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Abstract

We give the explicit chaos representation of the solution of a class of one dimensional stochastic bilinear heat equations with drift in the first haos and driven by a space-time white noise. The solution is a stochastic process with finite moments of all orders.

The L^2-Lyapunov exponents of the solution are also estimated.

Key Words: Chaos expansion, stochastic heat equation, Lyapunov exponent, space-time white noise

AMS Classification: 60H15, 60H07

1 Introduction

Semilinear heat equations driven by space-time or space white noise has been studied by several authors (see [1], [4], [7], [8], [10], [11], [13], [14]).

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In one dimensions the solution is usually in L^2 and in many dimensions the solution is in some distribution spaces.
The chaos expansion is a convenient way to solve linear stochastic equations and it is very useful for obtaining many properties of the solution, such as: continuity, estimates of the Lyapunov exponents and for the moments, large deviations, etc.(see [4], [11], [12], [13]).
In the present paper we consider one-dimensional linear heat equations with the drift in the first chaos, deterministic initial condition and driven by a two-parameter Brownian sheet. This class includes those considered in the above mentioned papers. The solution is a process for which we compute explicitely the chaos decomposition of the unique weak solution. Then we use the chaos expansion in order show that the solution is an analitic functional with finite moments of all orders and also for obtaining upper estimates for the second moment Lyapunov exponent of the solution. The results can be extended to general elliptic operators.

2 Preliminaries

Let $\{W_{t,x}\}_{t\geq 0, x\in[0.1]}$ be a Brownian sheet defined on a complete probability space $(\Omega,\mathcal{F},P)$, where $\mathcal{F}$ is the completion of the Borel σ-algebra generated by W. That is W is a centered Gaussian random field with covariance

$$E\left(W_{s,x}W_{t,y}\right)=(s\wedge t)\left(x\wedge y\right),\ s,t\geq 0,\ x,y\in[0.1]\,.$$

In the sequel we recall the basic definitions and some results of the stochastic calculus of variations with respect to W. For details we refer to [8] and [9].
For $f\in L^2((R_+\times[0,1])^n),\ f$ symmetric, we denote by $I_n(f)$ the nth multiple Wiener-Itô integral. Recall that every $F\in L^2\left(\Omega,\mathcal{F},P\right)$ has a a unique orthogonal chaos decomposition

$$F=\sum_{n=0}^{\infty}I_n(f_n),$$

$f_0=I_0(f_0)=E(F)$ and $f_n\in L^2((R_+\times[0,1])^n),\ f_n$ symmetric.
For every $\lambda\geq 1$ we define

$$\|F\|_\lambda^2=\sum_{n=0}^{\infty}n!\lambda^{2n}\left\|f_n\right\|^2_{L^2((R_+\times[0,1])^n)}\,,$$

and

$$H_\infty = \left\{ F \in L^2(\Omega, \mathcal{F}, P) : \|F\|_\lambda < \infty, \ \forall \lambda \geq 1 \right\}.$$

The elements of H_∞ are called analitic functionals and H_∞ is included in the space D_∞ of smooth functionals introduced by Watanabe[15].
For a process $\{u_{t,x}\}_{t \geq 0, 0 \leq x \leq 1}$ with the chaos decomposition

$$u_{t,x} = \sum_{n=0}^{\infty} I_n(f_n^{t,x}),$$

$f_n \in L^2((R_+ \times [0,1])^{n+1})$, $f_n^{t,x}(.)$ symmetric and for $\lambda \geq 1$, we put

$$\|u\|_{\lambda,T}^2 = \sum_{n=0}^{\infty} n! \lambda^{2n} \int_0^T \int_0^1 \left\| f_n^{t,x} \right\|_{L^2((R_+ \times [0,1])^n)}^2 dxdt,$$

and we define $H_\infty(L^2([0,T]))$ as the class of all processes u such that $\|u\|_{\lambda,T} < \infty$ for all $\lambda \geq 1$. Also we denote by

$$D : \mathcal{D}^{1,2} \subset L^2(\Omega, \mathcal{F}, P) \to L^2(R_+ \times [0,1] \times \Omega)$$

the derivative operator and by

$$D^k_{t_1,x_1,\ldots,t_k,x_k} = D_{t_1,x_1} \ldots D_{t_k,x_k} :$$

$$\mathcal{D}^{k,2} \subset L^2(\Omega, \mathcal{F}, P) \to L^2\left((R_+ \times [0,1])^k \times \Omega \right)$$

the iterated derivative operator.
Let $\delta : Dom\ \delta \subset L^2(R_+ \times [0,1] \times \Omega) \to L^2(\Omega, \mathcal{F}, P)$ be the Skorohod integral operator, which is the adjoint of the derivative operator D. We will make use of the notation $\delta(u) = \int_0^\infty \int_0^1 u_{t,x} dW_{t,x}$ for any $u \in Dom\ \delta$.
Let $L^{k,2} = L^2(R_+ \times [0,1], \mathcal{D}^{k,2})$, that is, a process $u \in L^2(R_+ \times [0,1] \times \Omega)$ belongs to $L^{k,2}$ if $u_{t,x} \in \mathcal{D}^{k,2}$ for a.a. t, x and there exists a measurable version of $D^k_{t_1,x_1,\ldots,t_k,x_k} u_{t,x}$ veryfying

$$E\left[\int_{(R_+ \times [0,1])^{k+1}} \left| D^k_{t_1,x_1,\ldots,t_k,x_k} u_{t,x} \right|^2 dxdtdx_1..dx_ndt_1..dt_n \right] < \infty,$$

or equivalently, if

$$u_{t,x}=\sum_{n=0}^{\infty}I_n(f_n^{t,x}),$$

$f_n^{t,x}\in L^2((R_+\times[0,1])^n),\ f_n^{t,x}$ symmetric, $f_n\in L^2((R_+\times[0,1])^{n+1})$, to saying that

$$\sum_{n=2}^{\infty}n(n-1)...(n-k+1)n!\,\|f_n\|^2_{L^2((R_+\times[0,1])^{n+1})}<\infty.$$

