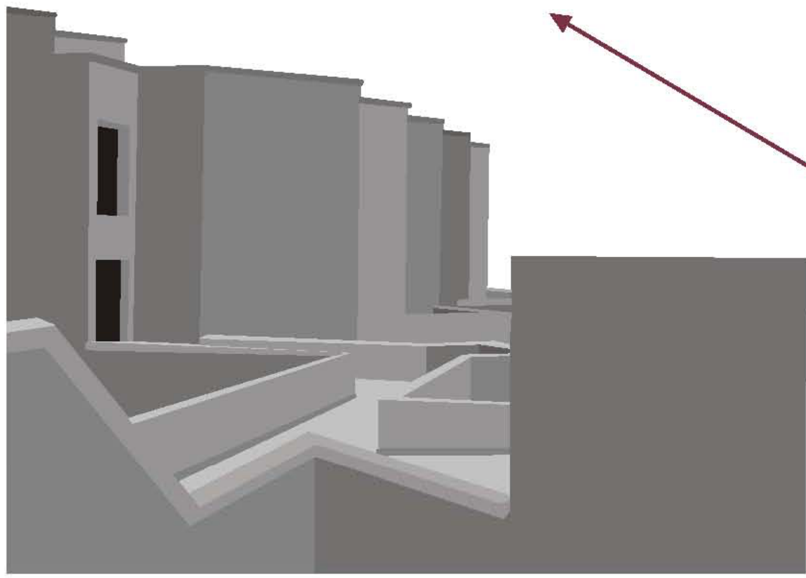

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Meromorphic Lévy processes and their fluctuation identities

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Abstract

The last couple of years has seen a remarkable number of new, explicit examples of the Wiener-Hopf factorization for Lévy processes where previously there had been very few. We mention in particular the many cases of spectrally negative Lévy processes in [19, 33], hyper-exponential and generalized hyper-exponential Lévy processes [22], Lamperti-stable processes in [8, 9, 12, 36], Hypergeometric processes in [32, 29, 10], β-processes in [28] and θ-processes in [27].

In this paper we introduce a new family of Lévy processes, which we call Meromorphic Lévy processes, or just M-processes for short, which overlaps with many of the aforementioned classes. A key feature of the M-class is the identification of their Wiener-Hopf factors as rational functions of infinite degree written in terms of poles and roots of the Lévy-Khintchin exponent, all of which appear on the imaginary axis of the complex plane. The specific structure of the M-class Wiener-Hopf factorization enables us to explicitly handle a comprehensive suite of fluctuation identities that concern first passage problems for finite and infinite intervals for both the process itself as well as the resulting process when it is reflected in its infimum. Such identities are of fundamental interest given their repeated occurrence in various fields of applied probability such as mathematical finance, insurance risk theory and queuing theory.

Keywords: Lévy processes, Wiener-Hopf factorization, exit problems, fluctuation theory.

AMS 2000 subject classifications: 60G51, 60G50.

1 Introduction

The theory of Lévy processes forms the cornerstone of an enormous volume of mathematical literature which supports a wide variety of applied and theoretical stochastic models. One of the most obvious and fundamental problems that can be stated for a Lévy process, particularly in relation to its role as a modelling tool, is the distributional characterization of the time at which a Lévy process first exists either an infinite or finite interval together with its overshoot beyond the boundary of the interval. As a family of stochastic processes, Lévy processes are now well understood and the exit problem has seen

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many different approaches dating back to the 1960s. See for example [2, 6, 7, 14, 37, 17, 12, 32] to name but a few.

Despite the maturity of this field of study it is surprising to note that, until very recently, there were less than a handful of examples for which explicit analytical detail concerning the first exit problem could be explored. Given the closeness in mathematical proximity of the first exit problem to the characterization of the Wiener-Hopf factorization, one might argue that the lack of concrete examples of the former was a consequence of the same being true for the latter. The landscape for both the Wiener-Hopf factorization problem and the first exit problem has changed quite rapidly in the last couple of years however with the discovery of a number of new, mathematically tractable families of Lévy processes. We mention in particular the many cases of spectrally negative Lévy processes in [19, 33], hyper-exponential and generalized hyper-exponential Lévy processes [22], Lamperti-stable processes in [8, 9, 12], Hypergeometric processes in [32, 29], β-processes in [28] and θ-processes in [27].

In this paper we introduce a new family of Lévy processes, which we call Meromorphic Lévy processes, or just M-class for short, that overlaps with many of the aforementioned classes. Our definition of the M-class of processes will allow us to drive features of their Wiener-Hopf factors through to many of the fluctuation identities which are of pertinence for a wide variety of applications. In the theory of actuarial mathematics, the problem of first exit from a half-line is of fundamental interest with regard to the classical ruin problem and is typically studied within the context of an expected discounted penalty function. The latter is also known as the Gerber-Shiu function following the first article [16] of a long series of papers found within the actuarial literature. In the setting of financial mathematics, the first exit of a Lévy process, as well as a Lévy process reflected in its infimum, from an interval is of interest in the pricing of barrier options and American-type options, [1], as well as certain credit risk models, [18, 30]. In queueing theory exit problems for Lévy processes play a central role in understanding the trajectory of the workload during busy periods as well as in relation to buffers, [15, 25]. Many optimal stopping strategies also turn out to boil down to first passage problems for both Lévy processes and Lévy processes reflected in their infimum; classic examples of which include McKean's optimal stopping problem, [35], as well as the Shepp-Shiryaev optimal stopping problem, [2, 3] .

It is not our purpose however to dwell on these applications. As alluded to above, the main objective will be expose a comprehensive suite of fluctuation identities in explicit form for the M-class of Lévy processes. We thus conclude the introduction with an overview of the paper.

In the next section we give a formal definition of meromorphic Lévy processes, connecting them to several classes of existing families of Lévy processes that have appeared in recent literature. Following that we begin a mathematical programme of analysis which begins in Section 3 by looking at how to re-write the (infinite) products that make up the Wiener-Hopf as (infinite) sums using an extension of the classical idea of partial fractions. From this the law of the maximum or minimum of the underlying Lévy process when sampled at an independent and exponentially distributed random time follows immediately. Next, in Section 4 we establish explicit identities for the exponentially discounted first passage problem. In particular we deal with (what is known in the actuarial literature as) the Gerber-Shiu measure describing the discounted joint triple law of the overshoot, undershoot and undershoot of the maximum at first passage over a level as well as the marginal thereof which specifies the law of the discounted overshoot. It is important to note that the discounting factor which appears in all of the identities means that one is never more than a Fourier transform away from the naturally associated space-time identity in which the additional law of the time to first passage is specified. This last Fourier inversion appears to be virtually impossible to produce analytically within the current context, but the expressions we offer are not difficult to work with in conjunction with straightforward Fourier inversion algorithms.

In Section 5 we look at the more complicated two-sided exit problem. Inspired by a technique of Rogozin [37] we solve a system of equations which characterize the discounted overshoot distribution on either side of the interval in question. The same technique also delivers explicit expressions for the discounted entrance law into an interval. In Section 6 we look in analytical detail at what can be said of the ascending and descending ladder variables. In particular we offer expressions for their joint Laplace exponent and jump measure. Section 7 mentions some additional examples of fluctuation identities which enjoy explicit detail and finally in Section 8 we describe some numerical experiments in order to exhibit the genuine practical applicability of our method.

2 Meromorphic Lévy processes

Recall that a general one-dimensional Lévy process is a stochastic process issued from the origin with stationary and independent increments and almost sure right continuous paths. We write $X = \{X_t : t \geq 0\}$ for its trajectory and $\mathbb{P}$ for its law. As X is necessarily a strong Markov process, for each $x \in \mathbb{R}$, we will need the probability $\mathbb{P}_x$ to denote the law of X when issued from x with the understanding that $\mathbb{P}_0 = \mathbb{P}$. The law $\mathbb{P}$ of Lévy processes is characterized by its one-time transition probabilities. In particular there always exist a triple (a, σ, Π) where $a \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and Π is a measure on $\mathbb{R}\backslash\{0\}$ satisfying the integrability condition $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$, such that, for all $z \in \mathbb{R}$

$$\mathbb{E}[e^{\mathrm{i}zX_t}] = \exp\{-\Psi(z)t\}, \tag{1}$$

where

$$\Psi(z) = \mathrm{i}az + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}}(1 - e^{\mathrm{i}zx} + \mathrm{i}zx\mathbf{1}_{\{|x|<1\}})\Pi(\mathrm{d}x).$$

There are four particular families of Lévy processes which have featured in recent literature, all of which have proved to have relevance to a number of applied probability models, all of which have exhibited some degree of mathematical tractability in the context that they have occurred. These processes are as follows.

- **Hyper-exponential Lévy processes:** The elementary but classical Kou model, [26], consists of a linear Brownian plus a compound Poisson process with two-sided exponentially distributed jump. The natural generalization of this model (which therefore includes the Kou model) is the hyper-exponential Lévy process for which the exponentially distributed jumps are replaced by hyper-exponentially distributed jumps. That is to say, the density of the Lévy measure is written

$$\pi(x) = \mathbb{I}_{\{x>0\}} \sum_{i=1}^{N} a_i \rho_i e^{-\rho_i x} + \mathbb{I}_{\{x<0\}} \sum_{i=1}^{\hat{N}} \hat{a}_i \hat{\rho}_i e^{\hat{\rho}_i x},$$

  where a_i, $\hat{a}_i$, ρ_i and $\hat{\rho}_i$ are positive numbers. One can verify that the characteristic exponent is a rational function and that hyper-exponential Lévy processes have finite activity jumps and paths of bounded variation unless $\sigma^2 > 0$.

- **β-class Lévy processes:** the β-class of Lévy processes was introduced by Kuznetsov [28]. The characteristic exponent is given by

$$\begin{aligned}\Psi(z) &= \mathrm{i}az + \frac{1}{2}\sigma^2 z^2 + \frac{c_1}{\beta_1}\left\{\mathrm{B}(\alpha_1, 1-\lambda_1) - \mathrm{B}(\alpha_1 - \frac{\mathrm{i}z}{\beta_1}, 1-\lambda_1)\right\} \\ &\quad + \frac{c_2}{\beta_2}\left\{\mathrm{B}(\alpha_2, 1-\lambda_2) - \mathrm{B}(\alpha_2 + \frac{\mathrm{i}z}{\beta_2}, 1-\lambda_2)\right\}\end{aligned}$$

where $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function, with parameter range $a \in \mathbb{R}, \sigma, c_i, \alpha_i, \beta_i > 0$ and $\lambda_1, \lambda_2 \in (0,3) \setminus \{1,2\}$. The corresponding Lévy measure Π has density

$$\pi(x) = c_1 \frac{e^{-\alpha_1\beta_1 x}}{(1-e^{-\beta_1 x})^{\lambda_1}} \mathbf{1}_{\{x>0\}} + c_2 \frac{e^{\alpha_2\beta_2 x}}{(1-e^{\beta_2 x})^{\lambda_2}} \mathbf{1}_{\{x<0\}}.$$

The large number of parameters allows one to choose Lévy processes within the β-class that have paths that are both of unbounded variation (when at least one of the conditions $\sigma \neq 0$, $\lambda_1 \in (2,3)$ or $\lambda_2 \in (2,3)$ holds) and bounded variation (when all of the conditions $\sigma = 0$, $\lambda_1 \in (0,2)$ and $\lambda_2 \in (0,2)$ hold) as well as having infinite and finite activity in the jump component (accordingly as both $\lambda_1, \lambda_2 \in (1,3)$ or $\lambda_1, \lambda_2 \notin (1,3)$). The β-class of Lévy processes includes another recently introduced family of Lévy processes known as Lamperti-stable processes, cf. [8, 9, 12].

- **θ-class Lévy processes:**

  The θ-class of Lévy processes was also introduced by Kuznetsov [27] as a family of processes with Gaussian component and two-sided jumps, characterized by the density of the Lévy measure

  $$\pi(x) = c_1\beta_1 e^{-\alpha_1 x}\Theta_k(x\beta_1)\mathbb{I}_{\{x>0\}} + c_2\beta_2 e^{\alpha_2 x}\Theta_k(-x\beta_2)\mathbb{I}_{\{x<0\}},$$

  where $\Theta_k(x)$ is the k-th order (fractional) derivative of the theta function $\theta_3(0, e^{-x})$:

  $$\Theta_k(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k}\theta_3(0, e^{-x}) = \mathbb{I}_{\{k=0\}} + 2\sum_{n\geq 1} n^{2k} e^{-n^2 x}, \quad x > 0$$

  The parameter $\chi = k + 1/2$ corresponds to the exponent of the singularity of $\pi(x)$ at $x = 0$, and when $\chi \in \{1/2, 3/2, 5/2\}$ ($\chi \in \{1,2\}$) the characteristic exponent of X_t is given in terms of trigonometric (digamma) functions.

