EXISTENCE OF GLOBAL POSITIVE SOLUTIONS FOR AN INHOMOGENEOUS REACTION-DIFFUSION EQUATION

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# Existence of Global Positive Solutions for an Inhomogeneous Reaction-Diffusion Equation

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**Abstract**

We investigate existence of non-negative global solutions to the semilinear equation $\frac{\partial u}{\partial t} = \Delta_\alpha u + a(x)\sum_{k=2}^{\infty} p_k u^k + (p_0 + p_1 u)\phi(x)$, $u(x,0) = f(x)$, $x \in \mathbb{R}^d$, where $\alpha \in (0,2]$, $\phi, a > 0$, $\phi \in C^\alpha(\mathbb{R}^d)$, $a$ and $\phi$ grow no faster than polynomially, $p > 1$, $p_0 > 0$ and $f \geq 0$.

## 1 Introduction and main results

In this paper we investigate existence of non-negative global solutions to semilinear equations of the type

$$\frac{\partial u}{\partial t} = \Delta_\alpha u + a(x)\sum_{k=2}^{\infty} p_k u^k + (p_0 + p_1 u)\phi(x), \quad u(x,0) = f \quad x \in \mathbb{R}^d, \tag{1}$$

where $\Delta_\alpha$ is the fractional power $-(-\Delta)^{\alpha/2}$ of the Laplacian, $\alpha \in (0,2]$, $a$, $\phi$ and $f$ are certain nice functions, and $p_k$, $k = 0, 1, \ldots$, are non-negative constants such that $\sum_k p_k = 1$, with $p_0 > 0$.

The asymptotic properties of equations of the form (1) are related to the large deviations behavior of super-Brownian motion (in the case $\alpha = 2$) and of other measure-valued processes, see [3] and the references therein. Moreover, the cumulant generating function of the occupation measures of a large family of superprocesses are governed by equations of the form (1), see e.g. [2] and [4]. Pinsky [5] and Zhang [6] studied both existence and non-existence of positive global solutions to the above equation when the sum in the reaction part consists of a single term $a(x)u^p$, where $p$ is greater than 1, $p_1 = 0$ and $p_0$ is a non-negative function of $x$.

In the present note we investigate existence of global solutions in a different setup, with the assumption that $p_0$ does not depend on $x$. We deal firstly the case in which the sum in (1) consists of a single term of the form $(a(x)u^p$, with $p > 1$. Our result is the contents of the following theorem.

**Theorem 1** *Consider the semilinear initial value problem*

$$\frac{\partial u}{\partial t} = \Delta_\alpha u + a(x)u^p + (p_0 + p_1 u)\phi(x) \quad u(x,0) = f \tag{2}$$

*where* $0 < a, \phi \in C^r(R^d)$, *$a$ and $\phi$ grow no faster than polynomially,* $p > 1$, $p_0$, $p_1 > 0$ *and* $f \geq 0$. *Assume that*

1. $0 < a(x) \leq c_1(1+|x|)^m$ *for* $c_1 > 0$,
2. $0 < \phi(x) \leq c_2(1+|x|)^{-q}$ *for* $c_2 > 0$, $q \in (\alpha, d]$,
3. $p > 1 + (\alpha+m)^+/(q-\alpha)$.

*Then for all sufficiently small* $p_0, p_1 > 0$ *and all sufficiently small* $f \geq 0$, *the mild solution to Eq. (2) exists globally in time. More specifically, for sufficiently small* $p_0$, $p_1 > 0$ *and each* $\epsilon > 0$, *there exists a constant* $c > 0$ *such that, if*

$$0 \leq f(x) \leq c(1+|x|)^{-(\alpha+m)^+/(p-1)-\epsilon},$$

*then the mild solution to (2) exists for all time.*

Our second theorem addresses existence of positive solutions to Equation (1).

**Theorem 2** *Consider the equation*

$$\partial u/\partial t = \Delta_\alpha u + a(x)\sum_{k=2}^{\infty} p_k u^k + (p_0 + p_1 u)\phi(x) \quad u(x,0) = f \quad x \in \mathbb{R}^d, \tag{3}$$

*where* $0 < a, \phi \in C^r(R^d)$, *a and* $\phi$ *grow no faster than polynomially,* $1 > p_0,\ p_1 > 0$, $\sum_{k=2}^{\infty} p_k = 1 - p_0 - p_1$ *and* $f \geq 0$. *Assume that*

1. $0 < a(x) \leq c_1(1 + |x|)^m$ *for* $c_1 > 0$,
2. $0 < \phi(x) \leq c_2(1 + |x|)^{-q}$ *for* $c_2 > 0$, $q \in (\alpha, d]$,
3. $q - \alpha > (\alpha + m)^+$.