Theorem 2.1[9](*Integration by parts formula*). *Let* $\{X_t\}_{t\in[0,T]}$ *be a process of the form*

$$X_t=\int_0^t\int_0^1 u_s(y)dW_{s,y},$$

such that
(i_1) $u\in L^{2,2}$.
(i_2) There exists $p>4$ *with*

$$\int_{([0,T]\times[0,1])^2}\left|E\left(D_{s,y}u_t\left(x\right)\right)\right|^p dxdtdyds+$$

$$E\left(\int_{([0,T]\times[0,1])^3}\left|D^2_{s_1,y_1,s_2,y_2}u_t\left(x\right)\right|^p dxdtdx_ids_i\right)<\infty.$$

Let $\{V_t\}_{t\in[0,T]}$ *be a continuous process with a.s. finite variation such that*
(j_1) $V\in L^{1,2}$ *and*

$$E\left[\int_0^T\int_0^T\int_0^1\left|D_{s,y}V_t\right|^4 dtdyds\right]<\infty.$$

(j_2) The mapping $t\to D_{s,x}V_t$ *is continuous with values in* $L^4\left(\Omega\right)$, *uniformly with respect to* s,x. *Then we have*

$$X_tV_t=\int_0^t\int_0^1 u_s(y)V_sdW_{s,y}+$$

$$\int_0^t V_s' X_s ds + \int_0^t \int_0^1 u_s(y) D_{s,y} V_s dy ds.$$

Let $p_t(x,y)$ be the Green function to the heat equation on the interval $[0.1]$ with Dirichlet boundary conditions, that is

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} \left[e^{-\frac{(y-x-2n)^2}{2t}} - e^{-\frac{(y+x-2n)^2}{2t}} \right].$$

For a function $g : R \to R$ denote

$$P_t g(x) = \int_0^1 p_t(x,y) g(y) dy.$$

Let $a^t(s) : R_+^2 \to R,\ b : R_+ \to R,\ \eta : [0,1] \to R$ be measurable functions.
We introduce the hypothesis *(H)*:
(H) b, η are bounded and

$$\int_0^T \int_0^\infty \left| a^t(s) \right| ds dt < \infty, \ \int_0^T \int_0^\infty \left| a^t(s) \right|^2 ds dt < \infty \text{ for all } T > 0.$$

We consider the linear stochastic heat equation

$$\left\{ \begin{array}{lcl} \frac{\partial u_t(x)}{\partial t} & = & \frac{1}{2} \frac{\partial^2 u_t(x)}{\partial x^2} + I_1(a^t) u_t(x) + b(t) u_t(x) \frac{\partial^2 W_{t,x}}{\partial t \partial x} \\ u_t(0) & = & u_t(1) = 0 \\ u_0 & = & \eta. \end{array} \right. \tag{2.1}$$

Definition 2.2. We say that a measurable process $\{u_t(x)\}_{t \geq 0, x \in \in [0,1]}$ is a weak solution of (2.1) if for every $t \in [0,1]$ and $\varphi \in C^2([0,1])$ with $\varphi(0) = \varphi(1) = 0$ we have $1_{[0,t]}(.) b(.) u_.(*) \in Dom\ \delta$ and

$$\int_0^1 \varphi(x) u_t(x) dx - \int_0^1 \varphi(x) \eta(x) dx = \int_0^t \int_0^1 \frac{1}{2} \frac{\partial^2 \varphi(x)}{\partial x^2} u_s(x) dx ds +$$

$$\int_0^t \int_0^1 \varphi(x) I_1(a^s) u_s(x) dx ds + \int_0^t \int_0^1 \varphi(x) b(s) u_s(x) dW_{s,x}. \tag{2.2}$$

By using integration by parts formula (Theorem 2.1) and stochastic Fubini theorem (deterministic and stochastic) we obtain as in Gyöngy[2], [3](in the adapted case) the following result.

Theorem 2.3. *Assume the hypothesis (H) holds. Let* $\{u_t(x)\}_{t\geq 0, x\in\in[0,1]}$ *be a measurable process which satisfies the assumptions of Theorem 2.1.*
Then the following statements are equivalent.
(1) u is a weak solution of (2.1).
(2) For every $t \geq 0$ *and* $\psi(s,x) \in C^{1,2}([0,t]\times[0,1])$ *with* $\psi(s,0)=\psi(s,1)=0,\ s\in[0,t]$ *we have*

$$\int_0^1 \psi(t,x)u_t(x)dx - \int_0^1 \psi(0,x)\eta(x)dx =$$

$$\int_0^t\int_0^1 \left[\frac{1}{2}\frac{\partial^2\psi(s,x)}{\partial x^2} + \frac{\partial\psi(s,x)}{\partial s}\right] u_s(x)dxds +$$

$$\int_0^t\int_0^1 \psi(s,x)I_1(a^s)u_s(x)dxds + \int_0^t\int_0^1 \psi(s,x)b(s)u_s(x)dW_{s,x}. \tag{2.3}$$

(3) For almost every $\omega\in\Omega$ *and for all* t

$$u_t(x) = P_t\eta(x) + \int_0^t\int_0^1 p_{t-s}(x,y)I_1(a^s)u_s(y)dyds +$$

$$\int_0^t\int_0^1 p_{t-s}(x,y)b(s)u_s(y)W_{s,y}, \tag{2.4}$$

for almost all x.

3 Main Results

We introduce the following notation (later we give the hypotheses which guarantee that the quantities introduced below are well-defined):
Given $t_1,...,t_n$ and a symmetric function of $n-j$ variables ($j<n$), we denote by $f_{n-j}(\hat{t}_{i_1},..,\hat{t}_{i_j})$ the function f_{n-j} evaluated in t's other that $t_{i_1},..,t_{i_j}$,

$$\Delta_{j,n} = \{(i_1,..,i_j) : i_k \neq i_l \text{ if } k\neq l, i_k = 1,...,n\},\ j\leq n,$$

$$h_t(u) = \int_0^t a^r(u)dr,\ Y_t = \exp\left\{\frac{1}{2}\left\|h^t(.)\right\|^2_{L^2(R_+)}\right\},$$

$$C_t = \exp\left\{\int_0^t \left\langle a^s(.), 1_{[0,s]}(.)b(.)\right\rangle_{L^2(R_+)} ds\right\},$$

