A NOTE ON THE GEOMETRY OF SUPERCURVES AND SUPERLINE BUNDLES

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Abstract. We discuss some geometric properties of complex supercurves and line bundles over them. In particular, we obtain the existence of a cohomological obstruction to express $N = 2$ line bundles as tensor products of $N = 1$ bundles. The motivation behind this paper is an attempt at understanding the $N = 2$ super KP equation via Baker functions, which are special sections of line bundles on supercurves.

Introduction.

Complex supercurves are certainly a fascinating subject, both for mathematical and physical reasons. On the one hand, they appear as basic building blocks of superstring theory, currently the best candidate for a unified theory of all physical interactions, and certainly one of the most active fields in all of science. On the other hand, although many of the classical results in the geometry of complex curves can be rephrased in this context, the presence of nilpotent elements makes for several fundamental differences between the theories, and many of the results are not as sharp as for classical curves (to name one of the most dramatic, there is no analog for arbitrary supercurves of the Riemann-Roch theorem), so there are still many questions, not altogether of the same nature as in the classical case, that remain to be answered (cf. [FR]).

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The motivation for this paper is our interest in understanding the geometry of $N = 2$ super KP equations (cf. [DG]) via Baker functions (this is however work still in progress and will be reported elsewhere). Roughly speaking, Baker functions are special unique sections with parameters of certain families of line bundles, satisfying the condition that any section of the corresponding line bundle is given by a differential operator applied to the Baker function. From the geometric point of view, the relevance of such a construction is that it allows us to reinterpret equations such as the KP (or its super analogs), as describing deformations of line bundles over curves (or supercurves).

Now, for supercurves one of the subtleties of the problem is that, since one is interested in families and not in single curves, one is forced to deal with curves that are not necessarily split and therefore might have complicated cohomological properties. In spite of this, the situation for the case $N = 1$ is quite well understood: the cohomological conditions on the supercurve ensuring the existence and uniqueness of such a section are known and simple, and there are several options, leading to various hierarchies (Jacobian, Manin-Radul, Kac-van de Leur; cf. [R1]), depending on how large the family of deformations is allowed to be, etc. But for the case $N = 2$ the results are much less complete—especially from the geometric point of view—and some further conditions, both on the equations and on the curves, might become necessary. (There is indeed a precedent for this in the case $N = 1$, where it is known that only the Manin-Radul hierarchy admits a formulation as a Lax pair.)

For the $N = 2$ case, one of the possible and natural approaches to the problem is to consider only equations determining deformations on bundles over the $N = 2$ super Riemann surfaces canonically attached to $N = 1$ supercurves; thus, in this paper we study some geometric properties of supercurves related to the possibilty of constructing Baker functions for such cases. We do this along two lines: the first one is to impose some additional conditions on the $N = 1$ curve, that in some cases give rise to relatively simple cohomological properties; the second one is to analyse the relation between line bundles on the $N = 2$ curve and bundles lifted from the $N = 1$ curve and its dual.

To conclude this introduction, we mention that we use [R2] and [BR] as our basic references, including most of our notation and conventions; in particular the latter contains a much more detailed study of supercurves than that made here.

1. Projected and injected curves.

A (complex) supercurve M is a family of ringed spaces $(M_{\text{red}}, \mathcal{O}_M)$, over the base $(\bullet, \Lambda)$, where Λ is a Grassmann algebra (in some generators, say $\eta_1, \ldots, \eta_n$), M_{red} is an ordinary complex curve, and $\mathcal{O} = \mathcal{O}_M$ is a sheaf of $\mathbb{Z}_2$-graded algebras, locally isomorphic (as sheaves of $\mathbb{Z}_2$-graded algebras) to $\mathcal{O}_{\text{red}} \otimes \Lambda[\theta^1, \ldots, \theta^N]$, $\mathcal{O}_{\text{red}}$ being the structure sheaf of M_{red}, and θ^α odd generators nilpotent of order two.

The point to be made here is simply that the geometry of classical curves and line bundles is bound to play an important role in the study of supercurves, but also the fundamental difference between the two cases, namely the presence of the nilpotent (or odd) elements (the θ^α being variables, and the η_γ global constants), poses different problems than those encountered in the classical case.

Concretely, supermanifolds can be described by giving coordinate changes of the form

$$z_j = F_{ji}(z_i, \theta_i^\alpha),\ \theta_j^\beta = \Psi_{ji}^\beta(z_i, \theta_i^\alpha); \quad \alpha, \beta = 1, \ldots, N;$$

where the F's are even, and the Ψ's are odd (super)functions satisfying some appropriate compatibility conditions, analogous to the usual changes of coordinates for manifolds. Here, and henceforth, when using coordinates there is implicit an open cover $\{U_i\}$ of M_{red}, which will be assumed to have contractible multiple (finite) intersections.

