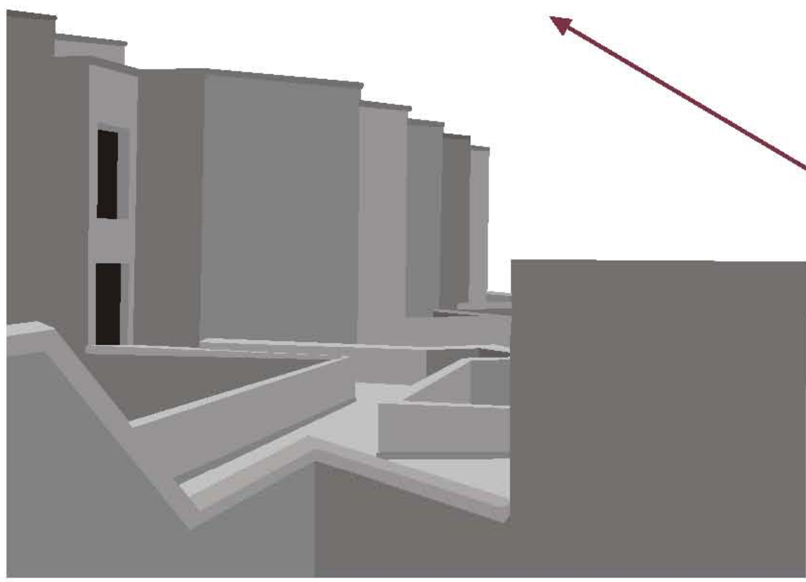

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Fluctuation limit for the occupation time of a two-type particle system

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Abstract

We consider the occupation time of a two-type particle system in the real line, where a particle of type $i \in \{1, 2\}$ moves following a symmetric α_i-stable Lévy process and changes its type at exponentially distributed holding times. We prove that the properly normalized time-rescaled occupation time fluctuations of the system converge to a process involving a fractional Brownian motion with Hurst parameter $H = 1 - 1/2 \max\{\alpha_1, \alpha_2\}$. This extends previous work of [3] to a multitype scenario.

Key words: Multitype particle system, occupation time fluctuations, fractional Brownian motion, weak convergence

Mathematics Subject Classifications: primary 60F17; secondary 60G20, 60G18

1 Background and limit theorem

In some stochastic particle systems in Euclidean spaces involving motions according to symmetric α-stable Lévy processes with two or more values of α, it has been of interest to investigate which one of the values plays a determining role for certain behaviors of the systems. In multitype critical branching particle systems and superprocesses, the persistence (convergence towards a non-trivial equilibrium state), and the asymptotics of solutions of related systems of non-linear partial pseudo-differential equations, the smallest of the α's plays the determining role [5], [7], [10]. In existence of self-intersection local time of a high-density limit of a system of particles with motion mutating between two values of α, the smallest one also has the dominant role [6]. Notice that a smaller value of α represents a larger "mobility" of the motion, since at any fixed time the position of a particle has finite moments of order only strictly smaller than α (if $\alpha < 2$). In this paper we consider a different type of question regarding a two-type particle system. The system consists of particles moving in $\mathbb{R}$ following symmetric α-stable Lévy processes with two different values, $0 < \alpha_1 < \alpha_2 \leq 2$, switching between them at exponentially distributed holding times independently of each other, and starting from a random Poisson configuration. A particle is called of type i when it moves according to $\alpha_i, i = 1, 2$. We study the large time asymptotics of rescaled occupation time fluctuations of the system. This problem has been investigated in [3] for the system with a single α, with and without critical branching, in the case when the limit process has long memory, which is also the phenomenon

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that interests us here. It turns out that it is the largest α, i.e., the less mobile of the motions, that has the dominant role. This is not unexpected by the following heuristic argument. In the single type model a larger α requires a larger norming because the less mobile particles spend more time in any given bounded set, and the corresponding norming is too strong for the occupation time of the more mobile particles (see [3]). However, the analysis of the system with two α's poses new technical difficulties because, whereas in the case of a single α the particle motion is self-similar, which plays a crucial role in the proofs, with two α's there are two different self-similarities that are constantly interwound. We consider a simple situation as a test case, namely, $1 < \alpha_1 < \alpha_2$, and holding times between mutations so that the motion switching between types is stationary; this simplifies the analysis. However, on the basis of the result obtained in this case, it is natural to surmise that analogous results (i.e., the largest α dominates) will hold in more general multitype models for this type of limits, including non-stationary models and models with critical multitype branching.

We suppose that the holding times for mutations of particles of type $i \in \{1, 2\}$ are exponentially distributed with parameter $V_i > 0$. We denote by Δ_{α_i} the infinitesimal generator of the symmetric α_i-stable process in $\mathbb{R}$, $i = 1, 2$. The mean matrix $M = (m_{ij})_{i,j=1,2}$ for this special form of multitype branching is given by $M = (1 - \delta_{ij})_{i,j=1,2}$, where δ_{ij} is Kronecker's delta.

The population of particles is modelled by measures $\mu \in M_f(\mathcal{E})$, where $M_f(\mathcal{E})$ denotes the space of counting measures on the product space $\mathcal{E} = \mathbb{R} \times \{1, 2\}$. The first component of $(x, i) \in \mathcal{E}$ stands for the position and the second one for the type of a particle. We denote by $N = \{N_t, t \geq 0\}$ the empirical measure process of the system, i.e., $N_t(A)$ is the number of particles in the Borel set $A \subset \mathcal{E}$ at time t. We assume that the initial population N_0 is a Poisson random measure on $\mathcal{E}$ such that $\mathbb{E}N(0) = \Lambda := \gamma_1(\lambda \times \delta_1) + \gamma_2(\lambda \times \delta_2)$, where

$$\gamma_1 = \frac{V_2}{V_1 + V_2}, \gamma_2 = \frac{V_1}{V_1 + V_2}, \tag{1.1}$$

and λ stands for Lebesgue measure on $\mathbb{R}$. The choice (1.1) for γ_1 and γ_2 makes the type process stationary [5]. The rescaled occupation time process $L_T = \{L_T(t), t \geq 0\}$ of N is given by

$$L_T(t) = \int_0^{Tt} N(s)\, ds,$$

and the fluctuation process $X_T = \{X_T(t), t \geq 0\}$ of L_T is defined by

$$X_T(t) = \frac{1}{F_T} \int_0^{Tt} (N_s - \mathbb{E}\, N_s)\, ds, \tag{1.2}$$

where F_T is a norming. We use test functions $\Phi(x, i)$ in $\mathcal{S}(\mathcal{E})$ and $\Phi(x, i, t)$ in $\mathcal{S}(\mathcal{E} \times \mathbb{R})$, where $\mathcal{S}(\mathcal{E})$ (respectively, $\mathcal{S}(\mathcal{E} \times \mathbb{R})$) is the space of measurable functions $\Phi : \mathcal{E} \to \mathbb{R}$ such that $x \mapsto \Phi(x, i)$ belongs to $\mathcal{S}(\mathbb{R})$ (respectively, $(x, t) \mapsto \Phi(x, i, t)$ belongs to $\mathcal{S}(\mathbb{R}^2)$), $i = 1, 2$, and $\mathcal{S}(\mathbb{R}^d)$ denotes the space of rapidly decreasing C^∞ functions on $\mathbb{R}^d$. Due to our assumption on Λ and (1.1), $\mathbb{E}N_s = \Lambda$ for all $s > 0$.

