MULTIVARIATE ANALYSIS OF VARIANCE UNDER MULTIPLICITY

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Abstract

This work studies the behavior of certain test criteria in multivariate analysis of variance (MANOVA), under the existence of multiplicity in the sample eigenvalues of the matrix $\mathbf{S}_{\mathbf{E}}^{-1}\mathbf{S}_{\mathbf{H}}$; where $\mathbf{S}_H$ is the matrix of sum of squares and sum of products due to the hypothesis and $\mathbf{S}_E$ is the matrix of sum of squares and sum of products due to the error.

Key words: Multiplicity, MANOVA, Wilks' criteria, Lawley-Hotelling criterion, Pillai's criterion, Roy criteria.
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1 Introduction

Let $\mathbf{A}$ be a $m \times m$ symmetric matrix with spectral decomposition

$$\mathbf{A} = \mathbf{HLH}', \tag{1}$$

where $\mathbf{H}$ is a $m \times m$ orthogonal matrix and $\mathbf{L}$ is a diagonal matrix, such that $\mathbf{L} = \operatorname{diag}(l_1, \dots, l_m)$. The representation (1) is unique if the eigenvalues $l_1, \dots, l_m$ are distinct and the sign of the first element in each column is non negative, Muirhead (1982, p. 588).

For $\mathbf{A}$ a positive definite matrix ($\mathbf{A} > \mathbf{0}$), i.e. for $l_1 > \cdots > l_m > 0$, the jacobian of the transformation (1) has been computed by different authors,

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James (1954), Muirhead (1982, pp. 104-105) and Anderson (1984, Section 13.2.2), among many others. Similarly, when $\mathbf{A}$ is a positive semidefinite matrix ($\mathbf{A} \geq \mathbf{0}$), i.e. when $l_1 > \cdots > l_r > 0$ and $l_{r+1} = \cdots = l_m = 0$, $r < m$, the respective jacobian was computed by Uhlig (1994), see also Díaz-García *et al.* (1997). Recently, Zhang (2007), generalized the last jacobian for: non positive definite matrices, $-\mathbf{A} > \mathbf{0}$ or $-\mathbf{A} \geq \mathbf{0}$; indefinite matrices; and matrices with multiplicity in their eigenvalues. Unfortunately, these generalizations have some inconsistences, but they have been corrected in Díaz-García (2007b).

Note that under the spectral decomposition, the Lebesgue measure defined on the homogeneous space of $m \times m$ positive definite symmetric matrices $\mathcal{S}_m^+$ (and implicitly the jacobian of the transformation (1)) is given by

$$(d\mathbf{A}) = 2^{-m} \prod_{i<j}^{m} (l_i - l_j)(\mathbf{H}'d\mathbf{H}) \wedge (d\mathbf{L}), \tag{2}$$

see Muirhead (1982, pp. 104-105), where

$$(\mathbf{H}'d\mathbf{H}) = \bigwedge_{i<j}^{m} h_j' dh_i, \qquad (d\mathbf{L}) = \bigwedge_{i=1}^{m} dl_i.$$

and $(d\mathbf{B})$ denotes the exterior product of the distinct elements of the matrix differentials (db_{ij}) and in particular $(\mathbf{H}'d\mathbf{H})$ denotes the Haar measure, see James (1954) and Muirhead (1982, Chapter 2).

By applying the definition of exterior product, it is easy to see that under multiplicity in the eigenvalues of the matrix $\mathbf{A}$, i.e., $l_i = l_j$ at least for a $i \neq j$, we obtain that $(d\mathbf{L}) = 0$, moreover, in (2)

$$\prod_{i<j}^{m} (l_i - l_j) = 0,$$

then $(d\mathbf{A}) = 0$. This happens because the multiplicity of the eigenvalues of $\mathbf{A}$ forces it to live in a $(ml - l(l-1)/2)$-dimensional manifold of rank l on the homogeneous space of $m \times m$ $\mathcal{S}_{m,l}^+ \subset \mathcal{S}_m^+$.