Actually, the only relevant cases for us are $N = 1$ and $N = 2$, so we will concentrate our attention on them in what follows (there are some physical reasons to do so, but also, from the mathematical point of view, for $N > 2$ one must face the complexities of the theory of vector bundles of higher rank over algebraic curves, which remains a challenging problem even today).

In particular, for the case $N = 1$ which we will now discuss, the change of coordinates takes on the form

$$\begin{aligned} z_j &= f_{ji}(z_i) + \theta_i \gamma_{ji}(z_i) \\ \theta_j &= \mu_{ji}(z_i) + \theta_i n_{ji}(z_i), \end{aligned}$$

where f_{ji}, n_{ji} are even holomorphic functions, while γ_{ji}, μ_{ji} are odd (and this notation will be kept throughout this paper).

Before proceeding any further, let us make the following simple observation: Since a part of the necessary conditions for the coordinate changes is that the even functions $f'_{ji} = \partial_z f_{ji}$ and n_{ji} be nowhere vanishing, we immediately see from the equations above the following three special cases:

Projected curves, defined by the condition that coordinate changes can be chosen so that $\gamma_{ji} \equiv 0 \quad \forall i, j$.

Injected curves, defined by $\mu_{ji} \equiv 0 \quad \forall i, j$.

Quasi-split curves: Projected and Injected.

Also, a curve is said to be split if, moreover, the f_{ji} and n_{ji} do not involve the odd constants η_α. (Projected, but especially split, curves have already been discussed in the literature, cf [BR]; the notion of "injectedness" for supercurves seems however to be new.)

Now, to write down the remaining conditions on the transition functions, we recall that a change of coordinates on a superfunction $a(z)$ is given by

$$a(z_j) = a(f_{ji}(z_i)) + \theta_i \gamma_{ji}(z_i) a'(f_{ji}(z_i)). \tag{1}$$

Since in what follows only composition with f_{ji} will appear, to abbreviate the notation we will write $\tilde{a}$ for the composition $a \circ f_{ji}$, and, unless otherwise explicitly stated, all functions will depend on the variable z_i. With this notation, the full compatibility conditions for $N = 1$ curves are as follows:

$$f_{ki} = \tilde{f}_{kj} - \tilde{\gamma}_{kj}\mu_{ji}\ ; \quad \gamma_{ki} = \tilde{\gamma}_{kj} n_{ji} + \left(\widetilde{f'}_{kj} - \widetilde{\gamma'}_{kj}\mu_{ji}\right)\gamma_{ji}\ ;$$
$$\mu_{ki} = \tilde{\mu}_{kj} + \tilde{n}_{kj}\mu_{ji}\ ; \quad n_{ki} = \tilde{n}_{kj} n_{ji} - \left(\widetilde{\mu'}_{kj} - \widetilde{n'}_{kj}\mu_{ji}\right)\gamma_{ji}.$$

The apparent complexity of these expressions already suggests that even the problem of studying $N = 1$ curves in full generality is not trivial. However, for projected and injected curves the formulas above reduce considerably to give:

$$\begin{array}{llll} & M \text{ projected} & f_{ki} = \tilde{f}_{kj} & -- \\ & & \mu_{ki} = \tilde{\mu}_{kj} + \tilde{n}_{kj}\mu_{ji} & n_{ki} = \tilde{n}_{kj} n_{ji} \\ (2) & & & \\ & M \text{ injected} & f_{ki} = \tilde{f}_{kj} & \gamma_{ki} = \tilde{\gamma}_{kj} n_{ji} + \widetilde{f'}_{kj}\gamma_{ji} \\ & & -- & n_{ki} = \tilde{n}_{kj} n_{ji} \end{array}$$

One remarks at once that for both projected and injected supercurves, the f_{ji} define a new $(1,0)$ supercurve, that we denote $M_0 = (M_{\text{red}}, \mathcal{O}_0)$, and then the n_{ji} are the transition functions for a line bundle $\mathcal{N}$ over M_0 (i.e., a rank 1, locally free sheaf of $\mathcal{O}_0$-modules), "sitting" inside $\mathcal{O}$. In fact, there are exact sequences (of sheaves of $\mathcal{O}_0$-modules):