Notice that for $\Phi \in \mathcal{S}(\mathcal{E})$,

$$\langle X_T(t), \Phi \rangle := \int_{\mathcal{E}} \Phi(x, i)\, X_T(t, d(x, i)) = \sum_{i=1}^{2} \int_{\mathbb{R}} \Phi(x, i) X_T^{(i)}(t, dx), \tag{1.3}$$

where $X_T^{(i)}(t, dx) := \delta_{ij} X_T(t, d(x, j))$, $i = 1, 2$, $t \geq 0$, are signed Radon measures on $\mathbb{R}$. Hence (1.3) can be written as

$$\langle X_T(t), \Phi \rangle = \left\langle (X_T^{(1)}(t), X_T^{(2)}(t)), (\Phi(\cdot, 1), \Phi(\cdot, 2)) \right\rangle, \tag{1.4}$$

where $\langle\ ,\ \rangle$ on the right-hand side denotes the duality on $(\mathcal{S}'(\mathbb{R}))^2 \times \mathcal{S}(\mathbb{R})^2$.

Our goal is to find a suitable norming F_T such that X_T, or equivalently the vector process $(X_T^{(1)}, X_T^{(2)})$, converges in distribution as $T \to \infty$ to a non-degenerate limit, and to identify the limit process. We will prove the following result.

Theorem *Let $1 < \alpha_1 < \alpha_2$, and $F_T = T^{1-1/2\alpha_2}$. Then, for any $\tau > 0$, $(X_T^{(1)}, X_T^{(2)}) \Rightarrow (X^{(1)}, X^{(2)})$ in the space of continuous functions $C([0,\tau], (\mathcal{S}'(\mathbb{R}))^2)$ as $T \to \infty$, where $X = (X^{(1)}, X^{(2)})$ is a centered Gaussian process with covariance functional given by*

$$\begin{aligned} \mathrm{Cov}(\langle X(s), \Phi\rangle, \langle X(t), \Psi\rangle) &= \frac{V_1}{V_1+V_2}\frac{\Gamma(2-h)}{\pi\alpha_2 h(h-1)}\langle\lambda,\varphi_2\rangle\langle\lambda,\psi_2\rangle(t^h + s^h - |t-s|^h), \quad (1.5)\\ &\quad \Phi = (\varphi_1,\varphi_2), \Psi = (\psi_1,\psi_2) \in (\mathcal{S}(\mathbb{R}))^2, \end{aligned}$$

where $h = 2 - 1/\alpha_2 \in (1, 3/2]$.

Remark 1. The limit process $X = (X^{(1)}, X^{(2)})$ is given by $X^{(1)} = 0$, and

$$X^{(2)} = \left(\frac{V_1}{V_1+V_2}\frac{\Gamma(2-h)}{\pi\alpha_2 h(h-1)}\right)^{1/2}\lambda\xi, \qquad (1.6)$$

where $\xi = (\xi(t))_{t\geq 0}$ is fractional Brownian motion with Husrt parameter $H = h/2$.
2. Although the limit process X is measure-valued, we have made use of the advantages of the topology of $\mathcal{S}'(\mathbb{R})$, which is commonly done in this type of problems.

2 Proof

In order to prove the theorem we extend the space-time method introduced in [2] and employed in [3] to a two-type particle setting.

We start by recalling some facts. Consider the Markov generator $\mathcal{A}$ defined by

$$\mathcal{A}\Phi(x,i) = \Delta_{\alpha_i}\Phi(x,i) + V_i\sum_{j=1}^{2}(m_{ij} - \delta_{ij})\Phi(x,j), \quad (x,i) \in \mathcal{E}, \quad \Phi(\cdot,i) \in \mathrm{Dom}(\Delta_{\alpha_i}), \qquad (2.1)$$

and denote by $\{(W_t,\eta_t),\ t \geq 0\}$ the Markov process on $\mathcal{E}$ with generator $\mathcal{A}$. The *type component* $\eta = \{\eta_t, t \geq 0\}$ follows a Markov chain with Q-matrix $(V_i(m_{ij} - \delta_{ij}))_{1\leq i,j\leq 2}$, and the *position component* $W = \{W_t, t \geq 0\}$ follows an α_i-stable motion as long as η is in state i. We write $\{U(t), t \geq 0\}$ for the semigroup in $L^2(\mathcal{E}, \Lambda)$ with generator $\mathcal{A}$.

Let P_i^η denote the distribution of the type chain η starting in i, $i = 1,2$. We write $L_i(t,\eta)$ for the amount of time that η spends in i during the time interval $(0,t]$, $t \geq 0$. We denote by $\{J_t^{(x,k)}, t > 0\}$ the transition kernels of the process $\{(W_t,\eta_t),\ t \geq 0\}$ starting in $(x,k) \in \mathcal{E}$. Let $\{q_t^{\alpha_i}, t > 0\}$ denote the transition densities of the symmetric α_i-stable process in $\mathbb{R}$, $i = 1,2$. We recall the following result from [9], where $D_{[0,\infty)}(\{1,2\})$ is the Skorohod space of right-continuous with left-limits functions $h : [0,\infty) \to \{1,2\}$.

Lemma *The transition kernels $J_t^{(x,k)}$, $t > 0$, are given by*

$$\begin{aligned} J_t^{(x,k)}(C \times \{i\}) &= \int_{D_{[0,\infty)}(\{1,2\})} 1_{\{\eta_0=k,\eta_t=i\}}(\eta)\int_C \left(q^{\alpha_1}_{L_1(t,\eta)} * q^{\alpha_2}_{L_2(t,\eta)}\right)(x,z)\,dz\,P_k^\eta(d\eta), \quad (2.2)\\ &\quad i,k \in \{1,2\},\ x \in \mathbb{R},\ C \in \mathcal{B}(\mathbb{R}). \end{aligned}$$

Moreover, $J_t^{(x,k)} \ll \Lambda$ *for all* $t > 0$ *and* $(x,k) \in \mathcal{E}$, *and* $(dJ_t^{(x,k)}/d\Lambda)(z,i) = J_t((x,k),(z,i))$, *where*

$$J_t((x,k),(z,i)) := \int_{D_{[0,\infty)}(\{1,2\})} \left(q^{\alpha_1}_{L_1(t,\eta)} * q^{\alpha_2}_{L_2(t,\eta)} \right)(x,z) 1_{\{\eta_0 = k,\ \eta_t = i\}}(\eta) P_k^\eta(d\eta). \tag{2.3}$$

Notice that $J_t((x,k),(z,i))$ depends on x and z only through $x - z$.

Due to (1.4), we will mostly use the process X_T rather than its vector counterpart $(X_T^{(1)}, X_T^{(2)})$.

