QUASI-JORDAN ALGEBRAS

Raúl Velásquez and Raúl Felipe

Comunicación Técnica No I-06-14/28-09-2006
(MB/CIMAT)

Quasi-Jordan algebras

Raúl Velásquez[a,b] **and Raúl Felipe**[a,c]

[a]CIMAT
Callejón Jalisco s/n Mineral de Valenciana
Guanajuato, Gto, México

[b]DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE ANTIOQUIA
Calle 67 N 53-108, Apartado Areo 1226
Medellín, Colombia

[c]ICIMAF
Calle F esquina a 15, No 309. Vedado.
Ciudad de la Habana, Cuba

Abstract

In this paper we introduce a new algebraic structure of a Jordan type and we show several examples. This new structure called quasi-Jordan algebras appear in the study of the product

$$x \triangleleft y := \frac{1}{2}(x \dashv y + y \vdash x),$$

where x, y are elements in a dialgrebra $(D, \dashv, \vdash)$. The quasi-Jordan algebras are a generalization of the Jordan algebras for which the commutative law is changed by a quasi-commutative identity and a special form of the Jordan identity is retained. The quasi-Jordan algebras are not contained in the generalizations of Jordan algebras, in particular with respect to noncommutative Jordan algebras. We show a few results about the relationship between Jordan algebras and quasi-Jordan algebras. Also, we compare quasi-Jordan algebras with some structures. In particular, we found a special relation with the Leibniz algebras. We attach a quasi-Jordan algebra L_x to any ad-nilpotent element x with an index of nilpotence at most 3 in a Leibniz algebra L. In this part we extended the results of Kostrikin and Benkart-Isaacs about nilpotent elements to Leibniz algebras and we show that L_x is nondegenerated if L is nondegenerated.

2000 Mathematics Subject Classification (MSC2000): 17C50, 17A32.
Key words: Jordan algebras, Leibniz algebras, dialgebras.

Introduction

In algebra there are three strongly related classical algebras: Associative, Jordan and Lie algebras. It is known that any associative algebra A becomes a Jordan algebra A^+ under the symmetric product (Jordan product) $x \bullet y := \frac{1}{2}(xy + yx)$ and becomes a Lie algebra under the skew-symmetric product (Lie bracket) $[x, y] := xy - yx$. On the other hand we know from the works of J. Tits, I. Kantor and M. Koecher that the Jordan algebras are imbedded in the Lie algebras (see [9], [11] and [16]). In particular, Koecher showed that for a Jordan algebra J there is a Lie algebra $L(J)$ such that J is a subspace of $L(J)$ and the product of J can be expressed in terms of the bracket in the Lie algebra $L(J)$ (see [11]).

It is known that the universal enveloping algebra of a Lie algebra has the structure of an associative algebra. More recently, J. L. Loday introduced the notion of Leibniz algebras (see [12]), which is a generalization of the Lie algebras where the skew-symmetric of the bracket is dropped and the Jacobi identity is changed by the Leibniz identity. Loday also showed that the relationship between Lie algebras and associative algebras translate into an analogous relationship between Leibniz algebras and the so-called Dialgebras (see [13]) which are a generalization of associative algebras possessing two operations. In particular Loday showed that any dialgebra $(D, \dashv, \vdash)$ becomes a Leibniz algebra D_{Leib} under the Leibniz bracket $[x, y] := x \dashv y - y \vdash x$ and the universal enveloping algebra of a Leibniz algebra has the structure of a Dialgebra (see [13] or [14]).

Our aim is to discover a new generalization of the Jordan algebras. This new structure, called **quasi-Jordan algebra**, is noncommutative in general and satisfys a special Jordan identity. The quasi-Jordan algebras appear in the study of the product

$$x \triangleleft y := \frac{1}{2}(x \dashv y + y \vdash x),$$

where x and y are elements in a dialgebra D over a field K of the characteristic different from 2. We study other products defined over Jordan bimodules and the vector space of linear transformations $gl(V)$, where V is a vector space.

From this study, we obtain the fundamental axioms for the quasi-Jordan algebras. This definition is the following

Definition. *A* **quasi-Jordan algebra** *is a vector space $\Im$ over a field K with a characteristic different from 2 equipped with a bilinear product $\triangleleft : \Im \times \Im \to \Im$ that satisfys*

$$x \triangleleft (y \triangleleft z) = x \triangleleft (z \triangleleft y) \qquad \textit{(right commutativity)} \qquad \text{(QJ1)}$$

$$(y \triangleleft x) \triangleleft x^2 = (y \triangleleft x^2) \triangleleft x \qquad \textit{(right Jordan identity)} \qquad \text{(QJ2)}$$

for all $x, y, z \in \Im$, where $x^2 = x \triangleleft x$.

The quasi-Jordan algebras are a different generalization of the Jordan algebra, in particular these algebras are not equivalent to noncommutative Jordan algebras. On the other hand, we show that all Jordan elements in a Leibniz algebras are associated to with quasi-Jordan algebras.

In sections one, two and three we study different constructions of algebraic structures of the Jordan type. From these sections we obtain the definition of quasi-Jordan algebras in a constructive form. In the first section we work with dialgebras and obtain the initial axioms of the quasi-Jordan algebras. In the second section we present two algebraic structures of the Jordan type obtained from Jordan algebras and Jordan bimodules. In the third section we construct an algebraic structure defined by a vector space and its Jordan algebra of linear transformations. In this part we obtain the principal axioms of the quasi-Jordan algebras.

In the fourth section we introduce the definition of quasi-Jordan algebras and present other examples. In this section we study the relationship between quasi-Jordan algebras and noncommutative Jordan algebras.

In the last section we present the principal results of this paper. In this section we obtain new results about the existence of ad-nilpotent elements in Leibniz algebras. We extend well known results of Kostrikin and Benkart-Isaacs in the context of Leibniz algebras. We define the concept of a Jordan element in Leibniz algebras and we show how it is possible to attach a quasi-Jordan algebra to any Jordan element of a Leibniz algebra (over a field of a characteristic different from 2 and 3). This result generalizes a construction of the Jordan algebras of a Lie algebras from A. Fernández, E. García and M. Gómez (see [6]).

The fundamental problem that we are studying in a future paper is the imbedding of quasi-Jordan algebras into Leibniz algebras.

1 Algebraic structures of Jordan type generate by dialgebras

Around 1990, J. L. Loday introduced the notions of Leibniz algebras and diassociative algebras (dialgebras) (see [13]). The Leibniz algebras are a generalization of Lie algebras where the skew-symmetric of the bracket is suppressed and the Jacobi identity is changed by the Leibniz identity.

Definition 1 *A* **Leibniz algebra** *over a field K is a K-vector space L equipped with a binary operation, called Leibniz bracket, $[\cdot,\cdot]: L\times L\to L$ which satisfies the Leibniz identity*

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]\,, \quad \text{for all } \; x,y,z\in L \tag{L}$$

If the bracket is skew-symmetric, then L is a Lie algebra. Therefore Lie algebras are particular cases of Leibniz algebras.

Example 2 *Let L be a Lie algebra and let M be a L-module with action $M\times L\to M$, $(m,x)\mapsto mx$. Let $f: M\to L$ be a L-equivariant linear map, this is*

$$f(mx) = [f(m),x]\,, \quad \text{for all } \; m\in M \text{ and } x\in L,$$

then one can put a Leibniz structure on M as follows

$$[m,n]' := mf(n)\,, \quad \text{for all } \; m,n\in M.$$

Additionally, the map f defines a homomorphism between Leibniz algebras, since

$$f([m,n]') = f(mf(n)) = [f(m), f(n)]$$

Example 3 *Let (A,d) be a differential associative algebra. So, by hypothesis, $d(ab) = da\, b + a\, db$ and $d^2 = 0$. Define the bracket on A by the formula*

$$[a,b] := a\, db - db\, a$$

The vector space A equipped with this bracket is a Leibniz algebra.

It follows from the Leibniz identity (L) that in any Leibniz algebra we have

$$[x,[y,y]] = 0, \quad [x,[y,z]] + [x,[z,y]] = 0.$$

Let L be a Leibniz algebra. Let L^{ann} be the subspace of L spanned by elements of the form $[x,x]$, $x \in L$. For any $x, y \in L$ we have

$$ann(x,y) = [x,y] + [y,x] \in L^{ann}(L).$$

If we defined

$$Z^r(L) = \{z \in L | [x,z] = 0,\ \forall x \in L\},$$

we obtain that

1. $L^{ann} \subset Z^r(L)$.
2. L^{ann} and $Z^r(L)$ are two-side ideals of L.
3. $[Z^r(L), L] \subset L^{ann}$.

Quotienting the Leibniz algebra L by the ideal $L^{ann}(L)$ gives a Lie algebra denoted by L_{Lie}. Moreover, the ideal $L^{ann}(L)$ is the smallest two-sided ideal of L such that $L/L^{ann}(L)$ is a Lie algebra. The quotient map $\pi : L \to L_{Lie}$ is a homomorphism of Leibniz algebras, this is $\pi([x,y]) = [\pi(x), \pi(y)]$. Besides π is universal with respect to all homomorphisms from L to another Lie algebra L', this is equivalent to the following diagram commute

$$\begin{array}{ccc} L & \xrightarrow{\pi} & L_{Lie} \\ & \searrow & \downarrow \\ & & L' \end{array}$$

Since $L^{ann}(L) \subset Z^r(L)$, we see that

$$L^{Lie} := L/Z^r(L)$$

is also a Lie algebra.