*Then for sufficiently small* $p_0, p_1 > 0$ *and sufficiently small* $f \geq 0$, *the solution to the equation (3) is global. More specifically, for sufficiently small* $p_0,\ p_1 > 0$, *and each* $\epsilon > 0$, *there exists a constant* $c > 0$ *such that if* $0 \leq f(x) \leq c(1 + |x|)^{-(\alpha+m)^+-\epsilon}$, *then the solution to (3) is global in time.*

# 2 Proofs

Our method of proof is an adaptation of a technique used in [3]. We use the following lemma (which is proved in the final section of the present paper).

**Lemma 3** *There exists a* $C_* > 0$ *and for all* $\delta > 0$, *a* $C_\delta > 0$ *such that,*

$$C_*(1 + |x|)^{\alpha-q} \leq \int_{R^d} \frac{(1 + |y|)^{-q}}{|x - y|^{d-\alpha}}\, dy \leq C_\delta(1 + |x|)^{\alpha-q+\delta}, \quad q \in (\alpha, d]. \tag{4}$$

We define

$$v(x, t) \equiv v(x) = \mu \int_{R^d} \frac{(1 + |y|)^{-q}}{|x - y|^{d-\alpha}}\, dy, \quad \mu > 0. \tag{5}$$

## 2.1 Proof of Theorem 1

Our assumptions imply that

$$a(x) \leq c_1(1 + |x|)^m, \qquad \phi(x) \leq c_2(1 + |x|)^{-q}$$

for some positive constants $c_1, c_2 > 0$. Noticing [1] that the Green's function $G$ of the generator $-\Delta_\alpha$ satisfies, $G(x, y) = \gamma_{\alpha,d}/|x - y|^{d-\alpha}$ for an appropriate constant $\gamma_{\alpha,d} > 0$, it follows from the right-hand inequality in (4), and $p > 1 + (\alpha+m)^+/(q-\alpha)$, that there exists a $\delta > 0$ such that

$$\Delta_\alpha v - v_t + a(x)v^p + (p_0 + p_1 v)\phi(x) \leq \left( \frac{-\mu}{\gamma_{\alpha,d}} + c_1 C_\delta^p \mu^p + (p_{0+} p_1 C_\delta \mu) c_2 \right) (1 + |x|)^{-q}. \tag{6}$$

It is easy to show that

$$\Delta_\alpha v \leq -\mu/\gamma_{\alpha,d}(1+|x|)^{-q} \quad \text{and} \quad p_0\phi(x) \leq p_0 c_2 (1+|x|)^{-q}.$$

Moreover,

$$a(x)v^p \leq c_1 C_\delta^p \mu^p (1+|x|)^{p(\alpha-q+\delta)+m} \leq c_1 C_\partial^p \mu^p (1+|x|)^{-q}$$

because

$$\begin{aligned} p(\alpha-q+\delta)+m &= p(\alpha-q)+p\delta+m \\ &\leq (1+(\alpha+m)^+/(q-\alpha))(\alpha-q)+p\delta+m \\ &\leq (1+(\alpha+m)^+/(q-\alpha)+\delta')(\alpha-q)+p\delta+m \qquad (7) \end{aligned}$$

for some $\delta' \geq 0$. This is possible because $p$ is strictly greater than $1+(\alpha+m)^+/(q-\alpha)$.

We consider two cases.

Case 1: $(\alpha+m)^+ = 0$ and $\alpha+m<0$. In this case we choose $\delta'=0$ and the right hand side in the last inequality becomes $(\alpha-q)+p\delta+m$. Now we can choose $\delta$ so small as to make $\alpha+p\delta+m$ negative and, so, $(\alpha-q)+p\delta+m \leq -q$.