Remark 3.1. We have the following straightforward equalities:

$$\frac{d}{dt}h_t^{\otimes n}(t,...,t_n) = n\text{sym}\left(a^t \otimes h_t^{\otimes(n-1)})(t,...,t_n)\right), \tag{3.5}$$

$$\frac{dY_t}{dt} = \left\langle a^t(.), h_t(.)\right\rangle_{L^2(R_+)} Y_t, \tag{3.6}$$

$$\frac{dC_t}{dt} = \left\langle a^t, 1_{[0,t]}b\right\rangle_{L^2(R_+)} C_t. \tag{3.7}$$

Theorem 3.2. *Assume that (H) is satisfied. Then the equation (2.1) has the unique weak solution in* $H_\infty(L^2([0,T])$ *for all* $T > 0$, *given by the chaos expansion*

$$u_t(x) = \sum_{n=0}^{\infty} I_n(f_n^{t,x}), \tag{3.8}$$

$$f_0^{t,x} = C_t Y_t P_t \eta(x), \tag{3.9}$$

and for $n \geq 1$,

$$f_n^{t,x}(t_1,...,t_n,x_1,...,x_n) = C_t Y_t \left\{\frac{P_t\eta(x)h_t^{\otimes n}(t_1,...,t_n)}{n!}+\right.$$

$$sym\left[\sum_{j=1}^{n} 1_{t_1<...<t_j<t} p_{t-t_j}(x,x_j)b(t_j)p_{t_j-t_{j-1}}(x_j,x_{j-1})b(t_{j-1})...\right.$$

$$p_{t_2-t_1}(x_2,x_2)b(t_1)P_{t_1}\eta(x_1)h_t^{\otimes(n-j)}(t_{j+1},...,t_n)\Big]\Big\} =$$

$$\frac{1}{n!}C_tY_t\left[\sum_{j=1}^{n}\sum_{(i_1,..,i_j)\in\Delta_{j,n}}1_{t_{i_1}<...<t_{i_j}<t}\times\right.$$

$$\times p_{t-t_{i_j}}(x,x_{i_j})b(t_{i_j})p_{t_{i_j}-t_{i_{j-1}}}(x_{i_j},x_{i_{j-1}})b(t_{i_{j-1}})...p_{t_{i_2}-t_{i_1 1}}(x_{i_2},x_{i_1})b(t_{i_1})\times$$

$$\left.\times P_{t_{i_1}}\eta(x_{i_1})h_t^{\otimes(n-j)}(\hat{t}_{i_1},...,\hat{t}_{i_j})+P_t\eta(x)h_t^{\otimes n}(t_1,...,t_n)\right]. \tag{3.10}$$

Moreover for all $t\geq 0, x\in[0,1]$ *and* $n\geq 1$,

$$\left\|f_n^{t,x}\right\|^2_{L^2((R_+\times[0,1])^n)}\leq\|\eta\|^2_\infty C_t^2Y_t^2(n+1)\times$$

$$\times\frac{1}{n!}\sum_{j=0}^{n}\frac{\|h_t\|^{2(n-j)}_{L^2(R_+)}\left(\frac{t\|b\|^4_\infty}{4}\right)^{\frac{j}{2}}}{(n-j)!\Gamma\left(\frac{j}{2}+1\right)}. \tag{3.11}$$

Proposition 3.3. *Let* $u_t(x)$ *be the unique solution given in Theorem 3.2. Then the following properties are satisfied.*
(i) For all $t\geq 0, x\in[0,1]$, $u_t(x)\in H_\infty$ *and*

$$\sup_{0\leq t\leq T,0\leq x\leq 1}\|u_t(x)\|_\lambda<\infty \text{ for every } T>0,\lambda\geq 1. \tag{3.12}$$

(ii) The mapping

$$(t,x)\to u_t(x):R_+\times[0,1]\to H_\infty$$

is continuous.
(iii) For every $p\geq 1$, $T>0$,

$$\sup_{0\leq t\leq T,0\leq x\leq 1}E\left(|u_t(x)|^p\right)<\infty. \tag{3.13}$$

Proposition 3.4. *Assume the hypothesis (H) is satisfied and let $u_t(x)$ be the unique solution given in Theorem 3.2. Moreover suppose that there exist $\alpha \geq 1, h_1 \in R, h_2 > 0$, such that*

$$\limsup_{t\longrightarrow\infty}\frac{1}{t^\alpha}\int_0^t\int_0^r a^r(s)b(s)dsdr \leq h_1 \tag{3.14}$$

$$\limsup_{t\longrightarrow\infty}\frac{1}{t^\alpha}\left\|h_t\right\|^2_{L^2(R_+)} \leq h_2. \tag{3.15}$$

Then

$$\limsup_{t\longrightarrow\infty}\frac{1}{t^\alpha}\log\sup_{0\leq x\leq 1}E\left(\left|u_t(x)\right|^2\right) \leq$$

$$\begin{cases} 2\left(h_1+h_2\right)+\frac{\|b\|^4_\infty}{4}, & \alpha = 1 \\ 2\left(h_1+h_2\right), & \alpha > 1. \end{cases} \tag{3.16}$$

Corollary 3.5. *Assume the hypothesis (H) is satisfied and $a \in L^1(R_+) \cap L^2(R_+)$. Then (3.16) is fulfilled with $\alpha = 1, h_1 = 0, h_2 = \|a\|^2_{L^2(R_+)}$.*
Corollary 3.6. *Assume the hypothesis (H) is satisfied and*

$$\limsup_{t\longrightarrow\infty}\frac{1}{t^2}\int_0^t\int_0^r \left|a^r(s)\right|dsdr \leq k_1 < \infty, \tag{3.17}$$

$$\limsup_{t\longrightarrow\infty}\frac{1}{t}\int_0^t\int_0^\infty \left|a^r(s)\right|^2 dsdr \leq k_2 < \infty. \tag{3.18}$$

Then (3.16) is fulfilled with $\alpha = 2, h_1 = k_2\|b\|_\infty, h_2 = k_1$.
Corollary 3.7 *(Adapted case). Assume the hypothesis (H) is satisfied. Moreover let $a^r(s) = a^r(s)1_{[0,r)}(s)$ for all r, s and suppose that (3.17)*

$$\limsup_{t\longrightarrow\infty}\frac{1}{t^2}\int_0^t\int_0^r \left|a^r(s)\right|^2 dsdr \leq k < \infty. \tag{3.19}$$

Then (3.16) is fulfilled with $\alpha = 2, h_1 = k\|b\|_\infty, h_2 = k_1$.