Observe that $\mathcal{S}_m^+$ is a subset of the $m(m+1)/2$-dimensional $\mathcal{S}_m$ Euclidian space of $m \times m$ symmetric matrices, and, in fact, it forms an open cone described by the following system of inequalities, see Muirhead (1982, p. 61 and p. 77 Problem 2.6):

$$\mathbf{A} > 0 \Leftrightarrow a_{11} > 0, \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0, \cdots, \det(\mathbf{A}) > 0. \tag{3}$$

In particular, let $m = 2$, after factorizing the Lebesgue measure in $\mathcal{S}_m$ by the spectral decomposition, then the inequalities (3) are as follows

$$\mathbf{A} > 0 \Leftrightarrow l_1 > 0, l_2 > 0, l_1 l_2 > 0. \tag{4}$$

But if $l_1 = l_2 = \varrho$, (4) reduces to

$$\mathbf{A} > 0 \Leftrightarrow \varrho > 0, \varrho^2 > 0. \tag{5}$$

Which defines a curve (a parabola) in the space, over the line $l_1 = l_2 (= \varrho)$ in the subspace of points (l_1, l_2). Formally, we say that $\mathbf{A}$ has a density respect to the Hausdorff measure, Billingsley (1986).

When $\mathbf{A} \in \mathcal{S}_m^+$, the eigenvalue distributions have been studied by several authors, Srivastava & Khatri (1979), Muirhead (1982), Anderson (1984), among many others. If $\mathbf{A} \in \mathcal{S}_m^+(q)$, i.e. $\mathbf{A}$ is a positive semidefinite matrix with q distinct positive eigenvalues, the eigenvalue distributions have been founded by Díaz-García and Gutiérrez (1997), Díaz-García *et al.* (1997) Srivastava (2003), Díaz-García and Gitiérrez-Jáimez (2006) and Díaz-García (2007a).

In general, we can consider multiplicity in the eigenvalues of any symmetric matrix, but in some applied cases (MANOVA problems) the eigenvalues are always assumed distinct, for instance, Okamoto (1973) studies the matrix $\mathbf{S_E}$ assuming that; N (the sample size) $\geq m$ (the dimension) and the sample is independent, i.e. the population has an absolutely continuous distribution. However, recall that if $\mathbf{S_E}^{-1/2}\mathbf{S_H}\mathbf{S_E}^{-1/2} \geq \mathbf{0}$ of rank $r \leq m$, then $\mathbf{S_E}^{-1/2}\mathbf{S_H}\mathbf{S_E}^{-1/2}$ has an eigenvalue $\lambda = 0$ with multiplicity $m - r$.

In the present work, we will not assume such conditions and then we will study the test criteria for a general multivariate linear model. Explicitly, we will consider multiplicity in the eigenvalues of the matrix $\mathbf{S_E}^{-1}\mathbf{S_H}$. In such case, we propose the distribution of the non null distinct eigenvalues of the matrices $\mathbf{S_E}^{-1}\mathbf{S_H}$ or $(\mathbf{S_H} + \mathbf{S_E})^{-1}\mathbf{S_H}$, see Section 2. Finally, under certain conditions, we provide a modified list of the classical test criteria involving the multiplicity of the respective eigenvalues, see **?**.