We put $\tau = 1$ (without loss of generality), and we define the related random variables $\tilde{X}_T$ for each $T \geq 1$ by

$$\left\langle \tilde{X}_T, \Phi \right\rangle = \int_0^1 \langle X_T(s), \Phi(\cdot, s) \rangle \, ds, \quad \Phi \in \mathcal{S}(\mathcal{E} \times \mathbb{R}), \tag{2.4}$$

(which determines $\tilde{X}_T$ as an $\mathcal{S}'(\mathcal{E} \times \mathbb{R})$-valued random variable, see [2]). Our first task is to show that

$$\tilde{X}_T \ \Rightarrow \ \tilde{X} \ \text{ in } \ \mathcal{S}'(\mathcal{E} \times \mathbb{R}) \ \text{ as } \ T \to \infty, \tag{2.5}$$

where $\tilde{X}$ is given by

$$\left\langle \tilde{X}, \Phi \right\rangle = \int_0^1 \langle X(s), \Phi(\cdot, s) \rangle \ ds, \quad \Phi \in \mathcal{S}(\mathcal{E} \times \mathbb{R}). \tag{2.6}$$

Putting (1.2) into (2.4) and changing the order of integration, we get

$$\left\langle \tilde{X}_T, \Phi \right\rangle = \frac{T}{F_T} \left[\int_0^1 \langle N_{Ts}, \Psi(\cdot, s) \rangle \ ds - \left\langle \Lambda, \int_0^1 \Psi(\cdot, s) \, ds \right\rangle \right], \tag{2.7}$$

where

$$\Psi(z,s) = \int_s^1 \Phi(z,t) \, dt, \ z \in \mathcal{E}. \tag{2.8}$$

Since the initial configuration is Poisson with mean Λ, we obtain from (2.7) the Laplace functional

$$\begin{aligned} \mathbb{E} \exp\left\{ - \left\langle \tilde{X}_T, \Phi \right\rangle \right\} \quad &= \quad \exp\left\{ \int_{\mathcal{E}} \int_0^T \Psi_T(x,i,s) \, ds \, d\Lambda(x,i) \right\} \\ &\cdot \exp\left\{ \int_{\mathcal{E}} \left[\mathbb{E} \exp\left(- \int_0^T \left\langle N_s^{(x,i)}, \Psi_T(\cdot, s) \right\rangle ds \right) - 1 \right] d\Lambda(x,i) \right\}, \end{aligned} \tag{2.9}$$

where

$$\Psi_T(z,s) = \frac{1}{F_T} \Psi\left(z, \frac{s}{T}\right), \quad z \in \mathcal{E}, \tag{2.10}$$

and $N_s^{(x,i)}$ denotes the empirical measure at time s of the particle system started from one initial particle in (x,i). For any nonnegative $\Psi \in \mathcal{S}(\mathcal{E} \times \mathbb{R})$ we define

$$w_i(x,r,t) = \mathbb{E} \exp\left\{ - \int_0^t \left\langle N_s^{(x,i)}, \Psi(\cdot, r+s) \right\rangle \ ds \ \right\}, \quad (x,i) \in \mathcal{E}, \quad r,t \geq 0. \tag{2.11}$$

Hence $0 \leq w_i \leq 1$. We denote by $\xi_t^{(x,i)}$ the position at time $t \geq 0$ of a one-dimensional symmetric α_i-stable process starting in $x \in \mathbb{R}$. By a renewal argument we obtain

$$w_i(x,r,t) = e^{-V_i t} \mathbb{E}\text{exp}\left\{ - \int_0^t \Psi(\xi_s^{(x,i)}, i, s+r) \, ds \right\}$$

$$
\begin{aligned}
& + \quad V_i \int_0^t e^{-V_i s} \mathbb{E}\mathrm{exp} \left\{ - \int_0^s \Psi(\xi_u^{(x,i)}, i, r+u)\, du \right\} \\
& \qquad \cdot \Big[p_{i1} \mathbb{E}\mathrm{exp} \left\{ - \int_0^{t-s} \Big\langle N_u^{(\xi_s^{(x,i)},1)}, \Psi(\cdot, r+s+u) \Big\rangle \, du \right\} \\
& \qquad + p_{i2} \mathbb{E} \exp \left\{ - \int_0^{t-s} \Big\langle N_u^{(\xi_s^{(x,i)},2)}, \Psi(\cdot, r+s+u) \Big\rangle \, du \right\} \Big] \, ds \\
&= \quad e^{-V_i t} \mathbb{E}\mathrm{exp} \left\{ - \int_0^t \Psi(\xi_s^{(x,i)}, i, s+r)\, ds \right\} \\
& + \quad V_i \int_0^t e^{-V_i s} \mathbb{E}\mathrm{exp} \left\{ - \int_0^s \Psi(\xi_u^{(x,i)}, i, r+u)\, du \right\} \\
& \qquad \cdot \Big[p_{i1} w_1(\xi_s^{(x,i)}, r+s, t-s) + p_{i2} w_2(\xi_s^{(x,i)}, r+s, t-s) \Big] \, ds,
\end{aligned}
$$

where $p_{ij} = 1 - \delta_{ij}$. (Notice at this point that more general p_{ij}'s can be handled too). For Ψ as above and $r \geq 0$, let

$$
h_i(x,r,t) = \mathbb{E} \exp \left\{ - \int_0^t \Psi(\xi_s^{(x,i)}, i, s+r)\, ds \right\},
$$

and for $r, \sigma \geq 0$, let

$$
\begin{aligned}
& k_i(x,r,\sigma) \\
& \quad = \quad \mathbb{E}\mathrm{exp} \left\{ - \int_0^\sigma \Psi(\xi_u^{(x,i)}, i, r+u)\, du \right\} \Big[p_{i1} w_1(\xi_\sigma^{(x,i)}, r+\sigma, t-\sigma) + p_{i2} w_2(\xi_\sigma^{(x,i)}, r+\sigma, t-\sigma) \Big],
\end{aligned}
$$

(the dependence on t in the right-hand side is not relevant). Then, after an obvious change of variables we get

$$
w_i(x,r,t) = e^{-V_i t} h_i(x,r,t) + V_i e^{-V_i t} \int_0^t e^{V_i s} k_i(x,r,t-s)\, ds.
$$