- **Hypergeometric Lévy processes:** These processes were introduced in Kyprianou et al. [32, 29] as an example of how to use Vigon's theory of philanthropy. Their characteristic exponent is given by

  $$\Psi(z) = \mathtt{d}\mathrm{i}z + \frac{1}{2}\sigma^2 z^2 + \Phi_1(-\mathrm{i}z)\Phi_2(\mathrm{i}z),$$

  where $\mathtt{d}, \sigma \in \mathbb{R}$ and Φ_1, Φ_2 are the Laplace exponents of two subordinators from the β-class of Lévy processes. Note that such Laplace exponents necessarily take the form

  $$\Phi(\theta) = \mathtt{k} + \delta\theta + \frac{c}{\beta}\big\{\mathrm{B}\left(1-\alpha+\gamma, -\gamma\right) - \mathrm{B}\left(1-\alpha+\gamma+\theta/\beta, -\gamma\right)\big\}. \tag{2}$$

It turns out that all of the above four classes of Lévy processes fall under one umbrella through certain fundamental characteristics which we identify as a definition below.

Definition 1 (M-class)**.** A Lévy process X is said to belong to the Meromorophic class (M-class) if it respects the following five characteristics.

(i) The characteristic exponent $\Psi(z)$ is a meromorphic function which has poles at points $\{-\mathrm{i}\rho_n, \mathrm{i}\hat{\rho}_n\}_{n\geq 1}$, where ρ_n and $\hat{\rho}_n$ are positive real numbers.

(ii) For $q \geq 0$ function $q + \Psi(z)$ has roots at points $\{-\mathrm{i}\zeta_n, \mathrm{i}\hat{\zeta}_n\}_{n\geq 1}$ where ζ_n and $\hat{\zeta}_n$ are nonnegative real numbers (strictly positive if $q > 0$). We will write $\zeta_n(q)$, $\hat{\zeta}_n(q)$ if we need to stress the dependence on q.

(iii) The roots and poles of $q + \Psi(\mathrm{i}z)$ satisfy the following interlacing condition

$$... - \rho_2 < -\zeta_2 < -\rho_1 < -\zeta_1 < 0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < \hat{\rho}_2 < ... \tag{3}$$

(iv) There exists $\alpha > 1/2$ such that $\rho_n \sim cn^\alpha$ and $\hat{\rho}_n \sim \hat{c}n^\alpha$ as $n \to +\infty$.

(v) The Wiener-Hopf factors are expressed as convergent infinite products,

$$\phi_q^+(\mathrm{i}z) = \mathbb{E}\left[e^{-z\overline{X}_{\mathrm{e}(q)}}\right] = \prod_{n\geq 1} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n}}, \qquad \phi_q^-(-\mathrm{i}z) = \mathbb{E}\left[e^{z\underline{X}_{\mathrm{e}(q)}}\right] = \prod_{n\geq 1} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 + \frac{z}{\hat{\zeta}_n}}. \tag{4}$$

Henceforth we shall refer to (ζ, ρ) as the negative roots and poles of X and $(\hat{\zeta}, \hat{\rho})$ as the positive roots and poles of X.

Remark 1. Within the scope of Definition 1, when the process X_t has hyper-exponential positive jumps (a mixture of N exponentials), we have only a finite sequence $\{\rho_n\}_{n=1,2,..,N}$ and either N (or $N+1$) negative roots ζ_N. All the formulas presented in this paper are still valid in this case if we adopt notation $\rho_k = \infty$ for $k > N$ and $\zeta_k = \infty$ for $k > N$ (or $k > N+1$). For example, expression (4) for the Wiener-Hopf factor becomes a finite product

$$\phi_q^+(\mathrm{i}z) = \prod_{n=1}^{N} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n}}, \quad \text{or} \quad \phi_q^+(\mathrm{i}z) = \left[\prod_{n=1}^{N} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n}}\right] \frac{1}{1 + \frac{z}{\zeta_{N+1}}}$$

depending on whether we have N or $N+1$ negative roots ζ_n. The same remark holds true when we have hyper-exponential negative jumps.

As alluded to above, the hyper-exponential, β-, θ- and hypergeometric class of Lévy processes all conform to Definition 1. We note in particular that for condition (iv), $\alpha = 1$ for β-family and hypergeometric family, and $\alpha = 2$ for θ-family of processes.

From here on, unless otherwise stated, we shall always assume that X is a Lévy process belonging to the M-class and that X is not a compound Poisson process.

3 Partial fraction decomposition for infinite products

As indicated in the introduction, our goal will be to extract a family of fluctuation identities from the Wiener-Hopf factors of X which are familiar to a number of applications in both mathematical finance and insurance mathematics. To this end, it is important to understand how to write the infinite products that describe the latter as infinite sums. Naively speaking, with finite products of rational functions, one would normally look for a partial fraction decomposition. This is exactly the basis of the next lemma, even when the product is infinite.

Lemma 1. *Assume that we have two increasing sequences* $\rho = \{\rho_n\}_{n\geq 1}$ *and* $\zeta = \{\zeta_n\}_{n\geq 1}$ *of positive numbers which satisfy the following conditions.*

(i) Interlacing condition,

$$\zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \ldots \tag{5}$$

(ii) There exists $\alpha > 1/2$ *and* $\epsilon > 0$ *such that* $\rho_n > \epsilon n^\alpha$ *for all integer numbers* n*.*

Then we have the following partial fraction decompositions

$$\begin{aligned} \prod_{n\geq 1} \frac{1+\frac{z}{\rho_n}}{1+\frac{z}{\zeta_n}} &= \mathrm{a}_0(\rho,\zeta) + \sum_{n\geq 1} \mathrm{a}_n(\rho,\zeta)\frac{\zeta_n}{\zeta_n+z}, && (6)\\ \prod_{n\geq 1} \frac{1+\frac{z}{\zeta_n}}{1+\frac{z}{\rho_n}} &= 1 + z\mathrm{b}_0(\zeta,\rho) + \sum_{n\geq 1} \mathrm{b}_n(\zeta,\rho)\left[1-\frac{\rho_n}{\rho_n+z}\right], && (7) \end{aligned}$$

where

$$\begin{aligned} \mathrm{a}_0(\rho,\zeta) &= \lim_{n\to+\infty}\prod_{k=1}^{n}\frac{\zeta_k}{\rho_k}, \quad \mathrm{a}_n(\rho,\zeta) = \left(1-\frac{\zeta_n}{\rho_n}\right)\prod_{\substack{k\geq 1\\ k\neq n}}\frac{1-\frac{\zeta_n}{\rho_k}}{1-\frac{\zeta_n}{\zeta_k}}, && (8)\\ \mathrm{b}_0(\zeta,\rho) &= \frac{1}{\zeta_1}\lim_{n\to+\infty}\prod_{k=1}^{n}\frac{\rho_k}{\zeta_{k+1}}, \quad \mathrm{b}_n(\zeta,\rho) = -\left(1-\frac{\rho_n}{\zeta_n}\right)\prod_{\substack{k\geq 1\\ k\neq n}}\frac{1-\frac{\rho_n}{\zeta_k}}{1-\frac{\rho_n}{\rho_k}}. && (9) \end{aligned}$$

Proof. Consider an integer number m and define a rational function

$$f_m(z) = \prod_{n=1}^{m}\frac{1+\frac{z}{\rho_n}}{1+\frac{z}{\zeta_n}}.$$

Writing the partial fraction decomposition for this function we obtain

$$f_m(z) = \mathrm{a}_{0,m}(\rho,\zeta) + \sum_{n\geq 1}\mathrm{a}_{n,m}(\rho,\zeta)\frac{\zeta_n}{\zeta_n+z}, \tag{10}$$

where

$$\mathrm{a}_{0,m}(\rho,\zeta) = \prod_{k=1}^{m}\frac{\zeta_k}{\rho_k}, \qquad \mathrm{a}_{n,m}(\rho,\zeta) = \mathbb{I}(n\leq m)\left(1-\frac{\zeta_n}{\rho_n}\right)\prod_{\substack{k=1,2,..,m\\ k\neq n}}\frac{1-\frac{\zeta_n}{\rho_k}}{1-\frac{\zeta_n}{\zeta_k}}.$$

Taking limit as $m\to+\infty$ of both sides of (10) we, find for $z>0$, the product on the left side converges to a positive value due to condition (ii), $\mathrm{a}_{0,m}(\rho,\zeta)$ converges to a non-negative number $\mathrm{a}_0(\rho,\zeta)$, since its logarithm is a sum of strictly negative terms, thus we obtain

$$\prod_{n\geq 1}\frac{1+\frac{z}{\rho_n}}{1+\frac{z}{\zeta_n}} = \mathrm{a}_0(\rho,\zeta) + \lim_{m\to+\infty}\sum_{n\geq 1}\mathrm{a}_{n,m}(\rho,\zeta)\frac{\zeta_n}{\zeta_n+z}.$$

In order to interchange the limit and infinite series we use dominated convergence series and the fact that the interlacing condition (5) guarantees that for each $n \geq 1$ coefficients $\mathrm{a}_{n,m}(\rho,\zeta)$ are non-negative and increasing in m. Thus we have proved (6) for $z > 0$, and for general z the result follows by analytic continuation.

To prove the second partial fraction decomposition, rewrite the infinite product as

$$\left(1+\frac{z}{\zeta_1}\right)\prod_{n\geq 1}\frac{1+\frac{z}{\zeta_{n+1}}}{1+\frac{z}{\rho_n}}$$

and note that sequences $\{\zeta_{n+1}\}_{n\geq 1}$ and $\{\rho_n\}_{n\geq 1}$ satisfy interlacing condition, thus we can apply previous results. The details are left to the reader. $\square$

By applying simple Laplace inversion in the formulae (6) and (7) we obtain the following corollary.

Corollary 1. *Assume that conditions (i) and (ii) of Lemma 1 are satisfied. Define*

$$\phi(z) = \prod_{n\geq 1}\frac{1+\frac{z}{\rho_n}}{1+\frac{z}{\zeta_n}}.$$

Then coefficients $\mathrm{a}_n(\rho,\zeta)$ and $\mathrm{b}_n(\zeta,\rho)$ are nonnegative for all $n \geq 0$ and we have

$$\phi(z) = \mathrm{a}_0(\rho,\zeta) + \int\limits_{\mathbb{R}^+}\left[\sum_{n\geq 1}\mathrm{a}_n(\rho,\zeta)\zeta_n e^{-\zeta_n x}\right]e^{-zx}\mathrm{d}x, \tag{11}$$

$$\frac{1}{\phi(z)} = 1 + z\mathrm{b}_0(\zeta,\rho) + \int\limits_{\mathbb{R}^+}\left[\sum_{n\geq 1}\mathrm{b}_n(\zeta,\rho)\rho_n e^{-\rho_n x}\right]\left(1-e^{-zx}\right)\mathrm{d}x. \tag{12}$$

For the special case that we take the sequences ζ and ρ to be either the positive roots and poles or the negative roots and poles, we get an important corollary which plays a crucial role in establishing identities concerning first exit from a half line. Before stating it we recall that a Lévy process creeps upwards if for some (and then all) $x \geq 0$, $\mathbb{P}(X_{\tau_x^+} = x | \tau_x^+ < \infty) > 0$. Moreover, we say that 0 is irregular for $(0,\infty)$ if and only if $\mathbb{P}(\tau_0^+ > 0) = 1$. (Note that this probability can only be 0 or 1 thanks to the Blumenthal zero-one law). We refer to Bertoin [5], Doney [13] or Kyprianou [31] for more extensive discussion of these subtle path properties.

We also introduce some more notation. In the forthcoming text everything will depend on the coefficients $\{\mathrm{a}_n(\rho,\zeta),\mathrm{a}_n(\hat\rho,\hat\zeta)\}_{n\geq 0}$ defined using (8) and $\{\mathrm{b}_n(\zeta,\rho),\mathrm{b}_n(\hat\zeta,\hat\rho)\}_{n\geq 0}$ defined using (9). Note that all these coefficients are nonnegative. We define for convenience a column vector

$$\bar{\mathrm{a}}(\rho,\zeta) = [\mathrm{a}_0(\rho,\zeta),\mathrm{a}_1(\rho,\zeta),\mathrm{a}_2(\rho,\zeta),...]^T$$

and similarly for $\bar{\mathrm{a}}(\hat\rho,\hat\zeta)$, $\bar{\mathrm{b}}(\zeta,\rho)$ and $\bar{\mathrm{b}}(\hat\zeta,\hat\rho)$. Next, given a sequence of positive numbers $\zeta = \{\zeta_n\}_{n\geq 1}$, we define a column vector $\bar{\mathrm{v}}(\zeta,x)$ as a vector of distributions

$$\bar{\mathrm{v}}(\zeta,x) = \left[\delta_0(x),\zeta_1 e^{-\zeta_1 x},\zeta_2 e^{-\zeta_2 x},\dots\right]^T,$$

where $\delta_0(x)$ is the Dirac delta function at $x = 0$.

