JACK POLYNOMIALS OF SECOND ORDER

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Abstract

This work solves the partial differential equation for the Jack polynomials of second order. When the parameter α of the solution takes the values $1/2, 1$ and 2 we get explicit formulas for the quaternionic, complex and real zonal polynomials of second order, respectively.

Key words: Jack functions, Jack polynomials, Laplace-Beltrami operator, zonal polynomials, hermitian matrix, quaternionic matrix, hypergeometric differential equation.
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1 Introduction

The theory of Jack functions (and Jack polynomials) has been a vertiginous development on the computations of coefficients and combinatorial conjectures about them, see Goulden and Jackson (1996), Sawyer (1997), Koev (2004), Koev and Demmel (2004) and Dimitriu *et al.* (2005), among many others. Before Jack polynomials, the real and complex zonal polynomials have been studied extensively in statistical literature. They have important open problems to solve which could be handled with Jack polynomials theory, using the fact that the zonal polynomials of a symmetric matrix and the zonal polynomials of a hermitian matrix are Jack polynomials for $\alpha = 2$ and $\alpha = 1$, respectively, see James (1964), James (1968), Khatri (1970), Muirhead (1982), Díaz-García and Caro (2004) and Díaz-García and Caro (2005), among many other.

It is known that the real and complex zonal polynomials are eigenfunctions of the Laplace-Beltrami operator. From the resulting partial differential equation a recurrence relation among the coefficients is obtained and then the polynomials can be computed. Few explicit formulae to calculate the Jack polynomials appear in literature; specifically, in the real case, i.e. when $\alpha = 2$, James (1964, Section 9) proposes some expressions. For the same value of α, but only for the second order, James (1968) solves the partial differential equation for real zonal polynomials.

In general, Jack polynomials are also eigenfunctions of a Laplace-Beltrami operator type (see Dimitriu *et al.* (2005, Definition 2.10)), and as before, a general recurrence relation can be derived from a partial differential equation to compute the coefficients of the polynomials.

Following the idea of James (1968), this work finds an explicit formula for the Jack polynomials of second order. This is carried out by solving the general partial differential equation of parameter α when two eigenvalues are considered. Taking $\alpha = 1/2, 1$, formulae for cuaternionic and complex zonal polynomials of second order are obtained, respectively (for definitions of the quaternionic zonal polynomials see Gross and Richards (1987)). Also, the results derived in James (1968) for the real zonal polynomials of second order are ratified when $\alpha = 2$ is replaced in the Jack polynomial formula.

2 A Formula for Jack Polynomials of Second order

Let us characterize the Jack symmetric function $J_\kappa^{(\alpha)}(y_1, \dots, y_m)$ of parameter α, see Sawyer (1997). A decreasing sequence of nonnegative integers

$\kappa = (k_1, k_2, \ldots)$ with only finitely many nonzero terms is said to be a partition of $k = \sum k_i$. Let κ and $\lambda = (l_1, l_2, \ldots)$ be two partitions of k, we write $\lambda \leq \kappa$ if $\sum_{i=1}^t l_i \leq \sum_{i=1}^t k_i$ for each t. The conjugate of κ is $\kappa' = (k'_1, k'_2, \ldots)$ where $k'_i = \text{card}\{j : k_j \geq i\}$. The length of κ is $l(k) = \max\{i : k_1 \neq 0\} = k'_1$. If $l(\kappa) \leq m$, one often writes $\kappa = (k_1, k_2, \ldots, k_m)$. The partition $(1, \ldots, 1)$ of length m will be denoted by 1^m.
The monomial symmetric function $M_\kappa(\cdot)$ indexed by the partition κ can be seen as a function on an arbitrary number of variables such that all but a finite number are equal to 0: if $y_i = 0$ for $i > m \geq l(\kappa)$ then $M_\kappa(y_1, \ldots, y_m) = \sum y_1^{\sigma_1} \cdots y_m^{\sigma_m}$, where the sum is over all the distinct permutation $\{\sigma_1, \ldots, \sigma_m\}$ of $\{k_1, \ldots, k_m\}$, and if $l(\kappa) > m$ then $M_\kappa(y_1, \ldots, y_m) = 0$. A symmetric function f is a linear combination of monomial symmetric functions. If f is a symmetric function then $f(y_1, \ldots, y_m, 0) = f(y_1, \ldots, y_m)$. For each $m \geq 1$, $f(y_1, \ldots, y_m)$ is a symmetric polynomial in m variables.