Remark 3.8. Estimate from below of the L^2-Lyapunov exponents can be obtained in a similar way.

4 Proofs

Proof of Theorem 3.2. *Existence.* Let $u \in H_\infty([0,T])$, $T>0$, be a weak solution given by the expansion (3.8). Then, from the expression of the Skorohod integral and the product formula for multiple Wiener integrals(see [5] or [8]) and by identifying the kernels of multiple Wiener integrals in (2.4), it follows that the sequence of kernels $\{f_n^{t,x}\}_{n\geq 0}$ is solution of the infinite system of deterministic integral equations

$$f_0^{t,x} = P_t\eta(x) + \int_0^t \int_0^1 p_{t-s}(x,y) \left\langle a^s \otimes 1, f_1^{s,y} \right\rangle_{L^2(R_+\times[0,1])} dyds, \tag{4.20}$$

and for $n \geq 1$,

$$f_n^{t,x}(s_1,...,s_n,x_1,...,x_n) = \frac{1}{n}\sum_{i=1}^n \int_0^t \int_0^1 p_{t-s}(x,y) a^s(s_i) f_{n-1}^{s,y}(\hat{s}_i,\hat{x}_i) dyds +$$

$$(n+1)\int_0^t \int_0^1 p_{t-s}(x,y) \left\langle a^s \otimes 1, f_{n+1}^{s,y}(s_1,...,s_n,x_1,...,x_n) \right\rangle_{L^2(R_+\times[0,1])} dyds +$$

$$\frac{1}{n}\sum_{i=1}^n 1_{s_i<t} p_{t-s_i}(x,x_i) b(s_i) f_{n-1}^{s_i,x_i}(\hat{s}_i,\hat{x}_i). \tag{4.21}$$

We prove that $f_n^{t,x}$ given by (3.9), (3.10) is a solution of (4.20), (4.21).
We have

$$f_1^{s,x}(s_1,x_1) = A_1(s,x,s_1,x_1) + A_2(s,x,s_1),$$

$$A_1(s,x,s_1,x_1) = C_s Y_s p_{s-s_1}(x,x_1) b(s_1)) P_{s_1}\eta(x_1) 1_{s_1<s},$$

$$A_2(s,x,s_1) = C_s Y_s P_s \eta(x) h_s(s_1)).$$

Then

$$\int_0^t \int_0^1 p_{t-s}(x,y) \left\langle a^s \otimes 1, A_1(s,y) \right\rangle_{L^2(R_+\times[0,1]} dyds = \int_0^t \int_0^1 p_{t-s}(x,y) C_s Y_s \times$$

$$\times \left[\int_0^s \int_0^1 a^s(s_1) p_{s-s_1}(y,y_1) b(s_1)) P_{s_1}\eta(y_1) ds_1 dy_1 \right] dyds =$$

$$\int_0^t C_s Y_s \int_0^1 \int_0^s \left[\int_0^1 p_{t-s}(x,y) p_{s-s_1}(y,y_1) dy \right] a^s(s_1) b(s_1)) P_{s_1}\eta(y_1) dy_1 ds_1 ds =$$

$$\int_0^t C_s Y_s \int_0^s \left[\int_0^1 p_{t-s_1}(y,y_1) P_{s_1}\eta(y_1) dy_1 \right] a^s(s_1) b(s_1)) ds_1 ds =$$

$$P_t\eta(x) \int_0^t C_s Y_s \int_0^s a^s(s_1) b(s_1)) ds_1 ds = P_t\eta(x) \int_0^t Y_s C_s'. \tag{4.22}$$

Also we have

$$\int_0^t \int_0^1 p_{t-s}(x,y) \left\langle a^s \otimes 1, A_2(s,y) \right\rangle_{L^2(R_+ \times [0,1]} dyds =$$

$$\int_0^t \int_0^1 p_{t-s}(x,y) C_s Y_s \int_0^\infty \int_0^1 a^s(s_1) P_{s_1}\eta(y) h_s(s_1) dy_1 ds_1 dyds =$$

$$\int_0^t C_s Y_s \int_0^\infty \left[\int_0^1 p_{t-s}(x,y) P_{s_1}\eta(y) dy \right] a^s(s_1) h_s(s_1) ds_1 ds =$$

$$P_t\eta(x) \int_0^t Y_s' C_s ds. \tag{4.23}$$

From (4.22) and (4.23) we obtain (4.20).
Next using the expression for f_2 given by (4.21) we have that

$$2 \int_0^t \int_0^1 p_{t-s}(x,y) \left\langle a^s \otimes 1, f_2^{s,y}(s_1,x_1) \right\rangle_{L^2(R_+ \times [0,1]} dyds =$$

$$2 \int_0^t \int_0^1 p_{t-s}(x,y) \left[\int_0^\infty \int_0^1 a^s(\theta) f_2^{s,y}(\theta, s_1, z, x_1) dz d\theta \right] dyds = \sum_{j=1}^5 I_j, \tag{4.24}$$

$$I_1=\int_0^t\int_0^1 p_{t-s}(x,y)\int_0^\infty\int_0^1 a^s(\theta)C_sY_sp_{s-s_1}(y,x_1)b(s_1)\times$$

$$\times p_{s_1-\theta}(x_1,z)b(\theta)(P_\theta\eta)(z)1_{0<\theta<s_1<s}dzd\theta dyds,$$

$$I_2=\int_0^t\int_0^1 p_{t-s}(x,y)\int_0^\infty\int_0^1 a^s(\theta)C_sY_sp_{s-\theta}(y,z)b(\theta)\times$$

$$\times p_{\theta-s_1}(z,x_1)b(s_1)(P_{s_1}\eta)(x_1)1_{0<s_1<\theta<s}dzd\theta dyds,$$

$$I_3=\int_0^t\int_0^1 p_{t-s}(x,y)\int_0^\infty\int_0^1 a^s(\theta)C_sY_sp_{s-\theta}(y,z)b(\theta)1_{0<\theta<s_1}\times$$

$$\times(P_\theta\eta)(z)h_s(s_1)dzd\theta dyds,$$

$$I_4=\int_0^t\int_0^1 p_{t-s}(x,y)\int_0^\infty\int_0^1 a^s(\theta)C_sY_sp_{s-s_1}(y,x_1)b(s_1)1_{0<s_1<\theta}\times$$

$$\times(P_{s_1}\eta)(x_1)h_s(\theta)dzd\theta dyds,$$

$$I_5=(P_t\eta)(x)\int_0^t\int_0^1 a^s(\theta)C_sY_sh_s(\theta)h_s(s_1)d\theta ds.$$