For projected curves

$$0 \to \mathcal{O}_0 \to \mathcal{O} \to \mathcal{N} \to 0.$$

For injected curves

$$0 \to \mathcal{N} \to \mathcal{O} \to \mathcal{O}_0 \to 0.$$

In particular, we have the corresponding morphisms of supercurves: a projection $M \to M_0$ for projected curves and an injection $M_0 \to M$ for injected ones, thereby justifying the terminology. The proof of these facts is quite standard: Over a coordinate neighborhood the inclusions and projections are the more or less obvious ones; thus, for instance for projected curves, to get the injective map of sheaves $\mathcal{O}_0 \to \mathcal{O}$, we map a local section f_i in $\mathcal{O}_0(U_i)$ to f_i viewed as section in $\mathcal{O}(U_i)$, while to get the projection $\mathcal{O} \to \mathcal{N}$, a section $f_i + \theta_i \gamma_i$ in $\mathcal{O}(U_i)$ is mapped to γ_i in $\mathcal{N}(U_i)$. Moreover, by construction, this also makes exactness rather clear at this level (i.e., at the presheaf level). But then, by (2) both definitions are consistent on overlaps, so the assertion that the corresponding maps are well defined as sheaf morphisms is essentially the same thing as the definition of projectedness (or injectedness), together with the consequent existence of the sheaf $\mathcal{N}$.

Remark. The reduced parts of the transition functions define the split curve, M_{sp}, associated to M, whose structure sheaf is $\mathcal{O}_{\mathrm{sp}} = (\mathcal{O}_{\mathrm{red}}|\mathcal{N}_{\mathrm{red}}) \otimes \Lambda$, where $\mathcal{N}_{\mathrm{red}}$ is an ordinary line bundle, defined by the reduced part of n_{ji}. (Recall that a vertical bar means "direct sum of even and odd parts.") Thus, if the f_{ji} and n_{ji} do not involve the odd constants η_α, one has

$$\mathcal{O}_0 = \mathcal{O}_{\mathrm{red}} \otimes \Lambda \ ; \quad \mathcal{N} = \mathcal{N}_{\mathrm{red}} \otimes \Lambda.$$

In particular, for split curves the sequences above become

$$0 \to \mathcal{O}_{\mathrm{red}} \otimes \Lambda \to \mathcal{O} \to \mathcal{N} \otimes \Lambda \to 0 \ ; \quad 0 \to \mathcal{N} \otimes \Lambda \to \mathcal{O} \to \mathcal{O}_{\mathrm{red}} \otimes \Lambda \to 0,$$

and these sequences split.

2. Relation to duality.

There is clearly some kind of duality between the two exact sequences above, which can be neatly related to the $N = 2$ super Riemann surface M_2, and the dual curve $\hat{M}$, associated to M.

In order to see this, first recall that M_2 is constructed by adjoining an odd variable, say ρ, that essentially transforms according to the Berezinian sheaf of M. More precisely, the new variable transforms according to

$$\rho_j = \frac{\gamma_{ji}}{n_{ji}} - \theta_i \rho_i \left(\frac{\gamma_{ji}}{n_{ji}} \right)' + \rho_i \frac{f'_{ji} n_{ji} - \mu'_{ji} \gamma_{ji}}{n_{ji}^2},$$

the coefficient of ρ being the Berezinian of the change of coordinates in M. However, since we can add to the coefficient of ρ_i in this expression *anything* multiplied by $\theta_i \rho_i$, using a formula similar to eq. (1) (namely $f(z - \theta\rho) = f(z) - \theta\rho f'(z)$) this may be rewritten as

$$\rho_j = \frac{\gamma_{ji}}{n_{ji}}(\hat{z}_i) + \rho_i \frac{f'_{ji} n_{ji} - \mu'_{ji} \gamma_{ji}}{n_{ji}^2}(\hat{z}_i),$$

where $\hat{z}_i = z_i - \theta_i\rho_i$. The new variable $\hat{z}_i$ then becomes the even coordinate for the dual curve $\hat{M}$ (with ρ the odd coordinate), and one way to see that this makes global sense is to directly compute

$$\begin{aligned}\hat{z}_j &= z_j - \theta_j\rho_j\\ &= f_{ji} - \theta_i\rho_i f'_{ji} - \frac{\mu_{ji}\gamma_{ji}}{n_{ji}} + \theta_i\rho_i\left(\frac{\mu_{ji}\gamma_{ji}}{n_{ji}}\right)' + \rho_i\mu_{ji}\frac{f'_{ji}n_{ji} - \mu'_{ji}\gamma_{ji}}{n_{ji}^2},\end{aligned}$$

which, for the same reason as before, can be rewritten as

$$\hat{z}_j = f_{ji}(\hat{z}_i) - \frac{\mu_{ji}\gamma_{ji}}{n_{ji}}(\hat{z}_i) + \rho_i\left(\mu_{ji}\frac{f'_{ji}n_{ji} - \mu'_{ji}\gamma_{ji}}{n_{ji}^2}\right)(\hat{z}_i),$$

showing that there is no θ dependence in the transformation rules for $\hat{z}$ and ρ, and therefore that the supercurve $\hat{M}$ is well defined.

