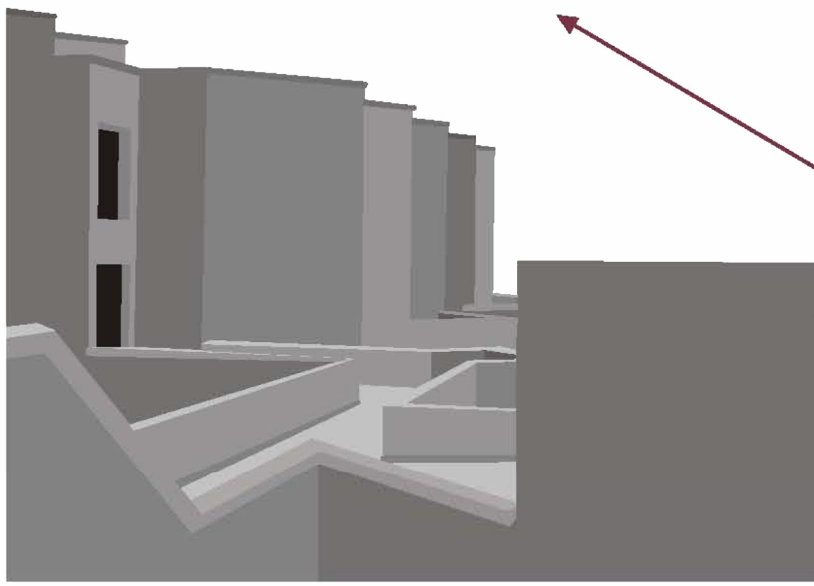

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Measures and Jacobians of Singular Random Matrices

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Abstract

This work studies the Jacobians of certain singular transformations and the corresponding measures which support the jacobian computations. In particular, a sort of asseverations about the validity of some results by Zhang (2007) are discussed.

Key words: Singular random matrices, Jacobian of transformation, Hausdorff measure, Lebesgue measure, multiplicity, nonnegative definite matrices, non-positive definite matrices, indefinite matrices.
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1 Introduction

First consider the following notation: Let $\mathcal{L}_{m,N}(q)$ be the linear space of all $N\times m$ real matrices of rank $q \leq \min(N,m)$ and $\mathcal{L}^{+}_{m,N}(q)$ be the linear space of all $N\times m$ real matrices of rank $q \leq \min(N,m)$, with q distinct singular values. The set of matrices $\mathbf{H}_1 \in \mathcal{L}_{m,N}$ such that $\mathbf{H}_1'\mathbf{H}_1 = \mathbf{I}_m$ is a manifold denoted $\mathcal{V}_{m,N}$, called Stiefel manifold. In particular, $\mathcal{V}_{m,m}$ is the group of orthogonal matrices $\mathcal{O}(m)$. Denote by $\mathcal{S}_m$, the homogeneous space of $m\times m$ positive definite symmetric matrices; and by $\mathcal{S}^{+}_m(q)$, the $(mq-q(q-1)/2)$-dimensional manifold of rank q positive semidefinite $m\times m$ symmetric matrices with q distinct positive eigenvalues.

Assuming that $\mathbf{X} \in \mathcal{L}^{+}_{m,N}(q)$, Díaz-García *et al.* (1997) proposed the Jacobian of non-singular part of the singular value decomposition, $\mathbf{X} = \mathbf{H}_1\mathbf{D}\mathbf{W}_1'$,

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where $\mathbf{H}_1 \in \mathcal{V}_{q,N}$, $\mathbf{D}$ is a diagonal matrix with $D_1 > D_2 > \cdots D_q > 0$ and $\mathbf{W}_1 \in \mathcal{V}_{q,m}$. Also, note that the jacobian itself defines the factorization of Hausdorff's measure $(d\mathbf{X})$ (or Lebesgue's measure defined on the manifold $\mathcal{L}^+_{m,N}(q)$, see Billingsley (1986, p. 249)). Analogous results for $\mathbf{V} \in \mathcal{S}^+_m(q)$ considering the non-singular part of the spectral decomposition of $\mathbf{V}$ were proposed by Uhlig (1994) and Díaz-García and Gutiérrez (1997). Based on these two results, Díaz-García and Gutiérrez-Jáimez (2005) and Díaz-García and Gutiérrez-Jáimez (2006) computed the jacobians of the transformations $\mathbf{Y} = \mathbf{X}^+$ and $\mathbf{W} = \mathbf{V}^+$, where $\mathbf{A}^+$ denotes the Moore-Penrose inverse of A, see Rao (1973, p.49).