The Leibniz algebras are in fact right Leibniz algebras. For the opposite structure (left Leibniz algebras), that is $[x,y]' = [y,x]$, the left Leibniz identity is

$$[[x,y]', z]' = [y,[x,z]']' - [x,[y,z]']'. \tag{L'}$$

The notion of dialgebras is a generalization of associative algebras, with two operations, which gives rise to Leibniz algebras instead of Lie algebras.

Definition 4 *A* **dialgebra** *over a field K is a K-vector space D equipped with two associative products*

$$\dashv : D \times D \to D$$

$$\vdash : D \times D \to D$$

satisfying the identities:

$$x \dashv (y \dashv z) = x \dashv (y \vdash z) \tag{D1}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z) \tag{D2}$$

$$(x \vdash y) \vdash z = (x \dashv y) \vdash z \tag{D3}$$

Observe that the analogue of formula $(D2)$, but with the product symbols pointing outward, is not valid in general in Dialgebras: $(x \dashv y) \vdash z \neq x \dashv (y \vdash z)$.

A morphism of dialgebras from D to D' is a linear map $f : D \to D'$ such that

$$f(x \dashv y) = f(x) \dashv f(y) \quad \text{and} \quad f(x \vdash y) = f(x) \vdash f(y),$$

for all x, y in D.

A bar-unit in D is an element e in D such that

$$x \dashv e = x = e \vdash x\,, \quad \text{for all} \;\; x \in D.$$

A bar-unit needs not to be unique. The subset of bar-units of D is called its **Halo**. A unital dialgebra is a dialgebra with a specified bar-unit e. The problem of adding a unit-bar to dialgebras remains open.

Observe that if a dialgebra has a unit ϵ, which satisfies $\epsilon \dashv x = x$ for any $x \in D$, then $\dashv = \vdash$ and D is an associative algebra with unit ϵ.

Example 5 *If A is an associative algebra, then the formula $x \dashv y = xy = x \vdash y$ define a structure of dialgebra on A*

Example 6 *If (A, d) is a differential associative algebra, then the formulas $x \dashv y = x\,dy$ and $x \vdash y = dx\,y$ define a structure of dialgebra on A.*

Example 7 *Let V be a vector space and fix $\varphi \in V'$ (the algebraic dual), then one can define a dialgebra structure on V by setting $x \dashv y = \varphi(y)x$ and $x \vdash y = \varphi(x)y$, denoted by V_φ. If $\varphi \neq 0$, then V_φ is a dialgebra with non-trivial bar-units. Moreover, its halo is an affine space modeled after the subspace $Ker\varphi$.*

If D is a dialgebra and we define the bracket $[\cdot,\cdot] : D \times D \to D$ by

$$[x, y] := x \dashv y - y \vdash x\,, \quad \text{for all } x, y \in D,$$

then $(D, [\cdot,\cdot])$ is a Leibniz algebra. Moreover, Loday show that the following diagram is commutative

$$\begin{array}{ccc} Dias & \stackrel{-}{\to} & Leib \\ \uparrow & & \uparrow \\ As & \stackrel{-}{\to} & Lie \end{array}$$

where *Dias*, *As*, *Lie* and *Leib* are denoted, respectively, as the categories of dialgebras, associative, Lie and Leibniz algebras (see [13]).

If we translate the quasi-multiplication (Jordan product) to the dialgebra framework, we obtain a new algebraic structure of a Jordan type. Let D be a dialgebra over a field K of the characteristic different from 2. We define the product $\lhd : D \times D \to D$ by

$$x \lhd y := \frac{1}{2}(x \dashv y + y \vdash x) , \quad \text{for all } x, y \in D. \tag{$\lhd$ 1}$$

A few calculus show that the product $\lhd$ satisfys the identities

$$x \lhd (y \lhd z) = x \lhd (z \lhd y) \tag{QJ1}$$

$$(y \lhd x) \lhd x^2 = (y \lhd x^2) \lhd x \tag{QJ2}$$

$$x^2 \lhd (x \lhd y) = x \lhd (x^2 \lhd y), \tag{QJ3}$$

but the product $\lhd$ is noncommutative in general.

Remark 8 *The other Jordan type identities are not satisfied by the product* $\lhd$*, besides the identity* $x^2 \lhd (y \lhd x) = x \lhd (y \lhd x^2)$*, but this identity is obvious by* $(QJ1)$ *and* $(QJ3)$*.*

Note 9 *F. Chapoton introduced the notion of commutative dialgebra. A dialgebra* D *is commutative if the Leibniz algebra* D_{Leib} *is zero, this is if* $x \dashv y = y \vdash x$*, for all* x, y *in* D *(see [4]). If we define* $x \circ y := x \dashv y$*, for all* x, y *in* D*, then the algebra* $(D, \circ)$ *is associative and satisfys the identity*

$$x \circ (y \circ z) = x \circ (z \circ y) , \quad \textit{for all } x, y, z \in D.$$

Therefore the algebra $(D, \circ)$ *satisfies the identities* $(QJ1)$*,* $(QJ2)$ *and* $(QJ3)$*. These algebras are called* **Perm algebras***.*

If D is a unital dialgebra, with a specific bar-unit e, we have that $x \lhd e = x$, for all x in D. This implies that e is a right unit for the algebra $(D, \lhd)$. In this case we have for $(QJ2)$ and $(QJ3)$ that

$$x^2 \lhd x = x \lhd x^2 \tag{1}$$

and

$$x^2 \lhd x^2 = (x^2 \lhd x) \lhd x, \tag{2}$$

for all x, y in D.

Then all algebras $(D, \lhd)$ that satisfy the identities $(QJ1)$, $(QJ2)$ and $(QJ3)$ with right unit e defined over a field of characteristic zero are power-associative. If D is a dialgebra without a bar-unit, then the product $\lhd$ only satisfies the $(QJ2)$ identity. To show that a nonassociative algebra is power-associative the previous identities are necessary conditions. In the following table we show the classical identities in non-associative algebras and we indicate which are true with respect to a product $\lhd$ over dialgebras:

Classical identities	True	Quasi-identities	True
$x \lhd y = y \lhd x$	Not	$x \lhd (y \lhd z) = x \lhd (z \lhd y)$	Yes
$x \lhd (y \lhd x) = (x \lhd y) \lhd x$	Not	$x \lhd (y \lhd (z \lhd y)) = x \lhd ((y \lhd z) \lhd y)$	Yes
$x \lhd (x \lhd y) = x^2 \lhd y$	Not	$x \lhd (y \lhd (y \lhd z)) = x \lhd (y^2 \lhd z)$	Not
$(y \lhd x) \lhd x = y \lhd x^2$	Not	$x \lhd ((y \lhd z) \lhd z) = x \lhd (y \lhd z^2)$	Not
$x^2 \lhd x = x \lhd x^2$	Not	$x \lhd (y \lhd y^2) = x \lhd (y^2 \lhd y)$	Yes
$x \lhd (y \lhd z) = (x \lhd y) \lhd z$	Not	$x \lhd ((y \lhd z) \lhd w) = x \lhd (y \lhd (z \lhd w))$	Not

Table 1

Remark 10 *If we review the definition of Leibniz algebra and we compare with Lie algebras, we see that Lie algebras satisfy the anti-symmetric identity $[x, y] = -[y, x]$, but this is not true in Leibniz algebras. The Leibniz algebras satisfy the identity*

$$[x, [y, z]] = [x, -[z, y]] \tag{3}$$

and we call this identity the **quasi-anti-symmetry identity**. *The Leibniz identity implies the* **quasi-skew-symmetric** *identity $[x, [y, y]] = 0$. These identities (quasi-anti and quasi-skew symmetric) are similar to quasi-commutativity identity $(QJ1)$.*

Summary 11 *In this section we find a new algebraic structure of the Jordan type. This structure is a generalization of Jordan algebras and it is noncommutative. Additionally, the product $\lhd$ over a dialgebra D satisfies for all $x, y \in D$*

$$x \lhd (y \lhd x^2) + 2(x^2 \lhd y) \lhd x = (x \lhd y) \lhd x^2 + 2x^2 \lhd (y \lhd x) \tag{QJ4}$$

If we introduce the associator operator $(x, y, z) = (x \lhd y) \lhd z - x \lhd (y \lhd z)$, then the identity $(QJ4)$ is equivalent to

$$(x, y, x^2) - 2(x^2, y, x) = 0 \tag{QJ4'}$$

The identity $(QJ4')$ is similar to identity

$$(x, y, x^2) + 2(x^2, y, x) = 0 \tag{4}$$

The last identity is satisfied in Jordan bimodules, for this we are going to study algebraic structures generated by Jordan bimodules.

2 The Jordan bimodules case

In this section we show two new constructions of algebraic Jordan structures generated by Jordan bimodules. First, we introduce the definitions of Jordan algebra and Jordan Bimodule.

Definition 12 *Let J be a vector space over a field K with characteristic different from 2. We say that J is a* **Jordan algebra** *if over J is defined a product $\bullet : J \times J \to J$ such that it satisfies the identities*

$$a \bullet b = b \bullet a \tag{5}$$

and

$$a^2 \bullet (b \bullet a) = (a^2 \bullet b) \bullet a\ , \tag{6}$$

for all $a, b \in J$, where $a^2 = a \bullet a$.