Case 2: $(\alpha+m)^+ \geq 0$ and $\alpha+m \geq 0$. The RHS in (7) becomes

$$\begin{aligned} &\alpha-q-(\alpha+m)^+ + \delta'(\alpha-q)+p\delta+m \\ &\qquad = \alpha-q-(\alpha+m)+\delta'(\alpha-q)+p\delta+m \\ &\qquad = -q+\delta'(\alpha-q)+p\delta. \end{aligned}$$

Now we can choose $\delta$ so small that $\delta'(\alpha-q)+p\delta$ is negative. Then the whole thing is less than or equal to $-q$. Thus, Case 2 is also done. In addition,

$$p_1 v\phi(x) \leq p_1 c_2 C_\partial \mu (1+|x|)^{\alpha-q+\partial-q} \leq p_1 c_2 C_\partial \mu (1+|x|)^{-q},$$

this being so because $\alpha-q+\partial$ is negative. In this way (6) is justified.

Now, by first choosing $\mu$ sufficiently small so that $C_\delta^p \mu^p \leq 1/2 \cdot \mu/\gamma_{\alpha,d}$, and then choosing $p_0, p_1 > 0$ sufficiently small, the RHS of (6) can be made negative. Using the maximum principle together with the first inequality in (4), it follows that for all sufficiently small $c, p_0, p_1 > 0$, the solution $u(x,t)$ to (1) with $f(x) \leq c(1+|x|)^{\alpha-q}$ satisfies $u(x,t) \leq v(x)$, and is thus a global solution.

Our next goal is to increase the exponent $\alpha - q$ appearing in the bound for $f$, up to the exponent $-(\alpha+m)^+/(p-1)-\epsilon$ figuring in the statement of the theorem.

We argue as follows. By assumption $p > 1 + (\alpha+m)^+/(q-\alpha)$. This implies

$$p - 1 > (\alpha+m)^+/(q-\alpha),$$

which in turn yields

$$\alpha - q < -(\alpha+m)^+/(p-1).$$

We can now choose $q \equiv q_0$ so that the last inequality becomes an equality. Let $q_1 \epsilon (q_0, q)$. Then $q_1$ satisfies the above inequality, and from the above results it follows that, if

$$\phi(x) \le c_2(1+|x|)^{-q_1}$$

for some $c_1 > 0$, then for sufficiently small $c, p_0, p_1 > 0$, the solution to (1) with $f(x) \le c(1+|x|)^{\alpha-q_1}$ is global. But then, by the maximum principle, the same holds true if we decrease $\phi$ so that it satisfies the bound in the statement of the theorem, namely $\phi(x) \le c_2(1+|x|)^{-q}$.

Subsuming, we have proved that (1) possesses a global solution provided that $c, p_0, p_1 > 0$ are sufficiently small, and $f(x) \le c(1+|x|)^{\alpha-q_1}$ for all $q_1 \in (q_0, q)$. Since $q_1$ can be chosen arbitrarily close to $q_0$, it follows that a global solution exists if

$$f(x) \le c(1+|x|)^{-(\alpha+m)^+/(p-1)-\epsilon}$$

for some $\epsilon > 0$ and $c > 0$ sufficiently small. This completes the proof of global existence.

Notice that, when $a(x) \equiv 1$, the first condition for global solution disappears and the exponent in the bound of $f$ we obtain$\alpha$ instead of $(\alpha+m)^+$. Also, it can easily be seen that when $p_1 = 0$ we can apply just the same proof.

## 2.2 Proof of Theorem 2

Our approach here is an adaptation of the technique used in the proof of Theorem 1. We use again Inequality (4). Let $v$ be given by (5). There exists $\delta > 0$ such that

$$\Delta_\alpha v - v_t + a(x)\sum_{k=2}^{\infty} p_k v^k + (p_0 + p_1 v)\phi(x)$$

$$\leq \left(\frac{-\mu}{\gamma_{\alpha,d}} + c_1 \sum_{k=2}^{\infty} p_k C_\delta^k \mu^k + (p_0 + p_1 v)c_2\right)(1+|x|)^{-q}. \quad (8)$$