By using the semigroup property of p we obtain easily that

$$I_1=1_{s_1<t}p_{t-s_1}(x,x_1)b(s_1)(P_{s_1}\eta)(x_1)\int_{s_1}^t C_sY_s\int_0^{s_1}a^s(\theta)b(\theta)d\theta ds,$$

$$I_2=1_{s_1<t}p_{t-s_1}(x,x_1)b(s_1)(P_{s_1}\eta)(x_1)\int_{s_1}^t C_sY_s\int_{s_1}^{s}a^s(\theta)b(\theta)d\theta ds,$$

and then

$$I_1 + I_2 = 1_{s_1<t} p_{t-s_1}(x, x_1) b(s_1)(P_{s_1}\eta)(x_1) \int_{s_1}^t \int_0^s C'_s Y_s ds. \tag{4.25}$$

Similarly we obtain

$$I_3 = (P_t\eta)(x) \int_0^t C'_s Y_s h_s(s_1) ds, \tag{4.26}$$

$$I_4 = 1_{s_1<t} p_{t-s_1}(x, x_1) b(s_1)(P_{s_1}\eta)(x_1) \int_{s_1}^t C_s Y'_s ds, \tag{4.27}$$

$$I_5 = (P_t\eta)(x) \int_0^t C_s Y'_s h_s(s_1) ds. \tag{4.28}$$

Replacing (4.25)-(4.28) and the expresion of f_1 given by (3.10)in (4.24) we obtain (4.21) for $n = 1$.
For arbitrary n write

$$f_{n+1}^{s,y}(s_1, ..., s_{n+1}, x_1, ..., x_{n+1}) = A^{s,y}(s_1, ..., s_{n+1}, x_{n+1}) +$$

$$\sum_{j=1}^{n} \sum_{(i_1,..,i_j)\in\Delta_{j,n}} \Big[A_1^{s,y,i_1,..,i_j}(s_1, ..., s_{n+1}, x_1, ..., x_{n+1}) +$$

$$\sum_{k=2}^{j} A_{2,k}^{s,y,i_1,..,i_j}(s_1, ..., s_{n+1}, x_1, ..., x_{n+1}) + A_3^{s,y,i_1,..,i_j}(s_1, ..., s_{n+1}, x_1, ..., x_{n+1}) +$$

$$B_j^{s,y,i_1,..,i_j}(s_1, ..., s_{n+1}, x_1, ..., x_{n+1}) \Big] + C^{s,y}(s_1, ..., s_{n+1}), \tag{4.29}$$

$$(n+1)! A^{s,y}(s_1, ..., s_{n+1}, x_{n+1}) =$$

$$C_s Y_s 1_{s_{n+1}<s} p_{s-s_{n+1}}(y, x_{n+1}) b(s_{n+1}) P_{s_{n+1}}\eta(x_{n+1}) h_s^{\otimes n}(s_1, ..., s_n),$$

$$(n+1)!A_j^{s,y}(s_1,...,s_{n+1},x_j)=$$

$$C_sY_s1_{s_j<s}p_{s-s_j}(y,x_j)b(s_j)P_{s_j}\eta(x_j)h_s(s_j)h_s^{\otimes n}(\hat{s}_j),$$

$$(n+1)!A_1^{s,y,i_1,..,i_j}(s_1,...,s_{n+1},x_1,...,x_{n+1})=$$

$$C_sY_s1_{s_{n+1}<s_{i_1}<...s_{i_j}<s}p_{s-s_{i_j}}(y,x_{i_j})b(s_{i_j})...p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1})b(s_{i_1})\times$$

$$\times p_{s_{i_1}-s_{n+1}}(x_{i_1},x_{n+1})b(s_{n+1})P_{s_{n+1}}\eta(x_{n+1})h_s^{\otimes(n-j)}(\hat{s}_{i_1},...,s_{\hat{i}_j},\hat{s}_{n+1}),$$

$$(n+1)!A_{2,k}^{s,y,i_1,..,i_j}(s_1,...,s_{n+1},x_1,...,x_{n+1})=$$

$$C_sY_s1_{s_{i_1}<...<s_{i_{k-1}}<s_{n+1}<s_{i_k}<...<s_{i_j}<s}p_{s-s_{i_j}}(y,x_{i_j})b(s_{i_j})...$$

$$...p_{s_{i_k}-s_{n+1}}(x_{i_k},y)b(s_{n+1})p_{s_{n+1}-s_{i_{k-1}}}(x_{n+1},x_{i_{k-1}})b(s_{i_{k-1}})...$$

$$...p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1})b(s_{i_1})P_{s_{i_1}}\eta(x_{i_1})h_s^{\otimes(n-j)}(\hat{s}_{i_1},...,s_{\hat{i}_j},\hat{s}_{n+1}),$$

$$(n+1)!A_3^{s,y,i_1,..,i_j}(s_1,...,s_{n+1},x_1,...,x_{n+1})=$$

$$C_sY_s1_{s_{i_1}<...<s_{i_j}<s_{n+1}<s}p_{s-s_{n+1}}(y,s_{n+1})b(s_{n+1})...$$

$$...p_{s_{n+1}-s_{i_j}}(x_{n+1},x_{i_j})b(s_{i_j})...P_{s_{i_1}}\eta(x_{i_1})h_s^{\otimes(n-j)}(\hat{s}_{i_1},...,\hat{s}_{\hat{i}_j},\hat{s}_{n+1}),$$

$$B_j^{s,y,i_1,..,i_j}(s_1,...,s_{n+1},x_1,...,x_{n+1})=C_sY_s1_{s_{i_1}<...<s_{i_j}<s}p_{s-s_{i_j}}(y,x_{i_j})b(s_{i_j})...$$

$$...P_{s_{i_1}}\eta(x_{i_1})h_s(s_{n+1})h_s^{\otimes(n-j)}(\hat{s}_{i_1},...,\hat{s}_{\hat{i}_j},\hat{s}_{n+1}),$$

$$(n+1)!C^{s,y}(s_1,...,s_{n+1})=C_sY_sP_s\eta(y)h_s^{\otimes(n+1)}(s_1,...,s_{n+1}).$$

