ROTATIONAL SYMMETRY OF SOLUTIONS OF A NONLINEAR ELLIPTIC EQUATION IN A BALL

Matthias Birkner, José A. López-Mimbela and Anton Wakolbinger

Comunicación Técnica No I-02-26/08-11-2002
(PE/CIMAT)

ROTATIONAL SYMMETRY OF SOLUTIONS OF A NONLINEAR ELLIPTIC EQUATION IN A BALL

Matthias Birkner   José A. López-Mimbela

Anton Wakolbinger

Abstract

We extend the method of moving planes, which is well known in the Laplacian case, to study symmetry of positive solutions of the elliptic equation $-(-\Delta)^{\alpha/2}u(x) + F(u(x)) = 0$, $x \in B$, $u = 0$ in ∂B. Essential tools (like Hopf's boundary lemma) are carried over to fractional powers of the Laplacian. The radial symmetry of positive solutions of the above equation is studied in the case of a ball $B \subset \mathbb{R}^d$ and a positive non-decreasing function F.

1 Introduction

Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing, not identically constant. Let $u : \mathbb{R}^d \to \mathbb{R}_+$ be a nonnegative bounded solution of

$$\Delta_\alpha u + F(u) = 0,\ x \in B_1(0), \quad u \equiv 0 \text{ in } B_1(0)^c, \tag{1.1}$$

i.e., for all x in the unit ball we have

$$u(x) = \int_0^\infty (T_s F(u))(x)\, ds = \int_{B_1(0)} G_\alpha(x,y) F(u(y))\, dy. \tag{1.2}$$

Here $\alpha \in (0,2)$, $\Delta_\alpha = -(-\Delta)^{(\alpha/2)}$ is the generator of the symmetric α-stable process (X_t), and $G_\alpha(x,y)$ is the Green's function for the unit ball of (X_t). Without loss of generality we assume that $u \not\equiv 0$. Our aim here is to show that

$$u \textbf{ is rotationally symmetric about the origin.} \tag{1.3}$$

Our approach is based on the celebrated *method of moving planes,* a device that goes back to Alexandrov [1] and has by now a venerable history in the study of symmetries of solutions of pde's, see also [3],[4], [5], [6] and [8]. The idea is as follows: Choose any direction in $\mathbb{R}^d$, wlog the x_1-direction, and

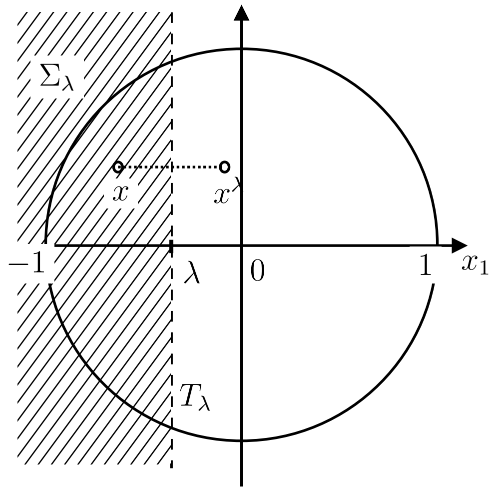

Figure 1: The moving planes method.

show that u is mirror symmmetric with respect to the hyperplane through the origin with this given direction as a normal vector. In order to achieve this let us define for $\lambda \in (-1,1)$

$$T_\lambda := \{x \in \mathbb{R}^d : x_1 = \lambda\}, \quad \Sigma_\lambda := \{x \in \mathbb{R}^d : x_1 < \lambda\}$$

and for $x \in \mathbb{R}^d$ let $x^\lambda := (2\lambda - x_1, x_2, \ldots, x_d)$ be the image under reflection along T_λ; see Figure 1.

Define the set Λ by

$$\Lambda := \left\{ \lambda \in (-1,0) : u(x^\lambda) \geq u(x) \;\forall\, x \in \Sigma_\lambda, \frac{\partial}{\partial x_1} u(x) > 0 \;\forall\, x \in T_\lambda \cap B_1(0) \right\}. \tag{1.4}$$

Observe that by the minimum principle we have $u(x^\lambda) > u(x)$ for $x \in \Sigma_\lambda \cap B_1(0)$ and $\lambda \in \Lambda$, i.e., $\lambda \in \Lambda$ means that reflection along T_λ (strictly) increases the value of u. In Section 3 we prove that

$$\sup \Lambda = 0 \tag{1.5}$$

so that by continuity $u(-x_1, x_2 \ldots, x_d) \geq u(x_1, x_2 \ldots, x_d)$ whenever $x_1 \leq 0$. By considering $\lambda > 0$ and working in the opposite direction we can then conclude the reversed inequality and hence obtain the desired symmetry.

