \geq 0$,*

$$\mathcal{L}((Z_{t_k})_{1 \leq k \leq n}) \in L_m(\mathbb{R}^{nd}).$$

Proof. Assume $\mathcal{L}(Z_1) \in L_m$. It follows from Lemma 8 (iv) that $\mathcal{L}(Z_t) \in L_m$ for all $t \geq 0$. Let $0 = t_0 \leq t_1 \leq \dots \leq t_n$. Then $\mathcal{L}(Z_{t_k} - Z_{t_{k-1}}) = \mathcal{L}(Z_{t_k - t_{k-1}}) \in L_m$. By Lemma 71 (ii), $\mathcal{L}((Z_{t_k} - Z_{t_{k-1}})_{1 \leq k \leq n}) \in L_m$. Hence, by Lemma 71 (i), $\mathcal{L}((Z_{t_k})_{1 \leq k \leq n}) \in L_m$. ■

Notes

This section is based on Sato [61] (1991) and Maejima, Sato, and Watanabe [43] (2000b). See also Section 16 of [S].

Theorem 67 was proved by Sato [61] (1991). Theorem 70 was given by Maejima, Sato, and Watanabe [43] (2000).

Properties of selfsimilar additive processes

Study of selfsimilar additive processes is an unexploited area. So far there are few papers in two directions. One is on path properties in one-dimensional increasing case by Watanabe [89] (1996). The other is on recurrence and transience by Sato and Yamamuro [70] (1998), [71] (2000) and Yamamuro [96] (2000a), [97] (2000b). Interesting sufficient conditions for recurrence or for transience have been discovered, but no necessary and sufficient condition has been found yet.

Properties of L_m related to degenerate linear transformations

The following results are interesting when compared with Lemma 71 (i).

Proposition 73 *Let $d \geq 2$. There is a distribution μ on \mathbb{R}^d having the following two properties.*

(a) $\mu \in ID(\mathbb{R}^d)$ but $\mu \notin L_0(\mathbb{R}^d)$.

(b) If X is a random variable with distribution μ , then, for any linear transformation T from \mathbb{R}^d to $\mathbb{R}^{d'}$ with $1 \leq d' < d$, $\mathcal{L}(TX)$ is in $L_0(\mathbb{R}^{d'})$.

This is by Sato [62] (1998); explicit construction of μ using “signed Lévy measure” is done. Generalization to L_m by Maejima, Suzuki, and Tamura [44] (1999) is as follows.

Proposition 74 *Let $d \geq 2$. Let $m \in \{1, 2, \dots\}$ ($m \neq \infty$). Then there is a distribution μ on \mathbb{R}^d having the following properties.*

(a) $\mu \in L_{m-1}(\mathbb{R}^d)$ but $\mu \notin L_m(\mathbb{R}^d)$.

(b) If X is a random variable with $\mathcal{L}(X) = \mu$, then, for any linear transformation T from \mathbb{R}^d to $\mathbb{R}^{d'}$ with $1 \leq d' < d$, $\mathcal{L}(TX)$ is in $L_m(\mathbb{R}^{d'})$.

Chapter 4

Multivariate subordination

Subordination consists of transforming a stochastic process $\{X_t\}$ to another one through random time change by an increasing Lévy process $\{Z_t\}$, called a subordinator, where $\{X_t\}$ and $\{Z_t\}$ are assumed to be independent. The process $\{Y_t\}$ obtained is said to be subordinate to $\{X_t\}$. Subordination has been an extensively studied area recently.

Subordination of a Lévy process on \mathbb{R}^d is studied in Chapter 6 of [S]. It is well known that in this case subordination provides a Lévy process; it means introducing a new Lévy process $\{Y_t\}$ defined by the composition as $Y(t) = X(Z(t))$. In Section 4.1 we recall the Lévy–Khintchine representation of characteristic functions of subordinators and the expression of the generating triplet of $\{Y_t\}$ in terms of those of $\{X_t\}$ and $\{Z_t\}$. Further, characterization of Lévy processes on a proper cone K is given. They are called K -valued subordinators.

In Section 4.2, the concept of Lévy processes is generalized by replacing the time parameter set $[0, \infty)$ by a proper cone K in \mathbb{R}^N . Thus a K -parameter Lévy process $\{X_s: s \in K\}$ on \mathbb{R}^d is defined to be a stochastic process such that it has independent increments $X_{s^j} - X_{s^{j-1}}$, $j = 1, \dots, n$, when $s^j - s^{j-1} \in K$, $j = 1, \dots, n$, and that it has stationary increment $X_{s^2} - X_{s^1} \stackrel{d}{=} X_{s^2 - s^1}$ when $s^2 - s^1 \in K$, with initial condition $X_0 = 0$ a. s. Certain continuity conditions are added in the definition. The concept of subordination is extended to substitution of s by a K -valued subordinator $\{Z_t\}$. It is proved that $Y(t) = X(Z(t))$ is a Lévy process.

The positive orthant \mathbb{R}_+^N is a proper cone. A deeper study in the case $K = \mathbb{R}_+^N$ is made in Section 4.3. Joint distributions of $\{X_s: s \in K\}$ are examined and the relations of the generating triplets involved in subordination are clarified.

The concepts discussed in this chapter were introduced by the recent work of Barndorff-Nielsen, Pedersen and Sato [4] (2001) in the case $K = \mathbb{R}_+^N$.

4.1 Subordinators and subordination

Basic results on subordination are presented here but their proofs are omitted (they can be found in the book [S]). From now on an increasing Lévy process $\{Z_t: t \geq 0\}$ on \mathbb{R} is called a subordinator. Further, in this section, we consider K -valued subordinators, that is, K -valued (or K -increasing) Lévy processes.

Theorem 75 *A Lévy process $\{Z_t: t \geq 0\}$ on \mathbb{R} with generating triplet (A_Z, ν_Z, γ_Z) is a subordinator if and only if*

$$A_Z = 0, \nu_Z((-\infty, 0)) = 0, \int_{(0,1]} x\nu_Z(dx) < \infty, \text{ and } \gamma_Z - \int_{(0,1]} x\nu_Z(dx) \geq 0, \quad (4.1)$$

where $\gamma_Z - \int_{(0,1]} x\nu_Z(dx) = \gamma_Z^0$, the drift of $\{Z_t\}$. For any $w \in \mathbb{C}$ with $\operatorname{Re} w \leq 0$,

$$E[e^{wZ_t}] = e^{t\Psi(w)}, \quad (4.2)$$

$$\Psi(w) = \gamma_Z^0 w + \int_{(0,\infty)} (e^{ws} - 1)\nu_Z(ds). \quad (4.3)$$

For a proof see [S] Theorem 21.5. Notice that (4.2)–(4.3) represent characteristic function if $w = iz$, $z \in \mathbb{R}$, and Laplace transform if $w = -u$, $u \geq 0$.

Theorem 76 *Let $\{Z_t: t \geq 0\}$ be a subordinator with Lévy measure ν_Z and drift γ_Z^0 . Let $\{X_t: t \geq 0\}$ be a Lévy process on \mathbb{R}^d with generating triplet (A_X, ν_X, γ_X) and $\mu = \mathcal{L}(X_1)$. Assume that $\{Z_t\}$ and $\{X_t\}$ are independent. Define $Y_t = X_{Z_t}$. Then $\{Y_t: t \geq 0\}$ is a Lévy process on \mathbb{R}^d and*

$$Ee^{i\langle z, Y_t \rangle} = e^{t\Psi(\log \hat{\mu}(z))}. \quad (4.4)$$

The generating triplet (A_Y, ν_Y, γ_Y) of $\{Y_t\}$ is as follows:

$$A_Y = \gamma_Z^0 A_X, \quad (4.5)$$

$$\nu_Y(B) = \int_{(0,\infty)} \mu^s(B)\nu_Z(ds) + \gamma_Z^0 \nu_X(B) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \quad (4.6)$$

$$\gamma_Y = \int_{(0,\infty)} \nu_Z(ds) \int_{|x| \leq 1} x\mu^s(dx) + \gamma_Z^0 \gamma_X. \quad (4.7)$$

If $\gamma_Z^0 = 0$ and $\int_{(0,1]} s^{1/2}\nu_Z(ds) < \infty$, then $A_Y = 0$, $\int_{|x| \leq 1} |x|\nu_Y(dx) < \infty$, and the drift of $\{Y_t\}$ is zero.

This is Theorem 30.1 of [S]. The procedure in Theorem 76 of getting $\{Y_t\}$ from $\{Z_t\}$ and $\{X_t\}$ is called (Bochner's) subordination. We say that $\{Y_t\}$ is subordinate to $\{X_t\}$ by $\{Z_t\}$. Sometimes we call $\{X_t\}$ subordinand and $\{Y_t\}$ subordinated.

In the proof of Theorem 76 the following fact is essential.

Lemma 77 *Let $\{X_t\}$ be a Lévy process on \mathbb{R}^d . Then there are constants $C(\varepsilon)$, C_1 , C_2 , C_3 such that*

$$\begin{aligned} P[|X_t| > \varepsilon] &\leq C(\varepsilon)t \quad \text{for } \varepsilon > 0, \\ E[|X_t|^2; |X_t| \leq 1] &\leq C_1t, \\ |E[X_t; |X_t| \leq 1]| &\leq C_2t, \\ E[|X_t|; |X_t| \leq 1] &\leq C_3t^{1/2}. \end{aligned}$$

This is Lemma 30.3 of [S].

Example 78 *Let $\{X_t\}$ be Brownian motion on \mathbb{R}^d and $\{Z_t\}$ a strictly α -stable subordinator, $0 < \alpha < 1$. Then $\{Y_t\}$ is a rotation invariant 2α -stable process.*

Indeed, by Theorem 75 and Remark 23, a nontrivial α -stable subordinator $\{Z_t\}$ has characteristic function

$$Ee^{i\langle z, Z_1 \rangle} = \exp \left[c_1 \int_{(0, \infty)} (e^{izx} - 1)x^{-1-\alpha} dx + i\gamma_0 z \right]$$

with $0 < \alpha < 1$, $\gamma_0 \geq 0$ and $c_1 > 0$. Or, equivalently, we can write

$$Ee^{i\langle z, Z_1 \rangle} = \exp \left[-c|z|^\alpha \left(1 - i \tan \left(\frac{\pi\alpha}{2} \operatorname{sgn}(z) \right) \right) + i\gamma_0 z \right]$$

with $c = c_1\alpha^{-1}\Gamma(1-\alpha)\cos(\pi\alpha/2) > 0$. Thus an α -stable process on \mathbb{R} with parameter (α, β, τ, c) of Definition 14.16 of [S] is a subordinator if and only if $0 < \alpha < 1$, $\beta = 1$, $\tau = \gamma_0 \geq 0$. The function $\Psi(w)$ in (4.3) for $w = -u \leq 0$ is given by

$$\Psi(-u) = -c'u^\alpha - \gamma_0 u$$

with $c' = c_1\alpha^{-1}\Gamma(1-\alpha)$ ([S] Example 24.12). $\{Z_t\}$ is a nontrivial strictly α -stable subordinator if and only if, in addition, $\gamma_0 = 0$.

Now let $\{Z_t\}$ be a nontrivial strictly α -stable subordinator. Then $\Psi(-u) = -c'u^\alpha$. From $\log \widehat{\mu}(z) = -(1/2)|z|^2$, we get

$$Ee^{i\langle z, Y_t \rangle} = \exp \left[-\frac{t}{2^\alpha} c' |z|^{2\alpha} \right]$$

by (4.4). Hence, $\{Y_t\}$ is a rotation invariant 2α -stable process. See [S], Theorem 14.14, for a characterization of a rotation invariant stable distribution.

Example 79 Let $\{X_t\}$ be a Lévy process on \mathbb{R}^d . Let $\{Z_t\}$ be Γ -process with parameter $q > 0$, that is, $\mathcal{L}(Z_1)$ is exponential distribution with parameter q . Then $\mathcal{L}(Y_1) = (1/q)V^q$, where V^q is the q -potential measure of $\{X_t\}$. For any t ,

$$Ee^{i\langle z, Y_t \rangle} = e^{t[-\log(1-q^{-1}\log \widehat{\mu}(z))]} = (1 - q^{-1}\log \widehat{\mu}(z))^{-t}, \quad z \in \mathbb{R}^d.$$

In particular, if $d = 1$ and $\{X_t\}$ is a Poisson process with parameter $c > 0$, then, for each $t > 0$, Y_t has negative binomial distribution with parameters t and $q/(c+q)$.

If $d = 1$ and $\{X_t\}$ is a symmetric α -stable process with $Ee^{izX_t} = e^{-t|z|^\alpha}$, $0 < \alpha \leq 2$, then

$$Ee^{izY_1} = (1 + q^{-1}|z|^\alpha)^{-1}, \quad z \in \mathbb{R}.$$

$\mathcal{L}(Y_1)$ is called Linnik distribution or geometric stable distribution.

Indeed, we have that

$$\Psi(-u) = \int_0^\infty (e^{-ux} - 1) \frac{e^{-qx}}{x} dx = -\log \left(1 + \frac{u}{q} \right), \quad u \geq 0,$$

see Example 25. The definition of subordination, $Y_t = X_{Z_t}$, gives

$$P(Y_t \in B) = \frac{q^t}{\Gamma(t)} \int_0^\infty P(X_s \in B) s^{t-1} e^{-qs} ds.$$

Hence $\mathcal{L}(Y_1) = q^{-1}V_q$.