By the semigroup property of p and Fubini's theorem we obtain

$$I:=(n+1)\int_0^t\int_0^1 p_{t-s}(x,y)\left\langle a^s\otimes 1,A^{s,y}(s_1,...,s_n)\right\rangle_{L^2(R_+\times[0,1]}dyds=$$

$$\frac{1}{n!}P_s\eta(x)\int_0^t Y_sC_s'h_s^{\otimes n}(s_1,...,s_n)ds, \tag{4.30}$$

$$I_j:=(n+1)\int_0^t\int_0^1 p_{t-s}(x,y)\left\langle a^s\otimes 1,A_j^{s,y}(s_1,...,s_{n+1})\right\rangle_{L^2(R_+\times[0,1])}dyds=$$

$$\frac{1}{n!}P_{s_j}\eta(x)\int_0^t Y_sC_s'h_s^{\otimes n}(\hat{s}_j)ds, \tag{4.31}$$

$$J_1^{i_1,..,i_j}:=(n+1)\int_0^t\int_0^1 p_{t-s}(x,y)\times$$

$$\times\left\langle a^s\otimes 1,A_{1,j}^{s,y,i_1,..,i_j}(s_1,...,s_n,x_1,...,x_n)\right\rangle_{L^2(R_+\times[0,1]}dyds=$$

$$\frac{1}{n!}1_{s_{i_1}<...<s_{i_j}<t}p_{t-s_{i_j}}(x,x_{i_j})b(s_{i_j})......P_{s_{i_1}}\eta(x_{i_1})\times$$

$$\int_{s_{i_j}}^t Y_sC_s\left[\int_0^{s_{i_1}}a^s(s_{n+1})b(s_{n+1})ds_{n+1}\right]h_s^{\otimes(n-j)}(\hat{s}_{i_1},...,\hat{s}_{\hat{i}_j})ds, \tag{4.32}$$

$$J_2^{i_1,..,i_j}:=(n+1)\int_0^t\int_0^1 p_{t-s}(x,y)\times$$

$$\times \left\langle a^s \otimes 1, A_{2,j,k}^{s,y,i_1,..,i_j}(s_1,...,s_n,x_1,...,x_n) \right\rangle_{L^2(R_+\times[0,1]} dyds =$$

$$\frac{1}{n!} 1_{s_{i_1}<...<s_{i_j}<t} p_{t-s_{i_j}}(x,x_{i_j}) b(s_{i_j}) ......P_{s_{i_1}} \eta(x_{i_1}) \times$$

$$\int_{s_{i_j}}^t Y_s C_s \left[\int_{s_{i_{k-1}}}^{s_{i_k}} a^s(s_{n+1}) b(s_{n+1}) ds_{n+1} \right] h_s^{\otimes(n-j)}(\hat{s}_{i_1},...,\hat{s}_{\hat{i}_j}) ds, \tag{4.33}$$

$$J_3^{i_1,..,i_j} := (n+1) \int_0^t \int_0^1 p_{t-s}(x,y) \times$$

$$\times \left\langle a^s \otimes 1, A_{3,j}^{s,y,i_1,..,i_j}(s_1,...,s_n,x_1,...,x_n) \right\rangle_{L^2(R_+\times[0,1]} dyds =$$

$$\frac{1}{n!} 1_{s_{i_1}<...<s_{i_j}<t} p_{t-s_{i_j}}(x,x_{i_j}) b(s_{i_j}) ...P_{s_{i_1}} \eta(x_{i_1}) \times$$

$$\int_{s_{i_j}}^t Y_s C_s \left[\int_{s_{i_j}}^{s} a^s(s_{n+1}) b(s_{n+1}) ds_{n+1} \right] h_s^{\otimes(n-j)}(\hat{s}_{i_1},...,\hat{s}_{\hat{i}_j}) ds. \tag{4.34}$$

From (4.32)-(4.34) we deduce

$$J_1^{i_1,..,i_j} + J_2^{i_1,..,i_j} + J_3^{i_1,..,i_j} =$$

$$\frac{1}{n!} 1_{s_{i_1}<...<s_{i_j}<t} p_{t-s_{i_j}}(x,x_{i_j}) b(s_{i_j}) ...P_{s_{i_1}} \eta(x_{i_1}) \times$$

$$\times \int_{s_{i_j}}^t Y_s C'_s h_s^{\otimes(n-j)}(\hat{s}_{i_1},...,\hat{s}_{\hat{i}_j}) ds. \tag{4.35}$$

Also we have

$$J_4^{i_1,..,i_j} := (n+1) \int_0^t \int_0^1 p_{t-s}(x,y) \times$$

$$\times \left\langle a^s \otimes 1, B_j^{s,y,i_1,..,i_j}(s_1,...,s_n,x_1,...,x_n)\right\rangle_{L^2(R_+\times[0,1]} dyds =$$

$$\frac{1}{n!} 1_{s_{i_1}<...<s_{i_j}<t} p_{t-s_{i_j}}(x,s_{i_j}) b(s_{i_j})......P_{s_{i_1}}\eta(x_{i_1}) \times$$

$$\int_{s_{i_j}}^t Y_s' C_s h_s^{\otimes(n-j)}(\hat{s}_{i_1},...,\hat{s}_{\hat{i}_j}) ds. \tag{4.36}$$