Using the Feynman-Kac formula as in [3] we deduce that

$$
\begin{aligned}
\frac{\partial}{\partial t} h_i(x,r,t) &= \left(\Delta_{\alpha_i} + \frac{\partial}{\partial r} - \Psi(x,i,r) \right) h_i(x,r,t), \\
h_i(x,r,0) &= 1
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial \sigma} k_i(x,r,t) &= \left(\Delta_{\alpha_i} + \frac{\partial}{\partial r} - \Psi(x,i,r) \right) k_i(x,r,t), \\
k_i(x,r,0) &= p_{i1} w_1(x,r,t) + p_{i2} w_2(x,r,t),
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
& \frac{\partial}{\partial t} w_i(x,r,t) \\
& \quad = \quad -V_i e^{-V_i t} h_i(x,r,t) + e^{-V_i t} \frac{\partial}{\partial t} h_i(x,r,t) \\
& \qquad\quad -V_i^2 e^{-V_i t} \int_0^t e^{V_i s} k_i(x,r,t-s)\, ds + V_i k_i(x,r,0) + V_i e^{-V_i t} \int_0^t e^{V_i s} \frac{\partial}{\partial t} k_i(x,r,t-s)\, ds
\end{aligned}
$$

$$
\begin{aligned}
&= -V_i e^{-V_i t} h_i(x,r,t) + e^{-V_i t}\left(\Delta_{\alpha_i} + \frac{\partial}{\partial r} - \Psi(x,i,r)\right) h_i(x,r,t) \\
&\quad -V_i^2 e^{-V_i t}\int_0^t e^{V_i s} k_i(x,r,t-s)\, ds + V_i k_i(x,r,0) \\
&\quad +V_i e^{-V_i t}\int_0^t e^{V_i s}\left(\Delta_{\alpha_i} + \frac{\partial}{\partial r} - \Psi(x,i,r)\right) k_i(x,r,t-s)\, ds \\
&= V_i k_i(x,r,0) - V_i w_i(x,r,t) + \left(\Delta_{\alpha_i} + \frac{\partial}{\partial r} - \Psi(x,i,r)\right) w_i(x,r,t) \\
&= \left(\Delta_{\alpha_i} + \frac{\partial}{\partial r} - \Psi(x,i,r)\right) w_i(x,r,t) + V_i[p_{i1} w_1(x,r,t) + p_{i2} w_2(x,r,t) - w_i(x,r,t)].
\end{aligned}
$$

Let

$$
v(x,i,r,t) = 1 - w_i(x,r,t). \tag{2.12}
$$

Then $v(x,i,r,t) \equiv v_\Psi(x,i,r,t)$ satisfies the equation

$$
\begin{aligned}
\frac{\partial}{\partial t} v(x,i,r,t) &= \left(\Delta_{\alpha_i} + \frac{\partial}{\partial r}\right) v(x,i,r,t) + \Psi(x,i,r)(1 - v(x,i,r,t)) \\
&\quad -V_i\left[v(x,i,r,t) - p_{i1} v(x,1,r,t) - p_{i2} v(x,2,r,t)\right], \\
v(x,i,r,0) &= 0.
\end{aligned}
$$

In our case, the infinitesimal generator $\mathcal{A}$ of $\{U(t),\, t \geq 0\}$, which is given by (2.1), has the form

$$
\mathcal{A} f(x,i) = \Delta_{\alpha_i} f(x,i) + V_i(f(x,j) - f(x,i)), \quad (x,i) \in \mathcal{E}, \quad f \in \mathrm{Dom}(\mathcal{A}),
$$

where $j \neq i$. Then, from the previous equation

$$
\begin{aligned}
v(x,i,r,t) &= \int_0^t \Big(U(t-s)[\Psi(\cdot, r+t-s)\ (1 - v(\cdot, r+t-s, s))]\Big)(x,i)\, ds && (2.13) \\
&\leq \int_0^t \Big(U(t-s)\ \Psi(\cdot, r+t-s)\Big)(x,i)\, ds. && (2.14)
\end{aligned}
$$

Using the invariance of Λ for $U(t)$ (due to our choice of γ_1, γ_2), we obtain

$$
\int_{\mathcal{E}} v(x,i,r,t)) d\Lambda(x,i) = \int_0^t \int_{\mathcal{E}} \Psi(x,i,r+t-s)(1 - v(x,i,r+t-s,s)) d\Lambda(x,i)\, ds. \tag{2.15}
$$

Now we are prepared to prove convergence of the Laplace functional (2.9):

$$
\lim_{T\to\infty} \mathbb{E}\exp\left\{-\left\langle \tilde{X}_T, \Phi\right\rangle\right\} = \mathbb{E}\exp\left\{-\left\langle \tilde{X}, \Phi\right\rangle\right\}, \tag{2.16}
$$

where $\widetilde{X}$ is given by (2.6) and (1.5).

Analogously to [3], in order to prove (2.5) it suffices take $\Phi \geq 0$. Using (2.11), (2.12) and (2.15), it follows from (2.9) that

$$
\begin{aligned}
&\mathbb{E}\exp\left\{-\left\langle \tilde{X}_T, \Phi\right\rangle\right\} \\
&= \exp\left\{\int_0^T \int_{\mathcal{E}} \Psi_T(x,i,s)\, d\Lambda(x,i)\, ds\right\} \exp\left\{-\int_{\mathcal{E}} v_{\Psi_T}(x,i,0,T)\, d\Lambda(x,i)\right\} \\
&= \exp\left\{\int_0^T \int_{\mathcal{E}} \Psi_T(x,i,T-s) v_{\Psi_T}(x,i,T-s,s)\, d\Lambda(x,i)\, ds\right\} \\
&= \exp(\gamma_1 I_1)\ \exp(\gamma_2 I_2),
\end{aligned} \tag{2.17}
$$

where

$$I_1 = \int_0^T \int_{\mathbb{R}} \Psi_T(x,1,T-s) v_{\Psi_T}(x,1,T-s,s)\, dx\, ds, \tag{2.18}$$

$$I_2 = \int_0^T \int_{\mathbb{R}} \Psi_T(x,2,T-s) v_{\Psi_T}(x,2,T-s,s)\, dx\, ds. \tag{2.19}$$

We may assume without loss of generality (as in [3]) that $\Phi \in \mathcal{S}(\mathcal{E} \times \mathbb{R})$ is of the form $\Phi(x,i,t) = \varphi(x,i)\psi(t)$, where $\varphi \in \mathcal{S}(\mathcal{E})$ and $\psi \in \mathcal{S}(\mathbb{R})$ are nonnegative functions. We put

$$\chi(t) = \int_t^1 \psi(s)\, ds \quad \text{and} \quad \chi_T(t) = \chi\left(\frac{t}{T}\right). \tag{2.20}$$

Using (2.13) we obtain for the integral (2.19)

$$\begin{aligned} I_2 &= \int_0^T \int_{\mathbb{R}} \Psi_T(x,2,T-s) v_{\Psi_T}(x,2,T-s,s)\, dx\, ds \\ &= \int_0^T \int_{\mathbb{R}} \int_0^s \Psi_T(x,2,T-s) \Big(U(s-u)[\Psi_T(\cdot,T-u)\ (1 - v_{\Psi_T}(\cdot,T-u,u))] \Big)(x,2)\, du\, dx\, ds \\ &= I_{21} - I_{22}, \end{aligned}$$

where

$$I_{21} = \frac{1}{F_T} \int_0^T \int_{\mathbb{R}} \int_0^s \Psi(x,2,1-s/T) \Big(U(u)\, \Psi_T(\cdot, T-s+u) \Big)(x,2)\, du\, dx\, ds, \tag{2.21}$$

$$\begin{aligned} &I_{22} \\ &= -\frac{1}{F_T} \int_0^T \int_{\mathbb{R}} \int_0^s \Psi(x,2,1-s/T) \Big(U(s-u)[(\Psi_T(\cdot,T-u)\ v_{\Psi_T}(\cdot,T-u,u)] \Big)(x,2)\, du\, dx\, ds. \end{aligned} \tag{2.22}$$