2 Multiplicity in MANOVA

Let $\delta_1, \ldots, \delta_m$ and $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of the matrices $\mathbf{S_E}^{-1}\mathbf{S_H}$ and $(\mathbf{S_H} + \mathbf{S_E})^{-1}\mathbf{S_H}$, respectively; where $\mathbf{S_H} : m \times m$ is Wishart distributed with ν_H degrees of freedom, $\mathbf{S_H} \sim \mathcal{W}_m(\nu_H, I_m)$ and $\mathbf{S_E} \sim \mathcal{W}_m(\nu_E, I_m)$. Various authors have proposed a number of different criteria for testing the multivariate general linear hypothesis, see Kres (1983)and Anderson (1984). Then all of the test statistics may be represented as functions of the $s = \min(m, \nu_H)$ non-zero eigenvalues $\lambda' s$ and/or $\delta' s$, observing that $\lambda_i = \delta_i/(1+\delta_i)$ and $\delta_i = \lambda_i/(1-\lambda_i)$, $i = 1, \dots, s$. Now, suppose that the eigenvalues $\lambda' s$ and $\delta' s$ have multiplicity, then, in particular we get: $\lambda_1, \dots, \lambda_l, \lambda_{l+1}, \dots, \lambda_m$, such that $1 > \lambda_1 > \cdots > \lambda_l > 0$ and $1 \geq \lambda_{l+1} \geq \cdots \geq \lambda_l \geq 0$, this is, $l \leq s \leq m$ denotes the number of non null distinct eigenvalues of the matrix $\mathbf{U} = (\mathbf{S_H}+\mathbf{S_E})^{-1/2}\mathbf{S_H}(\mathbf{S_H}+\mathbf{S_E})^{-1/2}$. Consider the spectral decomposition of $\mathbf{U}$, such that

$$\mathbf{U} = \mathbf{HLH}' = (\mathbf{H}_1\mathbf{H}_2)\begin{pmatrix} \mathbf{L}_1 & 0 \\ 0 & \mathbf{L}_1 \end{pmatrix}\begin{pmatrix} \mathbf{H}_1' \\ \mathbf{H}_2' \end{pmatrix} = \mathbf{H}_1\mathbf{L}_1\mathbf{H}_1' + \mathbf{H}_2\mathbf{L}_2\mathbf{H}_2' = \mathbf{U}_1 + \mathbf{U}_2.$$

We want to find the distribution of $\mathbf{U}_1$ and the distribution of $\mathbf{L}_1$, where $\mathbf{L}_1 = \operatorname{diag}(\lambda_1, \dots, \lambda_l)$, $\mathbf{H}_1 \in \mathcal{V}_{l,m} = \{\mathbf{H}_1 \in \Re^{m \times l} | \mathbf{H}_1'\mathbf{H}_1\}$ (the Stiefel manifold). Also observe that $\mathbf{U}_1 \in \mathcal{S}_{m,l}^+$, so if $l = \nu_H \leq m$, then by Uhlig (1994, Theorem 2)

$$(d\mathbf{U}_1) = 2^{-l}\prod_{i=1}^{l} l_i^{m-l}\prod_{i<j}^{l}(l_i - l_j)(\mathbf{H}_1' d\mathbf{H}_1) \wedge (d\mathbf{L}_1)$$

where

$$(\mathbf{H}_1' d\mathbf{H}_1) = \bigwedge_{i=1}^{m}\bigwedge_{j=i+1}^{m} h_j' dh_i, \quad (d\mathbf{L}_1) = \bigwedge_{i=1}^{l} dl_i;$$

for alternative expressions of $(d\mathbf{U}_1)$ in terms of other factorizations see Díaz-García and González-Farías (2005a) and Díaz-García and González-Farías (2005b). Under this context, the distribution of the non null distinct eigenvalues of $\mathbf{U}$ (the eigenvalues of $\mathbf{U}_1$) is given by Díaz-García and Gutiérrez (1997, Theorem 2). Alternatively, if $\mathbf{F} = \mathbf{S_E}^{-1/2}\mathbf{S_H}\mathbf{S_E}^{-1/2}$, the distribution of the non null distinct eigenvalues of $\mathbf{F}$ is given by Díaz-García and Gutiérrez (1997, Theorem 3).