Recently, Zhang (2007, Lemma 2 and 3) extends the above mentioned results by assuming multiplicity in the singular values (of $\mathbf{Y}$) and the eigenvalues (of $\mathbf{W}$); moreover, he proposes that his Lemma 3 is valid without the positive semidefinite assumption for $\mathbf{V}$ (or $\mathbf{W}$), respectively.

In the present work we proof that only some of the above propositions are right, in fact, we will derive the correct corresponding results for Zhang (2007, Lemma 2 and 3) under the same generalizations, and also we will determine the explicit measures with respect the jacobians are computed, see Díaz-García (2007b).

2 Some comments about Zhang's (2007) paper

The first observation about the Lemmas 2 and 3 in Zhang (2007, Last paragraph p. 6) and the corresponding results derived in Díaz-García and Gutiérrez-Jáimez (2005) and Díaz-García and Gutiérrez-Jáimez (2006) is inexact, because in the references the results are obtained by the singular value decomposition (SVD). The right explanation should indicate that in Díaz-García and Gutiérrez-Jáimez (2005) and Díaz-García and Gutiérrez-Jáimez (2006) the jacobians are computed properly by the exterior product, meanwhile Zhang (2007) proposes the same results by the jacobians of marginal transformations, without any proof but arguing that in general they are known from non singular transformations.

The second observation concerns Lemma 3 in Zhang (2007, Last paragraph p. 6): it is a general proposition, i.e. it has no explicit constrains, then we can consider Lemma 3 valid for every singular symmetric matrix, thus, it is valid for positive semidefinite matrices, for negative semidefinite matrices and for indefinite matrices. Those asseverations are **partially** correct as we will establish next and in Section 3.

The third observations involves the validity of Lemas 2 and 3 in Zhang (2007, Last paragraph p. 6) when there is multiplicity in the singular values of the matrix $\mathbf{X}$ or multiplicity in the eigenvalues of the matrix $\mathbf{V}$. In both cases the proposition is wrong, as we will prove in Section 4.

Two general comments:

i) With respect to the second observation, it is clear that the proposition holds for nonpositive definite matrices (i.e. for negative definite ($\mathbf{A} < 0$) and for negative semidefinite ($\mathbf{A} \leq 0$)). The particular lemma can be obtained directly from the corresponding result of nonnegative definite matrices by substituting the eigenvalues with their absolute value. A simple argument is the following: if $\mathbf{A} \in \mathcal{S}_m$ or $\mathbf{A} \in \mathcal{S}_m^+(q)$, then $-\mathbf{A}$ is a nonpositive definite matrix and $(d\mathbf{A}) = (-d\mathbf{A})$, where $(d\mathbf{A})$ denotes the exterior products of the mathematical independent elements in the differential matrix $d\mathbf{A} = [da_{ij}]$, Muirhead (1982) and James (1954).

ii) Unfortunately, Zhang (2007) does not specifies clearly the measure used in the computation of the jacobians, but according with the explicit comparisons in Zhang (2007, Last paragraph p. 6) of his results with the analogous results of Díaz-García and Gutiérrez-Jáimez (2005) and Díaz-García and Gutiérrez-Jáimez (2006), we infer that his measures are the same considered by Díaz-García and Gutiérrez-Jáimez (2005) and Díaz-García and Gutiérrez-Jáimez (2006). This fact is extremely important, just recall that those measures are not unique and the corresponding jacobians are not unique as well; but when a measure is particularly defined the respective jacobian is unique, see Khatri (1968), Díaz-García and González-Farías (1999) and Díaz-García and González-Farías (2005a).