For the integral I_{21} in (2.21), after the change of variables $s \mapsto T-s$ and using (2.20), we obtain

$$\begin{aligned} I_{21} &= \frac{1}{F_T^2} \int_0^T \int_{\mathbb{R}} \left[\int_0^{T-s} \varphi(x,2)\chi_T(s) \Big(U(u)\varphi(\cdot)\ \chi_T(s+u) \Big)(x,2)\, du \right] dx\, ds \\ &= \frac{1}{F_T^2} \int_0^T \int_s^T \left[\int_{\mathbb{R}} \varphi(x,2)\chi_T(s) \Big(U(u-s)\varphi(\cdot)\ \chi_T(u) \Big)(x,2)\, du\, dx \right] ds. \end{aligned}$$

Let $\{S_t^{\alpha_i},\ t \geq 0\}$ denote the α_i-stable semigroup, with generator Δ_{α_i}. Using the variation of parameters formula for perturbed semigroups (see e.g. [4], Chapter III, Corollary 1.7), it follows that

$$U(t)\Phi(x,i) = S_t^{\alpha_i}\Phi(x,i) + V_i \int_0^t S^{\alpha_i}(t-s)\, [U(s)\Phi(x,j) - U(s)\Phi(x,i)]\, ds, \tag{2.23}$$

Hence,

$$I_{21} = J_1 + J_2, \tag{2.24}$$

where

$$J_1 = \frac{1}{F_T^2} \int_0^T \int_{\mathbb{R}} \int_s^T \varphi(x,2)\chi_T(s) S_{u-s}^{\alpha_2}\varphi(x,2)\chi_T(u)\, du\, dx\, ds, \tag{2.25}$$

$$J_2 = \frac{1}{F_T^2} V_2 \int_0^T \int_s^T \int_0^{u-s} \int_{\mathbb{R}} \varphi(x,2) S_{u-s-r}^{\alpha_2} K(r,x)\, dx\, \chi_T(s)\chi_T(u)\, dr\, du\, ds, \tag{2.26}$$

and

$$K(r,x) := U(r)\varphi(x,1) - U(r)\varphi(x,2). \tag{2.27}$$

Let us write $\widehat{\varphi}(z,i)$ for the Fourier transform of $\varphi(\cdot,i)$, $i=1,2$, and recall that $F_T = T^{1-1/2\alpha_2}$. Applying Plancherel's theorem and the fact that the Fourier transform of the α-stable semigroup $S_t^\alpha\varphi$ is $e^{-|t|^\alpha}\widehat{\varphi}(z)$, we obtain for J_1 from (2.25)

$$\begin{aligned}
J_1 &= \frac{1}{F_T^2}\frac{1}{2\pi}\int_0^T\int_s^T\int_{\mathbb{R}}|\widehat{\varphi}(z,2)|^2 e^{-(u-s)|z|^{\alpha_2}}\,dz\,\chi_T(u)\chi_T(s)\,du\,ds\\
&= \frac{T}{F_T^2}\frac{1}{2\pi}\int_0^1\int_{sT}^T\int_{\mathbb{R}}|\widehat{\varphi}(z,2)|^2 e^{-(u-sT)|z|^{\alpha_2}}\,dz\,\chi_T(u)\chi_T(sT)\,du\,ds\\
&= \frac{T^2}{F_T^2}\frac{1}{2\pi}\int_0^1\int_s^1\int_{\mathbb{R}}|\widehat{\varphi}(z,2)|^2 e^{-T(u-s)|z|^{\alpha_2}}\,dz\,\chi(u)\chi(s)\,du\,ds\\
&= \frac{T^{2-1/\alpha_2}}{F_T^2}\frac{1}{2\pi}\int_0^1\int_s^1\int_{\mathbb{R}} e^{-|y|^{\alpha_2}}|\widehat{\varphi}(yT^{-1/\alpha_2}(u-s)^{-1/\alpha_2},2)|^2(u-s)^{-1/\alpha_2}\,dz\,\chi(u)\chi(s)\,du\,ds,
\end{aligned}$$

where in the second equality we made the substitution $s\mapsto sT$, in the third one $u\mapsto Tu$, and in the fourth one, $z=[T(u-s)]^{-1/\alpha_2}y$. Therefore,

$$\begin{aligned}
\lim_{T\to\infty} J_1 &= \frac{1}{2\pi}|\widehat{\varphi}(0,2)|^2\int_R e^{-|y|^{\alpha_2}}dy\int_0^1\int_s^1(u-s)^{-1/\alpha_2}\chi(s)\chi(u)\,ds\,du\\
&= |\widehat{\varphi}(0,2)|^2\,\frac{\Gamma(2-h)}{\pi\alpha_2 h(h-1)}\int_0^1\int_0^u\left[u^h+v^h+(u-v)^h\right]\,dv\,du,
\end{aligned} \tag{2.28}$$

where $h=2-1/\alpha_2$, the limit being finite because $1<\alpha_2$. For J_2 we have, using again Plancherel's theorem,

$$\begin{aligned}
J_2 &= \frac{1}{2\pi}\frac{1}{F_T^2}V_2\int_0^T\int_s^T\int_0^{u-s}\left[\int_{\mathbb{R}}\widehat{\varphi}(z,2)e^{-(u-s-r)|z|^{\alpha_2}}\overline{\widehat{K}(r,z)}\,dz\right]\chi_T(s)\chi_T(u)\,dr\,du\,ds\\
&= \frac{C_1T^3}{F_T^2}\int_0^1\int_s^1\int_0^{u-s}\left[\int_{\mathbb{R}}\widehat{\varphi}(z,2)e^{-T(u-s-r)|z|^{\alpha_2}}\overline{\widehat{K}(rT,z)}\,dz\right]\chi(s)\chi(u)\,dr\,du\,ds\\
&= \frac{C_1T^{3-1/\alpha_2}}{F_T^2}\int_0^1\int_s^1\int_0^{u-s}\left[\int_{\mathbb{R}}\widehat{\varphi}((u-s-r)^{-1/\alpha_2}T^{-1/\alpha_2}y,2)e^{-|y|^{\alpha_2}}\right.\\
&\qquad\cdot\left.\overline{\widehat{K}(rT,[(u-s-r)T]^{-1/\alpha_2}y)}\,dy\right](u-s-r)^{-1/\alpha_2}\chi(s)\chi(u)\,dr\,du\,ds,
\end{aligned}$$

where in the second equality we made the changes of variables $s\mapsto sT, u\mapsto uT, r\mapsto rT$, and in the last equality we made the substitution $z=[(u-s-r)T]^{-1/\alpha_2}y$, and $C_1=V_2/2\pi$.