If $\{X_t\}$ is a Poisson process on \mathbb{R} with parameter c , then $Ee^{-uX_t} = e^{tc(e^{-u}-1)}$ and

$$Ee^{-uY_t} = e^{-t \log(1-q^{-1}c(e^{-u}-1))} = p^t (1 - (1-p)e^{-u})^{-t}, \quad u \geq 0,$$

with $p = q/(c+q)$.

Definition 80 A subset K of \mathbb{R}^N is a cone if it is nonempty, closed, convex, and $K \neq \{0\}$ and if $s \in K$ and $a \geq 0$ imply $as \in K$. K is a proper cone if it is a cone and if no straight line through 0 is contained in K .

Assume, in the following, that K is a proper cone in \mathbb{R}^N . Then it determines a partial order.

Definition 81 Write $s^1 \leq_K s^2$ if $s^2 - s^1 \in K$. A sequence $\{s^n\}_{n=1,2,\dots} \subset \mathbb{R}^N$ is K -increasing if $s^n \leq_K s^{n+1}$ for each n ; K -decreasing if $s^{n+1} \leq_K s^n$ for each n . A mapping f from $[0, \infty)$ into \mathbb{R}^N is K -increasing if $f(t_1) \leq_K f(t_2)$ for $t_1 < t_2$; K -decreasing if $f(t_2) \leq_K f(t_1)$ for $t_1 < t_2$.

Lemma 82 A proper cone K has the following properties.

- (i) If $s^1 \in K$ and $s^2 \in K$, then $s^1 + s^2 \in K$.
- (ii) K does not contain any straight line.
- (iii) There is an $(N - 1)$ -dimensional linear subspace H of \mathbb{R}^N such that, for any $s \in K$, $(s + H) \cap K$ is a bounded set.
- (iv) If $\{s^n\}_{n=1,2,\dots}$ is a K -decreasing sequence in K , then it is convergent.

Proof (incomplete). (i) Notice that $s^1 + s^2 = 2(\frac{1}{2}s^1 + \frac{1}{2}s^2)$.

(ii) Suppose that a straight line $\{s^0 + as^1 : a \in \mathbb{R}\}$, $s^1 \neq 0$, is contained in K . Then $K \ni \frac{1}{n}(s^0 + ns^1) \rightarrow s^1$. Hence $s^1 \in K$. Similarly $-s^1 \in K$, since $K \ni \frac{1}{n}(s^0 - ns^1) \rightarrow -s^1$. Hence K contains the straight line $\{as^1 : a \in \mathbb{R}\}$, contradicting that K is a proper cone.

(iii) Let us admit the fact that there is an $(N - 1)$ -dimensional linear subspace H of \mathbb{R}^N such that $K \cap H = \{0\}$ (this fact is evident if $N = 1$ or 2 or if $K = \mathbb{R}_+^N$; in general case, books in convex analysis (e.g. Rockafellar [55]) will be helpful in giving a proof). We can choose $\gamma \neq 0$ such that $H = \{u : \langle u, \gamma \rangle = 0\}$ and $K \setminus \{0\} \subset \{u : \langle u, \gamma \rangle > 0\}$. We claim that $(s + H) \cap K$ is bounded for any $s \in K$. Suppose that there are $s^n \in (s + H) \cap K$ with $|s^n| \rightarrow \infty$. Then, $\langle s^n, \gamma \rangle > 0$ and $\langle s^n - s, \gamma \rangle = 0$. A subsequence of $|s^n|^{-1}s^n$ tends to some point $u \in K$ with $|u| = 1$. >From $\langle |s^n|^{-1}s^n - |s^n|^{-1}s, \gamma \rangle = 0$ we have $\langle u, \gamma \rangle = 0$, which contradicts that $K \cap H = \{0\}$.

(iv) We use H and γ in the proof of (iii). Let $\{s^n\}_{n=1,2,\dots}$ be a K -decreasing sequence in K . Let $K_1 = \{u : u \in K \text{ and } \langle u - s^1, \gamma \rangle \leq 0\}$. Then K_1 is bounded. Indeed, if there are $u^n \in K_1$ with $|u^n| \rightarrow \infty$, then a limit point v of $|u^n|^{-1}u^n$ satisfies $|v| = 1$, $v \in K$, and $\langle v, \gamma \rangle \leq 0$, which is absurd. Now let us show that $\{s^n\}$ is bounded. If $|s^n| \rightarrow \infty$, then $|s^1 + s^n| \rightarrow \infty$, $|s^1 - s^n| \rightarrow \infty$, and $s^1 + s^n, s^1 - s^n \in K$, and hence, for all large n , $s^1 + s^n \notin K_1$ and $s^1 - s^n \notin K_1$, which means that $\langle s^n, \gamma \rangle > 0$ and $\langle -s^n, \gamma \rangle > 0$, a contradiction. Similarly, if a subsequence $\{s^{n(k)}\}$ of $\{s^n\}$ satisfies $|s^{n(k)}| \rightarrow \infty$, we have a contradiction. Hence $\{s^n\}$ is bounded. If two subsequences $\{s^{n(k)}\}$ and $\{s^{m(l)}\}$ tend to u and v , respectively, then

$v - u \in K$ since $s^{m(l)} - s^{n(k)} \in K$ for $n(k) > m(l)$, and similarly $u - v \in K$, which shows $u = v$ by the properness of the cone K . Therefore $\{s^n\}$ is convergent.

Now let us extend Theorem 75 to higher dimensions.

Theorem 83 *Let $\{Z_t: t \geq 0\}$ be a Lévy process on \mathbb{R}^N with generating triplet (A, ν, γ) . Then the following three conditions are equivalent.*

- (a) *For any fixed $t \geq 0$, $Z_t \in K$ a. s.*
- (b) *Almost surely, $Z_t(\omega)$ is K -increasing in t .*
- (c) *The generating triplet satisfies*

$$A = 0, \nu(\mathbb{R}^N \setminus K) = 0, \int_{|x| \leq 1} |x| \nu(dx) < \infty, \text{ and } \gamma - \int_{|x| \leq 1} x \nu(dx) \in K, \quad (4.8)$$

where $\gamma - \int_{|x| \leq 1} x \nu(dx) = \gamma^0$, the drift of $\{Z_t\}$.

Proof. First, let us check the equivalence of (a) and (b). If (b) holds, then $Z_t = Z_t - Z_0 \in K$ a. s. and (a) holds. If (a) holds, then, for $0 \leq s \leq t$, $P[Z_t - Z_s \in K] = P[Z_{t-s} \in K] = 1$, hence

$$P[Z_t - Z_s \in K \text{ for all } s, t \text{ in } \mathbb{Q} \cap [0, \infty) \text{ with } s \leq t] = 1,$$

and thus (b) holds by right continuity of sample functions and by closedness of K .

Let us show that (c) implies (a). Assume (c). By the Lévy–Itô decomposition of sample functions in Theorem 19.3 of [S],

$$Z_t(\omega) = \lim_{n \rightarrow \infty} \int_{(0, t] \times \{|x| > 1/n\}} x J(d(s, x), \omega) + t\gamma^0 \quad \text{a. s.},$$

where, for $B \in \mathcal{B}((0, \infty) \times (\mathbb{R}^N \setminus \{0\}))$, $J(B, \omega)$ is defined to be the number of s such that $(s, Z_s(\omega) - Z_{s-}(\omega)) \in B$, and $J(B)$ is a Poisson random measure with intensity measure being the product of Lebesgue measure on $(0, \infty)$ with ν . Here we have used $A = 0$ and $\int_{|x| \leq 1} |x| \nu(dx) < \infty$. It follows from $\nu(\mathbb{R}^N \setminus K) = 0$ that

$$E \int_{(0, t] \times (\{|x| > 1/n\} \setminus K)} J(d(s, x)) = t\nu(\{|x| > 1/n\} \setminus K) = 0$$

and hence $\int_{(0, t] \times \{|x| > 1/n\}} x J(d(s, x), \omega)$ is the sum of a finite number of points in K . This, combined with $\gamma^0 \in K$ and with (i) of Lemma 82 and closedness of K , shows that $Z_t \in K$ a. s.

Conversely, assume (b) and let us show (c). We use a part of the general Lévy–Itô decomposition. Since all jumps $Z_s - Z_{s-}$ are in K , we have

$$\nu(\mathbb{R}^N \setminus K) = E [J((0, 1] \times (\mathbb{R}^N \setminus K))] = 0.$$

We deal with ω such that $Z_t(\omega)$ is K -increasing in t and $Z_0(\omega) = 0$, and we omit ω . If $0 \leq s < t$, then $Z_{t-} - Z_s = \lim_{\varepsilon \downarrow 0} Z_{t-\varepsilon} - Z_s \in K$. Hence, if $0 < s_1 < \dots < s_n \leq t$, then

$$Z_t - \sum_{k=1}^n (Z_{s_k} - Z_{s_k-}) = Z_t - Z_{s_n} + \sum_{k=2}^n (Z_{s_k-} - Z_{s_{k-1}}) + Z_{s_1} \in K.$$

Let $Z_t^{(n)} = \int_{(0,t] \times \{|x| > 1/n\}} xJ(d(s, x))$. It is the sum of jumps with size $> 1/n$ up to time t . It follows that $Z_t - Z_t^{(n)} \in K$ and that

$$(Z_t - Z_t^{(n)}) - (Z_t - Z_t^{(n+1)}) = Z_t^{(n+1)} - Z_t^{(n)} \in K,$$

that is, $Z_t - Z_t^{(n)}$ is a K -increasing sequence in K . Hence, by (iv) of Lemma 82, $Z_t - Z_t^{(n)}$ is convergent. Define $Z_t^1 = \lim_{n \rightarrow \infty} Z_t^{(n)}$ and $Z_t^2 = Z_t - Z_t^1$. We see that Z_t^1 and Z_t^2 take values in K . We claim that

$$x^n = \int_{1/n < |x| \leq 1} x\nu(dx) \text{ is convergent as } n \rightarrow \infty. \quad (4.9)$$

Using Proposition 19.5 of [S], we have

$$\begin{aligned} Ee^{i\langle z, Z_t^{(n)} \rangle} &= \exp \left[t \int_{|x| > 1/n} (e^{i\langle z, x \rangle} - 1) \nu(dx) \right] \\ &= \exp \left[t \left(\int_{1/n < |x| \leq 1} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle) \nu(dx) \right. \right. \\ &\quad \left. \left. + \int_{|x| > 1} (e^{i\langle z, x \rangle} - 1) \nu(dx) + i \int_{1/n < |x| \leq 1} \langle z, x \rangle \nu(dx) \right) \right]. \end{aligned}$$

As $n \rightarrow \infty$, $Ee^{i\langle z, Z_t^{(n)} \rangle} \rightarrow Ee^{i\langle z, Z_t^1 \rangle}$ and

$$\exp \left[t \int_{1/n < |x| \leq 1} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle) \nu(dx) \right] \rightarrow \exp \left[t \int_{|x| \leq 1} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle) \nu(dx) \right],$$

both uniformly in z in any compact set. Hence $\exp \left[it \int_{1/n < |x| \leq 1} \langle z, x \rangle \nu(dx) \right]$ is convergent uniformly in z in any compact set. That is, δ_{x^n} is convergent and, equivalently, (4.9). The meaning of (4.9) is that, componentwise, $\int_{1/n < |x| \leq 1} x_j \nu(dx)$ is convergent for $j = 1, \dots, N$. Starting from $Z_t^{(n),j} = \int_{(0,t] \times \{|x| > 1/n, x_j \geq 0\}} x J(d(s, x))$, we can see, in the same way, $\int_{\{1/n < |x| \leq 1, x_j \geq 0\}} x_j \nu(dx)$ is convergent as $n \rightarrow \infty$. Hence $\int_{\{1/n < |x| \leq 1, x_j < 0\}} x_j \nu(dx)$ is also convergent. It follows that $\int_{1/n < |x| \leq 1} |x_j| \nu(dx)$ is convergent. Hence $\int_{|x| \leq 1} |x| \nu(dx) < \infty$. We can now apply Theorem 19.3 of [S] and obtain that Z_t^2 is a Lévy process with triplet $(A, 0, \gamma^0)$. We know that $Z_t^2 \in K$ a.s. If A has rank $m > 0$, then, for $t > 0$, the support of $\mathcal{L}(Z_t^2)$ is an m -dimensional affine subspace of \mathbb{R}^N , which contradicts (ii) of Lemma 82. Hence $A = 0$. It follows that $Z_t^2 = t\gamma^0$ and hence $\gamma^0 \in K$. Thus all assertions in (c) are proved. ■

Definition 84 We call $\{Z_t : t \geq 0\}$ a K -increasing Lévy process, or K -valued Lévy process, or K -valued subordinator, if it satisfies the conditions in Theorem 83.

Example 85 $\mathbb{R}_+^N = [0, \infty)^N$ is a proper cone in \mathbb{R}^N . An \mathbb{R}_+^N -increasing Lévy process is sometimes called an N -variate subordinator.