Of course, one would still have to check that these are indeed coordinate changes, but we will omit the (mostly long and tedious) verification of this (well known, cf. [DRS]) fact because, anyway, the point in explicitly writing these formulas is most easily seen if we now go back to the special cases of projected and injected curves, to make the following table:

	Coordinates of M	Coordinates of $\hat{M}$
M projected	$z_j = f_{ji}$;	$\hat{z}_j = f_{ji} + \rho_i\mu_{ji}\frac{f'_{ji}}{n_{ji}}$;
	$\theta_j = \mu_{ji} + \theta_i n_{ji}$	$\rho_j = \rho_i\frac{f'_{ji}}{n_{ji}}$
M injected	$z_j = f_{ji} + \theta_i\gamma_{ji}$;	$\hat{z}_j = f_{ji}$;
	$\theta_j = \theta_i n_{ji}$	$\rho_j = \frac{\gamma_{ji}}{n_{ji}} + \rho_i\frac{f'_{ji}}{n_{ji}}$

(where for $\hat{M}$ the functions depend on $\hat{z}_i$). One sees immediately from this table that

Proposition 1. *The dual curve to a projected one is injected and vice versa.*

□

Remark. It is also worthwhile to remark that the bundle corresponding to $\mathcal{N}_{\mathrm{red}}$ for the dual curve $\hat{M}$ is its Serre dual, i.e., $\mathcal{K}\mathcal{N}_{\mathrm{red}}^{-1}$, where $\mathcal{K}$ is the canonical bundle of $M_{\rm red}$. Thus, duality for supercurves is in a sense an extension of ordinary Serre duality; in fact, for the special case of split supercurves, it is well known (and easy to see from what has been said) that the corresponding structure sheaves are just

$$\mathcal{O} = (\mathcal{O}_{\rm red}|\mathcal{N}_{\rm red}) \otimes \Lambda \ ; \quad \hat{\mathcal{O}} = (\mathcal{O}_{\rm red}|\mathcal{K}\mathcal{N}_{\rm red}^{-1}) \otimes \Lambda.$$

For the sake of completeness, let us finally recall that the $N = 2$ supercurve described here is in fact a super Riemann surface, or SRS, which essentially means that it has a global rank $0|2$ nonintegrable distribution, locally generated by $D^+ = \partial_\rho$ and $D^- = \partial_\theta + \rho\partial_z$; the sections annihilated by D^+ are called *chiral* sections, those annihilated by D^- *antichiral*.

Moreover, the structure sheaf of M_2 fits into the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}_2 \to \mathcal{B}er \to 0, \tag{3}$$

where $\mathcal{B}er$ denotes the Berezinian sheaf of M, and there is a similar sequence for $\hat{M}$, namely

$$0 \to \hat{\mathcal{O}} \to \mathcal{O}_2 \to \hat{\mathcal{B}}er \to 0.$$

The corresponding projections are just the operators D^+ and D^-, respectively.

(Actually, to be precise one should write the coordinates of M as (z, θ), those of $\hat{M}$ as $(\hat{z}, \rho)$, and those of M_2 as (x, ξ, η); then the projections

$$\pi_1 \colon M_2 \to M \ ; \ \pi_2 \colon M_2 \to \hat{M},$$

coming from the first arrows in the sequences above, become

$$\pi_1^* z = x \ ; \ \pi_1^*\theta = \xi \ ; \ \pi_2^*\hat{z} = x - \xi\eta \ ; \ \pi_2^*\rho = \eta.$$

To avoid cluttering the notation one usally does not write the π's and does not distinguish between the coordinates of M and those in the embedded image in M_2, etc., but in some instances this might become necessary.)

3. $N = 1$ line bundles.

