A NOTE ON SIRIVASTAVA'S PAPER "SINGULAR WISHART AND MULTIVARIATE BETA DISTRIBUTIONS": JACOBIANS

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A note on Sirivastava's paper "Singular Wishart and multivariate beta distributions ": Jacobians

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ABSTRACT

In this study we intend to clarify the differences between the densities and the Jacobians of the transforms of singular random matrices. Some comments on the results proposed by Srivastava (2003) are presented. Different definitions of the measure with respect to which a singular random matrix possesses a density are proposed and, under any of these measures, various Jacobians of different transforms are found. An alternative proof of Uhlig's first conjecture, Uhlig (1994), is proposed. Furthermore, we propose various extensions to this conjecture under different singularities. Finally, some applications of the theory of singular distributions are discussed.

1. INTRODUCTION

Several studies have recently been published concerning singular random matrices, their densities and transforms see Khatri (1968), Uhlig (1994), Díaz-García et al. (1997), Díaz-García and Gutiérrez-Jáimez (1997) and Srivastava (2003), among others. In the present study, we examine the reason for the differences between the densities and the Jacobians of transforms of the same singular random matrix; we also highlight the care that must

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be taken in using such results. Indeed, in his introduction, Srivastava (2003) explains the difference between his expression for the Wishart singular density and that proposed by Uhlig (1994). Srivastava (2003) concludes that this difference is due to the fact that the measures with respect to which the densities are calculated are different; furthermore, he claims that the measure, according to his definition $(dX) = (dX_I)$, in his notation, is the Lebesgue measure. This is the first qualification that should be made of Srivastava's results. Formally, the measure as defined by Srivastava is not the Lebesgue measure. To see this, note that given a full rank random matrix, $X : N \times m$, this has density with respect to the Lebesgue measure defined in $\mathbb{R}^{Nm}$. However, if $X : N \times m$ is of non-full rank, say $q < \min(N, m)$, X, then X has no density with respect to the Lebesgue measure in $\mathbb{R}^{Nm}$; indeed, X has a zero measure with respect to Lebesgue measure in $\mathbb{R}^{Nm}$, see p. 172 in Billingley (1986). Note, moreover, that the Lebesgue measure has the characteristic that it is invariant under rotations, see Theorem 12.2, p. 172 in Billingley (1986); however, according to Theorem 2.2 in Srivastava (2003), Lebesgue's measure $(dX) = (dX_I)$, as defined, is not invariant under rotations or a reflections.

We now see, perhaps, that the densities of singular random matrices given in the literature (see Khatri (1968), Uhlig (1994), Díaz-García et al. (1997), Díaz-García and Gutiérrez-Jáimez (1997)) and which are used by Srivastava (2003) in applying his results, do not exist with respect to the measure $(dX) = (dX_I)$ that he proposes. For proof of this, note the following example:

Assume that $X \sim \mathcal{N}_{N\times m}(0, \Sigma, I_N)$, with $r(\Sigma) = q < \min(N, m)$. Let now $H : N \times N$ be an orthogonal matrix and let us define $Y = HX$; then X and Y have the same density, explicitly,

$$dF_Y(Y) = \frac{1}{(2\pi)^{Nq/2}\prod_{q}^{i=1}\lambda^{N/2}} \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-}Y'Y\right)(dY) \tag{1}$$

where λ_i are the non-null eigenvalues of Σ.

Now, applying Theorem 2.2 in Srivastava (2003), we have

$$(dY) = (dY_I) = |H_{11}|^{m-q}(dX_I) = |H_{11}|^{m-q}(dX)$$

where $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$, $H_{11} : q \times q$. Then

$$\begin{aligned} dF_Y(Y) &= f_X(H'Y)J(X \to Y)(dY) \\ &= \frac{1}{(2\pi)^{Nq/2}\prod_{q}^{i=1}\lambda^{N/2}} \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-}Y'Y\right)|H_{11}|^{-(m-q)}(dY) \end{aligned} \tag{2}$$

the result of which contradicts (1). This contradiction means that the density (1) does not exist with respect to the measure (dY) defined in Srivastava (2003) as (dY_I). To obtain the correct result, we must determine density (1) with respect to the measure (dY_I), or otherwise find the explicit form or forms of the measure (dY) (see Khatri (1968)), with respect to which (1) is a density.

We can now establish a second qualification concerning the reasoning behind Srivastava's study (2003). He argued the need to evaluate the volume (dS), with $S = Y'Y$ (and therefore the volume (dY)) in order to perform practical applications. Basically, for such an effect, this action is not necessary. Furthermore, it is not even necessary to know the explicit forms of the measures (dY) and (dS), as is shown by Rao (1973) and Khatri (1968), among others. Specifically, given an independent random sample, $X_1, \cdots, X_N$, of a normal m-dimensional singular population, i.e., $X_i \sim \mathcal{N}_m(\mu, \Sigma)$, for all i, $i = 1, \dots, m$, with $r(\Sigma) = r < m$, p. 532 in Rao (1973), finds the maximum-likelihood estimators of the parameters μ and Σ. Similarly, Khatri (1968) studied the linear model of multivariate regression - maximum-likelihood estimators of the parameters, the proof of the hypothesis and simultaneous confidence intervals - assuming there exists a linear dependence between the elements of the sample and between the regression variables, but without any need to know the explicit value of the volumes (dY) and (dS). Where it is, in fact, essential to know the explicit form of the measures (dY) and (dS) is when we wish to calculate the Jacobian of the transforms on the matrices X and S, and naturally, in determining the corresponding densities of X and S or any transform of the latter, as is made clear in Srivastava (2003) and in contradiction (2).