2 Some preparatory lemmas

Lemma 2.1 *A bounded solution u of (1.2) satisfies $u \in C(\mathbb{R}^d) \cap C^\infty(B_1(0))$.*

Proof Use the explicit form of Green's kernel.

Lemma 2.2 (Minimum principle) *Let $D \subset \mathbb{R}^d$ be a bounded domain. Suppose $u : \mathbb{R}^d \to \mathbb{R}_+$ is continuous, $\Delta_\alpha u \leq 0$ on D, and satisfies $u \equiv 0$ on D^c. Then either $u \equiv 0$ or $u > 0$ on D.*

Proof Let $D_\varepsilon := \{x \in D : u(x) > \varepsilon\}$. By continuity D_ε is open. Assume that $D_\varepsilon \neq \emptyset$ for some $\varepsilon > 0$. Let (X_t) be the α-stable process, and $\tau := \inf\{s : X_s \notin D\}$ the hitting time of D^c. Then $M_t := u(X_{t\wedge\tau}) - u(X_0) - \int_0^{t\wedge\tau} \Delta_\alpha u(X_s)\, ds$ is a $\mathbf{P}_x$-martingale for each $x \in D$. Let furthermore $\tau' := \inf\{s : X_s \in D_\varepsilon\}$. For each $x \in D$ we have $\mathbb{E}_x M_{\tau'} = 0$, or

$$\begin{aligned} u(x) &= \mathbb{E}_x\left[u(X_{\tau'\wedge\tau})\right] + \mathbb{E}_x\left[\int_0^{\tau'\wedge\tau} (-\Delta_\alpha u)(X_s)\, ds\right] \\ &\geq \mathbb{E}_x\left[u(X_{\tau'\wedge\tau})\right] \;\geq\; \varepsilon \mathbf{P}_x(\tau' < \tau) > 0 \end{aligned}$$

because (X_t) hits any open subset of D with positive probability before exiting from D. □

We will have occasion to consider the behaviour of u at the boundary of the ball. In this respect, the following lemma is helpful:

Lemma 2.3 (Hopf's α-stable boundary lemma) *Let $D \subset \mathbb{R}^d$ be open, $u : \mathbb{R}^d \to \mathbb{R}_+$ continuous with $u \equiv 0$ on D^c, $\Delta_\alpha u \leq 0$ on D. Let $x_0 \in \partial D$*

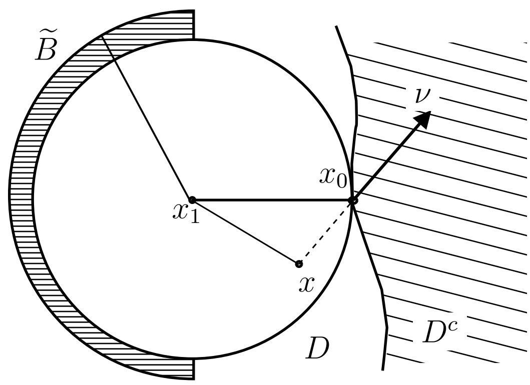

Figure 2: The Hopf's boundary lemma.

satisfy an interior sphere condition, *i.e. there exists a ball* $B_\delta(x_1) \subset D$ *with* $\overline{B_\delta(x_1)} \cap D^c = \{x_0\}$, *and let* ν *be an outward pointing unit vector at* x_0. *Then*

$$\frac{\partial}{\partial \nu} u(x_0) < 0$$

(in fact, $\lim_{\varepsilon \searrow 0}(u(x_0) - u(x_0 - \varepsilon\nu))/\varepsilon = -\infty$*).*