Definition 13 *Let J be a Jordan algebra and let M be a vector space over the same field as J. Then M is a* **Jordan Bimodule** *for J in case there are two bilinear compositions $(m, a) \mapsto ma$ and $(m, a) \mapsto am$, for all $m \in M$ and $a \in J$, satisfying*

$$ma = am \tag{7}$$

and

$$(a^2, m, a) = (a^2, b, m) + 2(ma, b, a) = 0\ , \tag{8}$$

for all $m \in M$ and $a, b \in J$, when (a, b, c) denotes the associator.

Let J be a Jordan algebra and let M be a Jordan bimodule. A linear map $f : M \to J$ is called J**-equivariant** over M if $f(am) = af(m)$, for all $m \in M$ and $a \in J$. If f is a J-equivariant map over M, then we define the product $\triangleleft : M \times M \to M$ by

$$m \triangleleft n = f(n)m\ , \quad \text{for all } m, n \in M \tag{$\triangleleft$ 2}$$

The product $\triangleleft$ satisfies the following identities

$$m \triangleleft (n \triangleleft s) = m \triangleleft (s \triangleleft n) \tag{QJ1}$$

$$(n \triangleleft m) \triangleleft m^2 = (n \triangleleft m^2) \triangleleft m \tag{QJ2}$$

$$m \triangleleft (n \triangleleft m^2) + 2(m^2 \triangleleft n) \triangleleft m = (m \triangleleft n) \triangleleft m^2 + 2m^2 \triangleleft (n \triangleleft m)\ , \tag{QJ4}$$

for all $m, n, s \in M$, but it is not commutative. The other Jordan type identities are not satisfied by the product $\triangleleft$.

If we compare the products $\triangleleft$ defined by ($\triangleleft$ 1) and ($\triangleleft$ 2), these products satisfy the identities ($QJ1$), ($QJ2$) and ($QJ4$).

Another way to define a product over Jordan algebras and Jordan bimodules is the following. Over the vector space $J \times M$, where J is a Jordan algebra and M is a Jordan bimodule over J, we define the product $\triangleleft$ by

$$(a, m) \triangleleft (b, n) = (ab, mb)\ , \quad \text{for all } a, b \in J \text{ and } m, n \in M. \tag{$\triangleleft$ 3}$$

For simplicity we write $(a, m)b = (ab, mb)$. The product defined by ($\triangleleft$ 3) satisfies the same conditions as the product defined by ($\triangleleft$ 2) and it is noncommutative. If we define the projection map $\pi_J : J \times M \to J$ by $\pi_J(a, m) = a$, then $\pi_J((am)b) = ab = a\pi_J(b, n)$ and this is equivalent to

$$\pi_J((a, m) \triangleleft (b, n)) = \pi_J(a, m) \bullet \pi_J(b, n)\ ,$$

for all $a, b \in J$ and $m, n \in M$.

We have that $J \times M$ is a Jordan bimodule over J with bilinear compositions defined by $a(b, n) = (ab, an)$ and $(b, m)a = (ba, ma)$. The map π_J defined over $J \times (J \times M)$ is J-equivariant since

$$\pi_J((a, m)b) = \pi_J(a, m) \bullet b\ , \quad \text{for all } b \in J \text{ and } (a, m) \in J \times M.$$

Summary 14 *The products defined by ($\triangleleft$ 1), ($\triangleleft$ 2) and ($\triangleleft$ 3) are noncommutative and satisfy the identities ($QJ1$), ($QJ2$) and ($QJ4$). The other Jordan type identities, different from ($QJ2$), are not satisfied by the products defined in this section.*

3 The $gl^+(V)$ case

In this section we are going to construct an algebraic structure of a Jordan type with respect to a vector space and its Jordan algebra of linear transformations.

Let V be a vector space over a field K with a characteristic different from 2 and let $gl^+(V)$ be a Jordan algebra of linear transformations over V with a product defined by

$$A \bullet B = \frac{1}{2}(AB + BA),$$

where AB denotes the composition of the maps A and B. We consider the vector space $gl^+(V) \times V$ and we define the product $\triangleleft : (gl^+(V) \times V) \times (gl^+(V) \times V) \to gl^+(V) \times V$ by

$$(A, u) \triangleleft (B, v) = (A \bullet B, Bu), \tag{$\triangleleft$ 4}$$

for all $A, B \in gl(V)$ and $u, v \in V$. This product satisfies the identities

$$(A, u) \triangleleft ((B, v) \triangleleft (C, w)) = (A, u) \triangleleft ((C, w) \triangleleft (B, v)) \tag{QJ1}$$

and

$$((B, v) \triangleleft (A, u)) \triangleleft (A, u)^2 = ((B, v) \triangleleft (A, u)^2) \triangleleft (A, u)\ , \tag{QJ2}$$

for all $A, B \in gl^+(V)$ and $u, v \in V$, where $(A, u)^2 = (A, u) \triangleleft (A, u)$.

This product does not satisfy other Jordan type identities and does not satisfy the identity ($QJ4$), but it satisfies the identities

$$(A, u)^2 \triangleleft (A, u) = (A, u) \triangleleft (A, u)^2 \tag{9}$$

and

$$(A, u) \triangleleft (Id, v) = (A, u), \tag{10}$$

where Id denotes the identity map over V. The last identity shows that (Id, v) is a right unit for all $v \in V$, but (Id, v) is not a left unit with respect to the product $\triangleleft$. If the field K is the characteristic zero, then $gl^+(V) \times V$ is a power-associative algebra.

Summary 15 *When we review the products $\triangleleft$ defined by ($\triangleleft$ 1), ($\triangleleft$ 2), ($\triangleleft$ 3) and ($\triangleleft$ 4) we find in common the identities* ($QJ1$) *and* ($QJ2$). *From these identities we are going to propose a new algebraic structure of a Jordan type in the following section.*

4 Quasi-Jordan algebras

In this section we introduce a new algebraic structure of a Jordan type and we study the relations with Jordan algebras and noncommutative Jordan algebras.

According to the previous summaries (11, 14 and 15), we introduce the following algebraic structure.

Definition 16 *A* **quasi-Jordan algebra** *is a vector space* $\Im$ *over a field* K *with a characteristic different from 2 equipped with a bilinear product* $\triangleleft : \Im \times \Im \to \Im$ *that satisfies*

$$x \triangleleft (y \triangleleft z) = x \triangleleft (z \triangleleft y) \qquad \textit{(right commutativity)} \tag{QJ1}$$

$$(y \triangleleft x) \triangleleft x^2 = (y \triangleleft x^2) \triangleleft x \qquad \textit{(right Jordan identity),} \tag{QJ2}$$

for all $x, y, z \in \Im$*, where* $x^2 = x \triangleleft x$.

In terms of the left and right multiplicative maps L_x and R_x, defined for $x \in \Im$ by $L_x(y) = x \triangleleft y$ and $R_x(y) = y \triangleleft x$, for all $y \in \Im$, the identities $(QJ1)$ and $(QJ2)$ are equivalent to

$$L_x L_y = L_x R_y \tag{QJ1*}$$

$$R_x R_{x^2} = R_{x^2} R_x \tag{QJ2*}$$

There is an analogous structure if we define a product $\triangleright : \Im \times \Im \to \Im$ by $x \triangleright y := y \triangleleft x$, for all $x, y \in \Im$. This product satisfies the identities

$$(x \triangleright y) \triangleright z = (y \triangleright x) \triangleright z \qquad \text{(left commutativity)} \tag{QJ1'}$$

$$x^2 \triangleright (x \triangleright y) = x \triangleright (x^2 \triangleright y)\ , \qquad \text{(left Jordan identity)} \tag{QJ2'}$$

for all $x, y \in \Im$, where $x^2 = x \triangleright x$. In the last three sections we define the product $\triangleright$ by

1. $x \triangleright y = \frac{1}{2}(x \vdash y + y \dashv x)$
2. $m \triangleright n = f(m)n$
3. $(A, u) \triangleright (B, v) = (A \bullet B, Av)$.

These products satisfy the identities $(QJ1')$ and $(QJ2')$. Then we have two quasi-Jordan algebras, the right and the left structures. We will only consider the right quasi-Jordan algebras.

Note 17 *The Jordan and Perm algebras are obvious examples of quasi-Jordan algebras.*

A right unit in a quasi-Jordan algebra $\Im$ is an element e in $\Im$ such that $x \triangleleft e = x$, for all $x \in \Im$.

Let $\Im$ be a quasi-Jordan algebra, if there is an element ϵ in $\Im$ such that $\epsilon \triangleleft x = x$ then $\Im$ is a classical Jordan algebra and ϵ is a unit. For this reason we only consider right units over quasi-Jordan algebras.

It is possible to attach a unit to any Jordan algebra, but in quasi-Jordan algebras the problem of attaching a right unit is a open problem. Additionally, the right units in quasi-Jordan algebras are not unique (see example 18).

We denote by $U_r(\Im)$ the set of all right units of a quasi-Jordan algebra $\Im$. A **right unital quasi-Jordan algebra** is a quasi-Jordan algebra with a specified right unit e.