Finally, let $\overline{X}_t = \sup_{s\leq t} X_s$, $\underline{X}_t = \inf_{s\leq t} X_s$ and denote by $\mathrm{e}(q)$ an independent, exponentially distributed random variable with rate $q > 0$.

Corollary 2.

(i) For $x \geq 0$

$$\mathbb{P}(\overline{X}_{\mathrm{e}(q)} \in \mathrm{d}x) = \bar{\mathrm{a}}(\rho,\zeta)^T \times \bar{\mathrm{v}}(\zeta,x)\mathrm{d}x, \quad \mathbb{P}(-\underline{X}_{\mathrm{e}(q)} \in \mathrm{d}x) = \bar{\mathrm{a}}(\hat{\rho},\hat{\zeta})^T \times \bar{\mathrm{v}}(\hat{\zeta},x)\mathrm{d}x. \tag{13}$$

(ii) $\mathrm{a}_0(\rho,\zeta)$ (equiv. $\mathrm{a}_0(\hat{\rho},\hat{\zeta})$) *is nonzero if and only if* 0 *is irregular for* $(0,\infty)$ (equiv. $(-\infty,0)$).

(iii) $\mathrm{b}_0(\zeta,\rho)$ (equiv. $\mathrm{b}_0(\hat{\zeta},\hat{\rho})$) *is nonzero if and only if the process X_t creeps upwards.* (equiv. downwards)

Proof. We start by proving (i). This follows as an immediate consequence of the assumed structure of the Wiener-Hopf factorization (4) together with (11) in the previous corollary. Next we note that 0 is irregular for $(0,\infty)$ if and only if, for any $q>0$, $\overline{X}_{\mathrm{e}(q)}$ has an atom in its distribution at 0. From part (i) this is clearly the case if and only if $\mathrm{a}_0(\rho,\zeta)$ is non-zero. Finally for part (iii) of the proof, we note that we may necessarily write

$$\phi_q^+(\mathrm{i}z) = \frac{\kappa(q,0)}{\kappa(q,z)},$$

where $\kappa(q,z)$ is the Laplace exponent of the bivariate subordinator which describes the ascending ladder process of X. It is also known that a Lévy process creeps upwards if and only if the subordinator describing its ladder height process has a linear drift component. The drift coefficient is then described by $\lim_{z\uparrow\infty}\kappa(q,z)/z \in [0,\infty)$ where the limit is independent of the value of $q \geq 0$. Inspecting (12) we see that the required drift coefficient is non-zero if and only if $\mathrm{b}_0(\zeta,\rho)$ is non-zero. The conclusion thus follows. □

Remark 2. When the process X has hyper-exponential positive jumps, calculation of coefficients $\mathrm{a}(\rho,\zeta)$ and $\mathrm{b}(\rho,\zeta)$ is easier, since the corresponding expressions in (8) and (9) are just finite products. Moreover, calculation of the coefficients $\mathrm{a}(\hat{\rho},\hat{\zeta})$ and $\mathrm{b}(\hat{\rho},\hat{\zeta})$ is also simplified considerably even if the sequence $\{\hat{\rho}_n\}_{n\geq 1}$ is infinite, since we can use Wiener-Hopf factorization and the fact that $\mathrm{i}\hat{\zeta}_n$ and $\mathrm{i}\hat{\rho}_n$ are simple roots (poles) of function $q+\Psi(z)$. For example we can compute coefficients $\mathrm{a}_n(\hat{\rho},\hat{\zeta})$ as follows:

$$\mathrm{a}_n(\hat{\rho},\hat{\zeta}) = \frac{1}{\hat{\zeta}_n}\mathrm{Res}\left(\phi_q^-(z) : z = \mathrm{i}\hat{\zeta}_n\right) = \frac{1}{\hat{\zeta}_n}\mathrm{Res}\left(\frac{q}{q+\Psi(z)}\frac{1}{\phi_q^+(z)} : z = \mathrm{i}\hat{\zeta}_n\right) = \frac{1}{\hat{\zeta}_n}\frac{q}{\Psi'(\mathrm{i}\hat{\zeta}_n)\phi_q^+(\mathrm{i}\hat{\zeta}_n)}$$

and this expression is more convenient than (8), since usually we know $\Psi'(z)$ explicitly and $\phi_q^+(z)$ is just a rational function.

The Lévy measure of β-, θ- and hypergeometric processes is known in closed form. However as the next corollary shows, it also can be reconstructed from roots and poles of $q+\Psi(z)=0$. This has the spirit of Vigon's theory of philanthropy (cf. [38]) in that we construct the Lévy measure from the Wiener-Hopf factors.

Corollary 3. *Assume $q>0$. The Lévy measure of an M-class Lévy process X has density which is an infinite mixture of exponential distributions,*

$$\pi(x) = \mathbb{I}_{\{x>0\}}\sum_{n\geq 1} a_n\rho_n e^{-\rho_n x} + \mathbb{I}_{\{x<0\}}\sum_{n\geq 1}\hat{a}_n\hat{\rho}_n e^{\hat{\rho}_n x}, \tag{14}$$

where a_n and $\hat{a}_n$ are positive coefficients defined as

$$a_n = \mathrm{b}_n(\zeta,\rho)\frac{q}{\phi_q^-(-\mathrm{i}\rho_n)}, \quad \hat{a}_n = \mathrm{b}_n(\hat{\zeta},\hat{\rho})\frac{q}{\phi_q^+(\mathrm{i}\hat{\rho}_n)}. \tag{15}$$

Proof. The rigorous proof can be obtained with the help of (7) and (4) by writing partial fraction decomposition of function

$$q + \Psi(\mathrm{i}z) = \frac{q}{\phi_q^+(\mathrm{i}z)\phi_q^-(\mathrm{i}z)}. \tag{16}$$

Algebraic computations are straightforward but lengthy and we leave them to the reader. We offer here instead a simple and intuitive justification of this result. Assume that (14) is true, then by removing the cut-off function from the integral in the Lévy-Khintchine formula, we may write

$$\Psi(\mathrm{i}z) = -\frac{\sigma^2}{2}z^2 + a'z - \sum_{n\geq 1} a_n \left[\frac{\rho_n}{\rho_n + z} - 1 + \frac{z}{\rho_n}\right] - \sum_{n\geq 1} \hat{a}_n \left[\frac{\hat{\rho}_n}{\hat{\rho}_n - z} - 1 - \frac{z}{\hat{\rho}_n}\right]$$

for an appropriate value of $a' \in \mathbb{R}$. Therefore the coefficients a_n are related to residues of $\Psi(\mathrm{i}z)$ by the relation

$$a_n = -\frac{1}{\rho_n}\mathrm{Res}(\Psi(\mathrm{i}z) : z = -\rho_n).$$

Moreover, since $\Psi(\mathrm{i}z)$ can be factorized as (16) and $z = -\rho_n$ is a simple pole of $\phi_q^+(\mathrm{i}z)$ while $\phi_q^-(\mathrm{i}z)$ is analytic at $z = -\rho_n$ we obtain

$$a_n = \frac{1}{\rho_n}\frac{q}{\phi_q^-(-\mathrm{i}\rho_n)}\mathrm{Res}(\phi_q^+(\mathrm{i}z)^{-1} : z = -\rho_n).$$

The partial fraction decomposition (7) tells us that $\mathrm{Res}(\phi_q^+(\mathrm{i}z)^{-1} : z = -\rho_n) = -\rho_n \mathrm{b}_n(\zeta, \rho)$ which proves (15). □

Remark 3. It is quite a simple result that if the Lévy measure is a mixture of exponential distributions then the roots and poles will satisfy the interlacing condition (3) (see [27] for the proof of this fact). Corollary 3 shows that the converse is also true. Thus instead of using property (iii) in the definition of the M-class we could require that the Lévy measure is a mixture of exponentials distributions.

Remark 4. Note that the representation of the Lévy measure in (14) depends on q on the right hand side but not on the left hand side. It is therefore tempting to take limits as $q \downarrow 0$ on the right hand side. This is not as straightforward as it seems as the limits of a_n and $\hat{a}_n$ are not easy to compute. It would be more straightforward to start with the Wiener-Hopf factorization for the case that $q = 0$ and then perform the same analysis as in Corollary 3. The next corollary provides us with the aforementioned Wiener-Hopf factorization.

Corollary 4. *Assume $\mathbb{E}X_1 = \mu > 0$. As $q \to 0^+$ we have $q/\zeta_1(q) \to \mu$ and for $n \geq 1$*

$$\hat{\zeta}_n(q) \to \hat{\zeta}_n(0) \neq 0, \quad \zeta_{n+1}(q) \to \zeta_{n+1}(0) \neq 0.$$

We have the Wiener-Hopf factorization $\Psi(\mathrm{i}z) = \kappa(0, z)\hat{\kappa}(0, -z)$, where

$$\begin{aligned} \kappa(0, z) &= \lim_{q\to 0^+} \frac{q}{\phi_q^+(\mathrm{i}z)} = \mu z \prod_{n\geq 1} \frac{1 + \frac{z}{\zeta_{n+1}(0)}}{1 + \frac{z}{\rho_n}}, \\ \hat{\kappa}(0, z) &= \lim_{q\to 0^+} \frac{1}{\phi_q^-(-\mathrm{i}z)} = \frac{1}{\phi_0^-(-\mathrm{i}z)} = \prod_{n\geq 1} \frac{1 + \frac{z}{\hat{\zeta}_n(0)}}{1 + \frac{z}{\hat{\rho}_n}}. \end{aligned} \tag{17}$$

Proof. Since $\mathbb{E}X_1 > 0$ we know that $\overline{X}_{\mathrm{e}(q)} \to +\infty$ and $\underline{X}_{\mathrm{e}(q)} \to \underline{X}_\infty$ as $q \to 0^+$, and also that $\underline{X}_\infty < \infty$ with probability one. Therefore the Wiener-Hopf factor $\phi_q^+(\mathrm{i}z) = \mathbb{E}[\exp(-z\overline{X}_{\mathrm{e}(q)})]$ must converge to 0 as $q \to 0^+$ while $\phi_q^-(-\mathrm{i}z) = \mathbb{E}[\exp(z\underline{X}_{\mathrm{e}(q)})]$ must converge to $\mathbb{E}[\exp(z\underline{X}_\infty)]$ as $q \to 0^+$. Since all roots $\zeta_n(q)$ and $\hat{\zeta}_n(q)$ have nonzero limits as $q \to 0^+$ if $n \geq 2$ (this true due to interlacing condition (3) and the fact that $\Psi(z)$ is a meromorphic function) we conclude that $\zeta_1(q)$ must go to zero while $\hat{\zeta}_1(q) \to \hat{\zeta}_1(0) \neq 0$. Function $\Psi(\mathrm{i}z)$ is analytic in the neighbourhood or $z = 0$, and it's derivative $\Psi'(0) = -\mathrm{i}\mu \neq 0$, thus using Implicit Function Theorem we conclude that $\zeta_1(q)$ is analytic function of q in some neighbourhood of $q = 0$, and taking derivative with respect to q of the equation $q + \Psi(\mathrm{i}z) = 0$ we find that

$$\frac{\mathrm{d}}{\mathrm{d}q}\zeta_1(q) = -\frac{\mathrm{i}}{\Psi'(-\mathrm{i}\zeta_1(q))}.$$

Since $\Psi'(0) = -\mathrm{i}\mu$ we conclude that $q/\zeta_1(q) \to \mu$ as $q \to 0^+$. Using this fact and the formulae in (4) for the Wiener-Hopf factors we obtain expressions (17). □

4 One-sided exit problem

Doney and Kyprianou [14] have given a detailed characterization of the one-sided exit problem through the so-called quintuple law. The latter can be easily integrated out to give the following discounted triple law which is also known in the actuarial mathematics literature as the Gerber-Shiu measure.