For any $w = (w_j)_{1 \leq j \leq N}$ and $v = (v_j)_{1 \leq j \leq N}$ in \mathbb{C}^N , we define $\langle w, v \rangle = \sum_{j=1}^N w_j v_j$. This is not the Hermitian inner product.

Remark 86 Let $\{Z_t : t \geq 0\}$ be a K -valued subordinator. Then we have

$$E[e^{\langle w, Z_t \rangle}] = e^{t\Psi(w)} \quad (4.10)$$

with

$$\Psi(w) = \langle \gamma^0, w \rangle + \int_K (e^{\langle w, s \rangle} - 1) \nu(ds) \quad (4.11)$$

for any $w \in \mathbb{C}^N$ satisfying $\operatorname{Re} \langle w, s \rangle \leq 0$ for all $s \in K$. If $\operatorname{Re} \langle w, s \rangle \leq 0$ for $s \in K$, then $|e^{\langle w, s \rangle}| = e^{\operatorname{Re} \langle w, s \rangle} \leq 1$ and both sides of (4.10) are definable. The equality is a special case of Theorem 25.17 of [S].

Example 87 Let $\{B_t^- : t \geq 0\}$ be a negative binomial subordinator with parameter $0 < p < 1$, that is, for $t > 0$,

$$P[B_t^- = n] = p^t \binom{n+t-1}{n} (1-p)^n, \quad n = 0, 1, \dots$$

For each $j = 1, \dots, N$ let $\{X_j(t): t \geq 0\}$ be a Lévy process on \mathbb{R} with $\mathcal{L}(X_j(t))$ being Γ -distribution with parameters λt and α ($\lambda > 0$ and $\alpha > 0$ do not depend on j):

$$P[X_j(t) \in B] = \int_{B \cap (0, \infty)} \frac{\alpha^{\lambda t}}{\Gamma(\lambda t)} x^{\lambda t - 1} e^{-\alpha x} dx, \quad B \in \mathcal{B}(\mathbb{R}).$$

Assume that $\{B_t^-\}, \{X_j(t)\}, \dots, \{X_N(t)\}$ are independent. Define

$$Y_t = (Y_j(t))_{1 \leq j \leq N} = (X_j(t + \lambda^{-1} B_{\lambda t}^-))_{1 \leq j \leq N}.$$

Then $\{Y_t\}$ is an N -variate subordinator whose components are not independent. Each component $\{Y_j(t)\}$ is a Lévy process with $\mathcal{L}(Y_j(t))$ being Γ -distribution with parameters λt and $p\alpha$. Indeed,

$$\begin{aligned} P[Y_j(t) \leq u] &= P[X_j(t + \lambda^{-1} B_{\lambda t}^-) \leq u] \\ &= \sum_{n=0}^{\infty} \int_0^u \frac{\alpha^{\lambda(t + \lambda^{-1}n)}}{\Gamma(\lambda(t + \lambda^{-1}n))} v^{\lambda(t + \lambda^{-1}n) - 1} e^{-\alpha v} dv p^{\lambda t} \binom{n + \lambda t - 1}{n} (1-p)^n \\ &= \int_0^u \left(\alpha^{\lambda t} v^{\lambda t - 1} e^{-\alpha v} p^{\lambda t} \sum_{n=0}^{\infty} \frac{\alpha^n}{\Gamma(\lambda t + n)} v^n (1-p)^n \binom{n + \lambda t - 1}{n} \right) dv \end{aligned}$$

and, since $\binom{n + \lambda t - 1}{n} = \frac{\Gamma(n + \lambda t)}{n! \Gamma(\lambda t)}$, we get

$$P[Y_j(t) \leq u] = \int_0^u \frac{(\alpha p)^{\lambda t}}{\Gamma(\lambda t)} v^{\lambda t - 1} e^{-\alpha p v} dv.$$

If $N = 2$, then, for each $t > 0$, we can find the distribution of Y_t has density

$$C_t (y_1 y_2)^{(\lambda t - 1)/2} e^{-\alpha(y_1 + y_2)} I_{\lambda t - 1} \left(2\alpha \sqrt{(1-p)y_1 y_2} \right)$$

on \mathbb{R}_+^2 , where C_t is a positive constant depending on t .

4.2 Subordination of cone-parameter Lévy processes

Proper cones are multidimensional analogues of $[0, \infty)$. We extend, in a natural way, the concept of a Lévy process to a process with parameter set being a proper cone.

Let K be a proper cone in \mathbb{R}^N .

Definition 88 Let f be a mapping from K into \mathbb{R}^d .

(i) Let $s^0 \in K$. We say that f is K -right continuous at s^0 , if, for every K -decreasing sequence $\{s^n\}_{n=1,2,\dots}$ in K with $|s^n - s^0| \rightarrow 0$, we have $|f(s^n) - f(s^0)| \rightarrow 0$. We say that f is K -right continuous if f is K -right continuous at every $s^0 \in K$.

(ii) Let $s^0 \in K \setminus \{0\}$. We say that f has K -left limit at s^0 , if, for every K -increasing sequence $\{s^n\}_{n=1,2,\dots}$ in $K \setminus \{s^0\}$ satisfying $|s^n - s^0| \rightarrow 0$, $\lim_{n \rightarrow \infty} f(s^n)$ exists in \mathbb{R}^d . We say f has K -left limits if it has K -left limit at every $s^0 \in K \setminus \{0\}$.

Remark 89 We should keep in mind that, if f has K -left limit at s^0 , $\lim_{n \rightarrow \infty} f(s^n)$ may depend on the choice of the sequence $\{s^n\}$. For example, if $K = \mathbb{R}_+^2$, $s^0 = (s_j^0)_{j=1,2} \in \mathbb{R}_+^2 \setminus \{0\}$, and $f(s) = f_1(s_1) + f_2(s_2)$ for $s = (s_j)_{j=1,2}$ and if, for each j , $f_j(s_j)$ is a step function with a jump at s_j^0 , then, for an \mathbb{R}_+^2 -increasing sequence $s^n = (s_j^n)_{j=1,2}$ in $\mathbb{R}_+^2 \setminus \{s^0\}$ satisfying $|s^n - s^0| \rightarrow 0$,

$$\lim_{n \rightarrow \infty} f(s^n) = \begin{cases} f_1(s_1^0-) + f_2(s_2^0-) & \text{if } s_j^n < s_j^0 \text{ for } j = 1, 2 \text{ for all } n, \\ f_1(s_1^0-) + f_2(s_2^0) & \text{if } s_1^n < s_1^0, s_2^n = s_2^0 \text{ for all } n, \\ f_1(s_1^0) + f_2(s_2^0-) & \text{if } s_1^n = s_1^0, s_2^n < s_2^0 \text{ for all } n. \end{cases}$$

Definition 90 A K -parameter Lévy process $\{X_s: s \in K\}$ on \mathbb{R}^d is a collection of random variables on \mathbb{R}^d satisfying the following conditions.

(a) If $n \geq 3$, $s^1, \dots, s^n \in K$, and $s^k \leq_K s^{k+1}$ for $k = 1, \dots, n-1$, then $X_{s^{k+1}} - X_{s^k}$, $k = 1, \dots, n-1$, are independent.

(b) If $s^1, \dots, s^4 \in K$ and $s^2 - s^1 = s^4 - s^3 \in K$, then $X_{s^2} - X_{s^1} \stackrel{d}{=} X_{s^4} - X_{s^3}$.

(c) For each $s \in K$, $X_{s'} \rightarrow X_s$ in probability as $|s' - s| \rightarrow 0$ with $s' \in K$.

(d) $X_0 = 0$ a. s.

(e) Almost surely, $X_s(\omega)$ is K -right continuous with K -left limits in s .

Lemma 91 Let $\{X_s: s \in K\}$ be a K -parameter Lévy process on \mathbb{R}^d and let $\mu_s = \mathcal{L}(X_s)$. Then the following are true.

(i) $\mu_{s^1+s^2} = \mu_{s^1} * \mu_{s^2}$ for all $s^1, s^2 \in K$.

(ii) $\{X_{ts^0}: t \geq 0\}$ is a Lévy process on \mathbb{R}^d for any $s^0 \in K$.

(iii) μ_s is infinitely divisible for all $s \in K$.

Proof. (i) $\mathcal{L}(X_{s^1} + X_{s^2}) = \mathcal{L}((X_{s^1+s^2} - X_{s^2}) + X_{s^2}) = \mathcal{L}(X_{s^1+s^2} - X_{s^2}) * \mathcal{L}(X_{s^2}) = \mathcal{L}(X_{s^1}) * \mathcal{L}(X_{s^2})$, since $X_{s^1+s^2} - X_{s^2}$ and X_{s^2} are independent and $X_{s^1+s^2} - X_{s^2}$ and X_{s^1} have the same distribution.

(ii) Fix $s^0 \in K$. If $0 \leq t_1 < \dots < t_n$ with $n \geq 3$, then $X_{t_{j+1}s^0} - X_{t_j s^0}$, $j = 1, \dots, n-1$, are independent. If $0 \leq s < t$, then $X_{ts^0} - X_{ss^0} \stackrel{d}{=} X_{(t-s)s^0}$. Note that $\lim_{t' \rightarrow t} P[|X_{t's^0} - X_{ts^0}| > \varepsilon] = 0$

and $X_{0s^0} = X_0 = 0$ a. s. Finally, almost surely X_{ts^0} is right continuous with left limits in t from the property (e).

(iii) Fix $s^0 \in K$. Then μ_{s^0} is the distribution at time 1 of the Lévy process $\{X_{ts^0}: t \geq 0\}$ in (ii). Hence $\mu_{s^0} \in ID$. ■

Now let us give an analogue of the first half of Theorem 76 on subordination.

Theorem 92 *Let $\{Z_t: t \geq 0\}$ be a K -valued subordinator and $\{X_s: s \in K\}$ a K -parameter Lévy process on \mathbb{R}^d . Suppose that they are independent. Define $Y_t = X_{Z_t}$. Then $\{Y_t: t \geq 0\}$ is a Lévy process on \mathbb{R}^d .*

Proof. Let $n \geq 2$ and f^1, \dots, f^{n-1} be measurable and bounded from \mathbb{R}^d to \mathbb{R} , and let $0 \leq t_1 \leq \dots \leq t_n$. Let $s^k \in K$, $k = 1, \dots, n$, with $s^1 \leq_K s^2 \leq_K \dots \leq_K s^n$ and let

$$G(s^1, \dots, s^n) = E \left[\prod_{k=1}^{n-1} f^k(X_{s^{k+1}} - X_{s^k}) \right].$$

Since $X_{s^{k+1}} - X_{s^k}$, $k = 1, \dots, n-1$, are independent, we have

$$G(s^1, \dots, s^n) = \prod_{k=1}^{n-1} E[f^k(X_{s^{k+1}} - X_{s^k})].$$

Next, let $g^k(s) = E[f^k(X_s)]$ for $s \in K$. Since $X_{s^{k+1}} - X_{s^k} \stackrel{d}{=} X_{s^{k+1}-s^k}$, we have

$$E[f^k(X_{s^{k+1}} - X_{s^k})] = g^k(s^{k+1} - s^k).$$

It follows that

$$G(s^1, \dots, s^n) = \prod_{k=1}^{n-1} g^k(s^{k+1} - s^k).$$

We use the standard argument for independence (based on Proposition 1.16 of [S]). As $\{X_s\}$ and $\{Z_t\}$ are independent, we obtain

$$E \left[\prod_{k=1}^{n-1} f^k(Y_{t_{k+1}} - Y_{t_k}) \right] = E[G(Z_{t_1}, \dots, Z_{t_n})] = \prod_{k=1}^{n-1} E[g^k(Z_{t_{k+1}} - Z_{t_k})], \quad (4.12)$$

noting that Z_{t_1}, \dots, Z_{t_n} make a K -increasing sequence. Choosing $f^j = 1$ for all $j \neq k$ and using that $\{Z_t\}$ has stationary increments, we see that

$$E[f^k(Y_{t_{k+1}} - Y_{t_k})] = E[g^k(Z_{t_{k+1}} - Z_{t_k})] = E[g^k(Z_{t_{k+1}-t_k})] = E[f^k(Y_{t_{k+1}-t_k})]. \quad (4.13)$$

Equations (4.12) and (4.13) say that $\{Y_t\}$ has independent increments and stationary increments, respectively. Evidently $Y_0 = 0$ a. s. Since Z_t is right continuous with left limits in t and since $\{X_t\}$ has property (e) in Definition 90, Y_t is right continuous with left limits in t a. s. Here notice that, when $t_n < t$ and $t_n \uparrow t$, we have $Z_{t_n} \leq_K Z_{t_{n+1}}$ and $Z_{t_n} \rightarrow Z_{t-}$, but Z_{t_n} can be equal to Z_{t-} ; even if $Z_{t_n} = Z_{t-}$ for large n , $Y_{t_n} = X(Z_{t_n})$ is convergent. Now $\mathcal{L}(Y_t - Y_s) = \mathcal{L}(Y_{t-s}) \rightarrow \delta_0$ as $t \downarrow s$ or $s \uparrow t$, which shows stochastic continuity of $\{Y_t\}$. Thus $\{Y_t\}$ is a Lévy process on \mathbb{R}^d . ■

Definition 93 *We call this procedure to get $\{Y_t\}$ from $\{X_s\}$ and $\{Z_t\}$ multivariate subordination if $N \geq 2$.*

Multivariate subordination in Theorem 92 shows that our definition of K -parameter Lévy processes is harmonious with the notion of K -valued subordinators.