Example 18 *Let* V *be a vector space and fix* $\varphi \in V'$ *with* $\varphi \neq 0$*. We define the product* $\triangleleft : V \times V \to V$ *by* $x \triangleleft y = \varphi(y)x$*, for all* $x, y \in V$*. Then* (V, φ) *is a quasi-Jordan algebra and all elements* x *in* V *such that* $\varphi(x) \neq 0$ *define a right unit* $x/\varphi(x)$.

Example 19 *Let V be a 2-dimensional vector space with base $\{e_1, e_2\}$. If we define the product $\lhd : V \times V \to V$ with respect to e_1 and e_2 by $e_i \lhd e_j = e_i$, for $i = 1, 2$, and extend the product to V for linearity, we have that $(V, \lhd)$ is a noncommutative quasi-Jordan algebra.*

Now, if we consider the symmetric product $x \bullet y = x \lhd y + y \lhd x$, for all $x, y \in V$, then $(V, \bullet)$ is a Jordan algebra.

Example 20 *Let J be a Jordan algebra and let M be a Jordan bimodule such that the identity $(a, b, am) = 0$ is not satisfied. Then the quasi-Jordan algebra $(J \times M, \lhd)$, with a product $\lhd$ defined by ($\lhd$ 3), is not a Jordan algebra with respect to symmetric product*

$$(a, m) \bullet (b, n) = \frac{1}{2}((a, m) \lhd (b, n) + (b, n) \lhd (a, m))$$

In this point it is important to remember the definition of a noncommutative Jordan algebra and see then relation with quasi-Jordan algebras.

A **noncommutative Jordan algebra** is a vector space J_n over a field K of a characteristic different from 2 equipped with a product $\cdot : J_n \times J_n \to J_n$ satisfying the flexible law and the Jordan identity, this is

$$x \cdot (y \cdot x) = (x \cdot y) \cdot x \tag{11}$$

$$x^2 \cdot (y \cdot x) = (x^2 \cdot y) \cdot x\ , \tag{12}$$

for all $x, y \in J_n$. The following lemma gives necessary and sufficient conditions such that an algebra can be a noncommutative Jordan algebra (see [3]).

Lemma 21 *An algebra A is a noncommutative Jordan algebra if and only if it is flexible (satisfys the flexible law) and the corresponding plus-algebra A^+ is a Jordan algebra ($A^+ = (A, \bullet)$ is a Jordan algebra, with $x \bullet y = \frac{1}{2}(x \cdot y + y \cdot x)$).*

Remark 22 *The previous lemma and the last example imply that the quasi-Jordan algebras are not noncommutative Jordan algebras. Moreover, the noncommutative Jordan algebras are not quasi-Jordan algebras. In effect, let A be an associative and noncommutative algebra over a field K (characteristic $\neq 2$) and let $a \in K$ with $a \neq \frac{1}{2}$. We define a new product on A as follows*

$$x \bullet_a y = axy + (1 - a)yx$$

and we denote the resulting algebra by A^a. The algebra A^a is a noncommutative Jordan algebra, but it is not a quasi-Jordan algebra.

The previous remark implies that the quasi-Jordan algebras are a different generalization of the Jordan algebras, besides the noncommutative Jordan algebras and their generalizations.

Remark 23 *Let $\Im$ be a quasi-Jordan algebra and we define the Lie bracket $[\cdot, \cdot] : \Im \times \Im \to \Im$ by $[x, y] = x \lhd y - y \lhd x$. Then this bracket is skew-symmetric, but does not satisfy the Leibniz and Jacobi identities. This implies that $(\Im, [\cdot, \cdot])$ is not a Leibniz algebra. If we define the associator $(x, y, z) := x \lhd (y \lhd z) - (x \lhd y) \lhd z$, then $(\Im, [\cdot, \cdot], (\cdot, \cdot, \cdot))$ is an Akivis algebra (see [3], section 8).*

Following Kinyon and Weinstein's ideas (see [10]), in the rest of this section we will obtain Jordan algebras from quasi-Jordan algebras and we show an universal property about homomorphisms from quasi-Jordan algebras to Jordan algebras.

It follows from the right commutativity $(QJ1)$ that in any quasi-Jordan algebra we have

$$x \triangleleft (y \triangleleft z - z \triangleleft y) = 0.$$

Let $\Im$ be a quasi-Jordan algebra. A subspace $I \subset \Im$ is called left (resp. right) ideal if for any $a \in I$ and $x \in \Im$ we have $a \triangleleft x \in I$ (resp. $x \triangleleft a \in I$). If I is both left and right ideal, then I is called two-side ideal. For a quasi-Jordan algebra $\Im$ we put

$$Z^r(\Im) = \{z \in \Im | x \triangleleft z = 0, \ \forall x \in \Im\}.$$

We denote by $\Im^{ann}$ the subspace of $\Im$ spanned by elements of the form $x \triangleleft y - y \triangleleft x$, with $x, y \in \Im$. Clearly we have

$$\Im^{ann} \subset Z^r(\Im).$$

Since $Z^r(\Im) \triangleleft \Im \subset \Im^{ann}$, we have that $\Im^{ann}$ and $Z^r(\Im)$ are two-side ideals of $\Im$.

Let $(\Im, \triangleleft)$ be a quasi-Jordan algebra. If we consider the quotient algebra $\Im_{Jor} := \Im / \Im^{ann}$ we see that $\Im_{Jor}$ is a Jordan algebra. Moreover, the ideal $\Im^{ann}$ is the smallest two-sided ideal in $\Im$ such that $\Im/\Im^{ann}$ is a Jordan algebra. In effect, let I be any two-sided ideal of $\Im$ such that $\Im/I$ is a Jordan algebra, then $x \triangleleft y - y \triangleleft x + I = I$ and this implies that $\Im^{ann} \subset I$. The quotient map $\pi : \Im \rightarrow \Im_{Jor}$ is a homomorphism of quasi-Jordan algebras, this is $\pi(x \triangleleft y) = \pi(x) \triangleleft \pi(y)$. Besides π is universal with respect to all homomorphisms from $\Im$ to another Jordan algebra J, this is equivalent to the following diagram commute

$$\begin{array}{ccc} \Im & \xrightarrow{\pi} & \Im_{Jor} \\ & \searrow & \downarrow \\ & & J \end{array}$$

Also, if $\Im$ is a right unital quasi-Jordan algebra, with a specific right unit e, we have that

$$\Im^{ann} = \{x \in \Im | e \triangleleft x = 0\}$$

and

$$U_r(\Im) = \{x + e | x \in \Im^{ann}\}$$

In future works we will search for find more specific relations between Jordan algebras (commutative and noncommutative) and quasi-Jordan algebras.

Furthermore, we will search for relations between quasi-Jordan and Lie algebras.

5 Quasi-Jordan algebras generated by Jordan elements in Leibniz algebras

In this section we show that it is possible to attach a quasi-Jordan algebra to any Jordan element x of a Leibniz algebra L. We define a product in L by $a \lhd b := \frac{1}{2}[a,[b,x]]$ and divide the nonassociative algebra $L^{(x)} := (L, \lhd)$ by the ideal $Ker_L(x) := \{a \in L | [[a,x],x] = 0\}$. Then $L_x := L^{(x)}/Ker_L(x)$ turns out to be a quasi-Jordan algebra.

This construction generalizes the results obtained by Fernández, García and Gómez (see [6] Theorem 2.4) in the context of Leibniz algebras and quasi-Jordan algebras.

First, we generalize the results of Kostrikin and Benkart-Isaacs about ad-nilpotent elements in Lie algebras to ad-nilpotent elements in Leibniz algebras. In particular, we obtain the following results:

1. Any nonzero ad-nilpotent element in a Leibniz algebra gives rise to a nonzero Jordan element.
2. Any nonzero finite dimensional Leibniz algebra over an algebraically closed field of an arbitrary characteristic necessarily contains a nonzero ad-nilpotent element, and therefore a nonzero Jordan element.

We begin by introducing the definition of an adjoint map in Leibniz algebras and we show a few properties.

Definition 24 *Let L be a Leibniz algebra. For all $x \in L$, we define the* **adjoint** *map $ad_x : L \to L$ by $ad_x y = [y,x]$, for all $y \in L$. Additionally, the Leibniz identity implies that ad_x is a derivation over L, since $ad_x[y,z] = [ad_x y, z] + [y, ad_x z]$, for all $y, z \in L$.*

Remark 25 *The map $ad : L \to gl(L)$, $x \mapsto ad_x$, where $gl(L)$ is the Lie algebra of linear maps over L with a Lie bracket $[T,S] = TS - ST$, it is an antihomomorphism of Leibniz algebras, this is*

$$ad_{[x,y]} = [ad_y, ad_x]\,, \quad \text{for all } x, y \in L \tag{13}$$

The set $ad(L) = \{ad_x | x \in L\}$ with the bracket defined by $[ad_x, ad_y] := ad_x ad_y - ad_y ad_x$ turns out to be a Lie algebra, in particular it is a Lie subalgebra of $gl(L)$.