The above considerations are intended to reinforce the fact established by Khatri (1968), i.e. that (1) exists with respect to the measure (dY), which is not unique, but which cannot be just any measure, either.

In the present article, we propose alternative ways of defining the measure (dY), which, as will be established, is the Hausdorff measure, with its corresponding invariance with respect to rotations (and/or reflections), see Section 2. Some results presented in Srivastava (2003) are studied under these new measures, see Theorems 3.1 - 3.3. We also include an alternative proof to that given by Díaz-García and Gutiérrez-Jáimez (1997) concerning Uhlig's first conjecture, Uhlig(1994), see Theorem 3.4. An extension of this latter result is studied under other singularities, see Theorem 3.6. Two Jacobians that include the Moore-Penrose inverse are proposed in Theorems 3.5 and 3.7. The study concludes by applying some of the Jacobians studied in Section 3 to the theory of singular distributions.

2. NOTATION AND MEASURES

In the following, we establish alternative ways of defining the measure (dY). First, however, some notation should be established.

Let $\mathcal{L}_{m,N}(q)$ be the linear space of all $N \times m$ real matrices of rank $q \leq \min(N, m)$ and $\mathcal{L}^+_{m,N}(q)$ be the linear space of all $N \times m$ real matrices of rank $q \leq \min(N, m)$with q distinct singular values. The set of matrices $H_1 \in \mathcal{L}_{m,N}$ such that $H_1'H_1 = I_m$ is a manifold denoted $\mathcal{V}_{m,N}$, called Stiefel manifold. In particular, $\mathcal{V}_{m,m}$ is the group of orthogonal matrices $\mathcal{O}(m)$. Denote by $\mathcal{S}_m$, the homogeneous space of $m \times m$ positive definite symmetric matrices; $\mathcal{S}^+_m(q)$, the $(mq - q(q-1)/2)$-dimensional manifold of rank q positive semidefinite $m \times m$ symmetric matrices with q distinct positive eigenvalues. $\mathcal{T}^+_m$ is the group of $m \times m$ upper triangular matrices with positive diagonal elements; $\mathcal{T}^+_{m,N}$ the set of $N \times m$ upper quasi-triangular matrices such that $T = (T_1|T_2) \in \mathcal{T}^+_{m,N}$, with $T_1 \in \mathcal{T}^+_N$ and $T_2 \in \mathcal{L}_{m-N,N}(N)$

Note that in Srivastava (2003) for $Y \in \mathcal{L}^+_{m,N}(q)$ the subspace on which the measure

(dY_I) is defined can be defined by the axes identified by the elements in the submatrices Y_{11}, Y_{12}, Y_{21}, or a related subspace, in which

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \tag{3}$$

where $Y_{11} : q \times q$ of rank q. Therefore (dY_I) is defined as

$$(dY) = (dY_I) = \bigwedge_{i=1}^{q} \bigwedge_{j=1}^{m} dy_{ij} \wedge \bigwedge_{i=q+1}^{N} \bigwedge_{j=1}^{q} dy_{ij} = (dY_{11}) \wedge (dY_{12}) \wedge (dY_{21}). \tag{4}$$

The idea underlying the definition of the measure (dY) is to propose a base and its corresponding coordinate system for the subspace in which the mass of the density is concentrated. For the case of Srivastava's definition, Srivastava (2003), it is no simple matter to propose a matrix expression for the base and the coordinate system. However, if we begin with the vectorisation of Y by blocks, $\operatorname{vec} Y = (\operatorname{vec} Y_{11}', \operatorname{vec} Y_{21}', \operatorname{vec} Y_{12}', \operatorname{vec} Y_{22}')'$, we see that

$$\operatorname{vec} Y = M \operatorname{vec} Y_I. \tag{5}$$

In other words, the elements in the matrices Y_{11}, Y_{12}, Y_{21} define the coordinates and $M \in \mathbb{R}^{Nm \times q(mN+m-q)}$ is the base. This base is constituted as $M = (M_1' M_2')'$, where M_1 is the usual base for $\mathbb{R}^{q(N+m-q)}$ (perhaps with its rows permuted) and M_2 is a linear combination of M_1, such that $\operatorname{vec} Y_{22} = M_2 \operatorname{vec} Y_I$. For example, let us assume that $Y \in \mathbb{R}^2$, with $r(\operatorname{Cov}(Y)) = 1$, following Srivastava (2003), $(dX) = (dX_I) = dx_1$ (or $= dx_2$). With no loss of generality, let $(dX) = (dX_I) = dx_1$, then