Next

$$J := (n+1)\int_0^t \int_0^1 p_{t-s}(x,y) \left\langle a^s \otimes C^{s,y}(s_1,...,s_n)\right\rangle_{L^2(R_+\times[0,1]} dyds =$$

$$\frac{1}{n!} P_s\eta(x) \int_0^t C_s' Y_s h_s^{\otimes n}(s_1,...,s_n) ds. \tag{4.37}$$

Suming (4.30) and (4.37) we get

$$I + J = \frac{1}{n!} P_s\eta(x) \int_0^t (C_sY_s)' h_s^{\otimes n}(s_1,...,s_n) ds =$$

$$\frac{1}{n!} P_s\eta(x) C_t Y_t h_t^{\otimes n}(s_1,...,s_n) -$$

$$\frac{1}{n!}\int_0^t n\mathrm{sym}\left(a^s \otimes h^{\otimes(n-1)}\right)(s_1,..,s_n) ds, \tag{4.38}$$

and from and (4.36), (4.37) yield

$$J_1^{i_1,..,i_j} + J_2^{i_1,..,i_j} + J_3^{i_1,..,i_j} + J_4^{i_1,..,i_j} =$$

$$\frac{1}{n!} 1_{s_{i_1}<...<s_{i_j}<t} p_{t-s_{i_j}}(x,s_{i_j}) b(s_{i_j})......P_{s_{i_1}}\eta(x_{i_1}) \times$$

$$\times \int_{s_{i_j}}^t (Y_sC_s)' h_s^{\otimes(n-j)}(\hat{s}_{i_1},...,\hat{s}_{\hat{i}_j}) ds =$$

$$\frac{1}{n!}1_{s_{i_1}<...<s_{i_j}<t}p_{t-s_{i_j}}(x,s_{i_j})b(s_{i_j})...P_{s_{i_1}}\eta(x_{i_1})C_tY_th_t^{\otimes(n-j)}(\hat{s}_{\hat{i}_1},...,\hat{s}_{\hat{i}_j})-$$

$$\frac{1}{n!}1_{s_{i_1}<...<s_{i_j}<t}p_{t-s_{i_j}}(x,s_{i_j})b(s_{i_j})...P_{s_{i_1}}\eta(x_{i_1})C_{s_{i_j}}Y_{s_{i_j}}h_{s_{i_j}}^{\otimes(n-j)}(\hat{s}_{\hat{i}_1},...,\hat{s}_{\hat{i}_j})-$$

$$\frac{1}{n!}1_{s_{i_1}<...<s_{i_j}<t}p_{t-s_{i_j}}(x,s_{i_j})b(s_{i_j})...P_{s_{i_1}}\eta(x_{i_1})\times$$

$$\times\int_{s_{i_j}}^{t}C_sY_s(n-j)\mathrm{sym}\left(a^s\otimes h_s^{\otimes(n-j-1)}\right)(\hat{s}_{\hat{i}_1},...,\hat{s}_{\hat{i}_j})ds. \tag{4.39}$$

Finally (4.38), 4.31 and (4.39) prove the validity of (4.21) for n.
Next, for $t\geq 0,\ x\in[0,1],j\leq n$, we have

$$\left\|sym\left[1_{t_1<...<t_j<t}p_{t-t_j}(x,x_j)b(t_j)..P_{t_j}\eta(x_j)h_t^{\otimes(n-j)}(t_{j+1},..,t_n)\right]\right\|^2_{L^2((R_+\times[0,1])^n)}=$$

$$\frac{\left\|h1_{[0,t]}\right\|^{2(n-j)}_{L^2(R_+)}}{(n-j)!}\int_{t_1<...<t_j<t}\int_{[0,1]^j}\left|p_{t-t_j}(x,x_j)b(t_j)..P_{t_j}\eta(x_j)\right|^2dx_1..dx_jdt_1..dt_j$$

$$\leq\frac{\|\eta\|^2_\infty\|b\|^{2j}_\infty\left\|h_t1_{[0,t]}\right\|^{2(n-j)}_{L^2(R_+)}}{(n-j)!}\times$$

$$\times\int_{t_1<...<t_j<t}\int_{[0,1]^j}p^2_{t-t_j}(x,x_j)p^2_{t_j-t_{j-1}}(x_j,x_{j-1})...$$

$$...p^2_{t_2-t_1}(x_2,x1)dx_1..dx_jdt_1..dt_j=$$

$$\frac{\|\eta\|^2_\infty\left\|h_t1_{[0,t]}\right\|^{2(n-j)}_{L^2(R_+)}}{(n-j)!}\frac{\left(\frac{t\|b\|^4_\infty}{4}\right)^{\frac{j}{2}}}{\Gamma\left(\frac{j}{2}+1\right)},$$

which implies easily (3.11).
Now from (3.11) we have for $\lambda \geq 1$,

$$\|u_t(x)\|_\lambda^2 = \sum_n n!\lambda^{2n} \left\|f_n^{t,x}\right\|^2_{L^2((R_+\times[0,1])^n)} \leq C_t^2 Y_t^2 \|\eta\|_\infty^2 \times$$

$$\times \left\{ \sum_{n=1}^{\infty} (n+1)\,\lambda^{2n} \sum_{j=0}^{n} \frac{\|h_t\|^{2(n-j)}_{L^2(R_+)}}{(n-j)!} \frac{\left(\frac{t\|b\|_\infty^4}{4}\right)^{\frac{j}{2}}}{\Gamma\left(\frac{j}{2}+1\right)} \right\} =$$

$$C_t^2 Y_t^2 \|\eta\|_\infty^2 \left\{ \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} \frac{(n+1)\,\lambda^{2n} \|h_t\|^{2(n-j)}_{L^2(R_+)}}{(n-j)!} \frac{\left(\frac{t\|b\|_\infty^4}{4}\right)^{\frac{j}{2}}}{\Gamma\left(\frac{j}{2}+1\right)} + e^{\lambda^2 \|h_t\|^2_{L^2(R_+)}} \right\}$$

$$= C_t^2 Y_t^2 \|\eta\|_\infty^2 \left\{ \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \frac{(j+k+1)\lambda^{2(j+k)} \|h_t\|^{2k}_{L^2(R_+)}}{k!} \frac{\left(\frac{t\|b\|_\infty^4}{4}\right)^{\frac{j}{2}}}{\Gamma\left(\frac{j}{2}+1\right)} \right\} =$$