The following lemma is useful in considering examples.

Lemma 94 *Let $\{X_s^1: s \in K\}, \dots, \{X_s^n: s \in K\}$ be independent K -parameter Lévy processes on \mathbb{R}^d . Let*

$$X_s = X_s^1 + \dots + X_s^n.$$

Then $\{X_s: s \in K\}$ is a K -parameter Lévy process on \mathbb{R}^d .

Proof. It is straightforward to check the defining properties for a K -parameter Lévy process. ■

4.3 The case $K = \mathbb{R}_+^N$

In this section we assume $K = \mathbb{R}_+^N$. Denote the unit vectors $e^k = (\delta_{kj})_{1 \leq j \leq N}$ for $k = 1, \dots, N$, where $\delta_{kj} = 1$ or 0 according as $k = j$ or not. This cone is nicer than the general cone, as the vectors e^k play a special role and the partial order $s^1 \leq_K s^2$ is equivalent to componentwise order $s_j^1 \leq s_j^2, j = 1, \dots, N$, for $s^k = (s_j^k)_{1 \leq j \leq N} = s_1^k e^1 + \dots + s_N^k e^N, k = 1, 2$. First we give various examples of K -parameter Lévy processes. Then joint distributions of K -parameter Lévy processes are considered. Further, generating triplets appearing in multivariate subordination are described.

Example 95 Let $\{V_t: t \geq 0\}$ be a Lévy process on \mathbb{R}^d . Fix $c = (c_j)_{1 \leq j \leq N} \in K$. Define

$$X_s = V_{\langle c, s \rangle} = V_{c_1 s_1 + \dots + c_N s_N} \quad \text{for } s = (s_j)_{1 \leq j \leq N} \in K. \quad (4.14)$$

Then, $\{X_s: s \in K\}$ is a K -parameter Lévy process.

Indeed, let $s^1 \leq_K s^2 \leq_K \dots \leq_K s^n$ with $s^1, \dots, s^n \in K$. Then $X_{s^{k+1}} - X_{s^k} = V_{\langle c, s^{k+1} \rangle} - V_{\langle c, s^k \rangle}$, $k = 1, \dots, n$, are independent, since $\langle c, u^{k+1} \rangle - \langle c, s^k \rangle \geq 0$. If $s^1 \leq_K s^2$ and $s^3 \leq_K s^4$ such that $s^2 - s^1 = s^4 - s^3$, then $X_{s^2} - X_{s^1} = V_{\langle c, s^2 \rangle} - V_{\langle c, s^1 \rangle} \stackrel{d}{=} V_{\langle c, s^2 - s^1 \rangle} = V_{\langle c, s^4 - s^3 \rangle} \stackrel{d}{=} X_{s^4} - X_{s^3}$. If $s' \in K$ and $s' \rightarrow s$, then $X_{s'} = V_{\langle c, s' \rangle} \rightarrow V_{\langle c, s \rangle} = X_s$ in probability. If $\{s^k\}_{k \geq 1}$ is a K -decreasing sequence converging to $s \in K$, then $|X_{s^k} - X_s| = |V_{\langle c, s^k - s \rangle}| \rightarrow 0$, since $\langle c, s^k - s \rangle \rightarrow 0$. If $\{s^k\}_{k \geq 1}$ is K -increasing, $s^k \neq s$, and $s^k \rightarrow s$, then $\langle c, s^k \rangle \leq \langle c, s^{k+1} \rangle$, $\langle c, s^k \rangle \leq \langle c, s \rangle$ and $\langle c, s^k \rangle \rightarrow \langle c, s \rangle$, and hence $X_{s^k} = V_{\langle c, s^k \rangle}$ is convergent to $V_{\langle c, s \rangle -}$ or $V_{\langle c, s \rangle}$. Thus X_s is K -right continuous with K -left limits a. s. Finally $X_0 = V_{\langle c, 0 \rangle} = 0$ a. s.

Example 96 Let $\{V_t^j: t \geq 0\}$, $j = 1, \dots, N$, be independent Lévy processes on \mathbb{R}^d . Define

$$V_s = V_{s_1}^1 + V_{s_2}^2 + \dots + V_{s_N}^N \quad \text{for } s = (s_j)_{1 \leq j \leq N} \in K. \quad (4.15)$$

Then $\{V_s: s \in K\}$ is a K -parameter Lévy process on \mathbb{R}^d .

Indeed, for each j , $\{V_{s_j}^j: s \in K\}$ is a K -parameter Lévy process, as it is a special case of Example 95 with $c = (\delta_{jk})_{1 \leq k \leq N}$. Hence $\{V_s: s \in K\}$ is a K -parameter Lévy process by Lemma 94.

Example 97 For each $j = 1, \dots, N$, let $\{U_t^j: t \geq 0\}$ be a Lévy process on \mathbb{R}^{d_j} . Assume that they are independent. Let $d = d_1 + \dots + d_N$. Define

$$U_s = (U_{s_j}^j)_{1 \leq j \leq N} \quad \text{for } s = (s_j)_{1 \leq j \leq N} \in K, \quad (4.16)$$

that is, U_s is the direct product of $U_{s_j}^j$, $j = 1, \dots, N$. Then $\{U_s: s \in K\}$ is a K -parameter Lévy process on \mathbb{R}^d .

Indeed, for each k , let $\{X_s^k: s \in K\}$ be the process defined as $X_s^k = (X_{s_j}^{k,j})_{1 \leq j \leq N}$ for $s = (s_j)_{1 \leq j \leq N}$ with $X_{s_j}^{k,j} = 0$ in \mathbb{R}^{d_j} for $j \neq k$ and $X_{s_k}^{k,k} = U_{s_k}^k$. Then $\{X_s^k: s \in K\}$ is a K -parameter Lévy process on \mathbb{R}^d just by the same proof as each term of (4.15) in Example 96. Then $\{X_s^k\}$, $k = 1, \dots, N$, are independent, $U_s = X_s^1 + \dots + X_s^N$, and Lemma 94 applies.

Theorem 98 Let $\{X_s: s \in K\}$ be a K -parameter Lévy process on \mathbb{R}^d . Define $X_t^j = X_{te^j}$ and let $\{V_t^j: t \geq 0\}$, $j = 1, \dots, N$, be independent Lévy processes such that $\{V_t^j\} \stackrel{d}{=} \{X_t^j\}$ for each j . Define $\{V_s: s \in K\}$ by (4.15). Then, for every $n \in \mathbb{N}$ and $s^1, \dots, s^n \in K$ satisfying

$$s^1 \leq_K s^2 \leq_K \dots \leq_K s^n, \quad (4.17)$$

we have

$$(X_{s^k})_{1 \leq k \leq n} \stackrel{d}{=} (V_{s^k})_{1 \leq k \leq n}. \quad (4.18)$$

Proof. We claim that

$$X_s \stackrel{d}{=} V_s \quad \text{for } s \in K. \quad (4.19)$$

Indeed for $s = (s_j)_{1 \leq j \leq N} = s_1 e^1 + \dots + s_N e^N \in K$,

$$X_s = X_{s_1 e^1} + (X_{s_1 e^1 + s_2 e^2} - X_{s_1 e^1}) + \dots + (X_s - X_{s_1 e^1 + \dots + s_{N-1} e^{N-1}}).$$

The right-hand side is the sum of N independent terms by condition (a) in Definition 90 of a K -parameter Lévy process. Further, by condition (b)

$$\begin{aligned} X_{s_1 e^1} &= X_{s_1}^1 \stackrel{d}{=} V_{s_1}^1, \\ X_{s_1 e^1 + s_2 e^2} - X_{s_1 e^1} &\stackrel{d}{=} X_{s_2 e^2}^2 = X_{s_2}^2 \stackrel{d}{=} V_{s_2}^2, \end{aligned}$$

and so on. Hence we obtain (4.19) by (4.15). Now we claim that (4.18) holds for all $s^1, \dots, s^n \in K$ which satisfy (4.17). In order to prove this, it is enough to prove

$$(X_{s^k} - X_{s^{k-1}})_{1 \leq k \leq n} \stackrel{d}{=} (V_{s^k} - V_{s^{k-1}})_{1 \leq k \leq n}, \quad (4.20)$$

where $s^0 = 0$, since

$$(X_{s^k})_{1 \leq k \leq n} = T \left((X_{s^k} - X_{s^{k-1}})_{1 \leq k \leq n} \right), \quad (V_{s^k})_{1 \leq k \leq n} = T \left((V_{s^k} - V_{s^{k-1}})_{1 \leq k \leq n} \right)$$

with an $n \times n$ matrix T . Since the components of each side of (4.20) are independent and since

$$X_{s^k} - X_{s^{k-1}} \stackrel{d}{=} X_{s^k - s^{k-1}} \stackrel{d}{=} V_{s^k - s^{k-1}} \stackrel{d}{=} V_{s^k} - V_{s^{k-1}}$$

by virtue of (4.19), we have (4.20). Hence (4.18) holds. ■

Corollary 99 Let $\{X_s: s \in K\}$ be a K -parameter Lévy process on \mathbb{R}^d and define $X_t^j = X_{te^j}$. Then

$$Ee^{i\langle z, X_s \rangle} = \prod_{j=1}^N Ee^{i\langle z, X_{s_j}^j \rangle} \quad \text{for } s = (s_j)_{1 \leq j \leq N}, \quad z \in \mathbb{R}^d. \quad (4.21)$$

Proof. This is an expression of (4.18) when $n = 1$. ■

Remark 100 The theorem above tells us that joint distributions $\mathcal{L}((X_{s_k})_{1 \leq k \leq n})$ of a K -parameter Lévy process are determined by the distributions of X_{e^j} for $j = 1, \dots, N$, as long as (4.17) is satisfied. In particular, $\mathcal{L}(X_s)$ is determined for each s . However, general joint distributions are not determined by the distributions of X_{e^j} , $j = 1, \dots, N$. For example, suppose that $X_s = W_{s_1 + \dots + s_N}$ for $s = (s_j)_{1 \leq j \leq N} \in K$ with a Lévy process $\{W_t: t \geq 0\}$ as in Example 95 with $c_j = 1$. Then $X_{e^1} = X_{e^2} = \dots = X_{e^N}$ while $V_{e^1}, V_{e^2}, \dots, V_{e^N}$ are independent. Thus the distribution of $(X_{e^j})_{1 \leq j \leq N}$ and that of $(V_{e^j})_{1 \leq j \leq N}$ are different except in the trivial case.

Let us give description of generating triplets in multivariate subordination.

Theorem 101 Let $\{Y_t: t \geq 0\}$ be a Lévy process on \mathbb{R}^d obtained by multivariate subordination from a K -parameter Lévy process $\{X_s: s \in K\}$ on \mathbb{R}^d and K -valued subordinator $\{Z_t: t \geq 0\}$ as in Theorem 92. Let $X_t^j = X_{te^j}$.

(i) The characteristic function of Y_t is as follows:

$$Ee^{i\langle z, Y_t \rangle} = e^{t\Psi_Z(\psi_X(z))}, \quad z \in \mathbb{R}^d, \quad (4.22)$$

where Ψ_Z is the function Ψ of (4.11) in Remark 86 and

$$\psi_X(z) = (\psi_X^j(z))_{1 \leq j \leq N}, \quad \psi_X^j(z) = \log Ee^{i\langle z, X_1^j \rangle}. \quad (4.23)$$

(ii) Let ν_Z and $\gamma_Z^0 = (\gamma_{Z,j}^0)_{1 \leq j \leq N}$ be the Lévy measure and the drift of $\{Z_t\}$ and let $(A_X^j, \nu_X^j, \gamma_X^j)$ be the generating triplet of $\{X_t^j\}$. Let $\mu_s = \mathcal{L}(X_s)$. Then the generating triplet (A_Y, ν_Y, γ_Y) of $\{Y_t\}$ is as follows:

$$A_Y = \sum_{j=1}^N \gamma_{Z,j}^0 A_X^j, \quad (4.24)$$

$$\nu_Y(B) = \int_{\mathbb{R}_+^N} \mu_s(B) \nu_Z(ds) + \sum_{j=1}^N \gamma_{Z,j}^0 \nu_X^j(B), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \quad (4.25)$$

$$\gamma_Y = \int_{\mathbb{R}_+^N} \nu_Z(ds) \int_{|x| \leq 1} x \mu_s(dx) + \sum_{j=1}^N \gamma_{Z,j}^0 \gamma_X^j. \quad (4.26)$$

(iii) If $\int_{|s|\leq 1} |s|^{1/2} \nu_Z(ds) < \infty$ and $\gamma_Z^0 = 0$, then $A_Y = 0$, $\int_{|x|\leq 1} |x| \nu_Y(dx) < \infty$, and the drift γ_Y^0 of $\{Y_t\}$ is zero.