Let us now consider line bundles over M (i.e., $N = 1$ bundles). Recall that line bundles are defined as rank 1 locally free sheaves of $\mathcal{O}$-modules, and are determined by transition functions $\Gamma_{ji} = a_{ji} + \theta_i \alpha_{ji}$, with a_{ji} invertible, satisfying a cocycle condition. For an arbitrary bundle, the cocycle condition reads:

$$\begin{aligned} a_{ki} &= (\tilde{a}_{kj} - \tilde{\alpha}_{kj}\mu_{ji})a_{ji} \\ \alpha_{ki} &= (\tilde{a}_{kj} - \tilde{\alpha}_{kj}\mu_{ji})\alpha_{ji} + \big((\widetilde{a'}_{kj} - \widetilde{\alpha'}_{kj}\mu_{ji})\gamma_{ji} + \tilde{\alpha}_{kj} n_{ji}\big)a_{ji}. \end{aligned}$$

Again, we see that in general these conditions mix in a rather complicated way the functions defining the bundle and the curve and, for instance, it is known that the cohomology groups are Λ-modules, but in general not free. Thus, in order to get more detailed results we will have to impose some extra conditions on M. But first let us make the following observation: since $a_{ji} \neq 0$, the transition functions of a bundle can be factorized in the form

$$\Gamma_{ji} = a_{ji}(1 + \theta_i \frac{\alpha_{ji}}{a_{ji}})$$

This suggests a (naive) subdivision of line bundles into even ones, having transition functions of the form a_{ji}, and odd ones, having transition functions $1+\theta_i\alpha_{ji}$ (observe that $1 + \theta_i\alpha_{ji} = \exp(\theta_i\alpha_{ji})$).

In fact, in the case of split manifolds, the cocycle conditions simplify to

$$a_{ki} = \tilde{a}_{kj}a_{ji} \ ; \quad \alpha_{ki} = \alpha_{ji} + \tilde{\alpha}_{kj}n_{ji},$$

so that the conditions are indeed consistent (the former determining the even bundles and the latter the odd ones). This shows in particular that the even bundles are completely determined by an ordinary line bundle $\mathcal{L}$ over $M_{\rm red}$ (the reduced bundle, given by the reduced expressions of the a_{ji}); put another way, even line bundles are always of the form $\mathcal{O} \otimes \mathcal{L}$. In particular, this gives a proof of the well known result that the even cohomology group $H^1_{\rm ev}(\mathcal{O})$ of a split manifold is free, and its rank as a Λ module is the genus g of $M_{\rm red}$.

For odd bundles the situation is slightly more complicated: due to the unavoidable presence of nilpotent terms in the transition functions, odd bundles are not determined by their reduced part (put another way, the reduced bundle of an odd bundle is always trivial, even when the bundle itself is not). Nevertheless, as is also well known, $H^1_{\rm odd}(\mathcal{O})$ is also free, since odd bundles are also of the form $\mathcal{O} \otimes \mathcal{H}$, but where now $\mathcal{H}$ is not an $\mathcal{O}_{\rm red}$-locally free sheaf, but rather a quotient sheaf (this is because one has to "quotient out" the terms quadratic in θ in the transition functions $\exp(\theta_i\alpha_{ji})$). Its rank can be computed from the super Riemann-Roch theorem (which is valid in this case): for a locally free sheaf of $\mathcal{O}$-modules $\mathcal{L}$ the super Riemann-Roch formula reads

$$h^0(M, \mathcal{L}) - h^1(M, \mathcal{L}) = \deg \mathcal{L} + 1 - g | \deg \mathcal{L} + \deg \mathcal{N} + 1 - g,$$

so that for $\mathcal{L} = \mathcal{O}$ we get as rank for $H^1_{\rm odd}$, $g - 1 - \deg \mathcal{N}$.

Going back to the general case, if we try to subdivide bundles into even and odd ones, we would get as cocycle conditions

$$a_{ki} = \tilde{a}_{kj}a_{ji} \ ; \qquad \widetilde{\alpha'}_{kj}\gamma_{ji}a_{ji} = 0 \qquad \text{(even bundles)}$$

$$1 = 1 - \tilde{\alpha}_{kj}\mu_{ji} \ ; \quad \alpha_{ki} = (1 - \tilde{\alpha}_{kj}\mu_{ji})\alpha_{ji} + \tilde{\alpha}_{kj}n_{ji} - \widetilde{\alpha'}_{kj}\mu_{ji}\gamma_{ji} \qquad \text{(odd bundles).}$$

But we see that in general neither of these conditions is consistent (in each case at least one of the equations cannot hold), so such a subdivision makes no sense for arbitrary supercurves. In fact (cf. [BR], theorem 2.6.1), the result in the general case is that the cohomology groups $H^1(M)$ are quotients of the corresponding groups (free Λ-modules) of the associated split supermanifolds, and the point is that these quotients do not split into even and odd direct summands. However, by the very definition of projectedness and injectedness, it is also clear that the conditions for even bundles are consistent for projected curves, while for injected curves it is the conditions for odd bundles that are consistent.