Thus the Jack symmetric function $J_\kappa^{(\alpha)}(y_1, \ldots, y_m)$ of parameter α, satisfy the following conditions:

$$J_\kappa^{(\alpha)}(y_1, \ldots, y_m) = \sum_{\lambda \leq \kappa} j_{\kappa,\lambda} M_\lambda(y_1, \ldots, y_m), \tag{1}$$

$$J_\kappa^{(\alpha)}(1, \ldots, 1) = \alpha^k \prod_{i=1}^m \left(\frac{m - i + 1}{\alpha}\right)_{k_i}, \tag{2}$$

$$\sum_{i=1}^m y_i^2 \frac{\partial^2 J_\kappa^{(\alpha)}(y_1, \ldots, y_m)}{\partial y_i^2} + \frac{2}{\alpha} \sum_{i=1}^m y_i^2 \sum_{j \neq i} \frac{1}{y_i - y_j} \frac{\partial J_\kappa^{(\alpha)}(y_1, \ldots, y_m)}{\partial y_i} = \sum_{i=1}^m k_i(k_i - 1 + \frac{2}{\alpha}(m - i)) J_\kappa^{(\alpha)}(y_1, \ldots, y_m). \tag{3}$$

Where the constant $j_{\kappa,\lambda}$ is not dependent of $y_i's$ but of κ and λ, and $(a)_n = \prod_{i=1}^n (a + i - 1)$. Note that if $m < l(\kappa)$ then $J_\kappa^{(\alpha)}(y_1, \ldots, y_m) = 0$. The conditions accept the case $\alpha = 0$ and then $J_\kappa^{(0)}(y_1, \ldots, y_m) = e_{\kappa'} \prod_{i=1}^m (m - i + 1)^{k_i}$, with $e_\kappa(y_1, \ldots, y_m) = \prod_{i=1}^{l(\kappa)} e_{k_i}(y_1, \ldots, y_m)$ are the called elementary symmetric function indexed by the partition κ, and if $m \geq l(\kappa)$ then $e_r(y_1, \ldots, y_m) = \sum_{i_1 < i_2 < \cdots < i_r} y_{i_1} \cdots y_{i_r}$, or if $m < l(\kappa)$ then $e_r(y_1, \ldots, y_m) = 0$, see Sawyer (1997).

Now, from Koev and Demmel (2004), the Jack functions $J_\kappa^{(\alpha)}(Y) = J_\kappa^{(\alpha)}(y_1, \ldots, y_m)$, where $y_1, \ldots, y_m$ are the eigenvalues of the matrix Y, can be normalised in such way that

$$\sum_\kappa C_\kappa^\alpha(Y) = (\text{tr}(Y))^k$$

where $C_\kappa^\alpha(Y)$ denotes the Jack polynomials and they are related with the Jack functions by

$$C_\kappa^\alpha(Y) = \frac{\alpha^k k!}{j_\kappa} J_\kappa^\alpha(Y) \tag{4}$$

with

$$j_\kappa = \prod_{(i,j)\in\kappa} h_*^\kappa(i,j) h_\kappa^*(i,j)$$

and $h_*^\kappa(i,j) = k_j - i + \alpha(k_i - j + 1)$ and $h_\kappa^*(i,j) = k_j - i + 1 + \alpha(k_i - j)$ are the upper and lower hook lengths at $(i,j) \in \kappa$, respectively.

Then by applying (4), (3) can be written as

$$\sum_1^m y_i^2 \frac{\partial^2 C_\kappa^{(\alpha)}(Y)}{\partial y_i^2} + \frac{2}{\alpha} \sum_{i=1}^m y_i^2 \sum_{j\neq i} \frac{1}{y_i - y_j} \frac{\partial C_\kappa^{(\alpha)}(Y)}{\partial y_i} =$$
$$\sum_{i=1}^m k_i(k_i - 1 + \frac{2}{\alpha}(m-i)) C_\kappa^{(\alpha)}(Y). \tag{5}$$

3 Jack Polynomials of Second Order.

When $m = 2$ in (5) and denoting $C_\kappa^{(\alpha)}(Y)$ as $C_\kappa^{(\alpha)}$ we get the partial differential equation

$$y_1^2 \frac{\partial^2 C_\kappa^{(\alpha)}}{\partial y_1^2} + y_2^2 \frac{\partial^2 C_\kappa^{(\alpha)}}{\partial y_2^2} + \frac{2}{\alpha} y_1^2 (y_1 - y_2)^{-1} \frac{\partial C_\kappa^{(\alpha)}}{\partial y_1} - \frac{2}{\alpha} y_2^2 (y_1 - y_2)^{-1} \frac{\partial C_\kappa^{(\alpha)}}{\partial y_2}$$
$$- \left[k_1\left(k_1 - 1 + \frac{2}{\alpha}\right) + k_2(k_2 - 1)\right] C_\kappa^{(\alpha)} = 0. \tag{6}$$