Proof. (i) Let $\{V_t^j : t \geq 0\}$, $j = 1, 2, \dots, N$, and $\{V_s : s \in K\}$ be the processes defined in Theorem 98. Then, by (4.21) of Corollary 99,

$$Ee^{i\langle z, X_s \rangle} = \prod_{j=1}^N Ee^{i\langle z, X_{s_j}^j \rangle} = \prod_{j=1}^N e^{s_j \psi_X^j(z)} = e^{\langle s, \psi_X(z) \rangle} \quad (4.27)$$

for $z \in \mathbb{R}^d$ and $s \in K$. Use the standard argument for independence (based on Proposition 1.16 of [S]). We get

$$Ee^{i\langle z, Y_t \rangle} = E \left[\left(E \left[e^{i\langle z, X_s \rangle} \right] \right)_{s=Z_t} \right] = Ee^{i\langle Z_t, \psi_X(z) \rangle} = e^{t\Psi_Z(\psi_X(z))}$$

for $z \in \mathbb{R}^d$ by (4.10), since $\operatorname{Re} \langle \psi_X(z), s \rangle = \sum_{j=1}^N (\operatorname{Re} \psi_X^j(z)) s_j \leq 0$. This is (4.22).

(ii) Let $z \in \mathbb{R}^d$. We have

$$Ee^{i\langle z, Y_t \rangle} = e^{t\Psi_Z(\psi_X(z))} = \exp \left[t \left(\langle \gamma_Z^0, \psi_X(z) \rangle + \int_K (e^{\langle \psi_X(z), s \rangle} - 1) \nu_Z(ds) \right) \right]$$

by (4.11) since $\operatorname{Re} \langle \psi_X(z), s \rangle \leq 0$. Notice that

$$\langle \gamma_Z^0, \psi_X(z) \rangle = \sum_{j=1}^N \gamma_{Z,j}^0 \psi_X^j(z) = \sum_{j=1}^N \gamma_{Z,j}^0 \left(-\frac{1}{2} \langle z, A_X^j z \rangle + i \langle \gamma_X^j, z \rangle + \int_{\mathbb{R}^d} g(z, x) \nu_X^j(dx) \right)$$

with $g(z, x) = e^{i\langle z, x \rangle} - 1 - i \langle z, x \rangle 1_{\{|x|\leq 1\}}(x)$. Hence

$$\langle \gamma_Z^0, \psi_X(z) \rangle = -\frac{1}{2} \left\langle z, \sum_{j=1}^N \gamma_{Z,j}^0 A_X^j z \right\rangle + i \left\langle \sum_{j=1}^N \gamma_{Z,j}^0 \gamma_X^j, z \right\rangle + \int_{\mathbb{R}^d} g(z, x) \left(\sum_{j=1}^N \gamma_{Z,j}^0 \nu_X^j \right) (dx).$$

Next it follows from (4.27) that

$$\begin{aligned} \int_K (e^{\langle \psi_X(z), s \rangle} - 1) \nu_Z(ds) &= \int_K (Ee^{i\langle z, X_s \rangle} - 1) \nu_Z(ds) = \int_K \nu_Z(ds) \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1) \mu_s(dx) \\ &= \int_K \nu_Z(ds) \int_{\mathbb{R}^d} g(z, x) \mu_s(dx) + i \int_K \nu_Z(ds) \left\langle z, \int_{|x|\leq 1} x \mu_s(dx) \right\rangle. \end{aligned}$$

Here we used (4.30) of Lemma 102 below and $\int_{|s| \leq 1} |s| \nu_Z(ds) < \infty$. Define $\tilde{\nu}$ by $\tilde{\nu}(B) = \int_K \mu_s(B \setminus \{0\}) \nu_Z(ds)$, $B \in \mathcal{B}(\mathbb{R}^d)$. Then, by (4.28) and (4.29) of Lemma 102,

$$\begin{aligned} \int_{|x| \leq 1} |x|^2 \tilde{\nu}(dx) &\leq C_1 \int_K |s| \nu_Z(ds) < \infty, \\ \int_{|x| > 1} \tilde{\nu}(dx) &\leq C(1) \int_K |s| \nu_Z(ds) < \infty. \end{aligned}$$

Hence

$$\int_K (e^{\langle \psi_x(z), s \rangle} - 1) \nu_Z(ds) = \int_{\mathbb{R}^d} g(z, x) \tilde{\nu}(dx) + i \left\langle \int_K \nu_Z(ds) \int_{|x| \leq 1} x \mu_s(dx), z \right\rangle.$$

Thus we get (4.24), (4.25), and (4.26).

(iii) Assume $\int_{|s| \leq 1} |s|^{1/2} \nu_Z(ds) < \infty$ and $\gamma_Z^0 = 0$. Then $A_Y = 0$ by (4.24),

$$\int_{|x| \leq 1} |x| \nu_Y(dx) = \int_K \nu_Z(ds) \int_{|x| \leq 1} |x| \mu_s(dx) < \infty$$

by (4.25) of Lemma 102, and

$$\gamma_Y^0 = \gamma_Y - \int_{|x| \leq 1} x \nu_Y(dx) = \int_K \nu_Z(ds) \int_{|x| \leq 1} x \mu_s(dx) - \int_{|x| \leq 1} x \nu_Y(dx) = 0$$

by (4.26) and (4.25). ■

Lemma 102 *Let $\{X_s : s \in K\}$ be a K -parameter Lévy process on \mathbb{R}^d . Then, there are constants $C(\varepsilon)$, C_1 , C_2 , C_3 such that*

$$P[|X_s| > \varepsilon] \leq C(\varepsilon)|s| \quad \text{for } \varepsilon > 0, \quad (4.28)$$

$$E[|X_s|^2; |X_s| \leq 1] \leq C_1|s|, \quad (4.29)$$

$$|E[X_s; |X_s| \leq 1]| \leq C_2|s|, \quad (4.30)$$

$$E[|X_s|; |X_s| \leq 1] \leq C_3|s|^{1/2}. \quad (4.31)$$

Proof. We use Theorem 98. Since $X_s \stackrel{d}{=} V_s$, it is enough to show the estimates for V_s . Notice that $\sum_{j=1}^N |s_j| \leq \text{const} |s|$. Proof of (4.28) and (4.29) is as follows:

$$\begin{aligned} P[|V_s| > \varepsilon] &= P\left[\left|\sum_{j=1}^N V_{s_j}^j\right| > \varepsilon\right] \leq P\left[|V_{s_j}^j| > \varepsilon/N \text{ for some } j\right] \\ &\leq \sum_{j=1}^N P\left[|V_{s_j}^j| > \varepsilon/N\right] \leq \text{const} \sum_{j=1}^N s_j, \\ E[|V_s|^2; |V_s| \leq 1] &\leq E\left[\left|\sum_{j=1}^N V_{s_j}^j\right|^2; |V_{s_j}^j| \leq 1 \text{ for all } j\right] + P\left[|V_{s_j}^j| > 1 \text{ for some } j\right] \\ &\leq N^2 \sum_{j=1}^N E\left[|V_{s_j}^j|^2; |V_{s_j}^j| \leq 1\right] + \sum_{j=1}^N P\left[|V_{s_j}^j| > 1\right] \leq \text{const} \sum_{j=1}^N s_j. \end{aligned}$$

Here we have used Lemma 77 for $V_{s_j}^j$. To prove (4.30), we denote the k th component by putting the superscript (k) . We have

$$|E[V_s; |V_s| \leq 1]| \leq \sum_{k=1}^N \left|E\left[iV_s^{(k)}; |V_s| \leq 1\right]\right| = \sum_{k=1}^N |I_{k1} + I_{k2} + I_{k3}|,$$

where

$$\begin{aligned} I_{k1} &= E\left[e^{iV_s^{(k)}} - 1\right], & I_{k2} &= -E\left[e^{iV_s^{(k)}} - 1; |V_s| > 1\right], \\ I_{k3} &= -E\left[e^{iV_s^{(k)}} - 1 - iV_s^{(k)}; |V_s| \leq 1\right]. \end{aligned}$$

We have

$$\begin{aligned} I_{k1} &= E\left[e^{i\sum_{j=1}^N V_{s_j}^{j(k)}} - 1\right] \\ &= E\left[e^{i\sum_{j=1}^N V_{s_j}^{j(k)}} - e^{i\sum_{j=1}^{N-1} V_{s_j}^{j(k)}}\right] + \dots + E\left[e^{iV_{s_1}^{1(k)}} - 1\right] \end{aligned}$$

and hence

$$|I_{k1}| \leq \sum_{j=1}^N \left|E\left[e^{iV_{s_j}^{j(k)}} - 1\right]\right| = \sum_{j=1}^N \left|\left(E\left[e^{iV_{s_1}^{j(k)}}\right]\right)^{s_j} - 1\right| \leq \text{const} \sum_{j=1}^N s_j.$$

As we have

$$\begin{aligned} |I_{k2}| &\leq 2P[|V_s| > 1] \leq 2C(1)|s|, \\ |I_{k3}| &\leq \frac{1}{2}E[(V_s^{(k)})^2; |V_s| \leq 1] \leq \frac{1}{2}E[|V_s|^2; |V_s| \leq 1] \leq \frac{1}{2}C_2|s| \end{aligned}$$

by (4.28) and (4.29), we now obtain (4.30). Finally

$$E[|V_s|; |V_s| \leq 1] \leq \left(E[|V_s|^2; |V_s| \leq 1]\right)^{1/2} \leq C_2^{1/2}|s|^{1/2}$$

by Schwarz's inequality. ■

Remark 103 *Theorem 101 shows that the distribution of $\{Y_t\}$ subordinate to $\{X_s\}$ by $\{Z_t\}$ is determined by the distributions of $\{X_t^1\}, \dots, \{X_t^N\}$, and $\{Z_t\}$, although the joint distributions of $\{X_s\}$ are not determined by $\{X_t^1\}, \dots, \{X_t^N\}$ as Remark 100 says. This is because relevant joint distributions of $\{X_s\}$ are only those with K -increasing sequences of parameters and they are determined by $\{X_t^1\}, \dots, \{X_t^N\}$ as in Theorem 98.*

Notes

When $K = \mathbb{R}_+^N$, K -parameter Lévy processes and their subordination were introduced in Barndorff-Nielsen, Pedersen, and Sato [4] (2001). In this case of $K = \mathbb{R}_+^N$, all results in this chapter are found in [4] (2001). But the proof of Theorem 101 has been simplified. Theorem 83 in Section 4.1 on Lévy processes taking values in a proper cone is by Skorohod [80] (1991). In Section 4.2 the notion of subordination has been extended to the case of parameters in a general proper cone K . We mention that Bochner [7] (1955) already considered processes with parameter in a cone, under the name of multidimensional time variable. Example 87 is from [4] (2001); this paper contains several other examples of construction of N -variate subordinators.

In the Gaussian case the multiparameter Brownian motion $\{B_s: s \in \mathbb{R}^N\}$ and the Brownian sheet $\{W_s: s \in \mathbb{R}^N\}$ have been discussed for a long time. We mention Lévy [36] (1948), Chentsov [10] (1957), and McKean [45] (1963) for the former and Orey and Pruitt [49] (1973) and Khoshnevisan and Shi [29] (1999) for the latter. When the parameter s is restricted to a proper cone K not isomorphic to $[0, \infty)$, neither $\{B_s: s \in K\}$ nor $\{W_s: s \in K\}$ is a K -parameter Lévy process. Likewise, two-parameter Lévy processes in Vares [87] (1983) and Lagaize [33] (2001) are not K -parameter Lévy processes in our sense. But probabilistic potential theory for the \mathbb{R}_+^N -parameter Lévy process in Example 96 with $\{V_t^j\}$, $j = 1, \dots, N$, being a symmetric Lévy processes was studied by Hirsch [20] (1995) and, in the case where $\{V_t^j\}$ was a Brownian motion on \mathbb{R}^d for each j , Khoshnevisan and Shi [29] called it the (N, d) additive Brownian motion and studied its capacity.

Chapter 5

Inheritance of selfdecomposability in subordination

Once the general results about subordination of K -parameter Lévy processes by K -valued subordinators are established, it is now important to know what properties are inherited by the subordinate processes. Some inheritance results in relation to selfdecomposability and stability are presented in this chapter. In particular, it is proved in Section 5.1 that selfdecomposability, as well as strict stability, of the K -valued subordinators is inherited by the subordinated, under the condition that the original K -parameter Lévy process is strictly stable. Furthermore, if the subordinator is of class L_m , then the subordinated is of class L_m . In Section 5.2, generalization of these results to operator stability and operator selfdecomposability is discussed.

5.1 Inheritance of L_m property and strict stability

Halgreen [18] (1979) and Ismail and Kelker [21] (1979) proved part of the following results.

Theorem 104 *Let $\{Y_t: t \geq 0\}$ be a Lévy process on \mathbb{R}^d subordinate to a strictly α -stable process $\{X_t: t \geq 0\}$ on \mathbb{R}^d by a subordinator $\{Z_t: t \geq 0\}$.*

- (i) If $\{Z_t\}$ is selfdecomposable, then $\{Y_t\}$ is selfdecomposable.*
- (ii) More generally, let $m \in \{0, 1, \dots, \infty\}$. If $\{Z_t\}$ is of class $L_m(\mathbb{R})$, then $\{Y_t\}$ is of class $L_m(\mathbb{R}^d)$.*
- (iii) If $\{Z_t\}$ is strictly β -stable, then $\{Y_t\}$ is strictly $\alpha\beta$ -stable.*

Proof will be given as a special case of Theorem 110. We are interested in generalization of this theorem.