Case $m = 2$

Consider the case $m = 2$ such that the eigenvalues of the matrices $\mathbf{U}$ and $\mathbf{F}$ have multiplicity, namely, $\lambda_1 = \lambda_2 = \lambda$ and $\delta_1 = \delta_2 = \delta$, then from Díaz-García and Gutiérrez (1997, theorems 2 and 3),

$$f_\lambda(\lambda) = \frac{\Gamma[(\nu_E + 1)/2]}{\Gamma[(\nu_E - 1)/2]}(1 - \lambda)^{(\nu_E - 3)/2}, \qquad 0 < \lambda < 1, \tag{6}$$

and

$$f_\delta(\delta) = \frac{\Gamma[(\nu_E + 1)/2]}{\Gamma[(\nu_E - 1)/2]}(1 + \delta)^{(\nu_E + 1)/2}, \qquad 0 < \delta. \tag{7}$$

Now, recall the following test statistics of the literature, which are expressed in terms of the eigenvalues $\lambda' s$ and $\delta' s$, see Kres (1983),

(1) The likelihood ratio criterion Λ of Wilks,

$$\Lambda = \prod_{i=1}^{s}(1-\lambda_i) = \prod_{i=1}^{s}\frac{1}{(1+\delta_i)}.$$

(2) The trace criterion of Hotelling and Lawley,

$$V = \sum_{i=1}^{s}\frac{\lambda_i}{(1-\lambda_i)} = \sum_{i=1}^{s}\delta_i.$$

(3) The maximal root criterion of Roy,

$$\delta_{\max} = \frac{\lambda_{\max}}{(1-\lambda_{\max})}.$$

(4) The maximal root criterion of Pillai and Roy (Version due to Forster and Rees),

$$\lambda_{\max} = \frac{\delta_{\max}}{(1+\delta_{\max})},$$

(5) The trace criterion of Hotelling-Lawley-Pillai-Nanda-Bartlett,

$$V^{(s)} = \sum_{i=1}^{s}\lambda_i = \sum_{i=1}^{s}\frac{\delta_i}{(1+\delta_i)},$$

(6) Third criterion of Wilks (S-criterion of Olson)

$$S = \prod_{i=1}^{s}\frac{\lambda_i}{(1-\lambda_i)} = \prod_{i=1}^{s}\delta_i.$$

For our particular case ($m = 2$, $\nu_H = l = 1$), the test statistics are given by $\Lambda = (1-\lambda)^2$, $V = 2\delta$, $\delta_{\max} = \delta$, $\lambda_{\max} = \lambda$, $V^{(s)} = 2\lambda$ and $S = \delta^2$, and the associated density functions are respectively,

(1)

$$f_\Lambda(\Lambda) = \frac{\Gamma[(\nu_E+1)/2]}{2\Gamma[(\nu_E 11)/2]}\Lambda^{(\nu_E-5)/2}, \qquad 0 < \Lambda < 1,$$

(2)

$$f_V(V) = \frac{\Gamma[(\nu_E+1)/2]}{2\Gamma[(\nu_E-1)/2]}(1+V/2)^{-(\nu_E+1)/2}, \qquad 0 < V,$$

(3)

$$f_{\delta_{\max}}(\delta_{\max}) = \frac{\Gamma[(\nu_E+1)/2]}{\Gamma[(\nu_E-1)/2]}(1+\delta_{\max})^{-(\nu_E+1)/2}, \qquad 0 < \delta_{\max},$$

(4)

$$f_{\lambda_{\max}}(\lambda_{\max}) = \frac{\Gamma[(\nu_E+1)/2]}{\Gamma[(\nu_E 11)/2]}(1-\lambda_{\max})^{(\nu_E\text{-}3)/2}, \qquad 0 < \lambda_{\max} < 1,$$

(5)

$$f_{V^{(s)}}(V^{(s)}) = \frac{\Gamma[(\nu_E+1)/2]}{2\Gamma[(\nu_E-1)/2]}\left(1-V^{(s)}/2\right)^{(\nu_E-3)/2}, \qquad 0 < V^{(s)} < 2,$$

(6)

$$f_S(S) = \frac{\Gamma[(\nu_E+1)/2]}{2\Gamma[(\nu_E-1)/2]\sqrt{S}}\left(1+\sqrt{S}\right)^{-(\nu_E+1)/2}, \qquad 0 < S.$$

The following six tables resume results on the six mentioned criteria: the first two columns show the critical values of the corresponding criterion for $\alpha = 0.05$ (or $(1-\alpha) = 0.95$) and $\alpha = 0.01$ (or $(1-\alpha) = 0.99$), when we **do not** consider multiplicity in the eigenvalues; in contrast, the third and fourth columns present the critical values for $\alpha = 0.05$ and $\alpha = 0.01$, when we **do** consider multiplicity in the eigenvalues; and finally, the fifth and sixth columns show the p-values for which the null hypothesis could be rejected or accepted if the decision is taken in function of the critical values $\alpha = 0.05$ and $\alpha = 0.01$ computed without multiplicity of the eigenvalues, i.e. we use the criteria distributions involving multiplicity for computing the p-values associated to the critical values without multiplicity.