Lemma 2. *Fix $c > 0$. Define*

$$\tau_c^+ = \inf\{t > 0 : X_t > c\}.$$

For all $q, y, z > 0$ and $u \in [0, c \vee z]$ we have

$$\mathbb{E}\left[e^{-q\tau_c^+}\mathbb{I}(X_{\tau_c^+} - c \in \mathrm{d}y\ ;\ c - X_{\tau_c^+ -} \in \mathrm{d}z\ ;\ c - \overline{X}_{\tau_c^+ -} \in \mathrm{d}u)\right]$$
$$= \frac{1}{q}\mathbb{P}(\overline{X}_{\mathrm{e}(q)} \in c - \mathrm{d}u)\mathbb{P}(-\underline{X}_{\mathrm{e}(q)} \in \mathrm{d}z - u)\Pi(\mathrm{d}y + z).$$

Taking account of Corollary 2 this gives us the following immediate corollary for the M-class of Lévy processes.

Corollary 5. *Fix $c > 0$. For all $q, y, z > 0$ and $u \in [0, c \vee z]$ we have*

$$\mathbb{E}\left[e^{-q\tau_c^+}\mathbb{I}(X_{\tau_c^+} - c \in \mathrm{d}y\ ;\ c - X_{\tau_c^+ -} \in \mathrm{d}z\ ;\ c - \overline{X}_{\tau_c^+ -} \in \mathrm{d}u)\right]$$
$$= \frac{1}{q}\left[\bar{\mathrm{a}}(\rho, \zeta)^T \times \bar{\mathrm{v}}(\zeta, c - u)\right]\left[\bar{\mathrm{a}}(\hat{\rho}, \hat{\zeta})^T \times \bar{\mathrm{v}}(\hat{\zeta}, z - u)\right]\Pi(\mathrm{d}y + z)\mathrm{d}z\mathrm{d}u.$$

Often, one is only interested in the discounted overshoot distribution. In principle this can be obtained from the above formula by integrating out the variables z and u. However, it turns out to be more straightforward to prove this directly, particularly as otherwise it would require us to specify in more detail the Lévy measure in terms of poles and roots for M-class (cf. Corollary 3). The following result does precisely this, but in addition, it also gives us the probability of creeping.

Theorem 1. *Define a matrix* $\mathbf{A} = \{a_{i,j}\}_{i,j\geq 0}$ *as*

$$a_{i,j} = \begin{cases} 0 & \text{if } i=0,\ j\geq 0 \\ \mathrm{a}_i(\rho,\zeta)\mathrm{b}_0(\zeta,\rho) & \text{if } i\geq 1,\ j=0 \\ \dfrac{\mathrm{a}_i(\rho,\zeta)\mathrm{b}_j(\zeta,\rho)}{\rho_j-\zeta_i} & \text{if } i\geq 1,\ j\geq 1 \end{cases} \tag{18}$$

Then for $c>0$ *and* $y\geq 0$ *we have*

$$\mathbb{E}\left[e^{-q\tau_c^+}\mathbb{I}\left(X_{\tau_c^+}-c\in \mathrm{d}y\right)\right] \;=\; \bar{\mathrm{v}}(\zeta,c)^T\times \mathbf{A}\times \bar{\mathrm{v}}(\rho,y)\mathrm{d}y. \tag{19}$$

Proof. We start with the formula from Lemma 1 in [1]

$$\mathbb{E}\left[e^{-q\tau_c^+-z(X_{\tau_c^+}-c)}\right] = \frac{\mathbb{E}\left[e^{-z(\overline{X}_{\mathrm{e}(q)}-c)}\mathbb{I}(\overline{X}_{\mathrm{e}(q)}>c)\right]}{\mathbb{E}\left[e^{-z\overline{X}_{\mathrm{e}(q)}}\right]}. \tag{20}$$

Using formula (13) we find that

$$\mathbb{E}\left[e^{-z(\overline{X}_{\mathrm{e}(q)}-c)}\mathbb{I}(\overline{X}_{\mathrm{e}(q)}>c)\right] = \sum_{i\geq 1}\mathrm{a}_i(\rho,\zeta)\frac{\zeta_i}{z+\zeta_i}e^{-\zeta_i c}.$$

Next, using (4) and (7) after some algebraic manipulations we find that

$$\frac{1}{(z+\zeta_i)\mathbb{E}\left[e^{-z\overline{X}_{\mathrm{e}(q)}}\right]} = \frac{1}{(z+\zeta_i)}\prod_{n\geq 1}\frac{1+\frac{z}{\zeta_n}}{1+\frac{z}{\rho_n}} \;=\; \mathrm{b}_0(\zeta,\rho)+\sum_{j\geq 1}\frac{\mathrm{b}_j(\zeta,\rho)}{\rho_j-\zeta_i}\frac{\rho_j}{\rho_j+z}.$$

Combining the above three equations we find

$$\mathbb{E}\left[e^{-q\tau_c^+-z(X_{\tau_c^+}-c)}\right] = \sum_{i\geq 1}\mathrm{a}_i(\rho,\zeta)\mathrm{b}_0(\zeta,\rho)\zeta_i e^{-\zeta_i c}+\sum_{i\geq 1}\sum_{j\geq 1}\frac{\mathrm{a}_i(\rho,\zeta)\mathrm{b}_j(\zeta,\rho)}{\rho_j-\zeta_i}\zeta_i e^{-\zeta_i c}\frac{\rho_j}{\rho_j+z}$$

which allows for a straightforward inversion in z, thereby completing the proof. □

5 Entrance/exit problems for a finite interval

The two sided exit problem is a long standing problem of interest from the theory of Lévy processes and there are very few cases where explicit identities have been be extracted for the (discounted) overshoot distribution on either side of the interval. Classically the only examples with discounting which have been analytically tractable are those of jump diffusion processes with (hyper)-exponentially distributed jumps (see for example [24, 39]) and, up to knowing the so-called scale function, spectrally negative processes, [31]. Otherwise, the only other examples, without discounting, are those of stable processes, [37], and Lamperti-stable processes, [12].

Recall that

$$\tau_a^+ = \inf\{t>0 : X_t > a\}$$

and in addition define

$$\tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

For $f \in \mathcal{L}_\infty(\mathbb{R})$, the space of positive, measurable and uniformly bounded functions on $\mathbb{R}$, and $x \in \mathbb{R}$, define the following operators

$$\begin{aligned}(\mathcal{G}f)(x) &= \mathbb{I}(0 \le x \le a)\mathbb{E}_x\left[e^{-q\tau_a^+} f(X_{\tau_a^+})\mathbb{I}\left(\tau_a^+ < \tau_0^-\right)\right], \\ (\hat{\mathcal{G}}f)(x) &= \mathbb{I}(0 \le x \le a)\mathbb{E}_x\left[e^{-q\tau_0^-} f(X_{\tau_0^-})\mathbb{I}\left(\tau_0^- < \tau_a^+\right)\right]\end{aligned} \tag{21}$$

and for any subset $I \in \mathbb{R}$ define

$$\begin{aligned}(\mathcal{P}_I f)(x) &= \mathbb{I}(x \in I)\mathbb{E}_x\left[e^{-q\tau_a^+} f(X_{\tau_a^+})\right], \\ (\hat{\mathcal{P}}_I f)(x) &= \mathbb{I}(x \in I)\mathbb{E}_x\left[e^{-q\tau_0^-} f(X_{\tau_0^-})\right].\end{aligned} \tag{22}$$

These are bounded operators from $\mathcal{L}_\infty(\mathbb{R})$ into $\mathcal{L}_\infty(\mathbb{R})$. Note also that if $X_0 < 0$ then $\tau_0^- = 0$ and similarly if $X_0 > a$ then $\tau_a^+ = 0$ so, for example, when $x > a$, $\mathbb{E}_x\left[e^{-q\tau_a^+} f(X_{\tau_a^+})\right] = f(x)$ and when $x < 0$, $\mathbb{E}_x\left[e^{-q\tau_0^-} f(X_{\tau_0^-})\right] = f(x)$.

Following similar reasoning to Rogozin [37] (see also [23]), an application of the Markov property tells us that for all $x \in [0, a]$ we have

$$\begin{aligned}\mathbb{E}_x\left[e^{-q\tau_a^+} f(X_{\tau_a^+})\right] &= \mathbb{E}_x\left[e^{-q\tau_a^+} f(X_{\tau_a^+})\mathbb{I}\left(\tau_a^+ < \tau_0^-\right)\right] \\ &\quad +\mathbb{E}_x\left[e^{-q\tau_0^-}\mathbb{I}\left(\tau_0^- < \tau_a^+\right)\mathbb{E}_{X_{\tau_0^-}}\left[e^{-q\tau_a^+} f(X_{\tau_a^+})\right]\right].\end{aligned}$$

Thus we have the following operator identities:

$$\begin{cases}\mathcal{P}_{[0,a]} &= \mathcal{G} + \hat{\mathcal{G}}\mathcal{P}_{(-\infty,0]}, \\ \hat{\mathcal{P}}_{[0,a]} &= \hat{\mathcal{G}} + \mathcal{G}\hat{\mathcal{P}}_{[a,\infty)}.\end{cases} \tag{23}$$

Consider a space $\mathcal{H}$ of bounded linear operators on $\mathcal{L}_\infty(\mathbb{R})$ with the norm

$$||\mathcal{G}|| = \sup_{||f||_\infty \le 1} ||\mathcal{G}f||_\infty.$$

In the next theorem we give a series representation for $\mathcal{G}$ in terms of $\hat{\mathcal{P}}_{[0,a]}$, $\mathcal{P}_{(-\infty,0]}$, $\hat{\mathcal{P}}_{[a,\infty)}$, $\mathcal{P}_{(-\infty,0]}$ and a similar one for $\hat{\mathcal{G}}$. In the case of stable processes and general Lévy processes respectively, Rogozin [37] and Kadankov and Kadankova [23] have shown similar series representations. The presentation we give above only differs in that it takes the form of linear operators. This turns out to be more convenient for the particular application to the M-class of processes.

Theorem 2. *Assume $q > 0$. A pair of operators $(\mathcal{G}, \hat{\mathcal{G}})$ given by (21) is the unique solution in $\mathcal{H}$ to the system of equations (23). Moreover, This solution can also be represented in series form*

$$\mathcal{G} = \mathcal{P}_{[0,a]} - \hat{\mathcal{P}}_{[0,a]}\mathcal{P}_{(-\infty,0]} + \mathcal{P}_{[0,a]}\hat{\mathcal{P}}_{[a,\infty)}\mathcal{P}_{(-\infty,0]} - \hat{\mathcal{P}}_{[0,a]}\mathcal{P}_{(-\infty,0]}\hat{\mathcal{P}}_{[a,\infty)}\mathcal{P}_{(-\infty,0]} + \dots \tag{24}$$

and

$$\hat{\mathcal{G}} = \hat{\mathcal{P}}_{[0,a]} - \mathcal{P}_{[0,a]}\hat{\mathcal{P}}_{[a,\infty)} + \hat{\mathcal{P}}_{[0,a]}\mathcal{P}_{(-\infty,0]}\hat{\mathcal{P}}_{[a,\infty)} - \mathcal{P}_{[0,a]}\hat{\mathcal{P}}_{[a,\infty)}\mathcal{P}_{(-\infty,0]}\hat{\mathcal{P}}_{[a,\infty)} + \dots.$$

where, in both cases, convergence is exponential in the following sense. There exist $\eta \in (0,1)$ and a constant $C > 0$ such that if $\mathcal{S}_n$ is the sum of the first n terms in the series describing $\mathcal{G}$ then $||\mathcal{G} - \mathcal{S}_n|| \le C\eta^{n+1}$, with a similar statement holding for $\hat{\mathcal{G}}$.