Example 105 If $\{X_t\}$ is a strictly α -stable increasing process with $Ee^{-uX_t} = e^{-tu^\alpha}$, $0 < \alpha < 1$, and $\{Z_t\}$ is a Γ -process with $EZ_1 = 1$, then

$$P[Y_1 \leq x] = 1 - E_\alpha(-x^\alpha),$$

$$P[Y_t \leq x] = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(t+n)}{n! \Gamma(t) \Gamma(1+\alpha(t+n))} x^{\alpha(t+n)}.$$

Here $E_\alpha(x)$ is the Mittag-Leffler function,

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha + 1)}.$$

By Theorem 104 (i) $\mathcal{L}(Y_t)$ is selfdecomposable. See Pillai [52] (1990) or [S] E 34.4.

Example 106 If $\{X_t\}$ is a symmetric α -stable process on \mathbb{R} with $Ee^{izX_t} = e^{-t|z|^\alpha}$, $0 < \alpha \leq 2$, and $\{Z_t\}$ is a Γ -process with $EZ_1 = 1/q$, $q > 0$, then

$$Ee^{izY_t} = (1 + q^{-1}|z|^\alpha)^{-t}, \quad z \in \mathbb{R},$$

and $\mathcal{L}(Y_1)$ is Linnik distribution. Theorem 104 (i) shows that $\mathcal{L}(Y_t)$ is selfdecomposable. See Example 79.

Example 107 Let $\mu_{\gamma,\delta}$ be inverse Gaussian distribution on \mathbb{R} with parameters $\gamma > 0$, $\delta > 0$, that is,

$$\mu_{\gamma,\delta}(B) = \frac{\delta e^{\gamma\delta}}{\sqrt{2\pi}} \int_{B \cap (0,\infty)} x^{-3/2} e^{-(\delta^2 x^{-1} + \gamma^2 x)/2} dx, \quad B \in \mathcal{B}(\mathbb{R}).$$

This has Laplace transform

$$\begin{aligned} \int_{(0,\infty)} e^{-ux} \mu_{\gamma,\delta}(dx) &= \exp \left[-\delta \left(\sqrt{2u + \gamma^2} - \gamma \right) \right] \\ &= \exp \left[\frac{\delta}{2\sqrt{\pi}} \int_0^\infty \left(e^{-(2u+\gamma^2)x} - 1 \right) x^{-3/2} dx + \gamma\delta \right] \\ &= \exp \left[\frac{\delta}{2\sqrt{\pi}} \int_0^\infty \left(e^{-2ux} - 1 \right) x^{-3/2} e^{-\gamma^2 x} dx \right] \\ &= \exp \left[\frac{\delta}{\sqrt{2\pi}} \int_0^\infty \left(e^{-ux} - 1 \right) x^{-3/2} e^{-\gamma^2 x/2} dx \right], \quad u \geq 0. \end{aligned}$$

The last formula shows that $\mu_{\gamma,\delta}$ is infinitely divisible with Lévy measure density

$$(2\pi)^{-1/2}\delta x^{-3/2}e^{-\gamma^2 x/2}$$

on $(0, \infty)$. Hence $\mu_{\gamma,\delta}$ is selfdecomposable.

Now let $\{X_t\}$ be Brownian motion on \mathbb{R} and let $\{Z_t\}$ be the subordinator with $\mathcal{L}(Z_1) = \mu_{\gamma,\delta}$. Then $\mathcal{L}(Z_t) = \mu_{\gamma,t\delta}$. Let $\{Y_t\}$ be the Lévy process subordinate to $\{X_t\}$ by $\{Z_t\}$. Then

$$\begin{aligned} P[Y_t \in B] &= \int_0^\infty \mu_{\gamma,t\delta}(ds) \int_B (2\pi s)^{-1/2} e^{-x^2/(2s)} dx \\ &= (2\pi)^{-1} t\delta e^{t\gamma\delta} \int_B dx \int_0^\infty s^{-2} e^{-((x^2+t^2\delta^2)/(2s))-(\gamma^2 s/2)} ds \\ &= (4\pi)^{-1} t\gamma^2 \delta e^{t\gamma\delta} \int_B dx \int_0^\infty u^{-2} e^{-(\gamma^2(x^2+t^2\delta^2)/(4u))-u} du \\ &= \int_B \frac{\gamma e^{t\gamma\delta}}{\pi \sqrt{1+(x/(t\gamma))^2}} K_1\left(t\gamma\delta \sqrt{1+(x/(t\gamma))^2}\right) dx, \end{aligned}$$

where K_1 is the modified Bessel function of order 1. $\mathcal{L}(Y_t)$ is a special case of the so-called normal inverse Gaussian distribution. By Theorem 104, it is selfdecomposable. By Theorem 76 its characteristic function is:

$$Ee^{izY_t} = e^{t\Psi(-z^2/2)} = \exp\left[-t\delta\left(\sqrt{z^2+\gamma^2}-\gamma\right)\right]$$

with $\Psi(w) = -\delta\left(\sqrt{-2w+\gamma^2}-\gamma\right)$.

The distribution μ on $(0, \infty)$ with density

$$cx^{\lambda-1}e^{-(\chi x^{-1}+\psi x)/2}$$

is called generalized inverse Gaussian distribution with parameters λ, χ, ψ . Here c is a normalizing constant. The domain of the parameters is given by $\{\lambda < 0, \chi > 0, \psi \geq 0\}$, $\{\lambda = 0, \chi > 0, \psi > 0\}$, and $\{\lambda > 0, \chi \geq 0, \psi > 0\}$. Its Laplace transform $L_\mu(u)$, $u \geq 0$, is:

$$L_\mu(u) = \begin{cases} \left(\frac{\psi}{\psi+2u}\right)^{\lambda/2} \frac{K_\lambda(\sqrt{\chi(\psi+2u)})}{K_\lambda(\sqrt{\chi\psi})} & \text{if } \chi > 0 \text{ and } \psi > 0 \\ \frac{2^{1+(\lambda/2)} K_\lambda(\sqrt{2\chi u})}{\Gamma(-\lambda)(\chi u)^{\lambda/2}} & \text{if } \lambda < 0, \chi > 0, \text{ and } \psi = 0. \end{cases}$$

It is known that μ is infinitely divisible and, moreover, selfdecomposable. It belongs to a smaller class called generalized Γ -convolutions, which means that it is the limit of a sequence of convolutions of Γ -distributions.

This example continues to Example 114.

In order to extend Theorem 104 to multivariate subordination, we prepare two lemmas. Assume that K is a proper cone in \mathbb{R}^N .

Lemma 108 *Let $\{X_s: s \in K\}$ be a K -parameter Lévy process on \mathbb{R}^d . Let $0 < \alpha \leq 2$. Then $\mathcal{L}(X_s) \in \mathfrak{S}_\alpha^0$ if and only if $X_{ts} \stackrel{d}{=} t^{1/\alpha} X_s$ for every $t > 0$.*

Proof. Let $\mu_s = \mathcal{L}(X_s)$. The meaning of $\mu_s \in \mathfrak{S}_\alpha^0$ is that $\mu_s \in ID$ and $\widehat{\mu}_s(z)^t = \widehat{\mu}_s(t^{1/\alpha}z)$ for $t > 0$. See Definition 9 and Proposition 10. Since, by Lemma 91, $\{X_{ts}: t \geq 0\}$ is a Lévy process, $\widehat{\mu}_{ts}(z) = \widehat{\mu}_s(z)^t$. Hence the condition is written as $X_{ts} \stackrel{d}{=} t^{1/\alpha} X_s$. ■

Lemma 109 *Let $\{Z_t\}$ be a K -valued subordinator such that $\mathcal{L}(Z_t) \in L_0(\mathbb{R}^N)$ for $t \geq 0$. Let $\Psi(w)$ be the function in (4.11). For $b > 1$ define $\Psi_b(w)$ as*

$$\Psi(w) = \Psi(b^{-1}w) + \Psi_b(w). \quad (5.1)$$

Then $e^{t\Psi_b(iz)}$, $z \in \mathbb{R}^N$, is the characteristic function of a K -valued subordinator $\{Z_t^{(b)}\}$. Let $m \geq 1$. Then $\mathcal{L}(Z_t) \in L_m$ for $t \geq 0$ if and only if $\mathcal{L}(Z_t^{(b)}) \in L_{m-1}$ for $t \geq 0$.

Proof. Let $\mu = \mathcal{L}(Z_1)$ with generating triplet (A, ν, γ) . Its characteristic function is $\widehat{\mu}(z) = e^{\Psi(iz)}$. If $b > 1$, then by selfdecomposability there is a probability measure ρ_b such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z) \widehat{\rho}_b(z) \quad \text{for every } z \in \mathbb{R}^N.$$

Let $(\widetilde{A}_b, \widetilde{\nu}_b, \widetilde{\gamma}_b)$ and (A_b, ν_b, γ_b) be the generating triplets of μ_b and ρ_b , respectively, where $\mu = \mu_b * \rho_b$, and $\widehat{\mu}_b(z) = \widehat{\mu}(b^{-1}z)$. Recall, by Lemma 3, that μ_b and ρ_b are in ID . Then $A = \widetilde{A}_b + A_b$, $\nu = \widetilde{\nu}_b + \nu_b$ and $\gamma = \widetilde{\gamma}_b + \gamma_b$. By Theorem 83, $A = 0$, $\nu(\mathbb{R}^N \setminus K) = 0$, $\int_{|s| \leq 1} |s| \nu(ds) < \infty$, and $\gamma^0 \in K$. Hence $\nu_b \leq \nu$. Therefore $\nu_b(\mathbb{R}^N \setminus K) = 0$, and $\int_{|s| \leq 1} |s| \nu_b(ds) < \infty$. Also $A_b = 0$, as $0 \leq \langle z, A_b z \rangle \leq \langle z, A z \rangle = 0$. Further, $\gamma^0 = \widetilde{\gamma}_b^0 + \gamma_b^0$, where $\widetilde{\gamma}_b^0 = b^{-1} \gamma^0$. Thus $\gamma_b^0 = (1 - b^{-1}) \gamma^0 \in K$. Then, by Theorem 83, a Lévy process $\{Z_t^{(b)}\}$ with $\mathcal{L}(Z_1^{(b)}) = \rho_b$ is a K -valued subordinator. Since $e^{t\Psi_b(iz)} = \widehat{\rho}_b(z)^t$, it is the characteristic function of $\{Z_t^{(b)}\}$. Finally, $\mathcal{L}(Z_t)$ is of class L_m if and only if $\rho_b \in L_{m-1}$, that is, $\mathcal{L}(Z_t^{(b)})$ is of class L_{m-1} . ■

Theorem 110 *Let K be a proper cone in \mathbb{R}^N and let $0 < \alpha \leq 2$. Let $\{Z_t: t \geq 0\}$ be a K -valued subordinator and let $\{X_s: s \in K\}$ be a K -parameter Lévy process on \mathbb{R}^d such that $\mathcal{L}(X_s) \in \mathfrak{S}_\alpha^0$ for all $s \in K$. Let $\{Y_t: t \geq 0\}$ be a Lévy process on \mathbb{R}^d obtained by multivariate subordination from $\{X_s\}$ and $\{Z_t\}$.*

(i) *If $\{Z_t\}$ is selfdecomposable, then $\{Y_t\}$ is selfdecomposable.*

(ii) *Let $m \in \{0, 1, \dots, \infty\}$. If $\{Z_t\}$ is of class $L_m(\mathbb{R}^N)$, then $\{Y_t\}$ is of class $L_m(\mathbb{R}^d)$.*

(iii) *Let $0 < \beta \leq 2$. If $\mathcal{L}(Z_t) \in \mathfrak{S}_\beta^0$ for all $t \geq 0$, then $\mathcal{L}(Y_t) \in \mathfrak{S}_{\alpha\beta}^0$ for all $t \geq 0$.*

Proof. Let $\mu_s = \mathcal{L}(X_s)$.

(i) Let $\{Z_t\}$ be selfdecomposable, that is, $\mathcal{L}(Z_t) \in L_0$ for $t \geq 0$. Let $b > 1$. Using Lemma 109 and its notation

$$Z_t \stackrel{d}{=} b^{-1}Z_t + Z_t^{(b)},$$

where $b^{-1}Z_t$ and $Z_t^{(b)}$ are independent. Then,

$$Ee^{i\langle z, Y_t \rangle} = Ee^{i\langle b^{-1/\alpha}z, Y_t \rangle} E \left[\widehat{\mu}_{Z_t^{(b)}}(z) \right]. \quad (5.2)$$

Indeed we have, using Lemma 91 (i) and Lemma 108,

$$\begin{aligned} Ee^{i\langle z, Y_t \rangle} &= E \left[(Ee^{i\langle z, X_s \rangle})_{s=Z_t} \right] = E \left[\widehat{\mu}_{Z_t}(z) \right] = E \left[\widehat{\mu}_{b^{-1}Z_t + Z_t^{(b)}}(z) \right] \\ &= E \left[\widehat{\mu}_{b^{-1}Z_t}(z) \widehat{\mu}_{Z_t^{(b)}}(z) \right] = E \left[\widehat{\mu}_{b^{-1}Z_t}(z) \right] E \left[\widehat{\mu}_{Z_t^{(b)}}(z) \right] \\ &= E \left[\widehat{\mu}_{Z_t}(b^{-1/\alpha}z) \right] E \left[\widehat{\mu}_{Z_t^{(b)}}(z) \right], \end{aligned}$$

which is the right-hand side of (5.2). Since $b^{1/\alpha}$ can be an arbitrary real > 1 and since $E \left[\widehat{\mu}_{Z_t^{(b)}}(z) \right]$ is the characteristic function of a subordinated process by Lemma 109, this shows that $\{Y_t\}$ is selfdecomposable.