3 Jacobian of symmetric matrices

Consider again $\mathbf{A} \in \mathcal{S}_m$, it remains to study: $\mathbf{A}$ as a (nonsingular) indefinite matrix, i.e. $\mathbf{A} \in \mathcal{S}_m^{\pm}(m_1, m_2)$, with $m_1 + m_2 = m$, where m_1 is the number of positive eigenvalues and m_2 is the number of negative eigenvalues; and $\mathbf{A}$ as a (singular) semi-indefinite matrix, i.e. $\mathbf{A} \in \mathcal{S}_m^{\pm}(q, q_1, q_2)$, with $q_1 + q_2 = q$, here q_1 is the number of positive eigenvalues and q_2 is the number of negative eigenvalues.

First suppose $\mathbf{A} \in \mathcal{S}_m^{\pm}(m_1, m_2)$ such that $\mathbf{A} = \mathbf{HDH}'$, where $\mathbf{H} \in \mathcal{O}(m)$, $\mathbf{D}$ is a diagonal matrix. Without loss of generality, let $\lambda_1 > \cdots > \lambda_{m_1} > 0$ and

$0 > -\delta_1 > \cdots > -\delta_{m_2}$, explicitly

$$\mathbf{A} = \mathbf{H}\begin{bmatrix} \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{m_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\delta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -\delta_{m_2} \end{bmatrix}\mathbf{H}'.$$

Now let $\mathbf{A} \in \mathcal{S}_3^{\pm}(1,2)$ and let $A = \mathbf{HDH}'$ be its SD, then

$$d\mathbf{A} = d\mathbf{HDH}' + \mathbf{H}d\mathbf{DH}' + \mathbf{HD}d\mathbf{H}',$$

thus by the skew symmetry of $\mathbf{H}'d\mathbf{H}$ we have, see Muirhead (1982, p. 105)

$$\mathbf{H}'d\mathbf{AH} = \mathbf{H}'d\mathbf{HD} + d\mathbf{D} + \mathbf{D}d\mathbf{H}'\mathbf{H} = \mathbf{H}'d\mathbf{HD} + d\mathbf{D} - \mathbf{DH}'d\mathbf{H}$$

Moreover,

$$\begin{aligned}
\mathbf{H}'d\mathbf{AH} &= \begin{bmatrix} 0 & -h_2'dh_1 & -h_3'dh_1 \\ h_2'dh_1 & 0 & -h_3'dh_2 \\ h_3'dh_1 & h_3'dh_2 & 0 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & -\delta_1 & 0 \\ 0 & 0 & -\delta_2 \end{bmatrix} + \begin{bmatrix} d\lambda_1 & 0 & 0 \\ 0 & -d\delta_1 & 0 \\ 0 & 0 & -d\delta_2 \end{bmatrix} \\
&\quad - \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & -\delta_1 & 0 \\ 0 & 0 & -\delta_2 \end{bmatrix}\begin{bmatrix} 0 & -h_2'dh_1 & -h_3'dh_1 \\ h_2'dh_1 & 0 & -h_3'dh_2 \\ h_3'dh_1 & h_3'dh_2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \delta_1 h_2'dh_1 & \delta_2 h_3'dh_1 \\ \lambda_1 h_2'dh_1 & 0 & \delta_2 h_3'dh_2 \\ \lambda_1 h_3'dh_1 & -\delta_1 h_3'dh_2 & 0 \end{bmatrix} + \begin{bmatrix} d\lambda_1 & 0 & 0 \\ 0 & -d\delta_1 & 0 \\ 0 & 0 & -d\delta_2 \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 & -\lambda_1 h_2'dh_1 & -\lambda_1 h_3'dh_1 \\ -\delta_1 h_2'dh_1 & 0 & \delta_1 h_3'dh_2 \\ -\delta_2 h_3'dh_1 & -\delta_2 h_3'dh_2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} d\lambda_1 & (\lambda_1+\delta_1)h_2'dh_1 & (\lambda_1+\delta_2)h_3'dh_1 \\ (\lambda_1+\delta_1)h_2'dh_1 & -d\delta_1 & (\delta_2-\delta_1)h_3'dh_2 \\ (\lambda_1+\delta_2)h_3'dh_1 & (-\delta_1+\delta_2)h_3'dh_2 & -d\delta_2 \end{bmatrix}.
\end{aligned}$$