Let us replace $u = y_1 + y_2$ and $v = y_1 y_2$ in (6), then we find

$$(u^2 - 2v)\frac{\partial^2 C_\kappa^{(\alpha)}}{\partial u^2} + 2v^2 \frac{\partial^2 C_\kappa^{(\alpha)}}{\partial v^2} + 2uv \frac{\partial^2 C_\kappa^{(\alpha)}}{\partial u \partial v} + \frac{2u}{\alpha}\frac{\partial C_\kappa^{(\alpha)}}{\partial u} + \frac{2v}{\alpha}\frac{\partial C_\kappa^{(\alpha)}}{\partial v}$$
$$- \left[k_1\left(k_1 - 1 + \frac{2}{\alpha}\right) + k_2(k_2 - 1)\right] C_\kappa^{(\alpha)} = 0.$$

Substituting $z = \frac{u}{2\sqrt{v}}$ and $t = \sqrt{v}$ we obtain

$$(1 - z^2)\frac{\partial^2 C_\kappa^{(\alpha)}}{\partial z^2} - t^2 \frac{\partial^2 C_\kappa^{(\alpha)}}{\partial t^2} - \left(\frac{2}{\alpha} + 1\right) z \frac{\partial C_\kappa^{(\alpha)}}{\partial z} - \left(\frac{2}{\alpha} - 1\right) t \frac{\partial C_\kappa^{(\alpha)}}{\partial t}$$

$$+2\left[k_1\left(k_1-1+\frac{2}{\alpha}\right)+k_2(k_2-1)\right]C_\kappa^{(\alpha)}=0.$$

It is easy to see that the last equation is homogeneous in t. Thus, by taking

$$C_\kappa^{(\alpha)}=t^{(k_1+k_2)}f(z),$$

the next ordinary differential equation is obtained

$$(1-z^2)\frac{d^2f}{dz^2}-\left(\frac{2}{\alpha}+1\right)z\frac{df}{dz}+\left[(k_1-k_2)\left(k_1-k_2+\frac{2}{\alpha}\right)\right]f=0.$$

Now, taking $w=(1-z)/2$ as the independent variable, the differential equation becomes

$$w(1-w)\frac{d^2f}{dw^2}+\left(\frac{1}{\alpha}+\frac{1}{2}\right)(1-2w)\frac{df}{dw}+\rho\left(\rho+\frac{2}{\alpha}\right)f=0, \tag{7}$$

with $\rho=k_1-k_2$, a non negative integer, according to the definition of the partition κ.

Comparing with the general hypergeometric equation

$$w(1-w)\frac{d^2f}{dw^2}+[c-(a+b+1)w]\frac{df}{dw}-abf=0, \tag{8}$$

we see that the Jack polynomials are involved in the solution of an hypergeometric differential equation of parameters $a=-\rho$, $b=\rho+\frac{2}{\alpha}$ and $c=\left(\frac{1}{\alpha}+\frac{1}{2}\right)$.

Following Erdélyi *et al.* (1981), we know that a solution of (8) which is regular at $w=0$ is given by

$$f(w)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}w^n={}_2F_1(a,b;c;w),$$

where ${}_2F_1(a,b;c;w)$ is the classical hypergeometric function, which we will now denote as $F(a,b;c;w)$.

Thus a solution of (7) is

$$f(z)=F\left(-\rho,\rho+\frac{2}{\alpha};\left(\frac{1}{\alpha}+\frac{1}{2}\right);\frac{1-z}{2}\right),$$

Let us refine the above solution by applying properties of the hypergeometric functions. From Erdélyi *et al.* (1981, Section 2.11, p.111), equation (2), we see that

$$F\left(2d, 2e; d+e+\frac{1}{2}; t\right) = F\left(d, e; d+e+\frac{1}{2}; 4t(1-t)\right),$$

then

$$\begin{aligned} f(z) &= F\left(-\rho, \rho+\frac{2}{\alpha}; \left(\frac{1}{\alpha}+\frac{1}{2}\right); \frac{1-z}{2}\right) \\ &= F\left(-\frac{\rho}{2}, \frac{\rho}{2}+\frac{1}{\alpha}; \left(\frac{1}{\alpha}+\frac{1}{2}\right); 1-z^2\right). \end{aligned} \tag{9}$$

By Erdélyi *et al.* (1981, Section 2.10, p.108), equation (1),

$$\begin{aligned} F(a, b; c; t) = {} & A_1 F(a, b; a+b-c+1; 1-t) \\ & + A_2 (1-t)^{c-a-b} F(c-a, c-b; c-a-b+1; 1-t), \end{aligned}$$

where

$$A_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{and} \quad A_2 = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}.$$