Proof. It suffices to prove the result for $\mathcal{G}$, the proof for $\hat{\mathcal{G}}$ follows by duality. For uniqueness, we note first that the norm of operators $\mathcal{P}_{(-\infty,0]}$ is stricly less than one:

$$||\mathcal{P}_{(-\infty,0]}|| = \sup_{||f||_\infty \le 1} \left[\sup_{x\in\mathbb{R}} \mathbb{I}(x \le 0)\mathbb{E}_x \left[e^{-q\tau_a^+} f(X_{\tau_a^+})\right]\right] \le \sup_{x\in\mathbb{R}} \mathbb{I}(x \le 0)\mathbb{E}_x \left[e^{-q\tau_a^+}\right] = \mathbb{E}_0 \left[e^{-q\tau_a^+}\right] < 1$$

and similarly we find $||\hat{\mathcal{P}}_{[a,\infty)}|| < 1$. Uniqueness also follows from the latter fact. Indeed if we assume that we have another pair of solutions $(G, \hat{G})$, then using the system of equations (23) we find that the difference $\mathcal{G} - G$ must satisfy the equation

$$\mathcal{G} - G = (\mathcal{G} - G)\hat{\mathcal{P}}_{[a,\infty)}\mathcal{P}_{(-\infty,0]}.$$

Thus we find

$$||\mathcal{G} - G|| \le ||\mathcal{G} - G|| \times ||\mathcal{P}_{[a,\infty)}|| \times ||\mathcal{P}_{(-\infty,0]}||$$

and since $||\mathcal{P}_{[a,\infty)}|| \vee ||\mathcal{P}_{(-\infty,0]}|| < 1$ we conclude that $||\mathcal{G} - G|| = 0$ and $G = \mathcal{G}$. To prove series representation (24) we use (23) to find

$$\mathcal{G} = \mathcal{P}_{[0,a]} - \hat{\mathcal{P}}_{[0,a]}\mathcal{P}_{(-\infty,0]} + \mathcal{G}\hat{\mathcal{P}}_{[a,\infty)}\mathcal{P}_{(-\infty,0]}.$$

By iterating this equation we obtain series representation (24), which converges at an exponential rate as described in the statement of the theorem with $\eta = ||\hat{\mathcal{P}}_{[a,\infty)}|| \vee ||\mathcal{P}_{(-\infty,0]}|| \in (0, 1)$. □

The next theorem converts the above general setting to the specific setting of M-processes.

Theorem 3. (First exit from a finite interval) *Let $a > 0$ and define a matrix $\mathbf{B} = \mathbf{B}(\hat{\rho}, \zeta, a) = \{b_{i,j}\}_{i,j\ge 0}$ with*

$$b_{i,j} = \begin{cases} \zeta_j e^{-a\zeta_j} & \text{if } i = 0,\ j \ge 1 \\ 0 & \text{if } i \ge 0,\ j = 0 \\ \dfrac{\hat{\rho}_i\zeta_j}{\hat{\rho}_i + \zeta_j} e^{-a\zeta_j} & \text{if } i \ge 1,\ j \ge 1 \end{cases} \tag{25}$$

and similarly $\hat{\mathbf{B}} = \mathbf{B}(\rho, \hat{\zeta}, a)$. There exist matrices $\mathbf{C}_1$, $\mathbf{C}_2$ and $\hat{\mathbf{C}}_1$, $\hat{\mathbf{C}}_2$ such that for $x \in (0, a)$ we have

$$\begin{aligned} \mathbb{E}_x \left[e^{-q\tau_a^+}\mathbb{I}\left(X_{\tau_a^+} \in \mathrm{d}y\ ;\ \tau_a^+ < \tau_0^-\right)\right] &= \left[\bar{\mathrm{v}}(\zeta, a-x)^T \times \mathbf{C}_1 + \bar{\mathrm{v}}(\hat{\zeta}, x)^T \times \mathbf{C}_2\right] \times \bar{\mathrm{v}}(\rho, y-a)\mathrm{d}y \\ \mathbb{E}_x \left[e^{-q\tau_0^-}\mathbb{I}\left(X_{\tau_0^-} \in \mathrm{d}y\ ;\ \tau_0^- < \tau_a^+\right)\right] &= \left[\bar{\mathrm{v}}(\hat{\zeta}, x)^T \times \hat{\mathbf{C}}_1 + \bar{\mathrm{v}}(\zeta, a-x)^T \times \hat{\mathbf{C}}_2\right] \times \bar{\mathrm{v}}(\hat{\rho}, -y)\mathrm{d}y \end{aligned} \tag{26}$$

These matrices satisfy the following system of linear equations

$$\begin{cases} \mathbf{C}_1 &= \mathbf{A} - \hat{\mathbf{C}}_2\mathbf{B}\mathbf{A} \\ \hat{\mathbf{C}}_2 &= -\mathbf{C}_1\hat{\mathbf{B}}\hat{\mathbf{A}} \end{cases} \qquad \begin{cases} \hat{\mathbf{C}}_1 &= \hat{\mathbf{A}} - \mathbf{C}_2\hat{\mathbf{B}}\hat{\mathbf{A}} \\ \mathbf{C}_2 &= -\hat{\mathbf{C}}_1\mathbf{B}\mathbf{A} \end{cases} \tag{27}$$

where $\mathbf{A}$ and $\hat{\mathbf{A}}$ are defined by (18). The system of linear equations (27) can be solved iteratively with exponential convergence with respect to the matrix norm $||\cdot||_\infty$, where for any square matrix $\mathbf{M}$,

$$||\mathbf{M}||_\infty = \max_{i\ge 0} \sum_{j\ge 0} |M_{i,j}|. \tag{28}$$

Before proceeding to the proof of the above theorem, there is one technical lemma we must first address.

Lemma 3. *Let* $\mathbf{H} = \mathbf{B} \times \mathbf{A}$*. Then* $||\mathbf{H}||_\infty < 1$*.*

Proof. Using definitions (18) and (25) of $\mathbf{A}$ and $\mathbf{B}$ we find that for $i, j \geq 1$

$$\begin{aligned}
h_{0,0} &= \mathrm{b}_0(\zeta,\rho) \sum_{k\geq 1} \mathrm{a}_k(\rho,\zeta)\zeta_k e^{-a\zeta_k} \\
h_{0,j} &= \mathrm{b}_j(\zeta,\rho) \sum_{k\geq 1} \mathrm{a}_k(\rho,\zeta)\frac{\zeta_k e^{-a\zeta_k}}{\rho_j - \zeta_k} \\
h_{i,0} &= \hat{\rho}_i \mathrm{b}_0(\zeta,\rho) \sum_{k\geq 1} \mathrm{a}_k(\rho,\zeta)\frac{\zeta_k e^{-a\zeta_k}}{\zeta_k + \hat{\rho}_i} \\
h_{i,j} &= \hat{\rho}_i \mathrm{b}_j(\zeta,\rho) \sum_{k\geq 1} \mathrm{a}_k(\rho,\zeta)\frac{\zeta_k e^{-a\zeta_k}}{(\rho_j - \zeta_k)(\zeta_k + \hat{\rho}_i)}
\end{aligned}$$

Our first goal is to prove that $h_{i,j}$ are positive for all $i, j \geq 0$ (note that $h_{i,0} > 0$ for all $i \geq 0$). Let us consider a random variable $\xi = \mathrm{e}(\rho_j) + \overline{X}_{\mathrm{e}(q)}$, where as usual $\mathrm{e}(\rho_j)$ is an exponentially distributed random variable with rate ρ_j, independent of X and $\mathrm{e}(q)$. The Laplace transform of ξ is given by

$$\mathbb{E}\left[e^{-z\xi}\right] = \mathbb{E}\left[e^{-z\mathrm{e}(\rho_j)}\right] \times \mathbb{E}\left[e^{-z\overline{X}_{\mathrm{e}(q)}}\right] = \frac{\rho_j}{\rho_j + z} \prod_{n\geq 1} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n}}$$

and using partial fraction decomposition (6) after some algebraic manipulations we find that ξ has a probability density function

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(\xi \leq x) = \rho_j \sum_{k\geq 1} \mathrm{a}_k(\rho,\zeta)\frac{\zeta_k e^{-x\zeta_k}}{\rho_j - \zeta_k}$$

This proves that $h_{0,j}$ are all positive. To prove that $h_{i,j}$ are positive one can use identity

$$\mathbb{E}\left[e^{-\hat{\rho}_i(\xi - a)}\mathbb{I}(\xi > a)\right] = \rho_j \sum_{k\geq 1} \mathrm{a}_k(\rho,\zeta)\frac{\zeta_k e^{-a\zeta_k}}{(\rho_j - \zeta_k)(\zeta_k + \hat{\rho}_i)}$$

Next we need to prove that $||\mathbf{H}||_\infty < 1$. Define a row vector $\bar{\mathrm{h}}_i$ as the i-th row of the matrix $\mathbf{H}$. Using Theorem 1 we check that

$$\mathbb{E}\left[e^{-q\tau_a^+}\mathbb{I}(X_{\tau_a^+} - a \in \mathrm{d}y)|X_0 = -\eta_i\right] = \bar{\mathrm{h}}_i \times \bar{\mathrm{v}}(\rho, y)\mathrm{d}y$$

where $\eta_0 \equiv 0$ and η_i is independent and exponentially distributed with parameter $\hat{\rho}_i$. Thus we have

$$\sum_{j\geq 0} h_{i,j} = \mathbb{E}\left[e^{-q\tau_a^+}|X_0 = -\eta_i\right] < \mathbb{E}\left[e^{-q\tau_a^+}|X_0 = 0\right] < 1$$

and using the already established fact that $h_{i,j}$ are positive we conclude that $||\mathbf{H}||_\infty < 1$. □

Proof of Theorem 3. The operator $\mathcal{P}_I$ is an integral operator, with kernel

$$\mathbb{I}(x \in I)\mathbb{E}_x\left[e^{-q\tau_a^+}\mathbb{I}(X_{\tau_a^+} \in \mathrm{d}y)\right] = \mathbb{I}(x \in I, y \geq a)\bar{\mathrm{v}}(\zeta, a-x)^T \times \mathbf{A} \times \bar{\mathrm{v}}(\rho, y-a)\mathrm{d}y$$

(see Theorem 1). We also have a similar formula for $\hat{\mathcal{P}}_I$. Using the infinite series representation (24) we find that there exist some matrices of coefficients $\mathbf{C}_1$, $\mathbf{C}_2$ and $\hat{\mathbf{C}}_1$, $\hat{\mathbf{C}}_2$ so that integral kernels of operators $\mathcal{G}$ and $\hat{\mathcal{G}}$ can be represented in the form (26). Matrix equations (27) follow from operator identities (23) and the fact that

$$\int_0^\infty \bar{\mathrm{v}}(\hat{\rho}, z) \times \bar{\mathrm{v}}(\zeta, a+z)^T \mathrm{d}z = \mathbf{B}.$$

Equations (27) can be solved iteratively as follows:

$$\begin{aligned} \mathbf{C}_1^{(n+1)} &= \mathbf{A} + \mathbf{C}_1^{(n)}\hat{\mathbf{B}}\hat{\mathbf{A}}\mathbf{B}\mathbf{A}, \quad \mathbf{C}_1^{(0)} = \mathbf{0} \\ \hat{\mathbf{C}}_1^{(n+1)} &= \hat{\mathbf{A}} + \hat{\mathbf{C}}_1^{(n)}\mathbf{B}\mathbf{A}\hat{\mathbf{B}}\hat{\mathbf{A}}, \quad \hat{\mathbf{C}}_1^{(0)} = \mathbf{0} \end{aligned}$$

and the convergence $\mathbf{C}_1^{(n)} \to \mathbf{C}_1$ and $\hat{\mathbf{C}}_1^{(n)} \to \hat{\mathbf{C}}_1$ is exponential because of Lemma 3. Once we find $\mathbf{C}_1$ and $\hat{\mathbf{C}}_1$ we find $\mathbf{C}_2 = -\hat{\mathbf{C}}_1\mathbf{B}\mathbf{A}$ and $\hat{\mathbf{C}}_2 = -\mathbf{C}_1\hat{\mathbf{B}}\hat{\mathbf{A}}$. □

Remark 5. Assume that within the M-class the positive (negative) jumps come from a mixture of $N < \infty$ ($\hat{N} < \infty$) exponential distributions. Then matrices $\mathbf{B}$ and $\hat{\mathbf{B}}$ have size $(\hat{N}+1) \times (N+1)$ and $(N+1) \times (\hat{N}+1)$, while matrices $\mathbf{C}_1$ and $\hat{\mathbf{C}}_1$ have sizes $(N+1) \times (N+1)$ and $(\hat{N}+1) \times (\hat{N}+1)$ and can be found by the relations

$$\mathbf{C}_1 = \mathbf{A}\left(\mathbf{I} - \hat{\mathbf{B}}\hat{\mathbf{A}}\mathbf{B}\mathbf{A}\right)^{-1}, \quad \hat{\mathbf{C}}_1 = \hat{\mathbf{A}}\left(\mathbf{I} - \mathbf{B}\mathbf{A}\hat{\mathbf{B}}\hat{\mathbf{A}}\right)^{-1}.$$

The idea of writing down a pair of simultaneous equations (21) also works when considering the problem of first entrance *into* an interval. This problem has been considered earlier by [37] for the setting of stable processes and [24] for two sided Lévy processes whose upwards jumps are exponentially distributed.