Denoting $D=D_{[0,\infty)}(\{1,2\})$, from the lemma we obtain

$$\begin{aligned}
U(r)\varphi(x,1) &= \int_{\mathbb{R}}\varphi(z,1)\int_D 1_{\{\eta_0=1,\eta_r=1\}}(\eta)\,q^{\alpha_1}_{L_1(r,\eta)}*q^{\alpha_2}_{L_2(r,\eta)}(z-x)\,dzP_1^\eta(d\eta)\\
&\quad+\int_{\mathbb{R}}\varphi(z,2)\int_D 1_{\{\eta_0=1,\eta_r=2\}}(\eta)\,q^{\alpha_1}_{L_1(r,\eta)}*q^{\alpha_2}_{L_2(r,\eta)}(z-x)\,dzP_1^\eta(d\eta).
\end{aligned}$$

Hence, taking Fourier transform,

$$\begin{aligned}
\widehat{U(r)\varphi}(\theta,1) &= \int_D 1_{\{\eta_0=1,\eta_r=1\}}(\eta)\,\widehat{\varphi}(\theta,1)\widehat{q^{\alpha_1}_{L_1(r,\eta)}*q^{\alpha_2}_{L_2(r,\eta)}}(\theta)P_1^\eta(d\eta)\\
&\quad+\int_D 1_{\{\eta_0=1,\eta_r=2\}}(\eta)\,\widehat{\varphi}(\theta,2)\widehat{q^{\alpha_1}_{L_1(r,\eta)}*q^{\alpha_2}_{L_2(r,\eta)}}(\theta)P_1^\eta(d\eta)
\end{aligned}$$

$$
\begin{aligned}
&= \int_D 1_{\{\eta_0=1,\eta_r=1\}}(\eta)\,\widehat{\varphi}(\theta,1)e^{-L_1(r,\eta)|\theta|^{\alpha_1}}e^{-L_2(r,\eta)|\theta|^{\alpha_2}}P_1^\eta(d\eta)\\
&\quad+\int_D 1_{\{\eta_0=1,\eta_r=2\}}(\eta)\,\widehat{\varphi}(\theta,2)e^{-L_1(r,\eta)|\theta|^{\alpha_1}}e^{-L_2(r,\eta)|\theta|^{\alpha_2}}P_1^\eta(d\eta).
\end{aligned}
\tag{2.29}
$$

In the same way,

$$
\begin{aligned}
\widehat{U(r)\varphi}(\theta,2) &= \int_D 1_{\{\eta_0=2,\eta_r=1\}}(\eta)\,\widehat{\varphi}(\theta,1)\widehat{q^{\alpha_1}_{L_1(r,\eta)} * q^{\alpha_2}_{L_2(r,\eta)}}(\theta)P_2^\eta(d\eta)\\
&\quad+\int_D 1_{\{\eta_0=2,\eta_r=2\}}(\eta)\,\widehat{\varphi}(\theta,2)\widehat{q^{\alpha_1}_{L_1(r,\eta)} * q^{\alpha_2}_{L_2(r,\eta)}}(\theta)P_2^\eta(d\eta)\\
&= \int_D 1_{\{\eta_0=2,\eta_r=1\}}(\eta)\,\widehat{\varphi}(\theta,1)e^{-L_1(r,\eta)|\theta|^{\alpha_1}}e^{-L_2(r,\eta)|\theta|^{\alpha_2}}P_2^\eta(d\eta)\\
&\quad+\int_D 1_{\{\eta_0=2,\eta_r=2\}}(\eta)\,\widehat{\varphi}(\theta,2)e^{-L_1(r,\eta)|\theta|^{\alpha_1}}e^{-L_2(r,\eta)|\theta|^{\alpha_2}}P_2^\eta(d\eta).
\end{aligned}
\tag{2.30}
$$

Then, using similar changes of variables as before,

$$
J_2 = A_1 + A_2 + A_3 + A_4,
$$

where

$$
\begin{aligned}
A_1 &= -\frac{C_1T^{3-1/\alpha_2}}{F_T^2}\int_0^1\int_s^1\int_0^{u-s}\int_D\Big[\int_{\mathbb{R}}\widehat{\varphi}((u-s-r)^{-1/\alpha_2}T^{-1/\alpha_2}y,1)\exp\{-|y|^{\alpha_2}\}\\
&\quad\cdot\widehat{\varphi}((u-s-r)^{-1/\alpha_2}T^{-1/\alpha_2}y,1)\exp\left\{-L_1(rT,\eta)(u-s-r)^{-\alpha_1/\alpha_2}T^{-\alpha_1/\alpha_2}|y|^{\alpha_1}\right\}\\
&\quad\cdot\exp\left\{-L_2(rT,\eta)(u-s-r)^{-1}T^{-1}|y|^{\alpha_2}\right\}dy\ \Big]\\
&\quad\cdot 1_{\{\eta_0=1,\eta_{rT}=1\}}P_1^\eta(d\eta)(u-s-r)^{-1/\alpha_2}\chi(s)\chi(u)\,dr\,du\,ds,\\
A_2 &= -\frac{C_1T^{3-1/\alpha_2}}{F_T^2}\int_0^1\int_s^1\int_0^{u-s}\Big[\int_{\mathbb{R}}\hat{\varphi}((u-s-r)^{-1/\alpha_2}T^{-1/\alpha_2}y,1)\exp\{-|y|^{\alpha_2}\}\\
&\quad\cdot\hat{\varphi}((u-s-r)^{-1/\alpha_2}T^{-1/\alpha_2}y,2)\exp\left\{-L_1(rT,\eta)(u-s-r)^{-\alpha_1/\alpha_2}T^{-\alpha_1/\alpha_2}|y|^{\alpha_1}\right\}\\
&\quad\cdot\exp\left\{-L_2(rT,\eta)(u-s-r)^{-1}T^{-1}|y|^{\alpha_2}\right\}dy\ \Big]\\
&\quad\cdot 1_{\{\eta_0=1,\eta_{rT}=2\}}P_1^\eta(d\eta)(u-s-r)^{-1/\alpha_2}\chi(s)\chi(u)\,dr\,du\,ds,
\end{aligned}
$$

$$
\begin{aligned}
A_3 &= \frac{C_1T^{3-1/\alpha_2}}{F_T^2}\int_0^1\int_s^1\int_0^{u-s}\Big[\int_{\mathbb{R}}\widehat{\varphi}((u-s-r)^{-1/\alpha_2}T^{-1/\alpha_2}y,1)\exp\{-|y|^{\alpha_2}\}\\
&\quad\cdot\widehat{\varphi}((u-s-r)^{-1/\alpha_2}T^{-1/\alpha_2}y,1)\exp\left\{-L_1(rT,\eta)(u-s-r)^{-\alpha_1/\alpha_2}T^{-\alpha_1/\alpha_2}|y|^{\alpha_1}\right\}\\
&\quad\cdot\exp\left\{-L_2(rT,\eta)(u-s-r)^{-1}T^{-1}|y|^{\alpha_2}\right\}dy\ \Big]\\
&\quad\cdot 1_{\{\eta_0=2,\eta_{rT}=1\}}P_2^\eta(d\eta)(u-s-r)^{-1/\alpha_2}\chi(s)\chi(u)\,dr\,du\,ds,
\end{aligned}
$$