$$C_t^2 Y_t^2 \|\eta\|_\infty^2 \left\{ \sum_{k=0}^{\infty} \frac{\lambda^{2k} \|h_t\|^{2k}_{L^2(R_+)}}{k!} \sum_{j=0}^{\infty} \frac{j\left(\frac{t\lambda^4\|b\|_\infty^4}{4}\right)^{\frac{j}{2}}}{\Gamma\left(\frac{j}{2}+1\right)} + \right.$$

$$\lambda^2 \|h_t\|^2_{L^2(R_+)} \sum_{k=1}^{\infty} \frac{(\lambda\|h_t\|)^{2(k-1)}_{L^2(R_+)}}{(k-1)!} \sum_{j=0}^{\infty} \frac{\left(\frac{t\lambda^4\|b\|_\infty^4}{4}\right)^{\frac{j}{2}}}{\Gamma\left(\frac{j}{2}+1\right)} +$$

$$\left. \sum_{k=0}^{\infty} \frac{\lambda^{2k} \|h_t\|^{2k}_{L^2(R_+)}}{k!} \sum_{j=0}^{\infty} \frac{\left(\frac{t\lambda^4\|b\|_\infty^4}{4}\right)^{\frac{j}{2}}}{\Gamma\left(\frac{j}{2}+1\right)} \right\},$$

and then

$$\|u_t(x)\|_\lambda^2 \le C_t^2 Y_t^2 \|\eta\|_\infty^2 \left(2+\lambda^2 \|h_t\|_{L^2(R_+)}^2\right) \times$$

$$\times e^{\lambda^2 \|h_t\|_{L^2(R_+)}^2} \sum_{j=1}^{\infty} \frac{(j+1)\left(\frac{t\lambda^4 \|b\|_\infty^4}{4}\right)^{\frac{j}{2}}}{\Gamma\left(\frac{j}{2}+1\right)}. \tag{4.40}$$

Now it is clear that (4.40) implies that $u \in H_\infty(L^2([0,T])$ and u is a solution of (2.1). *Uniqueness.* Let $u \in H_\infty([0,T])$ be a weak solution. Define

$$v_t = I_1(h_t) - \int_0^t h_r(r)b(r)dr = \int_0^t I_1(a^r)dr - \int_0^t h_r(r)b(r)dr,$$

$$y_t(x) = e^{-v_t} u_t(x).$$

$$X_t = \int_0^t \int_0^1 \frac{1}{2}\frac{\partial^2 \varphi(x)}{\partial x^2} u_s(x)dxds +$$

$$\int_0^t \int_0^1 \varphi(x) I_1(a^s) u_s(x)dxds + \int_0^t \int_0^1 \varphi(x)b(s)u_s(x)dW_{s,x},$$

Of course we have

$$v_t' = I_1(a^t) = h_t(t)b(t), \;\; D_{t,x}e^{-v_t} = -h_t(t)e^{-v_t}.$$

From definition we have for $\varphi \in C^2([0,1])$ with $\varphi(0)=\varphi(1)=0$

$$X_t = \int_0^1 \varphi(x)u_t(x)dx - \int_0^1 \varphi(x)\eta(x)dx. \tag{4.41}$$

By integration by parts (Theorem 2.1) we obtain easily that

$$\int_0^1 \varphi(x)y_t(x)dx - \int_0^1 \varphi(x)\eta(x)dx =$$

$$\int_0^t \int_0^1 \frac{1}{2} \frac{\partial^2 \varphi(x)}{\partial x^2} y_s(x) dx ds + \int_0^t \int_0^1 \varphi(x) b(s) y_s(x) dW_{s,x}. \tag{4.42}$$

Thus (4.41), (4.42) show that y is a weak solution of the equation

$$\begin{cases} \frac{\partial y_t(x)}{\partial t} & = & \frac{1}{2} \frac{\partial^2 y_t(x)}{\partial x^2} + b(t) y_t(x) \frac{\partial^2 W_{t,x}}{\partial t \partial x} \\ y_t(0) & = & y_t(1) = 0 \\ y_0 & = & \eta, \end{cases}$$

which has a unique solution (see [1], [11]). ■

Proof of Proposition 3.3. The property (3.12) is a consequence of (4.40) and (3.13) follows from (4.40) and Proposition 2.2 of [13]. Consider now a sequence $\left(t^k, x^k\right)$ converging to (t, x).
We have $f_n^{t^k, x^k}(t_1, ..., t_n, x_1, ..., x_n)$ converges to $f_n^{t,x}(t_1, ..., t_n, x_1, ..., x_n)$ as $k \to \infty$, for almost all t_i, x_i. Then (3.11) and the dominated convergence theorem yield

$$\left\| f_n^{t^k, x^k} - f^{t,x} \right\|^2_{L^2((R_+ \times [0,1])^n)} \overset{k \to \infty}{\longrightarrow} 0. \tag{4.43}$$

Next (4.43) and (3.12) imply that

$$\left\| u_{t^k}(x^k) - u_t(x) \right\|^2_\lambda \overset{k \to \infty}{\longrightarrow} 0. \quad ■$$

Proof of Proposition 3.4. From (4.40) for $\lambda = 1$ and [4, page 50] we have for $\varepsilon > 1, t \to \infty$,

$$E\left(|u_t(x)|^2\right) \le C_t^2 Y_t^4 \|\eta\|^2_\infty \left(2 + \|h_t\|^2_{L^2(R_+)}\right) \times$$

$$\times \left\{ \sum_{j=1}^\infty \frac{\left(\frac{t\varepsilon^2 \|b\|^4_\infty}{4}\right)^{\frac{j}{2}}}{\Gamma\left(\frac{j}{2}+1\right)} + 1 \right\} \le C_t^2 Y_t^4 \|\eta\|^2_\infty \left(2 + \|h_t\|^2_{L^2(R_+)}\right) \times$$

$$\times \left\{ 2 \exp\left(\frac{t\varepsilon^2 \|b\|^4_\infty}{4}\right) + O(\frac{1}{t}) \right\},$$

where from it is easy to conclude taking $t \to \infty$ and then $\varepsilon \searrow 1$. ■

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