Theorem 4. (First entrance into a finite interval) *Let $a > 0$ and define $\tau = \inf\{t \geq 0 : X_t \in (0,a)\}$. Define a matrix $\mathbf{M} = \mathbf{B}(\hat{\zeta}, \rho, a)^T$ and similarly $\hat{\mathbf{M}} = \mathbf{B}(\zeta, \hat{\rho}, a)^T$ (see (25)). There exist matrices $\mathbf{N}_1$, $\mathbf{N}_2$ and $\hat{\mathbf{N}}_1$, $\hat{\mathbf{N}}_2$ such that for $y \in (0,a)$ we have*

$$\mathbb{E}_x\left[e^{-q\tau}\mathbb{I}\left(X_\tau \in \mathrm{d}y\right)\right] = \begin{cases} \bar{\mathrm{v}}(\zeta, -x)^T \times \left[\mathbf{N}_1 \times \bar{\mathrm{v}}(\rho, y) + \mathbf{N}_2 \times \bar{\mathrm{v}}(\hat{\rho}, a-y)\right] \times \mathrm{d}y, & x \leq 0 \\ \bar{\mathrm{v}}(\hat{\zeta}, x-a)^T \times \left[\hat{\mathbf{N}}_1 \times \bar{\mathrm{v}}(\hat{\rho}, a-y) + \hat{\mathbf{N}}_2 \times \bar{\mathrm{v}}(\rho, y)\right] \times \mathrm{d}y, & x \geq a \end{cases} \tag{29}$$

These matrices satisfy the following system of linear equations

$$\begin{cases} \mathbf{N}_1 &= \mathbf{A} + \mathbf{A}\mathbf{M}\hat{\mathbf{N}}_2 \\ \hat{\mathbf{N}}_2 &= \hat{\mathbf{A}}\hat{\mathbf{M}}\mathbf{N}_1 \end{cases} \qquad \begin{cases} \hat{\mathbf{N}}_1 &= \hat{\mathbf{A}} + \hat{\mathbf{A}}\hat{\mathbf{M}}\mathbf{N}_2 \\ \mathbf{N}_2 &= \mathbf{A}\mathbf{M}\hat{\mathbf{N}}_1 \end{cases} \tag{30}$$

where $\mathbf{A}$ and $\hat{\mathbf{A}}$ are defined by (18). This system of linear equations (30) can be solved iteratively with exponential convergence with respect to the matrix norm $||\cdot||_\infty$

The proof of this Theorem is very similar to the proof of Theorem 3, and we leave the details to the reader.

6 Ladder processes

In this Section we derive several results related to the Laplace exponent $\kappa(q,z)$ of the bivariate ladder process (L,H). We are interested in numerical evaluation of this object since it is the key to many important fluctuation identities, such as the quintuple law at the first passage which was introduced in [14].

The following Theorem is a generalization of the expression for the Wiener-Hopf factors which can be found in Lemma 4.2 in [34].

Theorem 5. *Assume that for some $\epsilon_1 > 0$*

$$|\Psi(z)| > |z|^{\epsilon_1}, \quad \text{as } z \to \infty \tag{31}$$

and

$$\int_{-\epsilon_2}^{\epsilon_2} \left| \frac{\Psi(z)}{z} \right| \mathrm{d}z \tag{32}$$

exists for some (and then for all) $\epsilon_2 > 0$. Then for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(z) > 0$ we have

$$\kappa(q,z) = \exp\left[\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \left(\frac{\ln(q+\Psi(u))}{u - \mathrm{i}z} - \frac{\ln(1+\Psi(u))}{u} \right) \mathrm{d}u \right] \tag{33}$$

Proof. We start with the following integral representation for $\kappa(q,z)$ (see Theorem 6.16 in [31])

$$\kappa(q,z) = \exp\left[\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \left(e^{-t} - e^{-qt - zx} \right) \frac{1}{t} \mathbb{P}(X_t \in \mathrm{d}x) \mathrm{d}t \right].$$

Assume that the process X_t has a non-zero Gaussian component ($\sigma \neq 0$), then $\mathbb{P}(X_t \in \mathrm{d}x)$ has a density which can be obtained as the inverse Fourier transform of the right hand side of (1). Using this fact and assuming that $\gamma > 0$ we obtain

$$\begin{aligned} &\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \left(e^{-t-\gamma x} - e^{-qt-zx} \right) \frac{1}{t} \mathbb{P}(X_t \in \mathrm{d}x) \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \left(e^{-t-\gamma x} - e^{-qt-zx} \right) \frac{1}{t} \int_{\mathbb{R}} e^{-t\Psi(u) - \mathrm{i}ux} \mathrm{d}u \mathrm{d}x \mathrm{d}t. \end{aligned} \tag{34}$$

Applying Fubini Theorem and performing integration in x variable we find that the above integral is equal to

$$\begin{aligned} &\frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{\mathrm{d}t}{t} \int_{\mathbb{R}} \mathrm{d}u \left(\frac{e^{-t(1+\Psi(u))}}{\gamma + \mathrm{i}u} - \frac{e^{-t(q+\Psi(u))}}{z + \mathrm{i}u} \right) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{\mathrm{d}t}{t} \int_{\mathbb{R}} \mathrm{d}u \left(\frac{1}{\gamma + \mathrm{i}u} (e^{-t(1+\Psi(u))} - e^{-t}) - \frac{1}{z + \mathrm{i}u} (e^{-t(q+\Psi(u))} - e^{-t}) \right), \end{aligned} \tag{35}$$

where in the last step we have used the fact that for $\gamma > 0$ and $\mathrm{Re}(z) > 0$

$$\int\limits_{\mathbb{R}} \left[\frac{1}{\gamma + \mathrm{i}u} - \frac{1}{z + \mathrm{i}u} \right] \mathrm{d}u = 0$$

(which can be easily verified by residue calculus). Next, we apply Fubini Theorem and Frullani integral to the last integral in (35) and combining this result with (34) we obtain

$$\int\limits_{\mathbb{R}^+} \int\limits_{\mathbb{R}^+} \left(e^{-t-\gamma x} - e^{-qt-zx} \right) \frac{1}{t} \mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \int\limits_{\mathbb{R}} \left(\frac{\ln(q + \Psi(u))}{u - \mathrm{i}z} - \frac{\ln(1 + \Psi(u))}{u - \mathrm{i}\gamma} \right) \mathrm{d}u. \tag{36}$$

Next we need to take the limit of the above identity as $\gamma \to 0^+$. For u small the integrand in the right hand side of (36) is bounded by

$$C \frac{\ln(1 + |\Psi(u)|}{|u|}$$

uniformly in $\gamma \in [0, \tilde{\gamma}]$ for each $\tilde{\gamma} > 0$, and for u large it can be bounded by

$$\begin{aligned} \left| \frac{\ln(q + \Psi(u))}{u - \mathrm{i}z} - \frac{\ln(1 + \Psi(u))}{u - \mathrm{i}\gamma} \right| &= \left| \mathrm{i}(z - \gamma) \frac{\ln(q + \Psi(u))}{(u - \mathrm{i}\gamma)(u - \mathrm{i}z)} - \frac{\ln\left(1 + \frac{1-q}{q+\Psi(u)}\right)}{u - \mathrm{i}\gamma} \right| \\ &= O(\ln(|\Psi(u)|)u^{-2}) + O((u\Psi(u))^{-1}) \end{aligned}$$

again uniformly in $\gamma \in [0, \tilde{\gamma}]$. Using the above two estimates and assumption (31) we conclude that the integrand in the right hand side of (36) is bounded by

$$\tilde{C} \frac{\ln(1 + |\Psi(u)|)}{|u|} (1 + |u|)^{-\epsilon_1}$$

for some $\tilde{C} > 0$ uniformly in $\gamma \in [0, \tilde{\gamma}]$, and the above function is integrable on $\mathbb{R}$ due to assumption (32). Thus we can apply the Dominated Convergence Theorem on the right hand side of (36) together with the Monotone Convergence Theorem on the left hand side whilst taking limits as $\gamma \to 0^+$. To finish the proof we only need to take the limit $\sigma \to 0^+$, which can be justified in exactly the same way using the Dominated Convergence Theorem on the right hand side of (36) and weak convergence on the right hand side of (36). Note in particular that the characteristic exponent of X is continuous in the Gaussian coefficient and hence, by the Continuity Theorem for Fourier transforms, the law of X is weakly continuous in the Gaussian coefficient. $\square$

Assumption (32) is a very mild one: it is satisfied by all processes except those which have unusually heavy tails of the Lévy measure. It is satisfied by all processes in M-class, and more generally, by all processes for which there exists an $\epsilon > 0$ such that $\Pi(\mathbb{R} \setminus (-x, x)) = O(x^{-\epsilon})$ as $x \to +\infty$ (for example, all stable processes have this property). While condition (31) is more restrictive, one can see that it only excludes compound Poisson processes and some processes of bounded variation which are equal to the sum of its jumps. One example of such a process is a pure jump Variance Gamma process with no linear drift $X_t = \sigma W_{\Gamma_t} + \mu \Gamma_t$, which satisfies $|\Psi(z)| \sim c \ln(|z|)$ as $z \to \infty$. One can see that condition (31) is satisfied for the processes of bounded variation with non-zero drift, processes with non-zero Gaussian component and all processes with the Lévy measure satisfying $\Pi(\mathbb{R} \setminus (-x,x)) > x^{-\epsilon}$ as $x \to 0^+$ for some $\epsilon > 0$. In particular this condition is satisfied for all processes in the β-, θ-, hypergeometric or hyperexponential family excluding compound Poisson processes.

Theorem 5 allows us to derive an expression for the Laplace exponent of the bivariate ladder process (L,H).

Theorem 6. *Assume that X belongs to β-, θ-, hyper-exponential or hypergeometric family of Levy processes and that X is not a compound Poisson process. Then*

$$\kappa(q,z) = \left[\phi_q^+(\mathrm{i}z)\right]^{-1} \prod_{n\geq 1} \frac{\zeta_n(q)}{\zeta_n(1)} \tag{37}$$

Proof. If X belong to β-, θ- or hypergeometric family of Levy processes, we can use asymptotics for ζ_n as $n \to +\infty$ (see [28], [27] and [29]) to find that $\zeta_n(q)/\zeta_n(1) = 1 + O(n^{-1-\epsilon})$ and $\hat{\zeta}_n(q)/\hat{\zeta}_n(1) = 1 + O(n^{-1-\epsilon})$, which implies that both infinite products

$$\prod_{n\geq 1} \frac{\zeta_n(q)}{\zeta_n(1)}, \qquad \prod_{n\geq 1} \frac{\hat{\zeta}_n(q)}{\hat{\zeta}_n(1)}$$

converge. Next, using identical technique as in the proof of Lemma 6 in [28] one can prove that

$$\prod_{n\geq 1} \frac{1 - \frac{\mathrm{i}z}{\zeta_n(q)}}{1 - \frac{\mathrm{i}z}{\zeta_n(1)}} \prod_{n\geq 1} \frac{1 + \frac{\mathrm{i}z}{\hat{\zeta}_n(q)}}{1 + \frac{\mathrm{i}z}{\hat{\zeta}_n(1)}} \to \prod_{n\geq 1} \frac{\zeta_n(1)}{\zeta_n(q)} \prod_{n\geq 1} \frac{\hat{\zeta}_n(1)}{\hat{\zeta}_n(q)}$$

as $z \to \infty$, $z \in \mathbb{R}$. Using Wiener-Hopf factorization $q/(q+\Psi(z)) = \phi_q^+(z)\phi_q^-(z)$ and (4) we obtain

$$\frac{q+\Psi(z)}{1+\Psi(z)} = q \prod_{n\geq 1} \frac{1 - \frac{\mathrm{i}z}{\zeta_n(q)}}{1 - \frac{\mathrm{i}z}{\zeta_n(1)}} \prod_{n\geq 1} \frac{1 + \frac{\mathrm{i}z}{\hat{\zeta}_n(q)}}{1 + \frac{\mathrm{i}z}{\hat{\zeta}_n(1)}}.$$

Since X is not a compound Poisson process, we have $|\Psi(z)| \to \infty$ as $z \to \infty$, $z \in \mathbb{R}$, thus we finally conclude that

$$\prod_{n\geq 1} \frac{\zeta_n(q)}{\zeta_n(1)} \prod_{n\geq 1} \frac{\hat{\zeta}_n(q)}{\hat{\zeta}_n(1)} = q.$$

The next step is to use Wiener-Hopf factorization $q/(q+\Psi(z)) = \phi_q^+(z)\phi_q^-(z)$, (4) and the above identity to rewrite $q+\Psi(z)$ as

$$q+\Psi(z) = \prod_{n\geq 1} \frac{\frac{\zeta_n(q)-\mathrm{i}z}{\zeta_n(1)}}{1 - \frac{\mathrm{i}z}{\rho_n}} \prod_{n\geq 1} \frac{\frac{\hat{\zeta}_n(q)+\mathrm{i}z}{\hat{\zeta}_n(1)}}{1 + \frac{\mathrm{i}z}{\hat{\rho}_n}}.$$