Indeed, as before, $\Delta_\alpha v \leq -\mu/\gamma_{\alpha,d}(1+|x|)^{-q}$ and $p_0\phi(x) \leq p_0 c_2(1+|x|)^{-q}$. Moreover,

$$a(x)\sum_{k=2}^{\infty} p_k v^k \leq c_1 \sum_{k=2}^{\infty} p_k C_\delta^k \mu^k (1+|x|)^{m+k(\alpha-q+\delta)}$$

for all $\delta > 0$. Since the inequality

$$m + k(\alpha - q + \delta) \leq m + 2(\alpha - q + \delta)$$

is valid for all $k$, it suffices to show that $m + 2(\alpha - q + \delta) \leq -q$. Again two cases arise: If $\alpha + m < 0$, then we choose $\delta > 0$ so small that $m + \alpha + \delta < 0$, and thus, $m + 2(\alpha - q + \delta) \leq -q$. In case of $\alpha + m \geq 0$, since $q - \alpha > (\alpha + m)^+ = \alpha + m$, we can choose $\delta > 0$ so small that $m + \alpha + (\alpha - q) + \delta$ is negative. Hence, $m + 2(\alpha - q + \delta) \leq -q$. In addition, because of $\alpha - q + \delta < 0$,

$$p_1 v\phi(x) \leq p_1 c_2 C_\delta \mu (1+|x|)^{\alpha-q+\delta-q} \leq p_1 c_2 C_\delta \mu (1+|x|)^{-q}.$$

Thus (8) is satisfied. Now, by first choosing $\mu$ so small so $C_\delta^p \mu^p \leq 1/2 \cdot \mu/\gamma_{\alpha,d}$ and then choosing $p_0, p_1 > 0$ sufficiently small, the RHS of (8) can be made negative. Hence, using the maximum principle together with the first inequality in (4), it follows that for sufficiently small $c, p_0, p_1 > 0$ the solution $u(x,t)$ to (3) with $f(x) \leq c(1+|x|)^{\alpha-q}$ satisfies $u(x,t) \leq v(x)$, and is thus a global solution.

In order to increase the exponent $\alpha - q$ appearing in the bound for $f$ to the exponent $-(\alpha + m)^+ - \epsilon$ appearing in the statement of the theorem, we argue as follows.

By assumption $q - \alpha > (\alpha + m)^+$ and so $\alpha - q < (\alpha + m)^+$. Let $q_0$ be such that the above inequality becomes an equality if we replace $q$ by $q_0$. Let $q_1 \in (q_0, q)$.

Then $q_1$ satisfies the above inequality. So, by the above results, it follows that if $\phi(x) \leq c_2(1+|x|)^{-q_1}$ for some $c_1 > 0$, then for sufficiently small $p_0, p_1 > 0$ and sufficiently small $c > 0$, the solution to (3) with $f(x) \leq c(1+|x|)^{\alpha-q_1}$ is global. But then, by the maximum principle, the same conclusion holds true if we decrease $\phi$ so that it satisfies the original bound, that is, $\phi(x) \leq c_2(1+|x|)^{-q}$. In this way we have proved that (3) possesses a global solution provided that $p_0, p_1 > 0$ and $c > 0$ are sufficiently small, and $f(x) \leq c(1+|x|)^{\alpha-q_1}$ for all $q_1 \in (q_0, q)$. Since $q_1$ can be chosen arbitrarily close to $q_0$, it follows that the solution to (3) is global if $f(x) \leq c(1+|x|)^{-(\alpha+m)^+-\epsilon}$ for some $\epsilon > 0$ and $c > 0$ small.

When $a(x) \equiv 1$, the first condition for existence of global solution disappears, and in the bound of $f$, we get $\alpha$ instead of $(\alpha+m)^+$. Finally, it is easy to see that the same argument also works when $p_1 = 0$.

# 3 Remaining proofs

In this section we prove the following statement:

There exists a $c_1 > 0$ and, for all $\delta > 0$, a $C_\delta > 0$ such that

$$c_1(1+|x|)^{\alpha-q} \leq \int_{R^d} (1+|y|)^{-q}/|x-y|^{d-\alpha} dy \leq C_\delta (1+|x|)^{\alpha-q+\delta}, \quad q \in (\alpha, d].$$

In order to prove the second inequality we need the following lemma.