(ii) Induction. If $m = 0$, then the assertion is true by (i). Suppose that the assertion is true for $m - 1$ in place of m . Let $\{Z_t\}$ be of class L_m , that is, $\mathcal{L}(Z_t) \in L_m$ for $t \geq 0$. Then $\{Z_t^{(b)}\}$ is a K -valued subordinator of class L_{m-1} by Lemma 109. Hence $E \left[\widehat{\mu}_{Z_t^{(b)}}(z) \right]$ is a characteristic function of class L_{m-1} . Thus $\mathcal{L}(Y_t) \in L_m$.

(iii) Let $\mathcal{L}(Z_t) \in \mathfrak{S}_\beta^0$ for $t \geq 0$. Then $Z_{at} \stackrel{d}{=} a^{1/\beta}Z_t$. Therefore, using Lemma 108,

$$\begin{aligned} Ee^{i\langle z, Y_{at} \rangle} &= E \left[(Ee^{i\langle z, X_s \rangle})_{s=Z_{at}} \right] = E \left[(Ee^{i\langle z, X_s \rangle})_{s=a^{1/\beta}Z_t} \right] \\ &= E \left[\widehat{\mu}_{a^{1/\beta}Z_t}(z) \right] = E \left[\widehat{\mu}_{Z_t}(a^{1/(\alpha\beta)}z) \right] = E \left[e^{i\langle z, a^{1/(\alpha\beta)}Y_t \rangle} \right]. \end{aligned}$$

Thus $Y_{at} \stackrel{d}{=} a^{1/(\alpha\beta)}Y_t$ for any $a > 0$. ■

When $d = 1$, Theorem 104 can be generalized to the case where $\{X_t: t \geq 0\}$ is Brownian motion with nonzero drift on \mathbb{R} . This is 2-stable, but not strictly 2-stable. So the assumption in Theorem 104 is not satisfied. Nevertheless, selfdecomposability is preserved in its subordination.

Theorem 111 Let $\{X_t: t \geq 0\}$ be Brownian motion with drift on \mathbb{R} . That is,

$$Ee^{izX_t} = e^{t(-z^2/2+i\gamma z)}, \quad z \in \mathbb{R}.$$

Let $\{Y_t\}$ be a Lévy process subordinate to $\{X_t\}$ by $\{Z_t\}$. If $\{Z_t\}$ is selfdecomposable, then $\{Y_t\}$ is selfdecomposable.

Proof is omitted.

Remark 112 We do not know whether Theorem 111 can be extended to the case where $\{X_t\}$ is an α -stable, not strictly α -stable process with $0 < \alpha < 2$ on \mathbb{R} .

Remark 113 If $d \geq 2$, then the situation is quite different and Theorem 111 cannot be generalized. It is known that, for $d \geq 2$, a Lévy process $\{Y_t\}$ on \mathbb{R}^d subordinate to Brownian motion with drift, $\{X_t\}$, by a selfdecomposable subordinator $\{Z_t\}$ is not necessarily selfdecomposable. Even if $\mathcal{L}(Z_1)$ is a generalized Γ -convolution, $\{Y_t\}$ is not necessarily selfdecomposable.

Example 114 A distribution on \mathbb{R} with density

$$\text{const exp} \left(-a\sqrt{1+x^2} + bx \right)$$

with parameters a, b satisfying $a > 0$ and $|b| < a$ or a scale change of this distribution is called hyperbolic distribution.

Let $\{X_t\}$ be Brownian motion with drift γ being zero or nonzero and let $\{Z_t\}$ be the subordinator with $\mathcal{L}(Z_1)$ being generalized inverse Gaussian of Example 107 with $\lambda = 1$, $\chi > 0$, $\psi > 0$. Let us calculate the distribution at $t = 1$ for the Lévy process $\{Y_t\}$ subordinate to $\{X_t\}$ by $\{Z_t\}$:

$$\begin{aligned} P[Y_1 \in B] &= c \int_0^\infty e^{-(\chi s^{-1} + \psi s)/2} ds \int_B \frac{1}{\sqrt{2\pi s}} e^{-(x-s\gamma)^2/(2s)} ds \\ &= \frac{c}{\sqrt{\psi + \gamma}} \int_B e^{-\sqrt{(\psi + \gamma)(\chi + x^2)} + \gamma x} dx \end{aligned}$$

by the calculation in Example 2.13 of [S]. Hence $\mathcal{L}(Y_1)$ is a hyperbolic distribution with $a = \sqrt{\chi(\psi + \gamma)}$ and $b = \sqrt{\chi}\gamma$.

More generally if we assume that $\mathcal{L}(Z_1)$ is generalized inverse Gaussian, then $\mathcal{L}(Y_1)$ is generalized hyperbolic distribution. For a proof, use the formula (30.28) of [S] for modified Bessel functions. The generalized hyperbolic distribution is defined by the density

$$\text{const} \left(\sqrt{1+x^2} \right)^{\lambda-(1/2)} K_{\lambda-(1/2)} \left(a\sqrt{1+x^2} \right) e^{bx}$$

or its scale change, where the domain of parameters is given by $\{\lambda \geq 0, a > 0, |b| < a\}$ and $\{\lambda < 0, a > 0, |b| \leq a\}$. It reduces to the hyperbolic distribution if $\lambda = 1$.

It follows from Theorem 104 (if $\gamma = 0$) and Theorem 111 (if $\gamma \neq 0$) that generalized hyperbolic distributions are selfdecomposable.

5.2 Operator generalization

For distributions on \mathbb{R}^d , $d \geq 2$, the concepts of stability, selfdecomposability, and L_m are generalized to the situation where multiplication by positive real numbers is replaced by multiplication by matrices of the form e^{aQ} .

For a set $J \subset \mathbb{R}$ let $M_J(d)$ be the set of real $d \times d$ matrices all of whose eigenvalues have real parts in J . Let $Q \in M_{(0,\infty)}(d)$.

Definition 115 A distribution μ on \mathbb{R}^d is called Q -selfdecomposable if, for every $b > 1$, there is $\rho_b \in \mathfrak{P}(\mathbb{R}^d)$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-Q'} z) \widehat{\rho}_b(z), \quad z \in \mathbb{R}^d, \quad (5.3)$$

where Q' is the transpose of Q and $b^{-Q'}$ is a $d \times d$ matrix defined by

$$b^{-Q'} = e^{-(\log b)Q'} = \sum_{n=0}^{\infty} (n!)^{-1} (-\log b)^n (Q')^n.$$

The class of all Q -selfdecomposable distributions on \mathbb{R}^d is denoted by $L_0(Q)$. For $m = 1, 2, \dots$ the class $L_m(Q)$ is defined to be the class of distributions μ on \mathbb{R}^d such that, for every $b > 1$, there exists $\rho_b \in L_{m-1}(Q)$ satisfying (5.3). Define $L_\infty(Q) = \bigcap_{m < \infty} L_m(Q)$.

Proposition 116 The classes just introduced form nested classes

$$ID \supset L_0(Q) \supset L_1(Q) \supset \dots \supset L_\infty(Q). \quad (5.4)$$

Proof can be given analogously to the proofs of Lemma 3 and Proposition 5. See Jurek [24] (1983) and Sato and Yamazato [75] (1985).

Definition 117 A distribution μ on \mathbb{R}^d is called Q -stable if, for every $n \in \mathbb{N}$, there is $c \in \mathbb{R}^d$ such that

$$\widehat{\mu}(z)^n = \widehat{\mu}(n^{Q'} z) e^{i\langle c, z \rangle}, \quad z \in \mathbb{R}^d. \quad (5.5)$$

It is called strictly Q -stable if, for all n ,

$$\widehat{\mu}(z)^n = \widehat{\mu}(n^{Q'} z), \quad z \in \mathbb{R}^d. \quad (5.6)$$

Let $\mathfrak{S}(Q)$ be the class of distributions on \mathbb{R}^d which are aQ -stable for some $a > 0$. Let $\mathfrak{S}^0(Q)$ be the class of distributions which are strictly aQ -stable for some $a > 0$.

Here we are obeying the usual terminology, but it is not harmonious with the usage of the word α -stable; μ is α -stable if and only if it is $\frac{1}{\alpha}I$ -stable, where I is the identity matrix. Similarly to Proposition 10, we have the following.

Proposition 118 *A distribution μ is Q -stable if and only if $\mu \in ID$ and, for every $a > 0$, there is $c \in \mathbb{R}^d$ such that*

$$\widehat{\mu}(z)^a = \widehat{\mu}(a^{Q'} z) e^{i\langle c, z \rangle}. \quad (5.7)$$

A distribution μ is strictly Q -stable if and only if $\mu \in ID$ and, for every $a > 0$,

$$\widehat{\mu}(z)^a = \widehat{\mu}(a^{Q'} z). \quad (5.8)$$

Proof is like E 18.4 of [S].

Remark 119 *If $\mu \in \mathfrak{S}(Q)$ for some $Q \in M_{(0,\infty)}(d)$, then μ is called operator stable and sometimes Q is called exponent of operator stability of μ . But Q is not uniquely determined by μ . If $\mu \in L_0(Q)$ for some $Q \in M_{(0,\infty)}(d)$, then μ is called operator selfdecomposable.*

Remark 120 *Operator stable and operator selfdecomposable distributions appear in a natural way when we study limit theorems for sums of independent random vectors, allowing normalization by linear transformations (matrices). Basic papers are Sharpe [77] (1969) and Urbanik [84] (1972a).*

Sharpe [77] (1969) found the following.

Proposition 121 *Suppose that μ is Q -stable and nondegenerate on \mathbb{R}^d . Then Q must be in $M_{[1/2,\infty)}(d)$ and, moreover, any eigenvalue of Q with real part $1/2$ is a simple root of the minimal polynomial of Q ; μ is Gaussian if and only if $Q \in M_{\{1/2\}}(d)$; μ is purely non-Gaussian if and only if $Q \in M_{(1/2,\infty)}(d)$.*

$\mathfrak{S}(Q)$ is a subclass of $L_\infty(Q)$. Moreover, Sato and Yamazato [75] (1985) proved the following.

Proposition 122 $L_\infty(Q)$ is the smallest class containing $\mathfrak{S}(Q)$ and closed under convolution and weak convergence.

Definition 123 A Lévy process $\{X_t: t \geq 0\}$ is called Q -selfdecomposable, Q -stable, or of class $L_m(Q)$, respectively, if $\mathcal{L}(X_1)$ (or, equivalently, (X_t) for every $t \geq 0$) is Q -selfdecomposable, Q -stable, or of class $L_m(Q)$.

Here are results on the inheritance of operator selfdecomposability, $L_m(Q)$ property, and strict operator stability in some cases. These partially extend Theorem 110. Propositions 121 and 122 are not used in the proof.

Let N and d be positive integers satisfying $d \geq N \geq 1$. Let d_j , $1 \leq j \leq N$, be positive integers such that $d_1 + \dots + d_N = d$. Every $x \in \mathbb{R}^d$ is expressed as $x = (x_j)_{1 \leq j \leq N}$ with $x_j \in \mathbb{R}^{d_j}$. We call x_j the j th component-block of x . The j th component-block of X_t is denoted by $(X_t)_j$. As in Section 4.3, we use the unit vectors $e^k = (\delta_{kj})_{1 \leq j \leq N}$, $k = 1, \dots, N$, in \mathbb{R}^N .

Theorem 124 Suppose that $\{X_s: s \in \mathbb{R}_+^N\}$ is a given \mathbb{R}_+^N -parameter Lévy process on \mathbb{R}^d with the following structure: for each $j = 1, \dots, N$,

$$(X_{te^j})_k = 0 \quad \text{for all } k \neq j. \quad (5.9)$$

Suppose that $\{Z_t: t \geq 0\}$ is a given N -variate subordinator and let $\{Y_t: t \geq 0\}$ be a Lévy process on \mathbb{R}^d obtained by multivariate subordination from $\{X_s\}$ and $\{Z_t\}$. That is, $\{X_s\}$ and $\{Z_t\}$ are independent and $Y_t = X_{Z_t}$. Let $Q_j \in M_{[1/2, \infty)}(d_j)$ and $c_j > 0$ for $1 \leq j \leq N$, and let $C = \text{diag}(c_1, \dots, c_N)$. Assume that, for each j , $\mathcal{L}((X_{te^j})_j)$ is strictly Q_j -stable. Define $D = \text{diag}(c_1 Q_1, \dots, c_N Q_N) \in M_{(0, \infty)}(d)$.

- (i) If $\{Z_t: t \geq 0\}$ is C -selfdecomposable, then $\{Y_t: t \geq 0\}$ is D -selfdecomposable.
- (ii) More generally, let $m \in \{0, 1, \dots, \infty\}$. If $\{Z_t: t \geq 0\}$ is of class $L_m(C)$ on \mathbb{R}^N , then $\{Y_t: t \geq 0\}$ is of class $L_m(D)$ on \mathbb{R}^d .
- (iii) If $\{Z_t: t \geq 0\}$ is strictly C -stable, then $\{Y_t: t \geq 0\}$ is strictly D -stable.