As a consequence of this, the even cohomology groups for projected curves, and the odd cohomology groups for injected curves, are well defined, and moreover, if the f_{ji} and n_{ji} do not involve the odd constants η_α, they are free Λ-modules.

More precisely:

Proposition 2. *Let M be a projected supercurve, then H^0 (respectively H^1) admits a submodule of even sections (a quotient submodule defining even line bundles). If the f_{ji} and n_{ji} do not involve the constants η_α, they are free and of the maximum possible rank, as given by the super Riemann-Roch formula stated above. For injected curves, replace even by odd.*

Proof: This follows easily from the discussion in section 1, by considering the cohomology sequences of the short exact sequences constructed there. Indeed, suppose for instance that the f_{ji} and n_{ji} do not involve the odd constants η_α, and that we want to prove the assertion regarding the even cohomology of projected curves. Consider the exact sequence

$$0 \to \mathcal{O}_{\text{red}} \otimes \Lambda \to \mathcal{O} \to \mathcal{N}_{\text{red}} \otimes \Lambda \to 0;$$

then, the two ends of the corresponding cohomology sequence are

$$0 \to H^0(M_{\text{red}}, \mathcal{O}_{\text{red}}) \otimes \Lambda \to H^0(M_{\text{red}}, \mathcal{O}) \to \dots$$

and

$$\dots \to H^1(M_{\text{red}}, \mathcal{O}) \to H^1(M_{\text{red}}, \mathcal{N}_{\text{red}}) \otimes \Lambda \to 0,$$

the first giving a submodule of $H^0(M_{\text{red}}, \mathcal{O})$ and the second a quotient module of $H^1(M_{\text{red}}, \mathcal{O})$; their ranks can be easily computed from the super Riemann-Roch formula above (or even from the standard Riemann-Roch theorem in this case) and are obviously maximal, since they coincide with the ranks of the even cohomology groups of M_{sp}, as claimed.

□

4. $N=2$ line bundles.

Let us now consider line bundles over the associated $N=2$ SRS, M_2. We ask what are the conditions for an $N=2$ line bundle to split as a tensor product of line bundles over M and $\hat{M}$.

First of all, we make the observation that any $N=2$ local superfunction

$$\Gamma = h + \theta_i\phi + \rho_i\psi + \theta_i\rho_i g$$

can be decomposed as a product of a function on M and a function on $\hat{M}$, i.e.,

$$\Gamma = \big(a(z_i) + \theta_i\alpha(z_i)\big)\big(b(\hat{z}_i) + \rho_i\beta(\hat{z}_i)\big).$$

In fact

$$h = ab\ ;\ \phi = \alpha b\ ;\ \psi = a\beta\ ;\ g = -(ab' + \alpha\beta)$$

(where according to our convention all functions depend on z_i); since we can obtain b'/b from them, these conditions actually determine b up to a multiplicative constant (and β also); a is then determined up to the *reciprocal* constant (and so is α).

Remark. There is a similar decomposition of $N=2$ local superfunctions as sums of $N=1$ superfunctions, $\Gamma(z_i,\theta_i,\rho_i) = F(z_i,\theta_i) + \hat{F}(\hat{z}_i,\rho_i)$, modulo an additive constant. This, of course, is essentially "taking the logarithm" of the decomposition above, and is a manifestation of the existence of an exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^{\times} \to 0,$$

where, as usual, $^{\times}$ denotes the invertible elements; however, we will not need this construction in what follows.

Now, if Γ_{ji} are the transition functions of an $N=2$ line bundle, written as above, since the a_{ji} and b_{ji} themselves are only determined up to a constant, say c_{ji}, what this actually gives is the existence of a short exact sequence of the form

$$0 \to \Lambda^{\times}_{\text{ev}} \to \mathcal{O}^{\times}_{\text{ev}} \times \hat{\mathcal{O}}^{\times}_{\text{ev}} \to \mathcal{O}^{\times}_{2,\text{ev}} \to 0, \tag{4}$$

where $\mathcal{O}^{\times}_{\text{ev}} \times \hat{\mathcal{O}}^{\times}_{\text{ev}}$ is the sheaf described at the presheaf level by taking direct products of the groups $\mathcal{O}^{\times}_{\text{ev}}(U_i)$ and $\hat{\mathcal{O}}^{\times}_{\text{ev}}(U_i)$. The second arrow is the map $c \mapsto (c, c^{-1})$ and the third is $(F,\hat{F}) \mapsto F\hat{F}$. Notice that all objects appearing in the construction of the sequence above are *even*; in particular the sheaves are sheaves of abelian groups. (And to be more precise, recall that we should write such sequences as

$$0 \to \Lambda^{\times}_{\text{ev}} \to \pi_1^*\mathcal{O}^{\times}_{\text{ev}} \times \pi_2^*\hat{\mathcal{O}}^{\times}_{\text{ev}} \to \mathcal{O}^{\times}_{2,\text{ev}} \to 0.)$$