Notation 26 *We will use capital letters to denote the adjoint maps (the elements of $ad(L)$): $X = ad_x$, $Y = ad_y$, etcetera. In this notation the last identity has the form*

$$ad_{[x,y]} = [Y, X] \tag{14}$$

Definition 27 *Let L be a Leibniz algebra and let x be an element in L. We say that x is an* **ad-nilpotent element** *if there is a positive integer m such that $ad_x^m = 0$. For any ad-nilpotent element $x \in L$ we define the* **ad-nilpotence index** *of x by the positive integer m such that $ad_x^m = 0$ and $ad_x^{m-1} \neq 0$.*

The elements in Leibniz algebra that satisfy the following definition are the central objects in the construction of quasi-Jordan algebras from Leibniz algebras.

Definition 28 *We say that an element x in a Leibniz algebra L is a* **Jordan element** *if x is an ad-nilpotent element of index at most 3.*

The previous definition is a generalization of the Jordan elements over Lie algebras.

Example 29 *A natural example of Jordan elements are the zero-square elements in a dialgebra. Let D be a dialgebra and let L be the Leibniz algebra D_{Leib}. For any $x, y \in L$, we have:*

1. $ad_x^2(y) = y \dashv (x \dashv x) - 2(x \vdash y) \dashv x + (x \vdash x) \vdash y$
2. $ad_x^3(y) = y \dashv x_{\dashv}^3 - 3(x \vdash y) \dashv (x \dashv x) + 3(x \vdash x) \vdash (y \dashv x) + x_{\vdash}^3 \dashv y$,

where $x_{\dashv}^3 = x \dashv (x \dashv x)$ and $x_{\vdash}^3 = (x \vdash x) \vdash x$. Thus, if x is an element in D such that $x \dashv x = 0$ or $x \vdash x = 0$ then $ad_x(y) = 0$, since $(x \vdash y) \dashv (x \dashv x) = (x \vdash y) \dashv (x \vdash x)$ and $(x \vdash x) \vdash (y \dashv x) = (x \dashv x) \vdash (y \dashv x)$. This implies that x is a Jordan element in the Leibniz algebra generated by D. Note also $ad_x^2(L) = x \vdash L \dashv x$.

We are going to show two results due to Konstrikin (see [1]) and Benkart-Isaacs (see [2]) about ad-nilpotent elements in Lie algebras.

Theorem 30 (Konstrikin) *Let $\mathfrak{g}$ be a Lie algebra and let a be a nonzero element in $\mathfrak{g}$ such that $ad_a^m = 0$, for $m \geq 4$. If $\mathfrak{g}$ is n-torsion free for all $n \leq m$, then*

$$\left(ad_{ad_a^{m-1}(c)}\right)^{m-1} = 0\,, \quad \text{for all } c \in \mathfrak{g}.$$

Therefore $\mathfrak{g}$ contains a nonzero Jordan element.

Theorem 31 *Any nonzero finite dimensional Lie algebra over an algebraically closed field of an arbitrary characteristic necessarily contains a nonzero ad-nilpotent element and therefore a nonzero Jordan element.*

In the next part of this section, we are going to show the results about Jordan elements in Leibniz algebras. We begin with the following technical lemma.

Lemma 32 *Let L be a Leibniz algebra. Then for all positive integers n we have*

$$AD_X^n(Y) = ad_{X^n(y)}\,, \quad \text{for all } x, y \in L\,, \tag{15}$$

where $AD_X : ad(L) \to ad(L)$, $Y \mapsto [X, Y]$ for all $X, Y \in ad(L)$, is the adjoint map over $ad(L)$.

Proof. If $n = 1$, for (14) we have

$$AD_X(Y) = [X, Y] = ad_{[y,x]} = ad_{ad_x(y)} = ad_{X(y)}.$$

We suppose that the property is true for $n = k$, this is

$$AD_X^k(Y) = ad_{X^k(y)}\,, \quad \text{for all } x, y \in L.$$

Because $X^{k+1} = X(X^k(y)) = [X^k(y), x]$, then

$$\begin{aligned} AD_X^{k+1}(Y) &= AD_X(AD_X^k(Y)) = AD_X(ad_{X^k(y)}) \\ &= [X, ad_{X^k(y)}] = [ad_x, ad_{X^k(y)}] \\ &= ad_{[X^k(y),x]} = ad_{X^{k+1}(y)} \end{aligned}$$

and the result is true for $n = k + 1$. ■

Definition 33 *A Leibniz algebra L is n-**torsion free** if, for $x \in L$, $nx = 0$ implies $x = 0$.*

Theorem 34 *Let L be a Leibniz algebra and let x be a nonzero ad-nilpotent element of an index $m > 3$ in L. If L is n-torsion free for all $n \leq m$, then there is a nonzero Jordan element in L.*

Proof. We have that $X^m(y) = 0$, for all $y \in L$, where $X = ad_x$ and $x \neq 0$. The last lemma implies that

$$AD_X^m(Y) = ad_{X^m(y)} = 0\,, \quad \forall Y \in ad(L).$$

Then X is an ad-nilpotent element in the Lie algebra $ad(L)$ with an ad-nilpotent index at most m. If $X = 0$ the result is true, then we suppose that $X \neq 0$. If $n \leq m$ and $nX = 0$ then $nX(y) = 0$, for all $y \in L$. This implies that $X(y) = 0$ for all $y \in L$, because L is n-torsion free, then $X = 0$ and therefore the Lie algebra $ad(L)$ is n-torsion free for all $n \leq m$. Let k be the ad-nilpotent index of X. We have two possibilities for k: $k \leq 3$ or $k > 3$.

If $k > 3$ we have, by the Kostrikin's theorem, that there is a nonzero Jordan element Z in $ad(L)$. Then there is a nonzero element in L such that $Z = ad_z$. If $k \leq 3$ we take $Z = X$. Then $AD_Z^l(Y) = 0$, for all $Y \in ad(L)$, when $1 \leq l \leq 3$. Here it is equivalent to have $ad_{Z^l(y)} = 0$, for all $y \in L$ and implies

$$0 = ad_{Z^l(y)}(w) = [w, Z^l(y)]\,, \quad \forall y, w \in L \tag{16}$$

There are two possibilities for Z^l:

1. If $Z^l = 0$, in this case we have proved the theorem.
2. If $Z^l \neq 0$, then there is a nonzero element $u \in L$ such that $Z^l(u) \neq 0$ and for (16) we have that $ad_{Z^l(u)}(w) = 0$, for all $w \in L$. Therefore $Z^l(u)$ is a nonzero Jordan element in L.

This proves the theorem. ■

Theorem 35 *Let L be a nonzero finite dimensional Leibniz algebra over an algebraically closed field of an arbitrary characteristic. Then L contains a nonzero ad-nilpotent element and therefore a nonzero Jordan element.*

Proof. $ad(L)$ is a finite dimensional Lie algebra over an algebraically closed field, since it is defined over the same field as L and it is a Lie subalgebra of the finite dimensional Lie algebra $gl(L)$. If $[x, y] = 0$, for all $x, y \in L$, then all elements nonzero in L are ad-nilpotent of index 1 and therefore the theorem is true. If there is a nonzero element $x \in L$ such that $[x, y] \neq 0$ for some $y \in L$, we have that X is a nonzero element in the Lie algebra $ad(L)$, then $ad(L)$ satisfies the hypothesis of Benkart-Issacs's theorem and therefore there is a nonzero ad-nilpotent element Z in $ad(L)$. Hence, there is a nonzero element $z \in L$ such that $Z = ad_z$.

Let m be the ad-nilpotent index of Z, this is $AD_Z^m = 0$ and $AD_Z^{m-1} \neq 0$. For all $y \in L$, we have

$$ad_{Z^m(y)} = AD_Z^m(Y) = 0$$

and this implies that

$$ad_{Z^m(y)}(w) = [w, Z^m(y)] = 0\,, \quad \forall y, w \in L.$$

If $Z^m = 0$, then the theorem is true. Suppose that $Z^m \neq 0$. There is a nonzero element $u \in L$ such that $Z^m(u) \neq 0$ and therefore

$$ad_{Z^m(u)}(w) = [w, Z^m(u)] = 0\,, \quad \forall w \in L.$$

This is equivalent to $ad_{Z^m(u)} = 0$, with $Z^m(u) \neq 0$. $\blacksquare$

Now, we are going to show that it is possible to obtain a quasi-Jordan algebra from any Jordan element in a Leibniz algebra. Throughout this section we will be dealing with Leibniz algebras over a field K containing 1/6 (there is K containing the elements 1/2 and 1/3). In particular, we have that L is 2 and 3-torsion free. First, we will show the following lemma.

Lemma 36 *Let x be a Jordan element of a Leibniz algebra L. For any $a, b \in L$ and $\alpha \in K$, we have*

1. $X^2AX = XAX^2$
2. $X^2AX^2 = 0$
3. $X^2A^2XAX^2 = X^2AXA^2X^2$
4. $[X^2(a), X(b)] = -[X(a), X^2(b)]$
5. $ad_x^2([a, [b, x]]) = [X(a), X^2(b)]$
6. $X^2 ad_{[a, X^2(b)]} = ad_{[X^2(a), b]} X^2$
7. $ad^2_{X^2(a)} = X^2A^2X^2$
8. *αx, $ad_x^2(a)$ are Jordan elements in L,*

where $A = ad_a$

Proof.