We know that $(\mathbf{H}'d\mathbf{A}\mathbf{H}) = (d\mathbf{A})$, then a column by column computation of the exterior product of the subdiagonal elements of $\mathbf{H}'d\mathbf{H}\mathbf{D} + d\mathbf{D} - \mathbf{D}\mathbf{H}'d\mathbf{H}$ gives, ignoring the sign,

$$(d\mathbf{A}) = (\lambda_1 + \delta_1)(\lambda_1 + \delta_2)(-\delta_1 + \delta_2)\left(\bigwedge_{i=1}^{3}\bigwedge_{j=i+1}^{3} h_j' dh_i\right) \wedge d\lambda_1 \wedge -d\delta_1 \wedge -d\delta_2.$$

Recall that, if for example, the first element in each column of $\mathbf{H}$ is nonnegative, so, the transformation $\mathbf{A} = \mathbf{H}\mathbf{D}\mathbf{H}'$ is $1-1$. Then the corresponding jacobian must be divided by 2^m, see Muirhead (1982, pp. 104-105). Thus we have

$$(d\mathbf{A}) = 2^{-3}(\lambda_1 + \delta_1)(\lambda_1 + \delta_2)(\delta_1 - \delta_2)(\mathbf{H}'d\mathbf{H}) \wedge (d\mathbf{D}),$$

where $(\mathbf{H}'d\mathbf{H})$ is the Haar measure on $\mathcal{O}(m)$ and

$$(\mathbf{H}'d\mathbf{H}) = \bigwedge_{i<j}^{m} h_j' dh_i, \quad (d\mathbf{D}) = d\lambda_1 \wedge d\delta_1 \wedge d\delta_2,$$

this is, $(d\mathbf{D})$ is a exterior product of all differentials $d\lambda_i$ and $d\delta_j$ ignoring the sign.

Analogously, if $\mathbf{A} \in \mathcal{S}_3^{\pm}(2,1)$,

$$(d\mathbf{A}) = 2^{-3}(\lambda_1 - \lambda_2)(\lambda_1 + \delta_1)(\lambda_2 + \delta_1)(\mathbf{H}'d\mathbf{H}) \wedge (d\mathbf{D}).$$

Similarly, let $\mathbf{A} \in \mathcal{S}_4^{\pm}(2,2)$, then

$$(d\mathbf{A}) = 2^{-4}(\lambda_1 - \lambda_2)(\delta_1 - \delta_2)(\lambda_1 + \delta_1)(\lambda_1 + \delta_2)(\lambda_2 + \delta_1)(\lambda_2 + \delta_2)(\mathbf{H}'d\mathbf{H}) \wedge (d\mathbf{D}).$$

By mathematical induction we have

Theorem 1 *Let $\mathbf{A} \in \mathcal{S}_m^{\pm}(m_1, m_2)$ such that $\mathbf{A} = \mathbf{H}\mathbf{D}\mathbf{H}'$, where $\mathbf{H} \in \mathcal{O}(m)$, $\mathbf{D}$ is a diagonal matrix with $\lambda_1 > \cdots > \lambda_{m_1} > 0$ and $0 > -\delta_1 > \cdots > -\delta_{m_2}$, $m_1 + m_2 = m$. Then*

$$(d\mathbf{A}) = 2^{-m}\prod_{i<j}^{m_1}(\lambda_i - \lambda_j)\prod_{i<1}^{m_1}(\delta_i - \delta_j)\prod_{i,j}^{m_1,m_2}(\lambda_i + \delta_j)(\mathbf{H}'d\mathbf{H}) \wedge (d\mathbf{D}).$$

where

$$\prod_{i,j}^{m_1,m_2}(\lambda_i + \delta_j) = \prod_{i=1}^{m_1}\prod_{j=1}^{m_2}(\lambda_i + \delta_j), \quad (\mathbf{H}'d\mathbf{H}) = \bigwedge_{i<j}^{m} h_j' dh_i, \quad (d\mathbf{D}) = \bigwedge_{i=1}^{m_1} d\lambda_i \bigwedge_{j=1}^{m_2} d\delta_j.$$

A similar procedure for $\mathbf{A} \in \mathcal{S}_m^{\pm}(q, q_1, q_2)$ gives:

Theorem 2 *Let $\mathbf{A} \in \mathcal{S}_m^{\pm}(q, q_1, q_2)$ such that $\mathbf{A} = \mathbf{H}_1\mathbf{D}\mathbf{H}_1'$, where $\mathbf{H}_1 \in \mathcal{V}_{q,m}$, $\mathbf{D}$ is a diagonal matrix with $\lambda_1 > \cdots > \lambda_{q_1} > 0$ and $0 > -\delta_1 > \cdots > -\delta_{q_2}$,*

$q_1 + q_2 = q$. *Then*

$$(d\mathbf{A}) = 2^{-q} \prod_{i=1}^{q_1} \lambda_i^{m-q} \prod_{j=1}^{q_2} \delta_j^{m-q} \prod_{i<j}^{q_1} (\lambda_i - \lambda_j) \prod_{i<1}^{q_1} (\delta_i - \delta_j) \prod_{i,j}^{q_1,q_2} (\lambda_i + \delta_j)(\mathbf{H}'d\mathbf{H}) \wedge (d\mathbf{D}).$$

where

$$\prod_{i,j}^{q_1,q_2} (\lambda_i + \delta_j) = \prod_{i=1}^{q_1} \prod_{j=1}^{q_2} (\lambda_i + \delta_j), \quad (\mathbf{H}_1'd\mathbf{H}_1) = \bigwedge_{i=1}^{m} \bigwedge_{j=i+1}^{q} h_j' dh_i, \quad (d\mathbf{D}) = \bigwedge_{i=1}^{q_1} d\lambda_i \bigwedge_{j=1}^{q_2} d\delta_j.$$

4 Jacobians of symmetric matrices with multiplicity in its eigenvalues

As a motivation of this section, consider a general random matrix $\mathbf{A} \in \Re^{m \times m}$, explicitly

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix}.$$

Any density function of this matrix can be expressed as

$$dF_{\mathbf{A}}(\mathbf{A}) = f_{\mathbf{A}}(\mathbf{A})(d\mathbf{A}),$$

where $d\mathbf{A}$ denotes the measure of Lebesgue in $\Re^{m^2}$, which can be written by using the exterior product, as

$$(d\mathbf{A}) = \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{m} da_{ij},$$

see Muirhead (1982).

However, if $\mathbf{A} \in \mathcal{S}_m$ and it is non singular, then the measure of Lebesgue defined in $\mathcal{S}_m$ is given by

$$(d\mathbf{A}) = \bigwedge_{i \leq j}^{m} da_{ij}. \tag{1}$$

Remark 3 *Note that the above product is the measure of Hausdorff on $\Re^{m^2}$ defined on the homogeneous space of positive definite symmetric matrices, see Billingsley (1986)*

In general, we can consider any factorization of the Lebesgue measure $(d\mathbf{A})$ on $\mathcal{S}_m$ as an alternative definition of $(d\mathbf{A})$ with respect to the corresponding coordinate system. For example, if we consider the spectral decomposition

(SD), $\mathbf{A} = \mathbf{HDH}'$, where $\mathbf{H} \in \mathcal{O}(m)$, and $\mathbf{D}$ is a diagonal matrix with $D_1 > \cdots > D_m > 0$ or we consider the Cholesky decomposition $\mathbf{A} = \mathbf{T}'\mathbf{T}$, where $\mathbf{T}$ is upper-triangular with positive diagonal elements, then we have respectively

$$d\mathbf{A} = \begin{cases} 2^{-m}\displaystyle\prod_{i<j}^{m}(D_i - D_j)(\mathbf{H}'d\mathbf{H}) \wedge (d\mathbf{D}), & \text{Spectral decomposition;} \\ 2^{m}\displaystyle\prod_{i=1}^{m} t_{ii}^{m+1-i}(d\mathbf{T}), & \text{Cholesky decomposition,} \end{cases}$$

see Díaz-García and González-Farías (2005a), where

$$(\mathbf{H}'d\mathbf{H}) = \bigwedge_{i<j}^{m} h_j' dh_i, \quad (d\mathbf{D}) = \bigwedge_{i=1}^{m} dD_i \text{ and } (d\mathbf{T}) = \bigwedge_{i\leq j}^{m} dt_{ij}.$$