$$X = \begin{pmatrix} 1 \\ a \end{pmatrix} x_1$$

with constant a, such that $x_2 = ax_1$. In matrix form, (5) defines a factorisation of Y and the explicit form of (dY) defines a factorisation of the measure (dY). Note, however, that (5) is not the only possible factorisation of Y. For example, we might consider the QR factorisation of Y. $Y = H_1 T$ where $H_1 \in \mathcal{V}_{q,N}$ and $T \in \mathcal{T}_{m,q}^{+}$, in which the elements of T denote the rectangular coordinates. Now observe that, because the measure $(H_1' dH_1')$ is invariant under orthogonal transforms, see p. 69 Muirhead (1982), then

$$(dX) = (QdY) = (dY) \tag{6}$$

for $Q \in \mathcal{O}(N)$. Note that $X = QY = QH_1T = R_1T$, with $R_1 = QH_1$ and $J(Y \to X) = J(dY \to dX)$. In other words, the measure (dY) defined under the QR factorisation is the Hausdorff measure, which, like Lebesgue's measure, must be invariant under rotations, see Theorem 19.2, p. 252 in Billingley (1986). Explicitly, the factorisation of the measure (dY) is given by:

$$(dY) = \prod_{i=1}^{q} t_{ii}^{N-i} (H_1' dH_1)(dT) \tag{7}$$

see Díaz-García and González-Farías (1999) and/or Díaz-García and González-Farías (2002a). The Jacobian of the QR factorisation is given by Srivastava (2003), Theorem 2.1 and Corollary 2.2, the expression of which does not coincide with that given in (7), because in Srivastava (2003) this Jacobian is determined with respect to the measure (dY_I). However, note that the equivalence between the two results can be obtained, observing that in our case $(H'dY) = (dY)$, while in Srivastava (2003), $(H'dY) = |H'_{11}|^{m-q}(dY)$, when $H \in \mathcal{O}(N)$. This situation is similar to that occurring between Theorem 2.3 in Srivastava (2003) and Theorem 2 in Uhlig (1994).

In general, note that it is possible to propose alternative definitions to those given in (7) for the measure (dY), with the additional condition that such measures should be invariant under leftward rotations, and even that some should be invariant under rightward rotations. Thus, among other possibilities for such a factorisation, including QR factorisation, we have the following:

$$Y = \begin{cases} H_1T & \text{QR factorisation} \\ Q_1R & \text{Polar factorisation} \\ P_1DW'_1 & \text{Singular value factorisation} \\ V_1\Lambda T_1 & \text{Modified QR factorisation} \\ LU & \text{LU factorisation} \end{cases} \tag{8}$$

where Q_1, P_1 and $V_1 \in \mathcal{V}_{q,N}$, $W_1 \in \mathcal{V}_{q,m}$, $R \in \mathcal{S}_m^+(q)$, D and Λ are diagonal matrices, with different elements, $U \in \mathcal{T}_{m,q}^+$ and $L' \in \mathcal{T}_{N,q}^+$. Under all these decompositions, the explicit form of the measure (dY) has been studied in Díaz-García and González-Farías G. (1999) and Díaz-García and González-Farías (2002a) and some application to the theory of distributions in Díaz-García and González-Farías (2002b).

Taking all the above into account, we seek to establish that the two approaches, that of this article and the one given by Srivastava (2003), are correct. However, when they are applied, it is necessary to be careful because both the Jacobian and the density in which we will apply these results must be expressed in terms of the same measure. Observe, also, that in general not all problems can easily be resolved by both approaches. For example, see the problem of finding the Pseudo-Wishart non-central singular density, see Theorem 5.1 in Srivastava (2003) and the last section in Díaz-García and González-Farías (2002b).

3. JACOBIANS

In this section we examine the version of Theorems 2.2. and 2.5 in Srivastava (2003) under our approach, and their extension to more general cases.

Given $X \in \mathcal{L}_{m,N}^+(q)$, and constant $A \in \mathcal{L}_{N,p}^+(r)$, and $Y \in \mathcal{L}_{m,p}^+(q)$ with $r \geq q$. We wish to determine the Jacobian of the transform $Y = AX$. Let us first consider the following case:

THEOREM 3.1. *Let $X \in \mathcal{L}_{m,N}^+(N)$, with $A \in \mathcal{L}_{N,p}^+(N)$ constant and $Y \in \mathcal{L}_{m,p}^+(N)$. If $Y = AX$, then*

$$(dY) = \prod_{i=1}^{N} \sigma_i(A)^m (dX) = \prod_{i=1}^{N} ch_i(AA')^{m/2}(dX) \tag{9}$$

where $ch_i(M)$ and $\sigma_i(M)$ are the i-th non-null eigenvalue and singular value of M, respectively.

PROOF. Let $A = H_1 D_A Q'$ be the non-singular part of the SVD of A, where $H_1 \in \mathcal{V}_{N,p}$, $D_A = \operatorname{diag}(\sigma_1(A), \cdots, \sigma_N(A))$, with $\sigma_i(A)$ the i-th singular value of A and $Q \in \mathcal{O}(N)$. Furthermore, note that $r(Y) = r(AX) = N$. By differentiating $Y = AX$, we obtain

$$\begin{aligned} dY &= AdX \\ &= H_1 D_A Q' dX. \end{aligned}$$

Now, let H_2 (a function of H_1) be such that $H = (H_1 \vdots H_2) \in \mathcal{O}(N)$, then

$$\begin{aligned} H'dY &= \begin{pmatrix} H_1' \\ H_2' \end{pmatrix} H_1 D_A Q' dX \\ &= \begin{pmatrix} H_1' H_1 D_A Q' dX \\ H_2' H_1 D_A Q' dX \end{pmatrix} \\ &= \begin{pmatrix} D_A Q' dX \\ 0 \end{pmatrix} \end{aligned}$$

as $H_2' H_1 = 0$. From (6), we have $(H'dY) = (dY)$ and that $(Q'dX) = (dX)$, then

$$\begin{aligned} (dY) &= |D_A|^m (dX) \\ &= \prod_{i=1}^{N} \sigma_i(A)^m (dX) \\ &= \prod_{i=1}^{N} ch_i(AA')^{m/2} (dX) \end{aligned}$$