$$\begin{aligned}
A_4 &= \frac{C_1 T^{3-1/\alpha_2}}{F_T^2}\int_0^1\int_s^1\int_0^{u-s}\Big[\int_{\mathbb{R}}\widehat{\varphi}((u-s-r)^{-1/\alpha_2}T^{-1/\alpha_2}y,1)\exp\{-|y|^{\alpha_2}\}\\
&\quad\cdot\widehat{\varphi}((u-s-r)^{-1/\alpha_2}T^{-1/\alpha_2}y,2)\exp\Big\{-L_1(rT,\eta)(u-s-r)^{-\alpha_1/\alpha_2}T^{-\alpha_1/\alpha_2}|y|^{\alpha_1}\Big\}\\
&\quad\cdot\exp\big\{-L_2(rT,\eta)(u-s-r)^{-1}T^{-1}|y|^{\alpha_2}\big\}\,dy\ \Big]\\
&\quad\cdot 1_{\{\eta_0=2,\eta_{rT}=2\}}P_2^{\eta}(d\eta)(u-s-r)^{-1/\alpha_2}\chi(s)\chi(u)\,dr\,du\,ds.
\end{aligned}$$

By the ergodic theorem ($L_i(t,\eta)/t\to\gamma_j$ a.s. as $t\to\infty, j\neq i$), and the hypothesis $1<\alpha_1<\alpha_2$, we obtain

$$\begin{aligned}
&\lim_{T\to\infty} A_1\\
&= -\lim_{T\to\infty}\frac{C_1T^{3-1/\alpha_2}}{F_T^2}\int_0^1\int_s^1\int_0^{u-s}\int_D\Big[\int_{\mathbb{R}}\widehat{\varphi}((u-s-r)^{-1/\alpha_2}T^{-1/\alpha_2}y,1)\exp\{-|y|^{\alpha_2}\}\\
&\quad\cdot\widehat{\varphi}((u-s-r)^{-1/\alpha_2}T^{-1/\alpha_2}y,1)\exp\left\{-\left(\frac{L_1(rT,\eta)}{rT}\right)rT(u-s-r)^{-\alpha_1/\alpha_2}T^{-\alpha_1/\alpha_2}|y|^{\alpha_1}\right\}\\
&\quad\cdot\exp\left\{-\left(\frac{L_2(rT,\eta)}{rT}\right)rT(u-s-r)^{-1}T^{-1}|y|^{\alpha_2}\right\}dy\Big]\\
&\quad\cdot 1_{\{\eta_0=1,\eta_{rT}=1\}}P_1^{\eta}(d\eta)(u-s-r)^{-1/\alpha_2}\chi(s)\chi(u)\,dr\,du\,ds\\
&= -\lim_{T\to\infty}C_1\int_0^1\int_s^1\int_0^{u-s}\int_D\Big[\int_{\mathbb{R}}\widehat{\varphi}((u-s-r)^{-1/\alpha_2}T^{-1/\alpha_2}y,1)^2\exp\{-|y|^{\alpha_2}\}\\
&\quad\cdot T\exp\left\{-\left(\frac{L_1(rT,\eta)}{rT}\right)rT^{1-\alpha_1/\alpha_2}(u-s-r)^{-\alpha_1/\alpha_2}|y|^{\alpha_1}\right\}\\
&\quad\cdot\exp\left\{-\left(\frac{L_2(rT,\eta)}{rT}\right)r\ (u-s-r)^{-1}|y|^{\alpha_2}\right\}dy\Big]\\
&\quad\cdot 1_{\{\eta_0=1,\eta_{rT}=1\}}P_1^{\eta}(d\eta)(u-s-r)^{-1/\alpha_2}\chi(s)\chi(u)\,dr\,du\,ds\\
&= 0, \qquad (2.31)
\end{aligned}$$

where we used the definition of F_T, Lebesgue's theorem, and $|\widehat{\varphi}(0,1)|<\infty$. In a similar fashion it is verified that

$$\lim_{T\to\infty}A_2=\lim_{T\to\infty}A_3=\lim_{T\to\infty}A_4=0. \qquad (2.32)$$

In this way we have proved that I_{21}, given by (2.21), converges at $T\to\infty$ to the integrated covariance function of the fractional Brownian motion given in (2.28).

We will prove that the term I_{22} in (2.22) tends to 0 as $T\to\infty$. Using (2.13), (2.23), and the fact that χ and φ are bounded, we obtain from (2.23)

$$\begin{aligned}
I_{12} &\leq \frac{C_2}{F_T^3}\int_0^T\int_0^s\int_0^u\int_{\mathbb{R}}\varphi(x,2)\left[U(s-u)\left(\varphi U(r)\varphi\right)\right](x,2)\,dx\,dr\,du\,ds,\\
&= B_1+B_2,
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= \frac{C_2}{F_T^3}\int_0^T\int_0^s\int_0^u\int_{\mathbb{R}}\varphi(x,2)S_{s-u}^{\alpha_2}[\varphi U(r)\varphi](x,2)\,dx\,dr\,du\,ds,\\
B_2 &= \frac{C_2}{F_T^3}\int_0^T\int_0^s\int_0^u\int_0^{s-u}\int_{\mathbb{R}}\varphi(x,2)\\
&\quad\cdot S_{s-u-l}^{\alpha_2}\Big([U(l)(\varphi U(r)\varphi)](x,1)-\ [U(l)(\varphi U(r)\varphi)](x,2)\Big)\,dx\,dl\,dr\,du\,ds,
\end{aligned}$$

where C_2 is a positive constant. It is straightforward to see, similarly as we did in the case of J_2, that $\lim_{T\to\infty} B_1 = \lim_{T\to\infty} B_2 = 0$. As a matter of fact, using the Plancherel theorem we obtain

$$
\begin{aligned}
B_1 &\leq \frac{C_2}{F_T^3}\int_0^T\int_0^s\int_0^u\int_{\mathbb{R}}\widehat{\varphi}(z,2)e^{-(s-u)|z|^{\alpha_2}}[\widehat{U(r)\varphi}](z,2)\,dz\,dr\,du\,ds\\
&= \frac{C_2T^3}{F_T^3}\int_0^1\int_0^s\int_0^u\int_R\widehat{\varphi}(z,2)e^{-T(s-u)|z|^{\alpha_2}}[\widehat{U(rT)\varphi}](z,2)\,dz\,dr\,du\,ds\\
&= \frac{C_2T^{3-1/\alpha_2}}{F_T^3}\int_0^1\int_0^s\int_0^u\int_{\mathbb{R}}\widehat{\varphi}(z,2)e^{-T(s-u)|z|^{\alpha_2}}[\widehat{U(rT)\varphi}](z,2)\,dz\,(s-u)^{-1/\alpha_2}dr\,du\,ds.
\end{aligned}
$$

Using the expression for $\widehat{U(r)\varphi}(z,2)$ given by (2.30), we get $\lim_{T\to\infty} B_1 = 0$ in the same way as we obtained (2.31).

Finally, to deal with the term I_1 given by (2.18) we use $1<\alpha_1<\alpha_2$, and $\lim_{T\to\infty} T^{2-1/\alpha_1}/F_T^2 = 0$. Proceeding as in the case of I_2, we conclude that $\lim_{T\to\infty} I_1 = 0$.