In some occasions is difficult to establish an explicit form of the Lebesgue o Hausdorff measures in the original coordinate system. In particular if $\mathbf{A} \in \mathcal{S}_m^+(q)$ some unsuccessful efforts have been trailed, see Srivastava (2003) and Díaz-García (2007a). A definition of such measure in terms of the SD is given by Uhlig (1994):

$$(d\mathbf{A}) = 2^{-q}\prod_{i=1}^{q} D_i^{m-q}\prod_{i<j}^{q}(D_i - D_j)(\mathbf{H}_1 d\mathbf{H}_1) \wedge (d\mathbf{D}), \tag{2}$$

where $\mathbf{H}_1 \in \mathcal{V}_{q,m}$, $\mathbf{D}$ is a diagonal matrix with $D_1 > \cdots > D_q > 0$ and

$$(\mathbf{H}_1'd\mathbf{H}_1) = \bigwedge_{i=1}^{m}\bigwedge_{j=i+1}^{m} h_j' dh_i, \quad (d\mathbf{D}) = \bigwedge_{i=1}^{q} dD_i;$$

for alternative expressions of $d\mathbf{A}$ in terms of other factorizations see Díaz-García and González-Farías (2005a) and Díaz-García and González-Farías (2005b).

Now suppose that one (or more) eigenvalue(s) of $\mathbf{A} \in \mathcal{S}_m$ has (have) multiplicity. Then consider $\mathbf{A} = \mathbf{HDH}'$, where $\mathbf{H} \in \mathcal{O}(m)$, $\mathbf{D}$ is a diagonal matrix with $D_1 \geq \cdots \geq D_m > 0$. Moreover, let $D_{k_1}, \ldots D_{k_l}$ be the l distinct eigenvalues of A, i.e. $D_{k_1} > \cdots > D_{k_l} > 0$, where m_j denotes the repetitions of the eigenvalue D_{k_j}, $j = 1, 2, \ldots, l$, and of course $m_1 + \cdots m_l = m$; finally denote the corresponding set of matrices by $\mathbf{A} \in \mathcal{S}_m^l$. It is clear that $\mathbf{A}$ exists in the homogeneous subspace of the symmetric matrices with dimension $m(m+1)/2$ of rank m; more accurately, when there exist multiplicity in the eigenvalues, $\mathbf{A}$ exists in the manifold of dimension $ml - l(l-1)/2$, even exactly for computations we say that $\mathbf{A} \in \mathcal{S}_m^+(l)$. For proving it, consider the matrix $\mathbf{B} \in \mathcal{S}_2$, such that $\mathbf{S} = \mathbf{HDH}'$, here $\mathbf{H} \in \mathcal{O}(2)$, and $\mathbf{D}$ is a diagonal matrix with $D_1 \geq D_2 > 0$ where $D_1 = D_2 = \kappa$, then the measure

$$(d\mathbf{B}) = 2^{-2}\prod_{i<j}^{2}(D_i - D_j)(\mathbf{H}'d\mathbf{H}) \wedge d\mathbf{D} = 2^{-2}(\kappa - \kappa)(\mathbf{H}'d\mathbf{H}) \wedge d\mathbf{D} = 0.$$

Also note that, in fact the measure $(d\mathbf{D}) = dD_1 \wedge dD_2 = d\kappa \wedge d\kappa = 0$. This is analogous to the following situation, to propose for a curve in the space $(\Re^3)$ the measure of Lebesgue defined by $dx_1 \wedge dx_2$.

Now, when we consider the factorization of the measure of Lebesgue in terms of the spectral decomposition, we do not have $2(2+1)/2 = 3$ but only $2(1) - 1(1+1)/2 + 1 = 2$ mathematical independent elements in $\mathbf{B}$, because in $\mathbf{D}$, $D_1 = D_2 = \kappa$ and then there is only one mathematical independent element.