We then have the following result:

Proposition 3. *Let M be an $N = 1$ supercurve. Then for any given line bundle $\mathcal{L}$ over the associated $N = 2$ SRS M_2, there exists a cohomology class in $H^2(M_{\rm red}, \Lambda_{\rm ev}^\times)$, that measures the obstruction to express $\mathcal{L}$ as a tensor product of line bundles over M and $\hat{M}$.*

Proof: Recall that in general, line bundles over a complex (super) manifold are classified by the first cohomology group, $H^1(\mathcal{O}_{\rm ev}^\times)$.

Now, taking the cohomology sequence of the short exact sequence (4) above, the last part reads

$$\cdots \to H^1(M_{\rm red}, \mathcal{O}_{\rm ev}^\times \times \hat{\mathcal{O}}_{\rm ev}^\times) \to H^1(M_{\rm red}, \mathcal{O}_{2,\rm ev}^\times) \xrightarrow{\beta} H^2(M_{\rm red}, \Lambda^\times) \to 0. \tag{5}$$

However, there is a canonical map of cohomology groups

$$H^1(M_{\rm red}, \mathcal{O}_{\rm ev}^\times) \times H^1(M_{\rm red}, \hat{\mathcal{O}}_{\rm ev}^\times) \to H^1(M_{\rm red}, \mathcal{O}_{\rm ev}^\times \times \hat{\mathcal{O}}_{\rm ev}^\times),$$

and this map is in fact an isomorphism, because the domain and range are both defined by equivalence relations on pairs of cocycles of transition functions. Indeed, the isomorphism is (essentially) determined by the assignment $\big(F(z), \hat{F}(\hat{z})\big) \mapsto \big(F(z), \hat{F}(z) - \theta\rho\hat{F}'(z)\big)$.

Therefore, we have a group morphism

$$\delta\colon H^1(M_{\rm red}, \mathcal{O}_{\rm ev}^\times) \times H^1(M_{\rm red}, \hat{\mathcal{O}}_{\rm ev}^\times) \to H^1(M_{\rm red}, \mathcal{O}_{2,\rm ev}^\times),$$

whose image, by construction, consists of the isomorphism classes of $N = 2$ bundles that can be expressed as tensor product of bundles over M and $\hat{M}$. Thus, a line bundle over M_2 can be decomposed as a tensor product if and only if the corresponding class is in the image of δ, which equals the kernel of β in (5); this gives the cohomology obstruction whose existence was asserted.

□

Let us make some remarks in relation to this result.

First of all, a more concrete proof of the proposition can be given along the following lines: Given a cocycle of transition functions for an $N = 2$ line bundle $\mathcal{L}$, we can form the quotients

$$\frac{D^+\Gamma_{ji}}{\Gamma_{ji}}\ ; \quad \frac{D^-\Gamma_{ji}}{\Gamma_{ji}},$$

and these are in fact sections of $\mathcal{B}er(M)$ and $\hat{\mathcal{B}}er(M)$ respectively. By indefinite integration of these local sections and then exponentiation, up to multiplicative constants one gets functions F_{ji} and $\hat{F}_{ji}$, and this gives a factorization of the form

$\Gamma = F\hat{F}$ for the transition functions of $\mathcal{L}$. Now, the cocycle conditions for Γ_{ji} show that on triple overlaps one has

$$\log F_{ji} + \log F_{kj} + \log F_{ki} = \log \hat{F}_{ji} + \log \hat{F}_{kj} + \log \hat{F}_{ki}$$

modulo some integers, representing the Chern class of the bundle $\mathcal{L}$. But the left hand side is chiral while the right hand side is antichiral, so that they are necessarily equal to a constant, say c_{kji}; then $\exp c_{kji}$ gives an explicit representative for the desired class in $H^2(M_{\mathrm{red}}, \Lambda_{\mathrm{ev}}^{\times})$.