The convergence of the Laplace functionals (2.16) is established.

We now show tightness for the real processes $\{\langle X_T,\Phi\rangle, T\geq 1\}$ for any $\Phi\in\mathcal{S}(\mathcal{E})$, which implies tightness of $\{X_T, T\geq 1\}$ by the theorem of Mitoma [11]. Let $s\leq t$ and $\Phi\in\mathcal{S}(\mathcal{E})$. We have

$$
\begin{aligned}
\mathbb{E}(\langle X_T(t),\Phi\rangle - \langle X_T(s),\Phi\rangle)^2 &= \frac{T^2}{F_T^2}\int_s^t\int_s^t \mathrm{Cov}(\langle N_{Tu},\Phi\rangle,\langle N_{Tv},\Phi\rangle)\,du\,dv\\
&= 2\frac{T^2}{F_T^2}\int_s^t\int_s^v \mathrm{Cov}(\langle N_{Tu},\Phi\rangle,\langle N_{Tv},\Phi\rangle)\,du\,dv. \qquad (2.33)
\end{aligned}
$$

Using [8] and the fact that Λ is invariant for the semigroup $U(t)$, we obtain

$$
\begin{aligned}
&\mathrm{Cov}(\langle N_{Tu},\Phi\rangle,\langle N_{Tv},\Phi\rangle)\\
&\quad= \langle\Lambda, U(Tu)[\Phi\, U(T(v-u))\Phi]\rangle\\
&\quad= \int_{\mathcal{E}}\Phi(x,i)(U(T(v-u))\Phi)(x,i)\,d\Lambda(x,i)\\
&\quad= \int_{\mathcal{E}}\Phi(x,i)\int_{\mathcal{E}}\Phi(z,j)J_{T(v-u)}((x,i),(z,j))\,d\Lambda(z,j)\,d\Lambda(x,i)\\
&\quad= \gamma_1\int_{\mathbb{R}}\Phi(x,1)\left(\int_{\mathcal{E}}\Phi(z,j)J_{T(v-u)}((x,1),(z,j))\,d\Lambda(z,j)\right)dx\\
&\qquad + \gamma_2\int_{\mathbb{R}}\Phi(x,2)\left(\int_{\mathcal{E}}\Phi(z,j)J_{T(v-u)}((x,2),(z,j))\,d\Lambda(z,j)\right)dx.
\end{aligned}
$$

The lemma and another use of Plancherel's theorem give

$$
\begin{aligned}
&\mathrm{Cov}(\langle N_{Tu},\Phi\rangle,\langle N_{Tv},\Phi\rangle)\\
&= \frac{1}{2\pi}\Bigg[\gamma_1\int_{\mathbb{R}}\int_D|\widehat{\Phi}(z,1)|^2e^{-L_1(T(v-u),\eta)|z|^{\alpha_1}-L_2(T(v-u),\eta)|z|^{\alpha_2}}1_{\{\eta_0=1,\eta_{T(v-u)}=1\}}P^\eta(d\eta)\,dz\\
&\quad+\gamma_1\int_{\mathbb{R}}\int_D\widehat{\Phi}(z,1)\overline{\widehat{\Phi}(z,2)}e^{-L_1(T(v-u),\eta)|z|^{\alpha_1}-L_2(T(v-u),\eta)|z|^{\alpha_2}}1_{\{\eta_0=1,\eta_{T(v-u)}=2\}}P^\eta(d\eta)\,dz\\
&\quad+\gamma_2\int_{\mathbb{R}}\int_D\widehat{\Phi}(z,2)\overline{\widehat{\Phi}(z,1)}e^{-L_1(T(v-u),\eta)|z|^{\alpha_1}-L_2(T(v-u),\eta)|z|^{\alpha_2}}1_{\{\eta_0=2,\eta_{T(v-u)}=1\}}P^\eta(d\eta)\,dz\\
&\quad+\gamma_2\int_{\mathbb{R}}\int_D|\widehat{\Phi}(z,2)|^2e^{-L_1(T(v-u),\eta)|z|^{\alpha_1}-L_2(T(v-u),\eta)|z|^{\alpha_2}}1_{\{\eta_0=2,\eta_{T(v-u)}=2\}}P^\eta(d\eta)\,dz\Bigg]. \qquad (2.34)
\end{aligned}
$$

Since $|z|^{\alpha_1} > |z|^{\alpha_2}$ on $\{z : |z| < 1\}$ and $|z|^{\alpha_1} < |z|^{\alpha_2}$ on $\{z : |z| \geq 1\}$, and obviously $L_1(t,\eta) + L_2(t,\eta) = t$, we have

$$\begin{aligned}
& e^{-L_1(T(v-u),\eta)|z|^{\alpha_1} - L_2(T(v-u),\eta)|z|^{\alpha_2}} \\
&= e^{-L_1(T(v-u),\eta)|z|^{\alpha_1} - L_2(T(v-u),\eta)|z|^{\alpha_2}} 1_{\{|z|<1\}} + e^{-L_1(T(v-u),\eta)|z|^{\alpha_1} - L_2(T(v-u),\eta)|z|^{\alpha_2}} 1_{\{|z|\geq 1\}} \\
&\leq e^{-T(v-u)|z|^{\alpha_2}} 1_{\{|z|<1\}} + e^{-T(v-u)|z|^{\alpha_1}} 1_{\{|z|\geq 1\}} \\
&\leq e^{-T(v-u)|z|^{\alpha_1}} + e^{-T(v-u)|z|^{\alpha_2}}. \qquad (2.35)
\end{aligned}$$

Substituting (2.35) into (2.34), we obtain

$$\begin{aligned}
& \mathrm{Cov}(\langle N_{Tu}, \Phi\rangle, \langle N_{Tv}, \Phi\rangle) \qquad (2.36) \\
&\leq C \int_{\mathbb{R}} (|\widehat{\Phi}(z,1)| + |\Phi(z,2)|)^2 (e^{-T(v-u)|z|^{\alpha_1}} + e^{-T(v-u)|z|^{\alpha_2}}) dz,
\end{aligned}$$

where C is a constant. Inserting (2.36) into (2.33), it follows, analogously as in Section 3.1 of [3], that

$$\mathbb{E}(\langle X_T(t), \Phi\rangle - \langle X_T(s), \Phi\rangle)^2 \leq C(\Phi) \left(\frac{(t-s)^{2-1/\alpha_1}}{T^{1/\alpha_1 - 1/\alpha_2}(2-1/\alpha_1)} + \frac{(t-s)^{2-1/\alpha_2}}{2-1/\alpha_2} \right).$$

for some constant $C(\Phi)$. According to [1], this finishes the proof of tightness.

Finally, we recall from [2] that convergence of $\tilde{X}_T$ in $\mathcal{S}'(\mathcal{E} \times \mathbb{R})$ as $T \to \infty$ and tightness of $\{X_T\}_{T\geq 1}$, implies the desired convergence in $C([0,1], (\mathcal{S}'(\mathbb{R}))^2)$. □

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