Proof Because of the interior sphere condition at x_0 we can find a ball $B_\delta(x_1) \subset D$ such that $\overline{B_\delta(x_1)} \cap D^c = \{x_0\}$ and also $\widetilde{B} \subset D$, where $\widetilde{B}$ is the "left half" of a spherical shell around x_1 with interior radius δ and exterior radius $\delta' > \delta$; see Fig. 2. Observe that $u > 0$ in D by Lemma 2.2, in particular $\inf_{\widetilde{B}} u > 0$ because $\widetilde{B}$ is compact and u continuous. Let $\tau := \inf\{t : X_t \notin B_\delta(x_1)\}$. Then

$$u(x) = \mathbb{E}_x \left[u(X_\tau) - \int_0^\tau (\Delta_\alpha u)(X_t)\, dt \right] \geq \mathbb{E}_x\, u(X_\tau)$$

for $x \in B_\delta(x_1)$. Take $x = x_1 - \varepsilon\nu$ with $\varepsilon > 0$ small enough, and denote by γ the angle between $\overline{x_1\, x_0}$ and $\overline{x\, x_0}$. Then $\gamma \in (-\pi/2, \pi/2)$ because ν is an outward pointing vector, hence $\cos\gamma > 0$. The cosine theorem gives $|x - x_1|^2 = \delta^2 + \varepsilon^2 - 2\delta\varepsilon\cos\gamma$. Using the explicit form of the Poisson kernel for the complement of a ball (see e.g. [2]) we can estimate

$$\begin{aligned} u(x) &\geq \mathbb{E}_x\, u(X_\tau) \geq \int_{\widetilde{B}} P(x,y) u(y)\, dy \\ &\geq \left(\inf_{\widetilde{B}} u\right) C_{\alpha,d} \int_{\widetilde{B}} \left(\frac{\delta^2 - |x-x_1|^2}{|y-x_1|^2-\delta^2}\right)^{\alpha/2} |x-y|^{-d}\, dy \\ &\geq C\left(\delta^2 - |x-x_1|^2\right)^{\alpha/2} \geq C'\varepsilon^{\alpha/2}. \end{aligned}$$

Thus we see that $\limsup_{\varepsilon\searrow 0}(u(x_0) - u(x_0 - \varepsilon\nu))/\varepsilon = -\infty$. $\square$

Lemma 2.4 *Let $w : \mathbb{R}^d \to \mathbb{R}$ be continuous, bounded with $w \geq 0$ on Σ_0, $w(x^0) = -w(x)$. Let $x_* \in T_0$ be such that there exists a $\delta > 0$ with $\Delta_\alpha w \leq 0$ on $B_\delta(x_*) \cap \Sigma_0$. Then either*

$$w \equiv 0, \quad \textit{or} \quad w > 0 \textit{ on } B_\delta(x_*) \cap \Sigma_0 \textit{ and } \frac{\partial}{\partial x_1} w(x_*) < 0.$$

Proof Assume $w \not\equiv 0$. Let $x \in B_\delta(x_*) \cap \Sigma_0$. If $w(x) = 0$ we would have (see e.g. [2])

$$\Delta_\alpha w(x) = c_{\alpha,d}\mathrm{PV}\int_{\mathbb{R}^d} \frac{w(x+y) - 0}{|y|^{d+\alpha}} dy > 0$$

by the non-triviality and symmetry of w in contradiction to the assumption.

To show that the derivative is non-zero choose $\delta' > 0$ such that

$$\sup_{z \in \Sigma_0 \setminus B_{\delta'}(x_*)} w(z) > 0.$$

Let $\tau := \inf\{t : X_t \notin B_{\delta'}(x_*)\}$ where (X_t) is the symmetric α-stable process. Define $v(x) := \mathbb{E}_x w(X_\tau)$, $\widetilde{v}(x) := \mathbb{E}_x \int_0^\tau (-\Delta_\alpha w)(X_s)\, ds$. Observe that

$$\Delta_\alpha v = 0 \text{ in } B_{\delta'}(x_*) \text{ and } v = w \text{ in } \mathbb{R}^d \setminus B_{\delta'}(x_*),$$

$$\Delta_\alpha \widetilde{v} = \Delta_\alpha w \text{ in } B_{\delta'}(x_*) \text{ and } \widetilde{v} = 0 \text{ in } \mathbb{R}^d \setminus B_{\delta'}(x_*).$$

Uniqueness of the Dirichlet problem for Δ_α in $B_{\delta'}(x_*)$ thus gives $w = v + \widetilde{v}$. We have $v(x) = \int_{|y-x_*|>\delta'} P(x,y)w(y)\, dy$ for $x \in B_{\delta'}(x_*)$, where the Poisson kernel is given by (see e.g. [2], formula (2.2))

$$P(x,y) = C_{\alpha,d}\left[\frac{\delta'^2 - |x-x_*|^2}{|y-x_*|^2-\delta'^2}\right]^{\alpha/2} |x-y|^{-d}, \quad x \in B_{\delta'}(x_*), y \notin B_{\delta'}(x_*).$$