Table 1. Comparisons for the criterion Λ of Wilks

	Critical value [a] (non multiplicity)		Critical value (multiplicity)		p-value	
ν_E	0.05	0.01	0.05	0.01	0.05	0.01
2	6.41E-4	2.5E-5	6.25e-6	1.00E-8	0.150	0.070
5	0.117368	0.049316	0.05000	0.01000	0.117	0.049
10	0.367038	0.245660	0.264098	0.129155	0.105	0.042
20	0.614483	0.505819	0.522230	0.379269	0.099	0.039
30	0.724899	0.637459	0.661527	0.529832	0.097	0.038
40	0.786433	0.714476	0.735463	0.623551	0.096	0.037
60	0.852599	0.799984	0.816196	0.731824	0.095	0.037
80	0.887496	0.846188	0.859261	0.792016	0.094	0.036
100	0.909051	0.875081	0.885999	0.880218	0.094	0.036
440	0.978644	0.970243	0.973073	0.958908	0.094	0.036
1000	0.990552	0.986804	0.988077	0.981730	0.093	0.036

[a] From Table 1 in Kres (1983)

Table 2. Table of comparisons for the maximal root criterion of Roy

	Critical value[a] (non multiplicity)		Critical value (multiplicity)		(1 - p)-value	
ν_E	0.95	0.99	0.95	0.99	0.95	0.99
13	12.23	20.36	3.885	6.926	0.99	0.999
21	10.78	16.90	3.492	5.840	0.99	0.999
26	10.24	15.64	3.385	5.688	0.99	0.999
31	9.95	14.98	3.315	5.390	0.99	0.999
41	9.59	14.20	3.231	5.178	0.99	0.999
51	9.39	13.77	3.182	5.056	0.99	0.999
61	9.27	13.50	3.150	4.977	0.99	0.999
81	9.12	13.17	3.110	4.880	0.99	0.999
101	9.04	13.01	3.087	4.823	0.99	0.999
301	8.79	12.49	3.025	4.676	0.99	0.999
1001	8.71	12.32	3.000	4.628	0.99	0.999

[a] From Table 3 in Kres (1983)

Table 3. Comparisons for maximum root criterion of Pillai and Roy (Version of Foster and Rees)

	Critical value[a] (non multiplicity)		Critical value (multiplicity)		(1 - p)-value	
ν_E	0.95	0.99	0.95	0.99	0.95	0.99
5	0.8577	0.9377	0.7763	0.9000	0.9797	0.9961
15	0.4475	0.5687	0.3481	0.4820	0.9843	0.9972
21	0.3427	0.4479	0.2588	0.3690	0.9849	0.9973
25	0.2960	0.3915	0.2209	0.3187	0.9851	0.9974
31	0.2457	0.3290	0.1810	0.2643	0.9854	0.9974
35	0.2206	0.2972	0.1615	0.2373	0.9855	0.9975
41	0.1912	0.2594	0.1391	0.2056	0.9856	0.9975
61	0.1324	0.1821	0.0950	0.1423	0.9858	0.9976
81	0.1013	0.1402	0.0721	0.1087	0.9860	0.9976
101	0.0820	0.1140	0.0581	0.0879	0.9861	0.9976
161	0.0521	0.0730	0.0367	0.0559	0.9861	0.9977

[a] From Table 5 in Kres (1983) and Anderson (1984, Table 4)