Now we use the above factorization and the following integral identity (which can be proved by shifting the contour of integration in the complex plane)

$$\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \left(\frac{\ln(\frac{1}{b}(a-u))}{u - \mathrm{i}z} - \frac{\ln(\frac{1}{b}(b-u))}{u} \right) \mathrm{d}u = \begin{cases} \ln\left(\frac{1}{b}(a-\mathrm{i}z)\right), & \text{if } \mathrm{Im}(a) < 0,\ \mathrm{Im}(b) < 0 \\ 0, & \text{if } \mathrm{Im}(a) > 0,\ \mathrm{Im}(b) > 0 \end{cases}$$

and deduce that for $\mathrm{Re}(z) > 0$

$$\frac{1}{2\pi\mathrm{i}} \int\limits_{\mathbb{R}} \left(\frac{\ln(q+\Psi(u))}{u - \mathrm{i}z} - \frac{\ln(1+\Psi(u))}{u} \right) \mathrm{d}u = \sum_{n\geq 1} \left[\ln\left(\frac{\zeta_n(q)+z}{\zeta_n(1)} \right) - \ln\left(1 + \frac{z}{\rho_n}\right) \right],$$

which is equivalent to (37). □

Corollary 6. *Let $\Pi(\mathrm{d}s, \mathrm{d}x)$ be the Lévy measure of the ascending ladder process (L_t^{-1}, H_t). Then for $x > 0$ we have*

$$\int\limits_{\mathbb{R}^+} e^{-qs}\Pi(\mathrm{d}s, \mathrm{d}x) = \left[\prod_{n\geq 1} \frac{\zeta_n(q)}{\zeta_n(1)} \right] \bar{\mathrm{b}}(\zeta, \rho)^T \times \bar{\mathrm{v}}(\rho, x)\mathrm{d}x. \tag{38}$$

Proof. Formula (38) is a corollary of (37), (12) and the fact that $\Pi(\mathrm{d}s, \mathrm{d}x)$ is related to $\kappa(q, z)$ through the formula

$$\kappa(q, z) = \kappa(q, 0) + az + \int\limits_0^\infty \left(1 - e^{-zx}\right) \int\limits_0^\infty e^{-qs}\Pi(\mathrm{d}s, \mathrm{d}x).$$

□

7 More Fluctuation identities

We offer some more fluctuation identities. Although they are slightly more complex, they are still equally straightforward for the purpose of numerical work.

We assume throughout this section that X is regular for both $(0, \infty)$ and $(-\infty, 0)$. Equivalently, we assume that $\mathrm{a}_0(\rho, \zeta) = \mathrm{a}_0(\hat{\rho}, \hat{\zeta}) = 0$. This is the case if, for example, X has paths of unbounded variation. It will be clear from the proofs of the results given below how this assumption may be removed.

For $a > 0$ and for $y \leq a$ we define resolvent for the half line as

$$R^{(q)}(a, \mathrm{d}y) := \int_0^\infty e^{-qt}\mathbb{P}(X_t \in \mathrm{d}y; t < \tau_a^+)\mathrm{d}t$$

Theorem 7. *Define a matrix $\mathbf{D} = \{d_{i,j}\}_{i,j\geq 0}$ as follows*

$$d_{i,j} = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{a}_i(\rho, \zeta) \dfrac{\zeta_i \hat{\zeta}_j}{\zeta_i + \hat{\zeta}_j} \mathrm{a}_j(\hat{\rho}, \hat{\zeta}) & \text{if } i \geq 1,\ j \geq 1 \end{cases}$$

Then if $y \leq a$ we have

$$qR^{(q)}(a, \mathrm{d}y) \quad = \quad \left[\bar{\mathrm{v}}(\zeta, 0 \vee y) \times \mathbf{D} \times \bar{\mathrm{v}}(\hat{\zeta}, 0 \vee (-y))) - \bar{\mathrm{v}}(\zeta, a) \times \mathbf{D} \times \bar{\mathrm{v}}(\hat{\zeta}, a - y) \right] \mathrm{d}y.$$

Proof. From the proof of Theorem 20 on p176 of [5], it can be seen that

$$
\begin{aligned}
q \int_{(-\infty,a]} f(y) R^{(q)}(a, \mathrm{d}y) &= \int_{[0,a]} \mathbb{P}(\overline{X}_{\mathrm{e}(q)} \in \mathrm{d}z) \int_{[0,\infty)} \mathbb{P}(-\underline{X}_{\mathrm{e}(q)} \in \mathrm{d}u) f(-u+z) \\
&= \int_{[0,a]} \mathbb{P}(\overline{X}_{\mathrm{e}(q)} \in \mathrm{d}z) \int_{-(\infty,z]} f(y) \mathbb{P}(\underline{X}_{\mathrm{e}(q)} \in -z + \mathrm{d}y) \\
&= \int_{(-\infty,a]} f(y) \int_{[0\vee y,a]} \mathbb{P}(\overline{X}_{\mathrm{e}(q)} \in \mathrm{d}z) \mathbb{P}(\underline{X}_{\mathrm{e}(q)} \in -z + \mathrm{d}y)
\end{aligned}
$$

Thus we obtain an alternative representation of the Spitzer-Bertoin identity

$$
qR^{(q)}(a, \mathrm{d}y) = \int_{[0\vee y,a]} \mathbb{P}(\overline{X}_{\mathrm{e}(q)} \in \mathrm{d}z) \mathbb{P}(\underline{X}_{\mathrm{e}(q)} \in -z + \mathrm{d}y)
$$

To finish the proof we have to use formulas (13) and perform the integration in the above expression (noting that some terms are lost on account of the fact that we have assumed $\mathrm{a}_0(\rho, \zeta) = \mathrm{a}_0(\hat{\rho}, \hat{\zeta}) = 0$). □

Next define

$$
\Theta^{(q)}(a, x, \mathrm{d}y) = \int_0^\infty e^{-qt} \mathbb{P}_x(X_t \in \mathrm{d}y, t < \tau_a^+ \wedge \tau_0^-) \mathrm{d}t.
$$

Theorem 8. *Assume $q > 0$ and $y \in [0, a]$, then*

$$
\begin{aligned}
& q\Theta^{(q)}(a, x, \mathrm{d}y) \\
&\quad = \Big[\bar{\mathrm{v}}(\zeta, 0 \vee y - x) \times \mathbf{D} \times \bar{\mathrm{v}}(\hat{\zeta}, 0 \vee (x - y))) - \bar{\mathrm{v}}(\zeta, a - x) \times \mathbf{D} \times \bar{\mathrm{v}}(\hat{\zeta}, a - y) \Big] \mathrm{d}y \\
&\quad - \Big[\mathrm{v}(\hat{\zeta}, x)^T \times \hat{\mathbf{C}}_1 + \mathrm{v}(\zeta, a - x)^T \times \hat{\mathbf{C}}_2 \Big] \big[\mathbf{B}(y) \times \mathbf{D} \times \bar{\mathrm{v}}(\hat{\zeta}, 0) - \mathbf{B}(a) \times \mathbf{D} \times \bar{\mathrm{v}}(\hat{\zeta}, a - y) \big] \mathrm{d}y.
\end{aligned}
$$

where matrix $\mathbf{B}(y) = \mathbf{B}(\hat{\rho}, \zeta, y)$ is defined in (25), $\mathbf{D}$ is defined in Theorem 7 while matrices $\hat{\mathbf{C}}_1$ and $\hat{\mathbf{C}}_2$ come from Theorem 3. Also, $\bar{\mathrm{v}}(\hat{\zeta}, 0)$ is intepreted as $[0, \hat{\zeta}_1, \hat{\zeta}_2, ...]^T$.

Proof. Denote

$$
\hat{g}^{(q)}(a, x, \mathrm{d}y) = \mathbb{E}_x \left[e^{-q\tau_0^-} \mathbb{I}(-X_{\tau_0^-} \in \mathrm{d}y \; ; \; \tau_0^- < \tau_a^+) \right] \tag{39}
$$

Note moreover that for f supported in $[0, a]$,

$$
\begin{aligned}
\int_{[0,a]} f(y) \Theta^{(q)}(a, x, \mathrm{d}y) &= \frac{1}{q} \mathbb{E}_x \left[f(X_{\mathrm{e}(q)}) \mathbb{I}(\mathrm{e}(q) < \tau_0^- \wedge \tau_a^+) \right] \\
&= \frac{1}{q} \mathbb{E}_x \left[f(X_{\mathrm{e}(q)}) \mathbb{I}(\mathrm{e}(q) < \tau_a^+) \right] - \frac{1}{q} \mathbb{E}_x \left[f(X_{\mathrm{e}(q)}) \mathbb{I}(\tau_0^- < \mathrm{e}(q) < \tau_a^+) \right] \\
&= \int_0^a f(y) R^{(q)}(a - x, \mathrm{d}y - x) - \frac{1}{q} \mathbb{E}_x \left[f(X_{\mathrm{e}(q)}) \mathbb{I}(\tau_0^- < \mathrm{e}(q) < \tau_a^+) \right]
\end{aligned}
$$

The second expectation in the above expression can be rewritten as

$$\begin{aligned}
&\int_0^\infty e^{-qt}\mathbb{E}_x\left[f(X_t)\mathbb{I}(\tau_0^- < t < \tau_a^+)\right]\mathrm{d}t \\
&= \mathbb{E}_x\left[\int_{\tau_0^-}^\infty e^{-qt}f(X_t)\mathbb{I}(t<\tau_a^+)\mathrm{d}t\right] \\
&= \mathbb{E}_x\left[e^{-q\tau_0^-}\mathbb{I}(\tau_0^- < \tau_a^+)\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^\infty e^{-qt}f(X_t)\mathbb{I}(t<\tau_a^+)\mathrm{d}t\right]\right] \\
&= \int_{\mathbb{R}^+}\hat{g}^{(q)}(a,x,\mathrm{d}z)\int_{\mathbb{R}^+}R^{(q)}(a+z,z+\mathrm{d}y)f(y)
\end{aligned}$$

So in conclusion we have for $y\in[0,a]$,

$$\Theta^{(q)}(a,x,\mathrm{d}y) = R^{(q)}(a-x,\mathrm{d}y-x) - \int_{\mathbb{R}^+}\hat{g}^{(q)}(a,x,\mathrm{d}z)R^{(q)}(a+z,z+\mathrm{d}y)$$

and to end the proof one should use results of Theorems 3 and 7 and compute the above integral. The details are left to the reader. □

Remark 6. The previous result allows us to write down the discounted joint overshoot, undershoot distribution for the two-sided exit problem:

$$g^{(q)}(a,x,\mathrm{d}y,\mathrm{d}z) = \mathbb{E}_x\left[e^{-q\tau_a^+}\mathbb{I}(\tau_a^+ < \tau_0^-\ ;\ X_{\tau_a^+}\in\mathrm{d}y\ ;\ X_{\tau_a^+-}\in\mathrm{d}z)\right]$$

(Again this relates to the so-called Gerber-Shiu measure for classical risk theory). Indeed, for $y>a$ and $z\in[0,a]$, using the compensation formula we have

$$\begin{aligned}
g^{(q)}(a,x,\mathrm{d}y,\mathrm{d}z) &= \mathbb{E}_x\left[\sum_{t\geq 0}e^{-qt}\mathbb{I}(\overline{X}_{t-}\leq a,\ \underline{X}_{t-}\geq 0,\ X_{t-}\in\mathrm{d}z)\mathbb{I}(X_t\in\mathrm{d}y)\right] \\
&= \mathbb{E}_x\left[\int_{\mathbb{R}^+}e^{-qt}\mathbb{I}(\overline{X}_{t-}\leq a,\ \underline{X}_{t-}\geq 0,\ X_{t-}\in\mathrm{d}z)\mathrm{d}t\right]\Pi(\mathrm{d}y-z) \\
&= \Theta^{(q)}(a,x,\mathrm{d}z)\Pi(\mathrm{d}y-z).
\end{aligned}$$