**Lemma 4** *The integral*

$$\int_{R^d} \frac{(1+|x|)^{q-\alpha-\partial}}{|x-y|^{d-\alpha}(1+|y|)^q}\, dy \tag{9}$$

*is a bounded function of $x$ for any $\delta \in (0, q-\alpha)$.*

**Proof.** Let $x \in \mathbb{R}^d$ be fixed. We split (9) into three integrals over the Borel sets

$$\begin{aligned} D_1 &= \{y \in R^d : |x-y| < |x|/2\} \\ D_2 &= \{y \in R^d : |x-y| > |x|/2, |x-y| < |y|/2\} \\ D_3 &= \{y \in R^d : |x-y| > |x|/2, |x-y| > |y|/2\} \end{aligned}$$

and proceed in two steps as follows.

**Step 1:** For the integral over $D_1$ we have,

$$\int_{D_1} \frac{(1+|x|)^{q-\alpha-\delta}}{|x-y|^{d-\alpha}(1+|y|)^q}\,dy \le \text{Const.}(1+|x|)^{-\alpha-\delta}\int_{D_1}|x-y|^{d-\alpha}\,dy. \qquad (10)$$

This is so because $|x-y|<|x|/2$ and $|x|-|y|<|x|/2$ imply $|y|>|x|/2$, and therefore

$$(1+|x|)^q/(1+|y|)^q \le (1+|x|)^q/(1+\frac{|x|}{2})^q \le \text{Const.}$$

At this stage we pass to polar co-ordinates

$$(y_1-x_1, y_2-x_2,\ldots,y_d-x_d) \longrightarrow (r\cos\theta_1, r\sin\theta_1\cos\theta_2,\ldots,r\sin\theta_1\sin\theta_2 ...\sin\theta_d),$$

the Jacobian of the transformation being equal to $r^{d-1}\sin^{d-2}\theta_1\cdots\sin\theta_{d-2}$. After the change of variables the RHS of (10) becomes

$$\begin{aligned}\text{Const.}(1+|x|)^{-\alpha-\delta}\int_0^{|x|/2}\int_{\theta_1}\cdots\int_{\theta_{d-1}} & r\sin^{d-2}\theta_1\cdots\sin\theta_{d-2}\,d\theta_{d-2}\cdots d\theta_1\,dr\\ &\le \text{Const.}(1+|x|)^{-\alpha-\delta}\cdot\int_0^{|x|/2} r dr\\ &\le \text{Const.}(1+|x|)^{-\alpha-\delta}\cdot|x|^2\\ &\le \text{Const.},\end{aligned}$$

which shows that the integral over $D_1$ is bounded in $x$.

**Step 2:** In order to bound from above the integral over $D_2$, we first assume that $|x|<1$. In this case $(1+|x|)^{q-\alpha-\delta}\le$ Const., and

$$\begin{aligned}\int_{D_2}\frac{(1+|x|)^{q-\alpha-\delta}}{|x-y|^{d-\alpha}(1+|y|)^q}\,dy &\le \text{Const.}\int_{D_2}\frac{dy}{|x-y|^{d-\alpha}(1+|y|)^q}\\ &\le \text{Const.}\int_{D_2}\frac{dy}{|x-y|^{d-\alpha}}\\ &\le \text{Const.}\int_{\{y\,:\,|x|/2<|x-y|<|x|\}}\frac{dy}{|x-y|^{d-\alpha}}\end{aligned}$$

because $|x-y|<|y|/2$ and $|y|-|x|<|y|/2$, and therefore $|x|>|y|/2$, which implies $|x-y|<|y|/2<|x|$ and thus $|x|/2<|x-y|<|x|$. Passing to polar coordinates renders

$$\begin{aligned}
\int_{D_2} \frac{(1+|x|)^{q-\alpha-\delta}}{|x-y|^{d-\alpha}(1+|y|)^q}\,dy &\leq \text{Const.} \int_{|x|/2}^{|x|} \int_{\theta_1} \ldots \int_{\theta_d} r \sin^{d-2}\theta_1 \ldots \sin\theta_{d-2} d\theta_{d-2} \ldots d\theta_1 dr \\
&\leq \text{Const.} \int_{|x|/2}^{|x|} r\,dr \\
&\leq \text{Const.}
\end{aligned}$$

Hence, the integral over $D_2$ is bounded in $x$ if $|x| \leq 1$.