One checks that $\frac{\partial}{\partial x_1}P(x_*,y) < 0$ for $y \in \Sigma_0$ and $\frac{\partial}{\partial x_1}P(x_*,y) > 0$ for $y \in -\Sigma_0$. The interchange of integration and differentation is justified because w is bounded, so we can compute

$$\frac{\partial}{\partial x_1}v(x_*) = \int_{|y-x_*|>\delta'} \frac{\partial}{\partial x_1}P(x_*,y)w(y)\,dy < 0.$$

Furthermore for $x \in B_{\delta'}(x_*)$

$$\widetilde{v}(x) = \int_{B_{\delta'}(x_*)} G(x,y)(-\Delta_\alpha w)(y)\,dy$$

where the Green kernel for $B_{\delta'}(x_*)$ is given by (see e.g. [2], formula (2.3), but also the remarks section)

$$G(x,y) = c_{\alpha,d}(|x-y|)^{\alpha-d}\int_0^{w_{\delta'}(x,y)} \frac{r^{\alpha/2-1}}{(r+1)^{d/2}}dr, \tag{2.6}$$

where $w_{\delta'}(x,y) = (\delta'^2 - |x-x_*|^2)(\delta'^2-|y-x_*|^2)/|x-y|^2$. Inspection shows that for $x,y \in B_{\delta'}(x_*) \cap \Sigma_0$ we have $G(x,y) \geq G(x,y^0)$, hence $\widetilde{v} \geq 0$ in $B_{\delta'}(x_*) \cap \Sigma_0$ and by symmetry $\widetilde{v} \leq 0$ in $B_{\delta'}(x_*) \cap (-\Sigma_0)$. We conclude that $(\partial/\partial x_1)\widetilde{v}(x_*) \leq 0$ and thus $\frac{\partial w(x_*)}{\partial x_1} \leq \frac{\partial v(x_*)}{\partial x_1} < 0$. □

3 Proof of (1.5)

We proceed in three steps and show that

1. Λ is not empty,
2. Λ is open, more precisely for $\lambda \in \Lambda$ there exists $\varepsilon > 0$ such that $[\lambda, \lambda+\varepsilon) \subset \Lambda$.

3. From 1. and 2. we conclude that $\Lambda = (-1, \lambda_{\max})$. We finally show that $\lambda_{\max} = 0$.

Step 1. Obviously $(-1, 0, \dots, 0)$ is an outward pointing direction for each $x \in \partial B_1(0) \cap \Sigma_{-1/2}$. By the boundary lemma and the fact that $(\partial/\partial x_1)u$ is continuous in $B_1(0)$ there is an open neighborhood D of $\partial B_1(0) \cap \Sigma_{-1/2}$ such that $(\partial/\partial x_1)u > 0$ on $D \cap B_1(0)$. Choose $\varepsilon > 0$ so small that $\Sigma_{-1+\varepsilon} \cap B_1(0) \subset D$. Then $-1 + \varepsilon/2 \in \Lambda$.

Step 2. We argue by contradiction. Assume there was $\lambda_* \in \Lambda$ and also a sequence $(\lambda_n) \subset (-1, 0) \setminus \Lambda$ with $\lambda_n \searrow \lambda_*$. From the definition of Λ, possibly passing to a suitable subsequence (which we again would denote by (λ_n)) we can always arrive at one of the following possibilities:

a) There exists a sequence $(x_n) \subset B_1(0)$, $x_n \in \Sigma_{\lambda_n}$, with $x_n \to x_* \in \overline{B_1(0)}$ and $u(x_n) \geq u(x_n^{\lambda_n})$ for all n, or

b) There exists a sequence $(x_n) \subset B_1(0)$, $x_n \in T_{\lambda_n}$, with $x_n \to x_* \in \overline{B_1(0)}$ and $(\partial/\partial x_1)u(x_n) \leq 0$ for all n.

Assume a) was true. We cannot have $x_* \in \Sigma_{\lambda_*}$ because u is continuous and $u(x^{\lambda_*}) > u(x)$ for $x \in \Sigma_{\lambda_*}$ by the above remark. Hence $x_* \in T_{\lambda_*} \cap \overline{B_1(0)}$. But then we have $(\partial/\partial x_1)u(x_*) = \lim_{n\to\infty}(u(x_n^{\lambda_n}) - u(x_n))/(2\mathrm{d}(x_n, T_{\lambda_n})) \leq 0$. By Hopf's boundary lemma, this forces x_* to be away from $\partial B_1(0)$, but then we obtain a contradiction to $\lambda_* \in \Lambda$.