Also, observe that the space of positive definite $m \times m$ matrices is a subset of Euclidian space of symmetric $m \times m$ matrices of dimension $m(m+1)/2$, and in fact it forms an open cone described by the following system of inequalities, see Muirhead (1982, p. 61 and p. 77 Problem 2.6):

$$\mathbf{A} > 0 \Leftrightarrow a_{11} > 0, \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0, \cdots, \det(\mathbf{A}) > 0. \tag{3}$$

In particular, let $m = 2$, after factorizing the measure of Lebesgue in $\mathcal{S}_m$ by the spectral decomposition, the inequalities (3) are as follows

$$\mathbf{A} > 0 \Leftrightarrow D_1 > 0, D_2 > 0, D_1 D_2 > 0. \tag{4}$$

But if $D_1 = D_2 = \kappa$, (4) it reduces to

$$\mathbf{A} > 0 \Leftrightarrow \kappa > 0, \kappa^2 > 0. \tag{5}$$

Which defines a curve (a parabola) in the space, over the line $D_1 = D_2 (= \kappa)$ in the subspace of points (D_1, D_2).

A similar situation appear in the following cases: i) When we consider multiplicity of the singular values in the SVD; such set of matrices will be denoted by $\mathbf{X} \in \mathcal{L}_{m,N}^{l}(q)$, $q \geq l$; or by $\mathbf{X} \in \mathcal{L}_{m,N}^{+}(q, l)$ $q \geq l$; ii) If we consider multiplicity in the eigenvalues; the corresponding set of matrices will be denoted by $\mathbf{A} \in \mathcal{S}_m^{+}(q, l)$, $q \geq l$; iii) And if A is nonpositive definite.

Thus, unfortunately, we must qualify as incorrect the asseverations of Zhang (2007) about the validity of his Lemmas 2, 3 and consequences, under multiplicity assumptions of singular values and eigenvalues.

As a summary we have the next results, which collect the main conclusions of Section 3 and the present section, and follow a similar proof of Theorem 1 in Díaz-García and Gutiérrez-Jáimez (2006):

Theorem 4 *Consider* $\mathbf{Y} \in \mathcal{L}_{m,N}(q)$ *and* $\mathbf{Y} = \mathbf{X}^+$*, then*

$$(d\mathbf{Y}) = \prod_{i=1}^{k} \sigma_i^{-2(N+m-k)} (d\mathbf{X})$$

where $\mathbf{X} = \mathbf{H}_1 \mathbf{D}_\sigma \mathbf{P}_1'$ *is the nonsingular part of SVD of* $\mathbf{X}$*, with* $\mathbf{H}_1 \in \mathcal{V}_{k,N}$*,* $\mathbf{P}_1 \in \mathcal{V}_{k,m}$*,* $\mathbf{D}_\sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_k)$*,* $\sigma_1 > \cdots > \sigma_k > 0$*, the measure* $(d\mathbf{X})$ *is*

$$(d\mathbf{X}) = 2^{-k} \prod_{i=1}^{k} \sigma_i^{(N+m-2k)} \prod_{i<j}^{k} (\sigma_i^2 - \sigma_j^2)(\mathbf{H}_1' d\mathbf{H}_1) \wedge (\mathbf{P}_1' d\mathbf{P}_1) \wedge (d\mathbf{D}_\sigma),$$

and

$$k = \begin{cases} q, & \mathbf{X} \in \mathcal{L}_{m,N}^{+}(q); \\ l, & \mathbf{X} \in \mathcal{L}_{m,N}^{+}(q,l). \end{cases}$$

Similarly, for symmetric matrices we have,

Theorem 5 *Let* $\mathbf{V} \in \Re^{m \times m}$ *be a symmetric matrix and let* $\mathbf{W} = \mathbf{V}^+$*, then*

(1)

$$(d\mathbf{W}) = \prod_{i=1}^{\beta} |\lambda_i|^{-2m+\beta-1} (d\mathbf{V}),$$

where $\mathbf{V} = \mathbf{H}_1 \mathbf{D}_\lambda \mathbf{H}_1'$ *is the nonsingular part of SD of* $\mathbf{V}$*, with* $\mathbf{H}_1 \in \mathcal{V}_{\beta,N}$*,* $\mathbf{D}_\lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_\beta)$*,* $|\lambda_1| > \cdots > |\lambda_\beta| > 0$*, the measure* $(d\mathbf{V})$ *is*

$$(d\mathbf{V}) = 2^{-\beta} \prod_{i=1}^{\beta} |\lambda_i|^{m-\beta} \prod_{i<j}^{\beta} (|\lambda_i| - |\lambda_j|)(\mathbf{H}_1 d\mathbf{H}_1) \wedge (d\mathbf{D}_\lambda),$$