1. Because $X^3(a) = 0$, then $ad_{X^3(a)} = 0$. For (15), we have

$$\begin{aligned} 0 &= ad_{X^3(a)} = AD_X^3(A) \\ &= [X, [X, [X, A]]] \\ &= X^3A - 3X^2AX + 3XAX^2 - AX^3 \\ &= 3(XAX^2 - X^2AX), \end{aligned}$$

which proves 1, since L is 3-torsion free.

2. From 1 we have that $X^2AX = XAX^2$. Then multiplying on the right side by X we obtain 2.

3. By 2 we have

$$\begin{aligned} 0 &= X^2([[[X, A], A], A])X^2 \\ &= X^2(XA^3 - 3AXA^2 + 3A^2XA - A^3X)X^2 \\ &= 3(X^2A^2XAX^2 - X^2AXA^2X^2) \end{aligned}$$

4. From $X^3 = 0$, using the Leibniz identity we get

$$\begin{aligned} 0 &= X^3([a,b]) \;=\; [[[[a,b],x],x],x] \\ &= [[[[a,x],b],x],x] + [[[a,[b,x]],x],x] \\ &= [[[[a,x],x],b],x] + 2[[[a,x],[b,x]],x] + [[a,[[b,x],x]],x] \end{aligned}$$

The Leibniz identity implies that

$$\begin{aligned} 0 &= 3([[[a,x],x],[b,x]] + [[a,x],[[b,x],x]]) \\ &= 3([X^2(a),X(b)] + [X(a),X^2(b)]). \end{aligned}$$

Then $[X^2(a),X(b)] = -[X(a),X^2(b)]$, because L is 3-torsion free.

5. From the Leibniz identity and 4, we have

$$\begin{aligned} ad_x^2([a,[b,x]]) &= [[[a,[b,x]],x],x] \\ &= [[[a,x],[b,x]],x] + [[a,[[b,x],x]],x] \\ &= [[[a,x],x],[b,x]] + 2[[a,x],[[b,x],x]] \end{aligned}$$

from the definition of X we obtain

$$\begin{aligned} ad_x^2([a,[b,x]]) &= [X^2(a),X(b)] + 2[X(a),X^2(b)] \\ &= -[X(a),X^2(b)] + 2[X(a),X^2(b)] \\ &= [X(a),X^2(b)]. \end{aligned}$$

6. Since $ad_{[a,X^2(b)]} = [[X,[X,B]],A]$, we get

$$\begin{aligned} X^2 ad_{[a,X^2(b)]} &= X^2\big((X^2B - 2XBX + BX^2)A - A(X^2B - 2XBX + BX^2)\big) \\ &= 2X^2AXBX - X^2ABX^2 \\ &= 2XAXBX^2 - X^2ABX^2 \end{aligned}$$

From the identities $(BX^2A - 2BXAX + BAX^2 - AX^2B)X^2 = 0$, $X^3 = 0$ and 2, it follows that

$$\begin{aligned} X^2 ad_{[a,X^2(b)]} &= \big(B(X^2A - 2XAX + AX^2) - (X^2A - 2XAX + AX^2)B\big)\,X^2 \\ &= [B,[X,[X,A]]]X^2 \\ &= ad_{[X^2(a),b]}X^2. \end{aligned}$$

7. From (15) and 2, we have

$$\begin{aligned} ad^2_{X^2(a)} &= (ad_{[[a,x],x]})(ad_{[[a,x],x]}) \\ &= ([X,[X,A]])\,([X,[X,A]]) \\ &= (X^2A - 2XAX + AX^2)(X^2A - 2XAX + AX^2) \\ &= X^2A^2X^2. \end{aligned}$$

8. $ad^3_{\alpha x} = \alpha^3 ad_x^3$ shows that αx is a Jordan element. Set $w := ad_x^2(a)$. Using 2 and 7 we get

$$\begin{aligned} ad_w^3 &= ad_{[[a,x],x]}ad^2_{ad_x^2(a)} = [X,[X,A]]X^2A^2X^2 \\ &= (X^2A - 2XAX + AX^2)X^2A^2X^2 \\ &= 0\,, \end{aligned}$$

so w is a Jordan element. ■

Theorem 37 *Let L be a Leibniz algebra and let x be a Jordan element of L. Then L with the new product defined by*

$$a \triangleleft b := \frac{1}{2}[a,[b,x]]$$

is a nonassociative algebra, denoted by $L^{(x)}$, such that

$$Ker_L(x) := \{a \in L | X^2(a) = 0\}$$

is an ideal of $L^{(x)}$.

Proof. Let $a \in Ker_L(x)$ and let $b \in L$. Using 5 and 6 in the previous lemma, we get

$$X^2([b,[a,x]]) = [X(b), X^2(a)] = 0$$

$$X^2([a,[b,x]]) = [X(a), X^2(b)] = -[X^2(a), X(b)] = 0\,,$$

since $X^2(a) = 0$. Therefore $a \triangleleft b$ and $b \triangleleft a$ are in $Ker_L(x)$. ■

Theorem 38 *Let L be a Leibniz algebra and let x be a Jordan element of L. Then $L_x := L^{(x)}/Ker_L(x)$ is a quasi-Jordan algebra. Moreover, L_x is a noncommutative algebra in general.*

Proof. Let $a, b \in L$ and $\overline{a}$ denotes the coset of a with respect to $ker_L(x)$. Because

$$ad_x^2([a,[b,x]] - [b,[a,x]]) = [X(a), X^2(b)] + [X^2(b), X(a)]$$

and L is not anti-symmetric in general, then $ad_x^2([a,[b,x]] - [b,[a,x]]) \neq 0$ in general. This implies that $\overline{a} \triangleleft \overline{b} \neq \overline{b} \triangleleft \overline{a}$ in general. We have

$$c \triangleleft (b \triangleleft a) = \frac{1}{4}[c,[[b,[a,x]],x]] \quad \text{and} \quad c \triangleleft (a \triangleleft b) = \frac{1}{4}[c,[[a,[b,x]],x]].$$

Since

$$\begin{aligned} ad_x^2\left([c,[[b,[a,x]],x]]\right) &= [X(c), X^2([b,[a,x]])] \\ &= [X(c), X^2([[b,a],x] - [[b,x],a])] \\ &= [X(c), X^2(-[[b,x],a])], \end{aligned}$$

by 4 in lemma 36 we have

$$\begin{aligned} ad_x^2\left([c,[[b,[a,x]],x]]\right) &= -[X^2(c), X(-[[b,x],a])] \\ &= -[X^2(c), [-[[b,x],a],x]] \\ &= -[X^2(c), [[a,[b,x]],x]] \\ &= -[X^2(c), X([a,[b,x]])]. \end{aligned}$$

Using 4 from the lemma 36 we obtain

$$\begin{aligned} ad_x^2\left([c,[[b,[a,x]],x]]\right) &= [X(c), X^2([a,[b,x]])] \\ &= ad_x^2([c,[[a,[b,x]],x]])\,. \end{aligned}$$

Then $c \triangleleft (b \triangleleft a) - c \triangleleft (a \triangleleft b) \in Ker_L(x)$, this is $\overline{c} \triangleleft (\overline{a} \triangleleft \overline{b}) = \overline{c} \triangleleft (\overline{b} \triangleleft \overline{a})$. We will verify the Jordan identity. Let $a, b \in L$ and put $w := [[a,[a,x]],x]$. Then

$$8(\overline{b} \triangleleft \overline{a}^2) \triangleleft \overline{a} = \overline{[[b,[[a,[a,x]],x]],[a,x]]} = \overline{[[b,w],[a,x]]}$$

and

$$
\begin{aligned}
8(\bar{b} \lhd \overline{a}) \lhd \overline{a}^2 &= \overline{[[b,[a,x]],[[a,[a,x]],x]]} = \overline{[[b,[a,x],w]} \\
&= \overline{[[b,w],[a,x]]} + \overline{[b,[[a,x],w]]} \\
&= 8(\bar{b} \lhd \overline{a}^2) \lhd \overline{a} + \overline{[b,[[a,x],w]]}
\end{aligned}
$$

Thus we only need to verify that $[b,[[a,x],w]]$ is in $Ker_L(x)$. In effect, because $[b,[[a,x],w]] = ad_{[[a,x],w]}(b)$ then

$$
\begin{aligned}
ad_x^2 ad_{[[a,x],w]} &= X^2[W,[X,A]] \\
&= X^2WXA - X^2WAX + X^2AXW,
\end{aligned}
$$

since $X^3 = 0$, and

$$
ad_w = W = X^2A^2 - 2XAXA + 2AXAX - A^2X^2.
$$

From

$$
\begin{aligned}
X^2WAX &= 0 \\
-X^2WAX &= -2X^2AXAXAX + X^2A^2X^2AX \\
X^2AXW &= 2X^2AXAXAX - X^2AXA^2X^2,
\end{aligned}
$$

we have $ad_x^2 ad_{[[a,x],w]} = 0$, i.e. $X^2([b,[[a,x],w]]) = 0$, for all $a, b \in L$. ∎

Definition 39 *For any Jordan element x of a Leibniz algebra L, the quasi-Jordan algebra L_x we have just introduced will be called the* **quasi-Jordan algebra of L at x**.