Assume now that $|x| \geq 1$. Then we have

$$\int_{D_2} \frac{(1+|x|)^{q-\alpha-\delta}}{|x-y|^{d-\alpha}(1+|y|)^q}\,dy \leq \text{Const.} \int_{D_2} \frac{(1+|x|)^{-\alpha-\delta}}{|x-y|^{d-\alpha}}\,dy. \tag{11}$$

This is so because $|x-y| < |y|/2$ together with the triangle inequality imply $|y| > 2/3|x|$, hence $(1+|x|)^q/(1+|y|)^q) \leq \text{Const.}$ Notice also that $|x-y| < |x|/2$ implies

$$|x-y| + 1/2 < 1/2 + |x|/2 = (1+|x|)/2,$$

which in turn gives $1/(1+|x|) < \text{Const.}(1/|x-y|)$. We can now bound from above the RHS of (11) by

$$\text{Const.} \int_{D_2} 1/|x-y|^{d-\alpha+\alpha+\delta} dy = \text{Const.} \int_{D_2} 1/|x-y|^{d+\delta}\,dy.$$

Using polar coordinates as before, we see that the LHS of (11) is bounded from above by

$$\text{Const.} \int_{\{y:\ |x|/2<|x-y|<|x|\}} r dr = \frac{1}{\delta |x|^\delta}.$$

This shows that the integral over $D_2$ is bounded in $x$ also in the case $|x| \geq 1$. We conclude that the integral over $D_2$ is bounded in $x$.

In order to bound the last integral we split $D_3$ into two parts $D_{31}$ and $D_{32}$, where

$$D_{31} = \{y : |x-y| > |x|/2,\ |x-y| > |y|/2,\ |x-y| \leq 1\},$$

$$D_{32} = \{y : |x-y| > |x|/2,\ |x-y| > |y|/2,\ |x-y| > 1\}.$$

We have $|x| \leq 2$ on $D_{31}$ as $|x-y| > |x|/2$ and $|x-y| \leq 1$ imply $|x|/2 \leq 1$. Therefore,

$$\begin{aligned}
\int_{D_{31}} \frac{(1+|x|)^{q-\alpha-\delta}}{|x-y|^{d-\alpha}(1+|y|)^q}\,dy &\leq \text{Const.} \int_{D_{31}} \frac{dy}{|x-y|^{d-\alpha}(1+|y|)^q} \\
&\leq \text{Const.} \int_{D_{31}} |x-y|^{\alpha-d}\,dy \\
&\leq \text{Const.} \int_{\{y:\ |x|/2 \leq |x-y| \leq 1\}} |x-y|^{\alpha-d}\,dy \\
&\leq \text{Const.} \int_{|x|/2}^{1} r\,dr \\
&\leq \text{Const.},
\end{aligned}$$

where we passed to polar coordinates after the third inequality, and used that $0 \leq |x| \leq 2$ to obtain the last inequality.

For the integral over $D_{32}$ we have

$$\int_{D_{32}} \frac{(1+|x|)^{q-\alpha-\delta}}{|x-y|^{d-\alpha}(1+|y|)^q}\,dy \leq \int_{D_{32}} \frac{|x-y|^{q-\alpha-\delta}}{|x-y|^{d-\alpha}(1+|y|)^q}\,dy.$$

This is because $|x-y| > |x|/2$ and $|x-y| > 1$ on $D_{32}$, and so $(1+|x|)/2 \leq \text{Const.}|x-y|$. Since $|x-y| > |y|/2$ and $|x-y| > 1$ yield both $|x-y| > \{1+|y|\}/2$ and $1+|y| < \text{Const.}|x-y|$, the above inequality becomes

$$\begin{aligned}
&\leq \text{Const.} \int_{D_{32}} 1/\{|x-y|^{d-q+\delta}(1+|y|)^q\}\,dy \\
&\leq \text{Const.} \int_{D_{32}} 1/\{(1+|y|)^{d-q+\delta}(1+|y|)^q\}\,dy \\
&\leq \text{Const.} \int_{D_{32}} 1/(1+|y|)^{d+\delta}\,dy,
\end{aligned}$$

which, after passing to polar coordinates renders

$$
\begin{aligned}
&\leq \text{Const.} \int_{R^d} 1/(1+|y|)^{d+\delta}\, dy \\
&\leq \text{Const.} \int_0^\infty r^{d-1}/(1+r)^{d+\delta}\, dr \\
&\leq \text{Const.} \int_0^\infty r^{d-1}/(1+r)^{d+\delta}\, dr \\
&\leq \text{Const.} \int_0^\infty 1/(1+r)^{1+\delta} dr \\
&\leq \text{Const.} \int_1^\infty 1/s^{1+\delta} ds = [-1/\delta s^\delta]_1^\infty \\
&= 1/\delta.
\end{aligned}
$$

Thus, the integral over $D_3$ is also bounded in $x$. Lemma (4) is proved.