■

REMARK 3.1. Note that, we can consider the QR decomposition instead of SVD of matrix A in Theorem 3.1. That is $A = H_1 T$, where $H_1 \in \mathcal{V}_{N,p}$ and $T \in \mathcal{T}_N^+$. Alternatively to (9) we have that

$$(dY) = \prod_{i=1}^{N} t_{ii}^m (dX). \tag{10}$$

The proof is parallel to that given in Theorem 3.1. Additionally note that, when $N = p$, (9) and (10) they agree.

THEOREM 3.2. *Let $X \in \mathcal{L}_{m,N}^+(q)$, with $A \in \mathcal{L}_{N,p}^+(r)$ constant, and $Y \in \mathcal{L}_{m,p}^+(q)$, with $\min(p, N) \geq r \geq q$. If $Y = AX$, then*

$$(dY) = \frac{\displaystyle\prod_{i=1}^{q} ch_i(ACC'A')^{m/2}}{\displaystyle\prod_{i=1}^{q} ch_i(CC')^{m/2}} (dX) \tag{11}$$

where $C \in \mathcal{L}_{q,N}^+(q)$.

PROOF. Let $C \in \mathcal{L}^+_{q,N}(q)$ such that $X = CZ$ where $Z \in \mathcal{L}^+_{m,q}(q)$ and let us denote $R = AC$. Then

$$\begin{aligned} Y &= AX \\ &= ACZ \\ &= RZ. \end{aligned}$$

Observing that $r(Y) = r(RZ) = r(Z) = q$, from Theorem 3.1 we have

$$\begin{aligned} (dY) &= \prod_{i=1}^{q} ch_i(RR')^{m/2}(dZ) \\ &= \prod_{i=1}^{q} ch_i(ACC'A')^{m/2}(dZ). \end{aligned} \tag{12}$$

Now, $X = BZ$, again applying Theorem 3.1, we obtain

$$(dX) = \prod_{i=1}^{q} ch_i(CC')^{m/2}(dZ)$$

from which, substituting $(dZ) = \prod_{i=1}^{q} ch_i(CC')^{-m/2}(dX)$ in (12), we obtain the desired result. ∎

REMARK 3.2. Note that, when $N = p = r$ in Theorem 3.2, we obtain a result analogous to Theorem 2.2 in Srivastava (2003). Under Theorem 3.2, note that if $A \in \mathcal{O}(N)$ and $Y = AX$, it is confirmed that $(dX) = (dY)$ and in consequence the result in (1) is obtained without the contradiction given in (2).

We now generalize the result from Theorem 3.2 to the case in which $Y = AXB$.

THEOREM 3.3. *Let $X \in \mathcal{L}^+_{m,N}(q)$, with constant $A \in \mathcal{L}^+_{N,p}(r_A)$ and $B \in \mathcal{L}^+_{n,m}(r_B)$ also constant, and let $Y \in \mathcal{L}^+_{n,p}(q)$, with $r \geq q$, such that $r_A \geq q$ and $r_B \geq q$. If $Y = AXB$, then*

$$(dY) = \frac{\displaystyle\prod_{i=1}^{r_C} ch_i(ACC'A')^{r_E/2} \prod_{j=1}^{r_E} ch_j(B'E'EB)^{r_C/2}}{\displaystyle\prod_{i=1}^{r_C} ch_i(CC')^{r_E/2} \prod_{j=1}^{r_E} ch_j(E'E)^{r_C/2}} (dX) \tag{13}$$

where $C \in \mathcal{L}^+_{r_C,N}(r_C)$, $E \in \mathcal{L}^+_{m,r_CE}(r_E)$ such that $X = CZE$ with $Z \in \mathcal{L}^+_{r_E,r_C}(q)$, $q = \min(r_C, r_E)$.

We now establish an alternative proof to that given by Díaz-García and Gutiérrez (1997) for Uhlig's first conjecture, see Theorem 4 Uhlig (1994). For this, consider the following preliminary results.

Lemma 3.1. *Let $Z \in \mathcal{L}^{+}_{m,N}(N)$, such that $Z = VDW'_1$ with $W_1 \in \mathcal{V}_{N,m}$, $V \in \mathcal{O}(N)$ and $D = \operatorname{diag}(d_1, \cdots, d_N)$, $d_1 > \cdots > d_N > 0$. Then*

$$(dZ) = 2^{-N}|D|^{m-N}\prod_{i<j}^{N}(d_i^2 - d_j^2)(dD)(V'dV)(W'_1dW_1) \tag{14}$$

where $(dD) \equiv \bigwedge_{i=1}^{N} dD_{ii}$, *and*

$$(W'_1dW_1) \equiv \bigwedge_{i=1}^{N}\bigwedge_{j=i+1}^{m} w'_j dw_i \quad \text{and} \quad (V'dV) \equiv \bigwedge_{i=1}^{N}\bigwedge_{j=i+1}^{N} v'_j dv_i$$

define an invariant measure on $\mathcal{V}_{N,m}$ and on $\mathcal{O}(m)$, respectively, see Uhlig (1994) and/or James (1954).