Remark 40 *If L is a Lie algebra then L_x is a Jordan algebra, since $[X(a), X^2(b)] = -[X^2(b), X(a)]$. The last theorem generalizes the result due to Fernández, García and Gómez for Lie algebras to Leibniz algebras (see [6], Theorem 2.4).*

Let L be a Leibniz algebra and let L^{ann} be the subspace of L spanned by elements of the form $[x,x]$, $x \in L$. Since L^{ann} is a two-side ideal of L, then if L is a Leibniz algebra that is non-Lie algebra we have that L^{ann} is a nonzero proper two-side ideal of L. In particular, $L^{ann} = \{0\}$ if and only if L is a Lie algebra.

This imply that Leibniz algebras cannot be nondegenerate in the classical sense, because all elements in L^{ann} are absolute zero divisors of L. We consider the following subspaces direct sum:

$$
L = L^{ann} \oplus Lie(L), \tag{LD}
$$

where the subspace $Lie(L)$ of L is the complement of the subspace L^{ann}. We write the elements $x \in L$ in the form $x = x_a + x_L$, where $x_a \in L^{ann}$ and $x_L \in Lie(L)$.

Therefore we introduce the following generalization of the definition of nondegenerated Lie algebra.

Definition 41 *An element x in a Leibniz algebra L is called an* **absolute zero divisor of** *L if $ad_x^2 = 0$. A Leibniz algebra L is said to be* **nondegenerate** *if it has no nonzero absolute zero divisor in $Lie(L)$ (if $x \neq 0$, then $ad_x^2 \neq 0$, for all $x \in Lie(L)$). This is, if $ad_x^2 = 0$, for $x \in Lie(L)$, then $x = 0$.*

It should be noted that the above definition agrees with the definition of nondegenerated Lie algebra, since $Lie(L) = L$ ($L^{ann} = \{0\}$), in this case.

Remark 42 *Let L be a Leibniz algebra and let x a Jordan element in L. From the definition of L^{ann} and $Lie(L)$, we have*

1. $[y, z] = [y, z_L]$, *for all* $y, z \in L$.
2. $[y_L, y_L] = 0$, *for all* $y \in L$.
3. x_L *is a Jordan element in* L.
4. *If L is a nondegenerate Leibniz algebra, then* $L^{ann} = Z^r(L)$.

The following lemma shows that the identity $[[y_L, x_L], x_L] = 2x_L$ implys the existence of a right unit in L_x.

Lemma 43 *Let L be a Leibniz algebra and let x be a Jordan element in L. If y is an element in L that satisfys the identity $[[y_L, x_L], x_L] = 2x_L$, then $\overline{y}$ is a right unit in L_x. Moreover, $\overline{z} + \overline{y}$ is a right unit in L_x, for all $z \in Z^r(L)$.*

Proof. For all $a \in L$, we have

$$\begin{aligned} X^2([a,[y,x]]) &= [X(a), X^2(y)] = [[a,x],[[y,x],x]] \\ &= [[a,x_L],[[y_L,x_L],x_L]] = 2[[a,x_L],x_L] \\ &= 2[[a,x],x] = 2X^2(a), \end{aligned}$$

since $[u, v] = [u, v_L]$ and $[a, [u, v]] = [a, -[v, u]]$, for all $u, v \in L$.

Therefore $X^2([a,[y,x]] - 2a) = 0$ and this is equivalent to $\overline{a} \triangleleft \overline{y} = \overline{a}$, for all $\overline{a} \in L_x$.

The identity $[a, z] = 0$, for all $z \in Z^r(L)$, implies that $\overline{z} + \overline{y}$ is a right unit in L_x, for all $z \in Z^r(L)$. ∎

The following lemma shows that the existence of the right unit is equivalent to the identity $([[y, x], x])_L = 2x_L$, for some $y \in L$ and for x a Jordan element in a nondegenerate Leibniz algebra L.

Lemma 44 *Let L be a nondegenerate Leibniz algebra and let x be a Jordan element of L. Then $\overline{y}$ is a right unit for the quasi-Jordan algebra L_x if and only if $w_L = 2x_L$, where $w = [[y, x], x]$.*

Proof. Suppose that $y \in L$ which $[[y, x], x] = 2x$, then for all $a \in L$

$$\begin{aligned} X^2([a,[y,x]]) &= [X(a), X^2(y)] = [[a,x],[[y,x],x]] \\ &= [[a,x],w] = [[a,x],w_L] = 2[[a,x],x_L] \\ &= 2[[a,x],x] = 2X^2(a). \end{aligned}$$

Therefore $X^2([a,[y,x]] - 2a) = 0$ and this is equivalent to $\overline{a} \triangleleft \overline{y} = \overline{a}$, for all $\overline{a} \in L_x$. Suppose, conversely, that $\overline{y}$ is a right unit for L_x. Put $z := X^2(y) - 2x$. For all $a \in L$,

$$\begin{aligned} [[a,z],z] &= [[a,X^2(y)],X^2(y)] - 2[[a,x],X^2(y)] - 2X^2([a,[y,x]] - 2a) \\ &= [[a,X^2(y)] - 2[a,x],X^2(y)] \\ &= [[a,X^2(y) - 2x],X^2(y)] \\ &= [[a,z],z] + 2[[a,z],x]\ , \end{aligned}$$

Then $[[a,z],x] = 0$, for all $a \in L$, since L is 2-torsion free. As $[z,x] = -2[x,x]$ we have

$$\begin{aligned} 0 &= [[a,z],x] = [[a,x],z] + [a,[z,x]] \\ &= [[a,x],z] - 2[a,[x,x]] \\ &= [[a,x],z] \end{aligned}$$

Finally, the Leibniz identity and the identities $[[a,z],x] = 0 = [[a,x],z]$ imply

$$\begin{aligned} [[a,z],z] &= [[a,X^2(y)],z] - 2[[a,x],z] \\ &= [[a,[[y,x],x],z] \\ &= [[[a,[y,x]],x],z] - [[[a,x],[y,x]],z] \\ &= -[[[a,x],z],[y,x]] - [[a,x],[[y,x],z]] \\ &= 0 \end{aligned}$$

Since $0 = [[a,z],z] = [[a,z_L],z_L]$ and $z_L = w_L - 2x_L$, therefore $z_L = 0$, because L is nondegenerate. Then $w_L = 2x_L$. ■

Remark 45 *The previous lemmas shows that if there is an right unit $\overline{y}$ in L_x, then $\overline{Z^r(L)} \subset L_x^{ann}$ and $\overline{Z^r(L)} + \overline{y_L} \subset U_r(L_x)$.*

We are going to define a special operator over quasi-Jordan algebras. This operator agrees with the U-operator over Jordan algebras.

Definition 46 *Let $\Im$ be a quasi-Jordan algebra and let R_a be a right multiplicative map by a over $\Im$ ($R_a : \Im \to \Im$, $b \mapsto b \triangleleft a$). For all $a \in \Im$ we define the U-operator $U_a : \Im \to \Im$ by*

$$U_a = 2R_a^2 - R_{a^2}\,, \quad \textit{where } a^2 = a \triangleleft a. \tag{17}$$

We will show a special formula for the U-operator over L_x.

Lemma 47 *Let x be a Jordan element of a Leibniz algebra L. Then the quasi-Jordan algebra L_x of L at x has a U-operator given by*

$$U_{\overline{a}}\overline{b} = \frac{1}{4}\overline{A^2X^2(b)}, \qquad \textit{for all } \overline{a}, \overline{b} \in L_x \tag{18}$$

Proof. For all $c \in L$ we have $[c,x] \in Ker_L(x)$, since $X^2([c,x]) = X^3(c) = 0$. Then $\overline{[c,x]} = \overline{0}$ and

$$\begin{aligned} \overline{A^2X^2(b)} &= \overline{[[[[b,x],x],a],a]} \\ &= \overline{[[[[b,x],a],x],a]} + \overline{[[[b,x],[x,a]],a]} \\ &= \overline{[[[[b,x],a],a],x]} + 2\overline{[[[b,x],a],[x,a]]} + \overline{[[b,x],[[x,a],a]]} \\ &= 2\overline{[[[b,a],x],[x,a]]} + 2\overline{[[b,[x,a]],[x,a]]} + \overline{[[b,[[x,a],a]],x]} \\ &\quad + \overline{[b,[x,[[x,a],a]]]} \\ &= 2\overline{[[[b,a],x],[x,a]]} + 2\overline{[[b,[a,x]],[a,x]]} - \overline{[b,[[a,[a,x]],x]]} \\ &= 4\left(2(\overline{b} \triangleleft \overline{a})\overline{a} - \overline{b} \triangleleft \overline{a}^2\right) \\ &= 4U_{\overline{a}}\overline{b}\,, \end{aligned}$$

since $\overline{[[[b,a],x],[x,a]]} = \overline{0}$, because

$$\begin{aligned} X^2([X([b,a]),[x,a]]) &= -X^2([X([b,a]),[a,x]]) \\ &= -[X(X([b,a]),X^2(a)] \\ &= [X^2(X([b,a]),X(a)] \\ &= [X^3([b,a]),X(a)] \\ &= 0\,, \end{aligned}$$

for all $b, a \in L$. ■

Let $\Im$ be a quasi-Jordan algebra and let $\Im^{ann}$ be the subspace of $\Im$ spanned by elements of the form $x \lhd y - y \lhd x$, $x, y \in \Im$. Since $\Im^{ann}$ is a two-side ideal of $\Im$, then if $\Im$ is a quasi-Jordan algebra that is non-Jordan algebra we have that $\Im^{ann}$ is a nonzero proper two-side ideal of $\Im$. In particular, $\Im^{ann} = \{0\}$ if and only if $\Im$ is a Jordan algebra.