and

$$\beta = \begin{cases} m, & \mathbf{V} \text{ or } -\mathbf{V} \in \mathcal{S}_m; \\ l, & \mathbf{V} \text{ or } -\mathbf{V} \in \mathcal{S}_m^l; \\ q, & \mathbf{V} \text{ or } -\mathbf{V} \in \mathcal{S}_m^+(q); \\ k, & \mathbf{V} \text{ or } -\mathbf{V} \in \mathcal{S}_m^+(q,k). \end{cases}$$

(2)

$$(d\mathbf{W}) = \prod_{i=1}^{\alpha_1} \lambda_i^{-2(m-\alpha_1/2-\alpha_2+1)} \prod_{j=1}^{\alpha_2} \delta_j^{-2(m-(\alpha-1)/2)} (d\mathbf{V}),$$

where $\alpha = \alpha_1 + \alpha_2$*,* $\mathbf{V} = \mathbf{H}_1 \mathbf{D} \mathbf{H}_1'$ *is the nonsingular part of SD of* $\mathbf{V}$*, with* $\mathbf{H}_1 \in \mathcal{V}_{\alpha,N}$*,* $\mathbf{D} = \operatorname{diag}(\lambda_1, \dots, \lambda_{\alpha_1}, -\delta_1, \dots, -\delta_{\alpha_1})$*,* $\lambda_1 > \cdots > \lambda_{\alpha_1} > 0$*;*

$|\delta_1| > \cdots > |\delta_{\alpha_2}| > 0$, *the measure* $(d\mathbf{V})$ *is*

$$(d\mathbf{V}) = 2^{-\alpha} \prod_{i=1}^{\alpha_1} \lambda_i^{m-\alpha} \prod_{j=1}^{\alpha_2} \delta_j^{m-\alpha} \prod_{i<j}^{\alpha_1}(\lambda_i-\lambda_j) \prod_{i<1}^{\alpha_1}(\delta_i-\delta_j) \prod_{i,j}^{\alpha_1,\alpha_2} (\lambda_i+\delta_j)(\mathbf{H}'d\mathbf{H})\wedge(d\mathbf{D}).$$

and

$$\alpha = \begin{cases} m, \ \mathbf{V} \in \mathcal{S}_m^{\pm}(m_1, m_2); \\ l, \quad \mathbf{V} \in \mathcal{S}_m^{\pm}(l_1, l_2); \\ q, \quad \mathbf{V} \in \mathcal{S}_m^{\pm}(q, q_1, q_2); \\ k, \quad \mathbf{V} \in \mathcal{S}_m^{\pm}(q, k_1, k_2), \end{cases}$$

and $\mathbf{V} \in \mathcal{S}_m^{\pm}(l_1, l_2)$ *denotes a nonsingular indefinite matrix with multiplicity in its eigenvalues and* $\mathbf{V} \in \mathcal{S}_m^{\pm}(q, k_1, k_2)$ *denotes a singular indefinite matrix with multiplicity in its eigenvalues.*

5 Conclusions

This work determines the jacobians of the SVD and the SD under multiplicity of the singular values and eigenvalues, respectively. For the SD case, we compute the jacobian for nonsingular and singular indefinite matrices with and without multiplicity in their eigenvalues. Also, we calculate the jacobians for a general matrix and its Moore-Penrose inverse, and for a symmetric matrix with all its variants (nonpositive, nonnegative and indefinite). In every case we specify the measures of Hausdorff which support the jacobian computations. These results detecte and correct some inconsistences in the validity of Lemmas 2 and 3 by Zhang (2007). We highlight that the results of this paper will be the foundations of an explored problem in literature: the test criteria in MANOVA when there exist multiplicities in: the matrix of sum of squares and sum of products, due to the hypothesis $\mathbf{S}_H$; the matrix of sum of square and sum of products, due to the error $\mathbf{S}_E$; the matrices $\mathbf{S}_H\mathbf{S}_E^{-1}$; $(\mathbf{S}_H + \mathbf{S}_E)^{-1}\mathbf{S}_H$; see Díaz-García (2007c).

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