Lemma 3.2. *Under the assumptions of Lemma 3.1, define $S = Z'Z = W'_1LW_1 \in \mathcal{S}^{+}_{m}(N)$, where $L = \operatorname{diag}(l_1, \cdots, l_N)$, $l_1 > \cdots > l_N > 0$. Then*

1. $(dS) = 2^{-N}|L|^{m-N}\prod_{i<j}^{N}(l_i - l_j)(dL)(W'_1dW_1)$

2. $(dZ) = 2^{-N}|L|^{(N-m-1)/2}(dS)(V'dV)$.

The proof follows from Lemma 3.1 and from Theorem 2 in Uhlig (1994).

Theorem 3.4 (First Uhlig's conjecture). *Let $X, Y \in \mathcal{S}^{+}_{m}(N)$, such that $X = B'YB$, with $B \in \mathcal{L}^{+}_{m,m}(m)$ fixed. Additionally, let $X = G_1KG'_1$ and $Y = H_1LH'_1$ with $G_1, H_1 \in \mathcal{V}_{N,m}$ and $K = \operatorname{diag}(k_1, \cdots, k_N)$, $k_1 > \cdots > k_N > 0$, $L = \operatorname{diag}(l_1, \cdots, l_N)$, $l_1 > \cdots > l_N > 0$. Then*

$$\begin{aligned}(dX) &= |G'_1BH_1|^{m+1-N}|B|^N(dY)\\ &= |H'_1B'G_1|^{m+1-N}|B|^N(dY)\\ &= |K|^{(m+1-N)/2}|L|^{-(m+1-N)/2}|B|^N(dY)\end{aligned} \tag{15}$$

Proof. Let $Z \in \mathcal{L}^{+}_{N,m}(N)$, such that $Y = Z'Z$. Then

$$X = B'YB = B'Z'ZB = \Lambda'\Lambda, \quad \text{with} \quad \Lambda = ZB \tag{16}$$

from Lemma 3.2

$$(dX) = 2^N|K|^{-(N-m-1)/2}(V'dV)^{-1}(d\Lambda). \tag{17}$$

In which $\Lambda = VDG'_1$, with $V \in \mathcal{O}(N)$ and $D^2 = K$. Note that $d\Lambda = dZB$, and so $(d\Lambda) = |B|^N(dZ)$, from which, substituting in (17), we obtain

$$(dX) = 2^N|K|^{-(N-m-1)/2}(V'dV)^{-1}|B|^N(dZ). \tag{18}$$

Now $Y = Z'Z$, from 3.2

$$(dZ) = 2^{-N}|L|^{(N-m-1)/2}(dY)(V_z'dV_z) \tag{19}$$

where $Z = V_zD_zH_1'$, where $D_z^2 = L$ and $V_z \in \mathcal{O}(N)$. Moreover, note that, due to the uniqueness of Haar's measure, $(V'dV) = (V_z'dV_z)$, see James (1954). Thus, substituting (19) in (18), we obtain

$$\begin{aligned}(dX) &= |L|^{(N-m-1)/2}|K|^{-(N-m-1)/2}|B|^N(dY)\\ &= |K|^{(m-N+1)/2}|L|^{-(m-N+1)/2}|B|^N(dY).\end{aligned}$$

Finally, note that

$$|K| = |G_1'XG_1| = |G_1'B'YBG_1| = |G_1'B'H_1LH_1'BG_1| = |G_1'B'H_1||L||H_1'BG_1|$$

from which, taking into account that $|A| = |A'|$, we obtain the other expressions for (dX). ■

Remark 3.3. Under the premises of Theorem 3.4, note that if $X \sim \mathcal{S}_m^+(N)$ and $X \sim \mathcal{W}_m(N, \Sigma)$, then X can be written as $X = U'U$, with $U \sim \mathcal{N}_{N\times m}(0, \Sigma, I_N)$, where $U = H_uD_uG_1$, is the SVD of U, where $H_u \in \mathcal{O}(N)$. Then $X = G_1LG_1'$, with $L = D_u^2$, see Uhlig (1994) and Srivastava (2003). However, note that this is not the only way in which the matrix X can be defined. Assume now that $U_1 \sim \mathcal{N}_{n\times m}(0, \Sigma, I_N)$, with $U_1 \in \mathcal{L}_{m,n}^+(N)$, such that $U_1 = H_{u_1}D_{u_1}G_1$ is the non-singular part of the SVD of U_1. Then we also obtain $X_1 = G_1LG_1'$, with $L = D_{u_1}^2$. Due to the uniqueness of the measure of $(G_1'dG_1)$, $(dX) = (dX_1)$, but this is not so for the corresponding measures (dU) and (dU_1), which are defined by Lemmas 3.1 and 3.4, respectively. When this situation is known, it must be taken into account when Theorem 3.4 is applied, otherwise the result obtained would be the following:

Consider the assumptions made under Theorem 3.4. Let $Z \in \mathcal{L}_{n,m}^+(N)$, such that $Y = Z'Z$, then

$$X = B'YB = B'Z'ZB = \Lambda'\Lambda, \quad \text{with} \quad \Lambda = ZB$$

from Lemma 3.4

$$(dX) = 2^N|K|^{-(n-m-1)/2}(V_1'dV_1)^{-1}(d\Lambda).$$

in which $\Lambda = V_1DG_1'$, with $V_1 \in \mathcal{V}_{N,n}$ and $D^2 = K$. Note that $d\Lambda = dZB$, and so, from Theorem 3.3

$$(d\Lambda) = \frac{\displaystyle\prod_{i=1}^{N} ch_i(B'C'CB)^{n/2}}{\displaystyle\prod_{i=1}^{N} ch_i(C'C)^{n/2}}\,(dZ)$$

where $Z = UC$, con $U \in \mathcal{L}_{N,n}^+(N)$ and $C \in \mathcal{L}_{m,N}^+(N)$ is fixed. Therefore

$$(dX) = 2^N|K|^{-(N-m-1)/2}(V'dV)^{-1}\frac{\displaystyle\prod_{i=1}^{N} ch_i(B'C'CB)^{n/2}}{\displaystyle\prod_{i=1}^{N} ch_i(C'C)^{n/2}}\,(dZ).$$

Now $Y = Z'Z$, from Lemma 3.4

$$(dZ) = 2^{-N}|L|^{(n-m-1)/2}(dY)(V_z'dV_z)$$

where $Z = V_zD_zH_1'$, with $D_z^2 = L$ and $V_z \in \mathcal{V}_{N,n}$. Additionally note that, due to the uniqueness, $(V1'dV_1) = (V_z'dV_z)$, see James (1954). Thus, finally, we obtain

$$(dX) = |K|^{(m-N+1)/2}|L|^{-(m-N+1)/2} \frac{\displaystyle\prod_{i=1}^{N} ch_i(B'C'CB)^{n/2}}{\displaystyle\prod_{i=1}^{N} ch_i(C'C)^{n/2}} (dY). \tag{20}$$

Finally, note that the result in Theorem 3.4 is obtained from (20) taking $n = N$, in which case $U = Z$ and $C = I$ in the proof.

Now, let us assume that A and B have a non-singular Wishart distribution; the matrix $R = A^{-1/2}BA'^{-1/2}$ then has a multivariate F (or beta type II) distribution, in which $A^{1/2}$ is a root of matrix A, such that $A = A^{1/2}A'^{1/2}$, see p. 92 in Srivastava and Khatri (1979) and p. 190 in Gupta and Nagar (2000), among others. An alternative definition, proposed by various authors, is given by the expression $R_1 = B^{1/2}A^{-1}B'^{1/2}$, see James (1964), p. 449 in Muirhead (1982) and p. 192 in Gupta and Nagar (2000), among others. A similar situation occurs in the case of the beta type I distribution, Srivastava (1968) and Díaz-García and Gutiérrez (2001).We now present various results that enable us to extend the densities of R and R_1 to the case in which both B and A are singular random matrices. First, however, consider the following lemma, the proof of which is given in Díaz-García et al. (2003).

Lemma 3.3. *Assume that $X \in \mathcal{S}_m^+(N)$ and let $Y = X^+$ (the inverse of Moore-Penrose of X, see Campbell and Meyer (1979)). Entonces,*

$$(dY) = |K|^{-2m+N-1}(dX)$$

with $X = G_1KG_1'$, $K = \operatorname{diag}(k_1, \cdots, k_N)$, $k_1 > \cdots > k_N > 0$.

Theorem 3.5. *Let $X, Y \in \mathcal{S}_m^+(N)$, such that $X = B'Y^+B$, with $B \in \mathcal{L}_{m,m}^+(m)$ fixed. Moreover, let $X = G_1KG_1'$ and $Y = H_1LH_1'$ with $G_1, H_1 \in \mathcal{V}_{N,m}$ y $K = \operatorname{diag}(k_1, \cdots, k_N)$, $k_1 > \cdots > k_N > 0$, $L = \operatorname{diag}(l_1, \cdots, l_N)$, $l_1 > \cdots > l_N > 0$. Then*

$$(dX) = |K|^{(m+1-N)/2}|L|^{-(5m+3-3N)/2}|B|^N(dY) \tag{21}$$

Proof. Denote $Z = Y^+$, then from Theorem 3.4

$$(dX) = |K|^{(m-N+1)/2}|L|^{-(m-N+1)/2}|B|^N(dZ).$$

The result follows from Lemma 3.3, observing that $(dZ) = |L|^{-2m+N-1}(dX)$. ■

Remark 3.4. An alternative version of Theorem 3.5 is obtained under the same idea as in Remark 3.3.