This imply that quasi-Jordan algebras cannot be nondegenerate in the classical sense, because all elements in $\Im^{ann}$ are absolute zero divisors of $\Im$. We consider the following subspaces direct sum:

$$\Im = \Im^{ann} \oplus Jor(\Im), \tag{QJD}$$

where the subspace $Jor(\Im)$ of $\Im$ is the complement of the subspace $\Im^{ann}$. We write the elements $a \in \Im$ in the form $a = a_a + a_J$, were $a_a \in \Im^{ann}$ and $a_J \in Jor(\Im)$.

Therefore we introduce the following generalization of the definition of nondegenerated Jordan algebra.

Definition 48 *Let $\Im$ be a quasi-Jordan algebra. An element a in $\Im$ is called an* **absolute zero divisor** *of $\Im$ if $U_a = 0$. A quasi-Jordan algebra $\Im$ is said to be* **nondegenerate** *if it has no nonzero absolute zero divisors in $Jor(\Im)$ (if $a \neq 0$, then $U_a \neq 0$, for all $a \in Jor(\Im)$). This is, if $U_a = 0$, for $a \in Jor(\Im)$, then $a = 0$.*

It should be noted that the above definition agrees with the definition of nondegenerated Jordan algebra, since $Jor(\Im) = \Im$ ($\Im^{ann} = \{0\}$), in this case.

Remark 49 *If $\Im$ is a nondegenerate quasi-Jordan algebra, then $\Im^{ann} = Z^r(\Im)$.*

Theorem 50 *Let L be a nondegenerate Leibniz algebra and let x be a Jordan element of L such that $x_L \neq 0$. Then L_x is a nondegenerate quasi-Jordan algebra.*

Proof. Since $ad_x = ad_{x_L}$, we can assume that $x = x_L$. Let $\overline{a}$ be an element in $Jor(L_x)$. If $U_{\overline{a}}\overline{b} = 0$ for every $\overline{b} \in L_x$, then $0 = ad_x^2 ad_a^2 ad_x^2(b) = X^2A^2X^2(b) = ad^2_{X^2(a)}(b)$. Since $[u, v] = [u, v_L]$, for all $u, v \in L$, we have that $0 = ad^2_{X^2(a)}(b) = ad^2_{(X^2(a))_L}(b)$ which implies that $(ad_x^2(a))_L = 0$, since L is nondegenerate. Therefore $ad_x^2(a) \in L_x^{ann}$.

We suppose that $\overline{a} \neq \overline{0}$, then we have that there is a nonzero element $\overline{c} \in L_x$ such that $\overline{c} \lhd \overline{a} \neq \overline{0}$, since $\overline{a} \in Jor(L_x)$. This implies that $[c, [a, x]] \notin Ker_L(x)$ and this is equivalent to $X^2([c, [a, x]]) \neq 0$. Because $X^2([c, [a, x]]) = [X(c), X^2(a)]$, then $[X(c), X^2(a)] \neq 0$ and therefore $ad_x^2(a) \notin L^{ann}$. It contradicts that $ad_x^2(a) \in L_x^{ann}$, then $\overline{a} = \overline{0}$. ■

Now, we will characterize the Jordan elements in Leibniz algebras by inner ideals. First, we define inner ideals in Leibniz algebras.

Definition 51 *Let L be a Leibniz algebra. A vector subspace B of L is an* **inner ideal** *if $[[L, B], B] \subseteq B$. Clearly, any ideal I of L is an inner ideal. Moreover subideals of L are also inner ideals.*

An **abelian inner ideal** *is an inner ideal B which is also an abelian subalgebra, this is $[B, B] = 0$.*

According to the last definition, we have the following characterization for Jordan elements in Leibniz algebras.

Lemma 52 *Let L be a Leibniz algebra and let $x \in L$. The following conditions are equivalent:*

1. $ad_x^3 = 0$
2. $x \in B$, *for B an abelian inner ideal.*

Proof. We suppose that $ad_x^3 = 0$ ($X^3(L) = 0$). First, we are going to show that $X^2(L)$ is an abelian inner ideal of L. It suffices to show $YZ(L) \subseteq X^2(L)$, for $Y = ad_y$ and $Z = ad_z$, where $y = X^2(v)$ and $z = X^2(w)$. Then, for Lemma 36, parts 1 and 2, we have

$$\begin{aligned} YZ &= [X,[X,V]][X,[X,W]] \\ &= (X^2V - 2XVX + VX^2)(X^2W - 2XWX + WX^2) \\ &= -2X^2VXWX + X^2VWX^2 + 4XVX^2WX - 2XVXWX^2 \\ &= X^2VWX^2 \end{aligned}$$

This implies that $YZ(L) \subseteq X^2(L)$. On the other hand

$$[y,z] = Z(y) = (X^2W - 2XWX + WX^2)(X^2(v)) = 0$$

therefore $X^2(L)$ is an abelian inner ideal of L. Then $Fx + X^2(L)$ is an abelian inner ideal of L, where F is the field over which L is defined.

Now, we suppose that $x \in B$, for B an abelian inner ideal of L. Then $ad_x^3(L) = [[[L,x],x],x] \subseteq [B,x] = 0$. ■

Let L be a Leibniz algebra. An element $x \in L$ is called *von Neumann regular* if $X^3 = 0$ and $x \in X^2(L)$. We finish with the following lemma.

Lemma 53 *Let x be a Jordan element of L.*

1. *If I is an ideal of L and $x \in I$ is von Newmann regular, then both quasi-Jordan algebras I_x and L_x agree.*
2. *If $L = I \oplus J$ is a direct sum of ideals and $x = i + j$ with respect to this decomposition, then $L_x \cong I_i \times J_j$.*
3. *For any inner ideal B of L, $B_x := (B/ker_L(x) \cap B, \triangleleft)$ is a subalgebra of L_x.*

Proof. The proofs of 2 and 3 are straightforward. Now to prove 1 it is sufficient to show that any coset $\overline{a}$ in L_x is equal to a coset $\overline{b}$ in I_x. Write $x = X^2(y)$ for some $y \in L$. Then, by Lemma 36 (part 7), $X^2 = ad^2_{X^2(y)} = X^2Y^2X^2$ and hence, for any $a \in L$, $X^2(a) = X^2(b)$, where $b = Y^2X^2(a) \in I$. ■

Acknowledgment

The first author thanks Esther García for sending her paper *The Jordan algebras of a Lie algebra*, which motivated the last section of the present work. Additionally, the first author gives acknowledgments to the Universidad de Antioquia for its support. The second author was supported in part under CONACYT grant 37558E and in part under the Cuba National Project "Theory and algorithms for the solution of problems in algebra and geometry".

References

[1] G. Benkart, *On inner ideals and ad-nilpotent elements on Lie algebras*, Trans. Amer. Math. Soc. **232** (1977), 61-81.

[2] G. Benkart, I. M. Isaacs, *On the existence of ad-nilpotent elements*, Proc. Amer. Math. Soc. **63** (1977), 39-40.

[3] M. R. Bremner, L. I. Murakami, I. P. Shestakov, *Nonassociative algebras* (preprint).

[4] F. Chapoton, *Un endofoncteur de la catégorie des opérades*, in: Dialgebras and related operads, Lectures Notes in Mathematics, vol. 1763, Springer Verlag, 2001, pp. 105-110.

[5] R. Felipe, N. López-Reyes, F. Ongay, R. Velásquez, *Yang-Baxter equations on matrix dialgebras with a bar unit*, Linear Alg. and its Appl., **403** (2005), 31-44.

[6] A. Fernández López, E. García, M. Gómez Lozano, *The Jordan algebras of a Lie algebra* (preprint).

[7] A. Fernández López, E. García, M. Gómez Lozano, *Abelian inner ideals and gradings in Lie algebras* (preprint).

[8] N. Jacobson, *Structure and representations of Jordan algebras*, Amer. Math. Soc. Vol. 39 (1968).

[9] I. L. Kantor, *Classification of irreducible transitive differential groups*, Dokl. Akad. Nauk SSR, Vol. 158, No. 6, 1271-1274 (1964).

[10] M. K. Kinyon, A. Weinstein, *Leibniz algebras, Courant algebroids and multiplications on reductive homogeneous spaces*, Amer. J. Math, **123** (2001), 525-550.

[11] M. Koecher, *Imbedding of Jordan algebras into Lie algebras I*, Amer. J. Math. **89** (1967), 787-816.

[12] J. L. Loday, *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Ens. Math. **39** (1993), 269-293.

[13] J. L. Loday, *Dialgebras*, in: Dialgebras and related operads, Lectures Notes in Mathematics, vol. 1763, Springer Verlag, 2001, pp. 7-66.

[14] J. L. Loday, T. Pirashvili, *Universal enveloping algebras of Leibniz algebras and (co)homology*, Math. Ann. **296** (1993), 139-158.

[15] K. McCrimmon, *A Teste of Jordan Algebras*, Springer Verlag, 2004.

[16] J. Tits, *Une classe d'algèbre de Lie en relation avec les algébres de Jordan*, Nederl. Akademic van Wetenschappen, Amsterdam, **65** (1962), 530-535.