Then (9) can be written as follows:

$$\begin{aligned} F\left(-\frac{\rho}{2}, \frac{\rho}{2}+\frac{1}{\alpha}; \left(\frac{1}{\alpha}+\frac{1}{2}\right); 1-z^2\right) = {} & A_1 F\left(-\frac{\rho}{2}, \frac{\rho}{2}+\frac{1}{\alpha}; \frac{1}{2}; z^2\right) \\ & + A_2\, z\, F\left(\frac{1}{\alpha}+\frac{1+\rho}{2}, \frac{1}{2}-\frac{\rho}{2}; \frac{3}{2}; z^2\right), \end{aligned}$$

where

$$A_1 = \frac{\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{\alpha}+\frac{1+\rho}{2}\right)\Gamma\left(\frac{1-\rho}{2}\right)} \quad \text{and} \quad A_2 = \frac{\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{\rho}{2}\right)\Gamma(\frac{\rho}{2}+\frac{1}{\alpha})}.$$

then the Jack polynomials of second order are given by

$$\begin{aligned} \frac{C^{(\alpha)}_{(k_1,k_2)}(Y)}{C^{(\alpha)}_{(k_1,k_2)}(I_2)} = {} & (y_1 y_2)^{(k_1+k_2)/2} \frac{\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{\alpha}+\frac{1+\rho}{2}\right)\Gamma\left(\frac{1-\rho}{2}\right)} F\left(-\frac{\rho}{2}, \frac{\rho}{2}+\frac{1}{\alpha}; \frac{1}{2}; \frac{(y_1+y_2)^2}{4y_1y_2}\right) \\ & + \frac{(y_1y_2)^{(k_1+k_2-1)/2}}{2(y_1+y_2)^{-1}} \frac{\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{\rho}{2}\right)\Gamma(\frac{\rho}{2}+\frac{1}{\alpha})} F\left(\frac{1}{\alpha}+\frac{1+\rho}{2}, \frac{1}{2}-\frac{\rho}{2}; \frac{3}{2}; \frac{(y_1+y_2)^2}{4y_1y_2}\right). \end{aligned} \tag{10}$$

Observe that ρ is a nonnegative integer, thus (10) can be simplified and put it in terms of the hypergeometric functions according to ρ be even or odd. For distinguishing the case under consideration, odd or even, we will put the upper indexes o or e on A_1 and A_2. Observing that

(1) $\Gamma(1/2+z)\Gamma(1/2-z) = \pi \sec(\pi z)$
(2) $\Gamma(z)\Gamma(-z) = -\pi z^{-1} \csc(\pi z)$
(3) $\Gamma(z+n) = z(z+1)(z+2)\cdots(z+n-1)\Gamma(z)$

The following results are obtained

Even case. If $\rho = k_1 - k_2 = 2n$, $n = 0, 1, 2, \ldots$ then

$$A_1^e = \frac{(-1)^n \prod_{i=0}^{n-1}(1+2i)}{\prod_{i=0}^{n-1}\left(1+2\left(\frac{1}{\alpha}+i\right)\right)} \quad \text{and} \quad A_2^e = 0$$

Odd case. If $\rho = k_1 - k_2 = 2n+1$, $n = 0, 1, 2, \ldots$ then

$$A_1^o = 0 \quad \text{and} \quad A_2^o = (2n+1)A_1^e$$

Three particular cases are of interest in the literature: the quaternionic case ($\alpha = 1/2$), the complex zonal polynomials ($\alpha = 1$) and the real zonal polynomials ($\alpha = 2$), these results are summarised in the following table:

α	ρ	a	b	c	A_1	A_2
$\frac{1}{2}$	even	$-n$	$n+2$	$\frac{1}{2}$	$\frac{(-1)^n 3}{(2n+1)(2n+3)}$	0
	odd	$n+3$	$-n$	$\frac{3}{2}$	0	$\frac{(-1)^n 3}{(2n+3)}$
1	even	$-n$	$n+1$	$\frac{1}{2}$	$\frac{(-1)^n}{(2n+1)}$	0
	odd	$n+2$	$-n$	$\frac{3}{2}$	0	$(-1)^n$
2	even	$-n$	$n+1/2$	$\frac{1}{2}$	$\frac{(-1)^n (2n)!}{2^{2n}(n!)^2}$	0
	odd	$n+3/2$	$-n$	$\frac{3}{2}$	0	$\frac{(-1)^n (2n+1)!}{2^{2n}(n!)^2}$

Finally, given that $F(a, b; c; z) = F(b, a; c; z)$, the above formula for the real zonal polynomials is in agreement with the expression derived by James (1968).

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