Table 4. Comparisons for trace criterion
of Hotelling and Lawley

	Critical value[a] (non multiplicity)		Critical value (multiplicity)		(1 - p)-value	
ν_E	0.95	0.99	0.95	0.99	0.95	0.99
2	985.9	24670	798	19998	0.955	0.991
5	6.2550	15.318	6.9443	18.0000	0.941	0.986
10	1.5818	2.7402	1.8919	3.5651	0.927	0.979
20	0.6019	0.9236	0.7414	1.2475	0.918	0.973
30	0.3693	0.5479	0.4589	0.7475	0.914	0.970
40	0.2661	0.3886	0.3321	0.5327	0.912	0.968
60	0.1706	0.2454	0.2137	0.3379	0.910	0.967
80	0.1255	0.1792	0.1576	0.2473	0.909	0.966
100	0.0993	0.1412	0.1248	0.1499	0.909	0.965
200	0.0485	0.0684	0.0611	0.0947	0.908	0.965

[a] From Table 6 in Kres (1983) and Anderson (1984, Table 2)

Table 5. Comparisons for trace criterion
of Hotelling-Lawley-Pillai-Nanda and Bartlett

	Critical value[a] (non multiplicity)		Critical value (multiplicity)		(1 - p)-value	
ν_E	0.95	0.99	0.95	0.99	0.95	0.99
13	0.5666	0.7212	0.7860	1.0710	0.864	0.931
15	0.5070	0.6516	0.6963	0.9641	0.870	0.936
23	0.3562	0.4694	0.4768	0.6841	0.884	0.947
33	0.2593	0.3474	0.3415	0.5002	0.891	0.952
43	0.2038	0.2756	0.2659	0.3938	0.895	0.955
63	0.1426	0.1948	0.1842	0.2761	0.899	0.985
83	0.1096	0.1507	0.1409	0.2125	0.900	0.964
123	0.0750	0.1036	0.0958	0.1454	0.902	0.964
243	0.0384	0.0535	0.0489	0.0747	0.904	0.962

[a] From Table 7 in Kres (1983) and Anderson (1984, Table 3)

Table 6. Comparisons for third criterion of Wilks
(Criterion of Olson)

	Critical value [a] (non multiplicity)		Critical value (multiplicity)		(1 - p)-value	
ν_E	0.95	0.99	0.95	0.99	0.95	0.99
2	361.00	9801.0	159201	1.0E8	0.804	0.900
5	1.2426	13.2611	12.0557	81.0000	0.776	0.953
10	0.1559	0.4463	0.8947	3.1775	0.776	0.899
20	0.0291	0.0752	0.1374	0.3891	0.776	0.899
30	0.0118	0.0296	0.0526	0.1397	0.775	0.899
40	0.0063	0.0157	0.0275	0.0709	0.774	0.899
60	0.0027	0.0065	0.0114	0.0285	0.775	0.898
80	0.0014	0.0036	0.0062	0.0153	0.765	0.899
100	0.0009	0.0022	0.0039	0.0095	0.768	0.896
440	5.0E-5	0.0001	0.0002	0.0004	0.787	0.898
1000	1.0E-5	2.1E-5	3.6E-5	8.0E-5	0.793	0.898

[a] From Tables 6 in Díaz-García and Caro-Lopera (2007)

For example with the criterion of Table 6, we conclude that a rejected (non multiplicity) null hypothesis with a significance level of $\alpha = 0.05$, really reaches an $\alpha \geq 0.2$ when we consider multiplicity in the eigenvalues. Similarly, for a rejected (non multiplicity) null hypothesis with $\alpha = 0.01$, we really obtain $\alpha \geq 0.1$ if we consider multiplicity in the eigenvalues . Analogous conclusions can be provided from Tables 1-5 for the remaining criteria.