We now generalize the results from Theorems 3.4 and 3.5 to the case in which B is a singular fixed matrix, such that $r(B) \geq N$. First, however, by analogy with Lemmas 3.1 and 3.2, we obtain the following result:

LEMMA 3.4. *Let $Z \in \mathcal{L}_{m,N}^{+}(q)$, such that $Z = V_1 D W_1'$ with $W_1 \in \mathcal{V}_{q,m}$, $V_1 \in \mathcal{V}_{q,N}$ y $D = \operatorname{diag}(d_1, \cdots, d_q)$, $d_1 > \cdots > d_q > 0$. Then*

1. $(dZ) = 2^{-N}|D|^{N+m-2q}\prod_{i<j}^{N}(d_i^2 - d_j^2)(dD)(V_1'dV_1)(W_1'dW_1)$
2. *If $S = Z'Z = W_1 L W_1'$ where $L = \operatorname{diag}(l_1, \cdots, l_N)$, $l_1 > \cdots > l_N > 0$. We then obtain*

$$(dZ) = 2^{-q}|L|^{(N-m-1)/2}(dS)(V_1'dV_1)$$

The proof can be found in Díaz-García et al. (1997).

THEOREM 3.6. *Let $X, Y \in \mathcal{S}_m^{+}(q)$, such that $X = B'YB$, with $B \in \mathcal{L}_{m,m}^{+}(r)$ fixed, such that $r \geq q$. Moreover, let $X = G_1 K G_1'$ and $Y = H_1 L H_1'$ with $G_1, H_1 \in \mathcal{V}_{q,m}$ and $K = \operatorname{diag}(k_1, \cdots, k_q)$, $k_1 > \cdots > k_q > 0$, $L = \operatorname{diag}(l_1, \cdots, l_q)$, $l_1 > \cdots > l_q > 0$. Then*

$$\begin{aligned}
(dX) &= |G_1'BH_1|^{m+1-n}\left(\frac{\displaystyle\prod_{i=1}^{q} ch_i(B'Q'QB)^{n/2}}{\displaystyle\prod_{i=1}^{q} ch_i(Q'Q)^{n/2}}\right)(dY)\\
&= |H_1'B'G_1|^{m+1-n}\left(\frac{\displaystyle\prod_{i=1}^{q} ch_i(B'Q'QB)^{n/2}}{\displaystyle\prod_{i=1}^{q} ch_i(Q'Q)^{n/2}}\right)(dY)\\
&= |K|^{(m+1-n)/2}|L|^{-(m+1-n)/2}\left(\frac{\displaystyle\prod_{i=1}^{q} ch_i(B'Q'QB)^{n/2}}{\displaystyle\prod_{i=1}^{q} ch_i(Q'Q)^{n/2}}\right)(dY)
\end{aligned}$$

where n is such that, $Y = Q'U'UQ$, with $U \in \mathcal{L}_{q,n}^{+}(q)$ and $Q \in \mathcal{L}_{m,q}^{+}(q)$.

PROOF. The proof is parallel to that given in Theorem 3.4, simply applying Theorem 3.4 when $(d\Lambda)$ is determined. ■

THEOREM 3.7. *In the Theorem 3.6, assume that $X = B'Y^{+}B$. Then*

$$(dX) = |K|^{(m+1-n)/2}|L|^{(n-5m-3+2q)/2}\left(\frac{\displaystyle\prod_{i=1}^{q} ch_i(B'Q'QB)^{n/2}}{\displaystyle\prod_{i=1}^{q} ch_i(Q'Q)^{n/2}}\right)(dY)$$

PROOF. The proof is immediate from Theorem 3.6 and Lemma 3.3. ■

REMARK 3.5. Once again, note that alternative results to those of Theorems 3.6 and 3.7 may be obtained under Remark 3.3. These new versions are derived by taking $n = N = q$ in Theorems 3.6 and 3.7.

4. SOME APPLICATIONS

Finally, in this section we present some applications of some of the results obtained in Section 3.

Assuming that $X \sim \mathcal{N}_{N\times m}(\mu, \Sigma, \Theta)$, $\Sigma \in \mathcal{S}_m^+(r)$, $r \leq m$ and $\Theta \in \mathcal{S}_N^+(k)$, $k \leq N$, using the characteristic function technique, we know that, $Y \sim \mathcal{N}_{p\times s}(A\mu B, B'\Sigma B, A\Theta A')$, if $Y = AXB$, with $A \in \mathcal{L}_{N,p}^+(r_A)$, $r_A \geq k$ and $B \in \mathcal{L}_{s,m}^+(r_B)$, $r_B \geq r$. We now see the proof, using the variable change theorem.

First, note that $X \sim \mathcal{N}_{N\times m}(\mu, \Sigma, \Theta)$ if and only if $X = CZE + \mu$, where $Z \sim \mathcal{N}_{k\times r}(0, I_r, I_k)$, $B \in \mathcal{L}_{k,N}^+(k)$ and $E \in \mathcal{L}_{m,r}^+(r)$. Moreover, $\Sigma = E'E$ y $\Theta = CC'$. Then, from Khatri (1968) or Díaz-García et al. (1997), the density of X is given by

$$dF_X(X) = \frac{1}{(2\pi)^{kr/2}\displaystyle\prod_{i=1}^{r} ch_i(\Sigma)^{k/2}\prod_{j=1}^{k} ch_j(\Theta)^{r/2}} \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^-(X-\mu)'\Theta^-(X-\mu)\right)(dX) \tag{22}$$