## 3.1 Proof of Lemma 3

The RHS inequality in (4) follows directly from Lemma (4).

We now prove the remaining inequality in (4). Notice that

$$
\int_{R^d} (1+|y|)^{-q}/|x-y|^{d-\alpha}\, dy \geq \text{Const.}(1+|x|)^{-q} \int_{D_1} |x-y|^{\alpha-d}\, dy. \tag{12}
$$

This is because $|x-y| < |x|/2$ on $D_1$, which implies $|y| < 3|x|/2$. Hence,

$$
(1+|x|)^q/(1+|y|)^q \geq (1+|x|)^q/(1+3|x|/2)^q \geq \text{Const.},
$$

yielding $(1+|y|)^{-q} \geq \text{Const.}(1+|x|)^{-q}$. We use once again polar coordinates to show that the LHS of (12) equals

$$\begin{aligned}
&\text{Const.}(1+|x|)^{-q}\int_{\{y:\ 0<|x-y|<|x|/2\}}|x-y|^{\alpha-d}dy\\
&\quad=\text{Const.}(1+|x|)^{-q}\int_0^{|x|/2}\int_{\theta_1}...\int_{\theta_{d-1}} r^{d-1}/r^{d-2}\sin^{d-2}\theta_1...\sin\theta_{d-2}d\theta_{d-2}...d\theta_1 dr\\
&\quad=\text{Const.}(1+|x|)^{-q}\int_0^{|x|/2} r.f(\pi,d)dr\\
&\quad=\text{Const.}(1+|x|)^{-q}\int_0^{|x|/2} rdr\\
&\quad=(1+|x|)^{-q}\cdot|x|^2.
\end{aligned}$$

Subsuming, we have proved that

$$\int_{R^d}(1+|y|)^{-q}/|x-y|^{d-\alpha}dy\geq\text{Const.}(1+|x|)^{-q}\cdot|x|^2.$$

From here, the left-hand inequality in (4) follows easily if $|x|\geq 1$. In the case $|x|<1$ one has to verify that $\int_{R^d}(1+|y|)^{-q}/|x-y|^{d-\alpha}dy$ is bounded away from zero. To prove this, notice that, as $|x|<1$,

$$\begin{aligned}
\int_{R^d}(1+|y|)^{-q}/|x-y|^{d-\alpha}dy &\geq \int_{R^d}(1+|y|)^{-q}/(|x|+|y|)^{d-\alpha}dy\\
&\geq \int_{R^d}(1+|y|)^{-q}/(1+|y|)^{d-\alpha}dy\\
&= \int_{R^d}(1/(1+|y|)^{d-\alpha+q}dy
\end{aligned}$$

which is a non-zero constant, and hence is no smaller than $C_*(1+|x|)^{-q}$ for a suitable constant $C_*>0$. This proves Lemma 3.

# References


[1] Bogdan, K. and Byczkowski, T. Potential theory for the $\alpha$-stable Schrödinger operator on bounded Lipschitz domains. *Studia Math.* **133** (1999), no. 1, 53–92.

[2] Iscoe, I. A weighted occupation time for a class of measure-valued branching processes. *Probab. Theory Relat. Fields* **71** (1986), no. 1, 85–116.

[3] Lee T.Y., Some limit theorems for super-Brownian motion and semilinear differential equations, *Annals of Probab.* **21** (1993), 979-995.

[4] Lee T.Y., Asymptotic Results for Super-Brownian Motions and Semilinear Differential Equations. *The Annals of Probability* **29**, No. 3, pp. 1047-1060.

[5] Pinsky, Ross G. Finite time blow-up for the inhomogeneous equation $u_t = \Delta u + a(x)u^p + \lambda\phi$ in $R^d$. *Proc. Amer. Math. Soc.* **127** (1999), no. 11, 3319–3327.

[6] Zhang, Qi S. A new critical phenomenon for semilinear parabolic problems. *J. Math. Anal. Appl.* **219** (1998), no. 1, 125–139.