Here $\text{diag}(c_1, \dots, c_N)$ denotes the diagonal matrix with diagonal entries c_1, \dots, c_N ; $\text{diag}(c_1 Q_1, \dots, c_N Q_N)$ denotes the blockwise diagonal matrix with diagonal blocks $c_1 Q_1, \dots, c_N Q_N$.

Proof. We use Theorem 101. Let $X_t^j = X_{te^j}$. Let $\psi_X^j(z) = \log Ee^{i\langle z, X_1^j \rangle}$, $z \in \mathbb{R}^{d_j}$, and $\psi_X(z) = (\psi_X^j(z))_{1 \leq j \leq N}$. Let $\mu_j = \mathcal{L}((X_1^j)_j) \in \mathfrak{P}(\mathbb{R}^{d_j})$. Then it follows from (5.9) that

$$e^{t\psi_X^j(z)} = Ee^{i\langle z, X_t^j \rangle} = Ee^{i\langle z_j, (X_t^j)_j \rangle} = \widehat{\mu}_j(z_j)^t,$$

where $z = (z_j)_{1 \leq j \leq N} \in \mathbb{R}^d$ with $z_j \in \mathbb{R}^{d_j}$. Thus

$$\psi_X(z) = (\log \widehat{\mu}_j(z_j))_{1 \leq j \leq N}.$$

We have

$$\widehat{\mu}_j(z_j)^a = \widehat{\mu}_j(a^{Q'_j} z_j), \quad a > 0$$

by the strict Q_j -stability of μ_j . Hence

$$a^C \psi_X(z) = (a^{c_j} \log \widehat{\mu}_j(z_j))_{1 \leq j \leq N} = (\log \widehat{\mu}_j(a^{c_j Q'_j} z_j))_{1 \leq j \leq N}. \quad (5.10)$$

(i) Assume $\{Z_t: t \geq 0\}$ is C -selfdecomposable. Let Ψ_Z be the function Ψ in (4.11) for $\{Z_t\}$. For $b > 1$ and $w = (w_j)_{1 \leq j \leq N} \in \mathbb{C}^N$ with $\operatorname{Re} w_j \leq 0$, Define $\Psi_{Z,b}(w)$ by

$$\Psi_Z(w) = \Psi_Z(b^{-C} w) + \Psi_{Z,b}(w).$$

Similarly to the proof of Lemma 3, we can show that $e^{\Psi_{Z,b}(iu)}$, $u \in \mathbb{R}^N$, is an infinitely divisible characteristic function. Further, as in Lemma 109, there is an \mathbb{R}_+^N -valued subordinator $\{Z_t^{(b)}\}$ such that $E e^{i\langle u, Z_t^{(b)} \rangle} = e^{t \Psi_{Z,b}(iu)}$. In the proof note that $\gamma_b^0 = (I - b^{-C}) \gamma^0 = \operatorname{diag}(1 - b^{-c_1}, \dots, 1 - b^{-c_N}) \gamma^0 \in \mathbb{R}_+^N$. Now we have

$$E e^{i\langle z, Y_t \rangle} = e^{t \Psi_Z(\psi_X(z))} = e^{t \Psi_Z(b^{-C} \psi_X(z))} e^{t \Psi_{Z,b}(\psi_X(z))}$$

and

$$b^{-C} \psi_X(z) = (\log \widehat{\mu}_j(b^{-c_j Q'_j} z_j))_{1 \leq j \leq N} = \psi_X(b^{-D'} z)$$

by (5.10), since

$$b^{-D'} z = \operatorname{diag}(b^{-c_1 Q'_1}, \dots, b^{-c_N Q'_N}) z = (b^{-c_j Q'_j} z_j)_{1 \leq j \leq N}.$$

Hence

$$E e^{i\langle z, Y_t \rangle} = E e^{i\langle b^{-D'} z, Y_t \rangle} e^{t \Psi_{Z,b}(\psi_X(z))}.$$

As the second factor in the right-hand side is the characteristic function of a subordinated process, we see that $\mathcal{L}(Y_t)$ is D -selfdecomposable.

(ii) Induction similar to (ii) of Theorem 110.

(iii) Assume that $\{Z_t\}$ is strictly C -stable, that is, $a\Psi_Z(w) = \Psi_Z(a^C w)$. Then, for $a > 0$,

$$Ee^{i\langle z, Y_{at} \rangle} = e^{at\Psi_Z(\psi_X(z))} = e^{t\Psi_Z(a^C \psi_X(z))}$$

and, as above,

$$a^C \psi_X(z) = \psi_X(a^{D'} z).$$

Hence

$$Ee^{i\langle z, Y_{at} \rangle} = Ee^{i\langle a^{D'} z, Y_t \rangle},$$

which shows D -stability of $\{Y_t\}$. ■

Remark 125 *If an \mathbb{R}_+^N -valued subordinator $\{Z_t\}$ is Q -selfdecomposable, then Q has a strong restriction. For example, let $N = 2$. Among the real Jordan forms, Q cannot be*

$$\begin{pmatrix} q_1 & 1 \\ 0 & q_1 \end{pmatrix} \quad \text{nor} \quad \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix}$$

with $q_1 > 0$, $q_2 > 0$. The only possibility of Q among the real Jordan forms is

$$\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}$$

with $q_1 > 0$, $q_2 > 0$. See Sato [59] (1985).

Notes

Halgreen [18] (1979) and Ismail and Kelker [21] (1979) proved assertion (i) of Theorem 104 in the case where $\{X_t\}$ is Brownian motion on \mathbb{R} . Assertion (iii) of Theorem 104 was essentially known to Bochner [7] (1955). Theorem 124 was given in Barndorff-Nielsen, Pedersen, and Sato [4] (2001), but the proof presented here is greatly simplified. Assertion (ii) of Theorem 104 is a special case of Theorem 124 (ii) with $N = 1$ and $Q = Q_1 = (1/\alpha)I$. Theorem 110 in this chapter and Theorem 92 in Chapter 4 are part of the work that Sato are preparing jointly with Jan Pedersen.

Examples 107 and 114 are from Barndorff-Nielsen and Halgreen [3] (1977) and Halgreen [18] (1979); see also Bondesson [8] (1992).

Theorem 111 was proved in Sato [65] (2001). Earlier Halgreen [18] (1979) and Shanbhag and Sreehari [76] (1979) proved it under the condition that $\mathcal{L}(Z_1)$ is a generalized Γ -convolution. Remark 113 is by Takano [81] (1989/90).

Various extensions of L_m and \mathfrak{S}_α

Extensions of the concepts of L_m , \mathfrak{S}_α , and selfsimilarity are being made in various directions. Here we give an incomplete list of related papers. You can find many others, consulting references cited in these papers.

1. Operator extensions on \mathbb{R}^d , $2 \leq d < \infty$.

Operator stable: Sharpe [77] (1969), Hahn and Klass [17] (1985), Sato [60] (1987), Jurek and Mason [26] (1993).

Operator selfdecomposable: Urbanik [84] (1972a), Jurek [23] (1982), Sato and Yamazato [74] (1984), Jurek and Mason [26] (1993).

Operator L_m : Jurek [24], [25] (1983a, 83b), Sato and Yamazato [73] (1983), [75] (1985).

OU type processes associated with matrices Q : Gravereaux [15] (1982), Sato and Yamazato [73] (1983), [74] (1984), Sato, Watanabe, Yamamuro, and Yamazato [68] (1996), Watanabe [90] (1998).

Operator stable processes: Sato [59] (1985). Processes with stable components are special cases; see [S] Remark 49.16.

Operator selfsimilar processes: Sato [61] (1991), Meerschaert and Scheffler [47] (1999).

2. "Semi" extensions on \mathbb{R}^d , $d \geq 1$.

Semi-stable: Lévy [34] (1925), [35] (1937), Shimizu [79] (1970), Pillai [51] (1971), Kruglov [30] (1972), Kagan, Linnik, and Rao [28] (1973), Meerschaert and Scheffler [46] (1996), Sato [S] (1999a). The class of semi-stable distributions is neither larger nor smaller than L_0 .

Semi-selfdecomposable: Loève [37] (1945), Bunge [9] (1997), Maejima and Naito [39] (1998), Sato [S] (1999a), Watanabe [92] (2000b).

Semi-stable processes: Rajput and Rama-Murthy [53] (1987), Watanabe [88] (1993), Choi [11] (1994), Sato [64] (1999b).

Semi-selfsimilar processes: Maejima and Sato [40] (1999).

Classes $L_m(b, \mathbb{R}^d)$ and $\tilde{L}_m(b, \mathbb{R}^d)$: Bunge [9] (1997), Maejima and Naito [39] (1998), Watanabe [91] (2000a).

3. Operator “semi” extensions on \mathbb{R}^d , $2 \leq d < \infty$.

Operator semi-stable: Jajte [22] (1977), Łuczak [38] (1981/87).

Operator semi-selfdecomposable, operator $L_m(b, \mathbb{R}^d)$, and operator $\tilde{L}_m(b, \mathbb{R}^d)$: Maejima, Sato, and Watanabe [41] (1999), [42] (2000a).

Operator semi-stable processes: Choi and Sato [12] (1995).

4. Zinger’s extension of stable distributions.

Given a positive integer k , Zinger [99] (1965) determined the class of limit distributions of normalized sums of independent random variables on \mathbb{R} which have at most k different distributions. These are selfdecomposable distributions which are convolutions of a finite number of semi-stable distributions of some kind.

5. Extensions on Banach spaces.

Some results are extended to distributions on Banach spaces. See the books Parthasarathy [50] (1967), Araujo and Giné [2] (1980) for basic material. There are many papers; e. g. Hahn and Klass [16] (1981).

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Notation

\mathbb{R} , \mathbb{Z} , \mathbb{N} , and \mathbb{C} are the sets of real numbers, integers, positive integers, and complex numbers, respectively. $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$.

\mathbb{R}^d is the d -dimensional Euclidean space and elements of \mathbb{R}^d are column vectors $x = (x_j)_{1 \leq j \leq d}$. The inner product is $\langle x, y \rangle = \sum_{j=1}^d x_j y_j$ for $x = (x_j)_{1 \leq j \leq d}$ and $y = (y_j)_{1 \leq j \leq d}$. The norm is $|x| = \langle x, x \rangle^{1/2}$. $S = \{\xi \in \mathbb{R}^d: |\xi| = 1\}$ is the unit sphere in \mathbb{R}^d .

$\mathfrak{P} = \mathfrak{P}(\mathbb{R}^d)$ is the class of probability measures (distributions) on \mathbb{R}^d . $ID = ID(\mathbb{R}^d)$ is the class of infinitely divisible distributions on \mathbb{R}^d . $\mathfrak{S} = \mathfrak{S}(\mathbb{R}^d)$ is the class of stable distributions on \mathbb{R}^d . $\mathfrak{S}_\alpha = \mathfrak{S}_\alpha(\mathbb{R}^d)$ is the class of α -stable distributions on \mathbb{R}^d .

$\widehat{\mu}(z) = \int_{\mathbb{R}^d} e^{i\langle z, x \rangle} \mu(dx)$, $z \in \mathbb{R}^d$, the characteristic function of $\mu \in \mathfrak{P}(\mathbb{R}^d)$.

For $\mu \in \mathfrak{P}$ and $n \in \mathbb{N}$, μ^n is the n -fold convolution of μ . For $\mu \in ID$ and $t \in \mathbb{R}_+$, μ^t is defined as $\widehat{\mu}^t(z) = e^{t \log \widehat{\mu}(z)}$, where $\log \widehat{\mu}$ is the distinguished logarithm of $\widehat{\mu}$ in [S], p. 33.

$\mathcal{L}(X)$ is the distribution (law) of a random variable X . $X \stackrel{d}{=} Y$ means that two random variables X and Y have a common distribution, that is, $\mathcal{L}(X) = \mathcal{L}(Y)$. $\{X_t\} \stackrel{d}{=} \{Y_t\}$ means that two stochastic processes $\{X_t\}$ and $\{Y_t\}$ are identical in law, that is, have a common system of finite-dimensional distributions. Note that $X_t \stackrel{d}{=} Y_t$ simply means that, for each t , X_t and Y_t have a common distribution.

For μ_n ($n = 1, 2, \dots$) and μ in \mathfrak{P} , $\mu_n \rightarrow \mu$ means weak convergence of μ_n to μ , that is, $\int f(x) \mu_n(dx) \rightarrow \int f(x) \mu(dx)$ for all bounded continuous functions f .

The words *increasing* and *decreasing* are used in the wide sense allowing flatness.

For any Borel set T in \mathbb{R}^d , $\mathcal{B}(T)$ is the class of Borel sets in T . Unless specifically mentioned, *measurable* means Borel measurable.

δ_c is a distribution concentrated at c ; it is called a *trivial* distribution. A random variable X is trivial if $\mathcal{L}(X)$ is trivial. A stochastic process $\{X_t\}$ is trivial if X_t is trivial for each t .

[S] refers to Sato's book [63] (1999a).