If b) was true we would again find a point $x_* \in T_{\lambda_*} \cap \overline{B_1(0)}$ with $\frac{\partial}{\partial x_1}u(x_*) \leq 0$ and arrive at a contradiction.

Step 3. From the preceding steps we know that $\Lambda = (-1, \lambda_{\max})$. If $u(x^{\lambda_{\max}}) \equiv u(x)$ for $x \in \Sigma_{\lambda_{\max}}$ then we have found a symmetry center. As u is continuous, $u = 0$ on $\partial B_1(0)$ and strictly positive inside $B_1(0)$ this can only be true for $\lambda_{\max} = 0$. Indeed, if $\lambda_{\max} < 0$ then by continuity we would have $u(x) \leq u(x^{\lambda_{\max}})$ for $x \in \Sigma_{\lambda_{\max}}$, but with $u(x) \not\equiv u(x^{\lambda_{\max}})$. Define $w(x) := u(x^{\lambda_{\max}}) - u(x)$. Observe that w is continuous and bounded, non-negative in $\Sigma_{\lambda_{\max}}$ and $w(x^{\lambda_{\max}}) = -w(x)$. For $x \in \Sigma_{\lambda_{\max}} \cap B_1(0)$ we have

$$\Delta_\alpha w(x) = (\Delta_\alpha)u(x^{\lambda_{\max}}) - (\Delta_\alpha)u(x) = -\left[F(u(x^{\lambda_{\max}})) - F(u(x))\right] \leq 0,$$

and we infer from Lemma 2.4 that $(\partial/\partial x_1)w(x) < 0$ for all $x \in T_{\lambda_{\max}} \cap B_1(0)$. In conclusion, $\lambda_{\max} < 0$ implies $\lambda_{\max} \in \Lambda$ which by Step 2 forces $\sup \Lambda > \lambda_{\max}$. This is a contradiction.

Acknowledgement J.A. López Mimbela appreciates the kind hospitality of Frankfurt University during his visit in summer 2002, and thanks CONACYT (Mexico) and DAAD (Germany) for support.

References

[1] A.D. Alexandrov (1956-1958). Uniqueness theorems for surfaces in the large I-V. *Vestnik Leningrad Univ.* **11** No. 19, 5-17 (1956); **12** No. 7, 15-44 (1957); **13** No. 7, 14-26 (1958); **13** No. 13, 27-34 (1958); **13** No. 19, 5-8 (1958). English transl. in *Am. Math. Soc. Transl.* **21**, 341-354, 354-388, 389-403, 403-411, 412-416 (1962).

[2] K. Bogdan, T. Byczkowski, Potential theory of Schrödinger operator based on fractional Laplacian. *Probab. Math. Statist.* **20** (2000), no. 2, 293–335.

[3] W.X. Chen and C. Li (1991). Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.* **63**, no. 3, 615–622.

[4] B. Gidas (1980). Symmetry properties and isolated singularities of positive solutions of nonlinear elliptic equations, in *Nonlinear Partial Differential Equations in Engineering and Applied Science,* Eds. R. Sternberg, A. Kalinowski and J. Papadakis, Proc. Conf. Kingston, R.I 1979. *Lect. Notes on Pure Appl. Math.* **54**, Decker, New York, 255-273.

[5] B. Gidas, W.M. Ni, and L. Nirenberg (1981). Symmetry of positive solutions of nonlinear elliptic equations in $\mathbf{R}^n$, *Mathematical analysis and applications, Part A, Adv. in Math. Suppl. Stud.* **7**, 369-402.

[6] B. Gidas and J. Spruck (1981). Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.* **34**, no. 4, 525–598.

[7] N.S. Landkof (1972). Foundations of modern potential theory. Springer-Verlag, Berlin.

[8] J. Serrin (1971). A symmetry problem in potential theory. *Arch. Rat. Mech. Anal.* **43**, 304-318.

Matthias Birkner
FB Mathematik, J.W. Goethe Universität
D-60054 Frankfurt am Main, Germany.

J. Alfredo López-Mimbela
Centro de Investigación en Matemáticas
Apartado Postal 402
36000 Guanajuato, México.

Anton Wakolbinger
FB Mathematik, J.W. Goethe Universität
D-60054 Frankfurt am Main, Germany.