On the other hand, the proof of the proposition given above sheds some light on an interesting phenomenon. Namely, it was known that for certain supercurves (nonprojected generic SKP curves, cf. below) there are examples of nontrivial $N = 1$ bundles $\mathcal{L}$ giving a nontrivial factorization $\mathcal{O}_2 = \mathcal{L} \otimes \hat{\mathcal{O}}$, in addition to the trivial one, $\mathcal{O}_2 = \mathcal{O} \otimes \hat{\mathcal{O}}$.

This can be explained as follows:

By going one further step backwards in the cohomology sequence, we get an exact sequence of the form

$$\cdots \to H^1(M_{\mathrm{red}}, \Lambda_{\mathrm{ev}}^{\times}) \to H^1(M_{\mathrm{red}}, \mathcal{O}_{\mathrm{ev}}^{\times}) \times H^1(M_{\mathrm{red}}, \hat{\mathcal{O}}_{\mathrm{ev}}^{\times}) \to H^1(M_{\mathrm{red}}, \mathcal{O}_{2,\mathrm{ev}}^{\times}) \to \dots$$

By construction, the second arrow in this sequence maps a flat line bundle (or, to be precise, its isomorphism class), say $\mathcal{L}$, defined by the cocycle of constant transition functions c_{ji}, to the pair of bundles $(\mathcal{L}, \mathcal{L}^{-1})$, defined by the pair (c_{ji}, c_{ji}^{-1}), where the bundles of the pair are regarded as being over M and $\hat{M}$ respectively.

Thus, the assertion above is that the cocycle c_{ji} might be trivial when seen as representing an element of $H^1(M_{\mathrm{red}}, \mathcal{O}_{\mathrm{ev}}^{\times})$ (i.e., as defining a bundle over M), *but not* when seen as an element of $H^1(M_{\mathrm{red}}, \hat{\mathcal{O}}_{\mathrm{ev}}^{\times})$ (i.e., as defining a bundle over $\hat{M}$), or vice versa. But this might very well happen, because, as we have already said, in general $H^1(M_{\mathrm{red}}, \mathcal{O}_{\mathrm{ev}}^{\times})$ and $H^1(M_{\mathrm{red}}, \hat{\mathcal{O}}_{\mathrm{ev}}^{\times})$ are both quotients of the free rank g Λ-module $H^1(M_{\mathrm{red}}, \mathcal{O}_{\mathrm{sp,ev}})$ (where g is the genus of M_{red}), and the two quotients *are different in general*. Moreover, this also shows that the bundles allowed in a nontrivial decomposition of $\mathcal{O}_2$ are necessarily flat (hence of degree zero).

Finally, let us point out that one can gain some further insight into the meaning of this cohomological obstruction by considering the relatively simple but important case of a generic SKP curve (which, by the way, are the curves needed for studying the super KP equation). Recall that generic SKP curves are curves for which $\deg \mathcal{N} = 0$, $\mathcal{N} \neq \mathcal{O}$, and the point is that they have free cohomology, so in this case $h^1(M, \mathcal{O}) = g|g-1$, and $h^1(M_2, \mathcal{O}_2) = g+1|g-1$, while $H^1(\hat{M}, \hat{\mathcal{O}})$ is in general a quotient of a free rank $g|0$ Λ-module. We know that all bundles over M and $\hat{M}$ lift to M_2, but the bundles coming from $\hat{M}$ lie in the same submodule as those lifted from M.

Thus, there is an "extra" even bundle on M_2 that does not come from bundles over M or $\hat{M}$, but rather from the group $H^1(M, \mathcal{B}er)$, appearing when one considers the long cohomology sequence associated to (3); this group is the Serre dual to $H^0(M, \mathcal{O})$ in the category of supercurves. Now, to identify this bundle, consider a covering of M_{red} consisting of an open disk D centered at a point P, with coordinate $z = 0$, and $M_{\text{red}} \setminus \{P\}$; then the extra line bundle has as transition function in the annulus

$$1 - k\frac{\theta\rho}{z} = \frac{1}{z^k}(z - \theta\rho)^k.$$

(That this has to be the form of the transition function can be justified by the fact that the residue mapping is integration of the principal parts representing elements of $H^1(M, \mathcal{B}er)$, which has rank $0|1$, so the dual of a constant function k should represent a bundle having a transition function with a pole of the form k/z; see [BR] for the details on these constructions). The point is that when k is not an integer, z^k *is not* single valued in $D \setminus \{P\}$, so the functions appearing in the decomposition in these cases cannot define line bundles over M and $\hat{M}$, and one sees that some (indeed, most) of these bundles cannot come from tensor products of bundles.

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