Case $m = 3$

Now, consider $m = 3$, $\nu_H = 2$, namely, the matrices $\mathbf{U}$ and $\mathbf{F}$ have rank 2. Also, assume that $l = 1$, i.e. the non null eigenvalues of $\mathbf{U}$ ($\mathbf{F}$) are equal, $\lambda_1 = \lambda_2 = \lambda$ and $\delta_1 = \delta_2 = \delta$. In particular, we will study in this section the behavior of the criterion Λ of Wilks. Then, by Díaz-García and Gutiérrez (1997), we obtain:

(1)

$$f_\lambda(\lambda) = \frac{2\Gamma[(\nu_E+1)/2]}{\sqrt{\pi}\Gamma[(\nu_E-2)/2]}\lambda^{1/2}(1-\lambda)^{(\nu_E-4)/2}$$

(2)

$$f_\delta(\delta) = \frac{2\Gamma[(\nu_E+1)/2]}{\sqrt{\pi}\Gamma[(\nu_E-2)/2]}\delta^{1/2}(1+\delta)^{-(\nu_E+1)/2}$$

Similar results can be derived for the joint distribution of λ_1, λ_2 and δ_1, δ_2 by using Díaz-García and Gutiérrez (1997). However, these are not necessary if use the statements by Díaz-García and Gutiérrez (2006) for the coincidence of the non null eigenvalue distribution, via singular distributions (Díaz-García and Gutiérrez, 1997), and the respective non singular distribution, see for example (Muirhead, 1982, Section 10.4, Case 2, pp.451-455). Then the critical values of the cited criteria can be computed from the existing tables ($\nu_H < m$) by making the parameter transformation $(m, \nu_H, \nu_E) \rightarrow (\nu_H, m, \nu_E + \nu_H - m)$, see Muirhead (1982, p. 455). Observe that for the criterion Λ of Wilks we do not need to perform that transformation, because the critical values coincide under both parameter definitions, see Anderson (1984, Theorem 8.4.2, p. 302).

Next we tabulate a comparison between the non multiplicity and multiplicity critical values, and we also provide the p-values for a sort of ν_E.

Table 7. Comparisons for the criterion Λ of Wilks

	Critical value [a] (non multiplicity)		Critical value (multiplicity)		p-value	
ν_E	0.05	0.01	0.05	0.01	0.05	0.01
2	0.000000	0.000000	0.00000	0.000000	0.000	0.000
5	0.243139	0.011210	0.009468	0.001078	0.131	0.056
10	0.514622	0.150746	0.156870	0.067583	0.113	0.046
20	0.647501	0.411734	0.429062	0.292612	0.105	0.042
30	0.723938	0.559656	0.577664	0.450770	0.102	0.040
40	0.807778	0.649620	0.666239	0.554541	0.101	0.040
60	0.852653	0.751990	0.765511	0.678456	0.100	0.039
80	0.880557	0.808282	0.819453	0.748957	0.099	0.039
100	0.971785	0.843804	0.853266	0.794241	0.099	0.039
440	0.978644	0.962428	0.964985	0.949572	0.098	0.038
1000	0.987475	0.983312	0.984469	0.977532	0.0938	0.038

[a] From Table 1 in Kres (1983)

From Table 7 we see that for a rejected (non multiplicity) null hypothesis with a significance level of 0.05 (0.01), we need a significance level of $\alpha \geq 0.09$ ($\alpha \geq 0.03$) for rejecting the same hypothesis if we consider multiplicity in the eigenvalues.

3 Conclusions

We highlight the variation of the criterion distributions, for testing hypothesis in a general linear model, when multiplicity of the eigenvalues is considered. The change is high in the sense that for rejecting a null hypothesis, in general, the significance level α increases. A practical way for handling the inclusion of multiplicity proposes the following modifications of the usual test statistics:

- Consider only the non null distinct eigenvalues in the computation of the different test statistics, namely, take l instead of ν_H; and compare those values with the tabulated critical values, but make the parameter transformation:

$$(m, \nu_H, \nu_E) \rightarrow (m, l, \nu_E), \quad 1 \leq l \leq \nu_H \leq m,$$

where l is the number of non null distinct eigenvalues.

Finally, note that the present work considers only the case when $\nu_H \leq m$, otherwise the procedure for finding the distribution of the non null distinct eigenvalues of the matrices $\mathbf{U}$ and $\mathbf{F}$ remains as an open problem.

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