where M^- is a symmetric generalized inverse of M, $MM^-M = M$ and $ch(M)_l$ are the non-null eigenvalues of M. Then

$$dF_Y(Y) = f_X(A^+YB^+)|J(X \to Y)|(dY).$$

Thus, from Theorem 3.3 we obtain

$$(dX) = \frac{\displaystyle\prod_{i=1}^{r_C} ch_i(CC')^{r_E/2}\prod_{j=1}^{r_E} ch_j(E'E)^{r_C/2}}{\displaystyle\prod_{i=1}^{r_C} ch_i(ACC'A')^{r_E/2}\prod_{j=1}^{r_E} ch_j(B'E'EB)^{r_C/2}}(dY) = \frac{\displaystyle\prod_{i=1}^{k} ch_i(\Theta)^{r/2}\prod_{j=1}^{r} ch_j(\Sigma)^{k/2}}{\displaystyle\prod_{i=1}^{k} ch_i(A\Theta A')^{r/2}\prod_{j=1}^{r} ch_j(B'\Sigma B)^{k/2}}(dY).$$

from which, substituting in (22), and denoting $\mu_y = A\mu B$, we have

$$dF_Y(Y) = \frac{1}{(2\pi)^{kr/2}\displaystyle\prod_{i=1}^{k} ch_i(A\Theta A')^{r/2}\prod_{j=1}^{r} ch_j(B'\Sigma B)^{k/2}}$$
$$\operatorname{etr}\left(-\tfrac{1}{2}\Sigma^- B'^+(Y-\mu_y)'A'^+\Theta^-A^+(Y-\mu_y)B^+\right)(dY).$$

Finally, therefore

$$dF_Y(Y) = \frac{1}{(2\pi)^{kr/2}\displaystyle\prod_{i=1}^{k} ch_i(A\Theta A')^{r/2}\prod_{j=1}^{r} ch_j(B'\Sigma B)^{k/2}}$$

$$\operatorname{etr}\left(-\tfrac{1}{2}(B'\Sigma B)^{-}(Y-\mu_y)'(A\Theta A')^{-}(Y-\mu_y)\right)(dY).$$

Once again, note that this result cannot be obtained by applying Theorem 2.2 in Srivastava (2003) directly in (22), for the reasons given in the Introduction.

Let us now assume that $S \in \mathcal{S}_m^+(r_{_S})$ has a singular Wishart or Pseudo-Wishart distribution, that is, $S \sim \mathcal{W}_m(n,\Sigma)$ with $S \sim \mathcal{W}_m(n,\Sigma)$. Additionally, assume that $A \in \mathcal{S}_m^+(r_{_A})$ is fixed, such that $r(A) \geq r(\Sigma)$ and $r(A'\Sigma A) = r(\Sigma)$. Then $V = A'SA \sim \mathcal{W}_m(n, A'\Sigma A)$, with $r(V) = r_{_V} = r(S)$. This result is well known and can be obtained from the characteristic function technique. Now the proof is obtained by applying the Jacobians found in Section 3.

From Díaz-García et al. (1997), the function of S is given by

$$dG_S(S) = \frac{\pi^{n\left(r_{_S}-r_\Sigma\right)/2}|L|^{(n-m-1)/2}}{2^{nr_\Sigma/2}\Gamma_{r_{_S}}\left[\frac{1}{2}n\right]\displaystyle\prod_{i=1}^{r_\Sigma} ch_i(\Sigma)^{n/2}} \operatorname{etr}\left(\tfrac{1}{2}\Sigma^{-}S\right)(dS)$$

where $S = G_1 L G_1'$ is the non-singular part of the spectral decomposition of S, with $G_1 \in \mathcal{V}_{r_{_S},m}$ and $L = \operatorname{diag}(l_1,\cdots,l_{r_{_S}})$, $l_1 > \cdots > l_{r_{_S}} > 0$. Let $\Sigma = Q'Q$, then from Theorem 3.6,

$$(dS) = |K|^{-(m+1-n)/2}|L|^{(m+1-n)/2}\left(\frac{\displaystyle\prod_{i=1}^{r_\Sigma} ch_i(\Sigma)^{n/2}}{\displaystyle\prod_{i=1}^{r_\Sigma} ch_i(B'\Sigma B)^{n/2}}\right)(dV)$$

With $K = \operatorname{diag}(k_1,\cdots,k_{r_{_V}})$, $k_1 > \cdots > k_{r_{_V}} > 0$ such that, $V = W_1 K W_1'$ is the non-singular part of the spectral decomposition of V, $W_1 \in \mathcal{V}_{r_{_V},m}$. Then

$$\begin{aligned} dG_V(V) &= g_{_S}(A'^{+}VA^{+})|J(S\to V)|(dV) \\ &= \frac{\pi^{n\left(r_{_V}-r_\Sigma\right)/2}|K|^{(n-m-1)/2}}{2^{nr_\Sigma/2}\Gamma_{r_{_V}}\left[\frac{1}{2}n\right]\displaystyle\prod_{i=1}^{r_\Sigma} ch_i(A'\Sigma A)^{n/2}} \operatorname{etr}\left(\tfrac{1}{2}(A'\Sigma A)^{-}V\right)(dV) \end{aligned}$$

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