Remark 7. Define the reflected process $Y := X - \underline{X}$ and its resolvent

$$\Lambda^{(q)}(a,x,\mathrm{d}y) = \int_0^\infty e^{-qt}\mathbb{P}_x(Y_t\in\mathrm{d}y,\ t<\sigma_a)\mathrm{d}t$$

where $\sigma_a = \inf\{s>0 : Y_s\geq a\}$. According to [4] one may write for all $q\geq 0$,

$$\Lambda^{(q)}(a,x,\mathrm{d}y) = \Theta^{(q)}(a,x,\mathrm{d}y) + \hat{g}^{(q)}(a,x)\cdot\lim_{z\downarrow 0}\frac{\Theta^{(q)}(a,z,\mathrm{d}y)}{1-\hat{g}^{(q)}(a,z)}.$$

where $\hat{g}^{(q)}(a,x) = \mathbb{E}_x[e^{-q\tau_0^-}\mathbb{I}(\tau_a^+ > \tau_0^-)]$. Moreover, note that we may then compute in a similar way to before with the help of the compensation formula,

$$\mathbb{E}_0\left[e^{-q\sigma_a}\mathbb{I}(Y_{\sigma_a} \in \mathrm{d}y\ ;\ Y_{\sigma_a -} \in \mathrm{d}z)\right] = \Lambda^{(q)}(a,0,\mathrm{d}z)\Pi(\mathrm{d}y - z)$$

and hence

$$\mathbb{E}_x\left[e^{-q\sigma_a}\mathbb{I}(Y_{\sigma_a} \in \mathrm{d}y\ ;\ Y_{\sigma_a -} \in \mathrm{d}z)\right] = g^{(q)}(a,x,\mathrm{d}y,\mathrm{d}z) + \hat{g}^{(q)}(a,x)\Lambda^{(q)}(a,0,\mathrm{d}z)\Pi(\mathrm{d}y - z).$$

8 Numerical Results

For all our numerical examples we will use a process X from the β-family (see the introduction or [28]) having parameters

$$(\sigma,\mu,\alpha_1,\beta_1,\lambda_1,c_1,\alpha_2,\beta_2,\lambda_2,c_2) = (\sigma,\mu,1,1.5,1.5,1,1,1.5,1.5,1)$$

Here $\mu = \mathbb{E}[X_1]$ and σ is the Gaussian coefficient, the other parameters define the density of a Lévy measure, which has exponentially decaying tails and $O(|x|^{-3/2})$ singularity at $x = 0$, thus this process has jumps of infinite activity but finite varation. We define the following four parameter sets

Set 1: $\sigma = 0.5, \mu = 1$ Set 2: $\sigma = 0.5, \mu = -1$
Set 3: $\sigma = 0, \mu = 1$ Set 4: $\sigma = 0, \mu = -1$

As the first illustration of the efficiency of our algorithms we will compute the following three quantities related to the double exit problem:

(i) density of the overshoot if the exit happens at the upper boundary

$$f_1(x,y) = \frac{\mathrm{d}}{\mathrm{d}y}\,\mathbb{E}_x\left[e^{-q\tau_1^+}\mathbb{I}\left(X_{\tau_1^+} \le y\ ;\ \tau_1^+ < \tau_0^-\right)\right]$$

(ii) probability of exiting from the interval $[0,1]$ at the upper boundary

$$f_2(x) = \mathbb{E}_x\left[e^{-q\tau_1^+}\mathbb{I}\left(\tau_1^+ < \tau_0^-\right)\right]$$

(iii) probability of exiting the interval $[0,1]$ by creeping across the upper boundary

$$f_3(x) = \mathbb{E}_x\left[e^{-q\tau_1^+}\mathbb{I}\left(X_{\tau_1^+} = 1\ ;\ \tau_1^+ < \tau_0^-\right)\right]$$

In order to compute these expressions we use methods described in Theorem 3. We truncate all the matrices $\mathbf{C}_i, \hat{\mathbf{C}}_i$ so that they have size 200×100 (this corresponds to truncating coefficients $\mathrm{a}_i(\rho,\zeta)$ and $\mathrm{a}_i(\hat{\rho},\hat{\zeta})$ at $i = 200$ and coefficients $\mathrm{b}_j(\zeta,\rho)$ and $\mathrm{b}_j(\hat{\zeta},\hat{\rho})$ at $j = 100$). In order to compute coefficients $\mathrm{a}_i(\rho,\zeta)$, $\mathrm{a}_i(\hat{\rho},\hat{\zeta})$, $\mathrm{b}_j(\zeta,\rho)$ and $\mathrm{b}_j(\hat{\zeta},\hat{\rho})$ we truncate infinite products in (8) and (9) at $k = 400$, thus all the computations depend on precomputing $\{\zeta_n, \hat{\zeta}_n\}$ for $n = 1,2,..,400$. All the code was written in Fortran and the computations were performed on a standard laptop (Intel Core 2 Duo 2.5 GHz processor and 3 GB of RAM).

We present the results for $q = 1$ in Figures 1 and 2. Computations required to produce graphs for each parameter set took around 0.15 seconds. The numerical results clearly show the effects that we would expect to see. In Figures 1b, 1d and 2b we see a positive probability of creeping, which is expected since the process X has a Gaussian component in the first two cases and a bounded variation and positive drift in the third case. Parameter Set 4 corresponds to a process with bounded variation and negative drift, thus we do not have any upward creeping, and this is exactly what we obtain in Figure 2d. Also, figures 1a, 1c and 2a show that $f_1(x, y) \to 0$ as $x \to 1^-$, which confirms our expectation, as in this case upper half line is regular and as $x \to 1^-$ the process will cross the barrier at 1 by creeping, not by jumping over it. This is different from figure 2c, where, because of bounded variation and negative drift, the upper half line is irregular and the only way to cross the barrier at 1 is by jumping over it. Next, Figures 1b and 1d show that $f_i(x) \to 0$ as $x \to 0^+$ and $f_i(x) \to 1$ as $x \to 1^-$, which again agrees with the theory, as for parameter Sets 1 and 2 the process X has a Gaussian component, therefore 0 is regular for $(-\infty, 0)$ and $(0, \infty)$. This is not so for parameter Sets 3 and 4, since now the process X has bounded variation and drift, and depending on the sign of the drift 0 is regular for either $(0, \infty)$ or $(-\infty, 0)$, and this is what we observe on figures 2b and 2d.

For the next example, we compute the density $u(s, x)$ of the bivariate renewal measure $\mathcal{U}(\mathrm{d}s, \mathrm{d}x)$ defined by

$$\mathcal{U}(\mathrm{d}s, \mathrm{d}x) = \int\limits_{\mathbb{R}^+} \mathbb{P}(L_t^{-1} \in \mathrm{d}s, H_t \in \mathrm{d}x)\mathrm{d}t$$

where (L, H) is the ascending ladder process, see [14], [31] and [32]. This measure is a very important object, as it gives us full knowledge of the quintuple law at the first passage, see [14]. Using formulas 6.18 and 7.10 in [31] we see that $\mathcal{U}(\mathrm{d}s, \mathrm{d}x)$ satisfies

$$\int\limits_{\mathbb{R}^+} e^{-qs}\mathcal{U}(\mathrm{d}s, \mathrm{d}x) = \frac{\mathbb{P}(\overline{X}_{\mathrm{e}(q)} \in \mathrm{d}x)}{\kappa(q, 0)}$$

therefore using Theorem 6 and Corollary 2 we find that the density of $\mathcal{U}(\mathrm{d}s, \mathrm{d}x)$ can be computed as

$$u(s, x) = \frac{1}{2\pi \mathrm{i}} \int\limits_{q_0 + \mathrm{i}\mathbb{R}} \prod_{n \geq 1} \frac{\zeta_n(1)}{\zeta_n(q)} \left[\bar{\mathrm{a}}(\rho, \zeta(q))^T \times \bar{\mathrm{v}}(\zeta(q), x) \right] e^{qs} \mathrm{d}q$$

for any $q_0 > 0$. It turns out that it is quite easy to compute the above integral numerically using technique discussed in [28] coupled with a Filon-type method (see [21]). Producing each graph on Figure 3 takes around 1.2 seconds. In order to compute $u(s, x)$ we truncated $\mathrm{a}_i(\rho, \zeta)$ at $i = 100$ and used 200 roots ζ_k to compute these coefficients using formulas (8). We chose $q_0 = 0.25$, truncated integal in q at $|\mathrm{Im}(q)| < 10^4$. The numerical results are presented in Figure 3. Note that both the ascending ladder height and time processes behind Figure 3 (c) have a linear drift where as in Figure 3 (d) they are both driftles compound Poisson processes. In the former case this explains the strong concentration of mass around a linear trend, and in the latter case there exists an atom at $x = s = 0$, which is not visible on the graph since we are only plotting the absolutely continuous part.

As we have mentioned in the introduction, all expressions related to fluctuation identities presented in this paper have the following property: (i) they are computed explicitly in terms of roots/poles of $q + \Psi(\mathrm{i}z)$ and possibly some linear algebra operations, (ii) all of them have the law of the space variables (for example, overshoot or location of the last maximum) in closed form and (iii) they involve Laplace transform of the first passage time τ_a^+ or τ_0^-. The third condition implies that if we want to compute joint distribution of both space and time functionals of the process (for example, joint density of the first passage time and the overshoot) we would have to perform a Fourier transform in q-variable, and it has to be done numerically. See [28] and [29] for examples of application of this technique. It turns out, that using exactly the same method we can also obtain similar reults for Lévy processes with stochastic volatility, which are very popular models in Mathematical Finance first, see for example [11]. We will briefly present this technique for the case of double exit problem considered above. Let T_s be an increasing continuous process satisfying $T_0 = 0$. We require that $\mathbb{E}[\exp(qT_s)]$ is known in closed form, a classical example is the integral of the Cox-Ingersoll-Ross diffusion process, however one can also choose several other processes, see [20]. Define a time-changed process $Z_s = X_{T_s}$, $s \ge 0$, where we assume that T_s is independent of X_t. As before, define s_0^- $\{s_a^+\}$ to be the first passage time of process Z_s below 0 {above a}. Since T_s is continuous we have $T_{s_a^+} = \tau_a^+$, thus we obtain for any positive q_0

$$\mathbb{E}_z\left[\mathbb{I}\left(Z_{s_a^+} \in \mathrm{d}y \; ; \; s_a^+ \le u \; ; \; s_a^+ < s_0^-\right)\right] = \mathbb{E}_z\left[\mathbb{I}\left(X_{\tau_a^+} \in \mathrm{d}y \; ; \; \tau_a^+ \le T_u \; ; \; \tau_a^+ < \tau_0^-\right)\right]$$
$$= \int\limits_{\mathbb{R}^+} \mathbb{E}_z\left[\mathbb{I}\left(X_{\tau_a^+} \in \mathrm{d}y \; ; \; \tau_a^+ \le t \; ; \; \tau_a^+ < \tau_0^-\right)\right] \mathbb{P}(T_u \in \mathrm{d}t) = \frac{1}{2\pi\mathrm{i}} \int\limits_{q_0 + \mathrm{i}\mathbb{R}} g^{(q)}(a, z, \mathrm{d}y) \mathbb{E}\left[e^{qT_u}\right] q^{-2} \mathrm{d}q$$

where we have used the fact that

$$\mathbb{E}_z\left[\mathbb{I}\left(X_{\tau_a^+} \in \mathrm{d}y \; ; \; \tau_a^+ \le t \; ; \; \tau_a^+ < \tau_0^-\right)\right] = \frac{1}{2\pi\mathrm{i}} \int\limits_{q_0 + \mathrm{i}\mathbb{R}} g^{(q)}(a, z, \mathrm{d}y) e^{qt} q^{-2} \mathrm{d}q$$

which follows from (39) by inverting the Laplace transform in q. Thus we see that if we choose the time change process $\{T_s : t \ge 0\}$ for which the Laplace transform $\mathbb{E}[\exp(qT_s)]$ is known in closed form, then computing quantities for the time-changed process Z is essentially identical to computing the same quantities for the process X itself.

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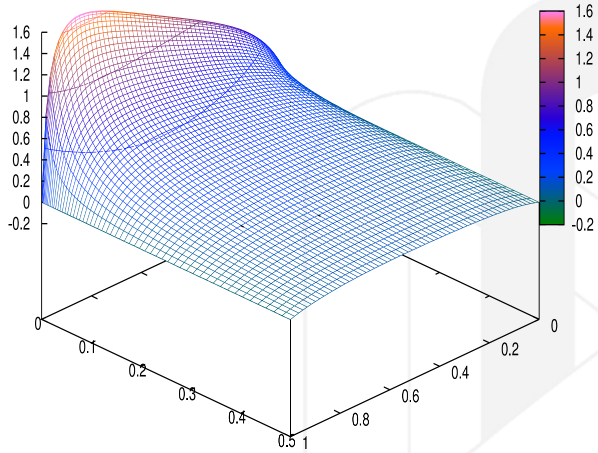
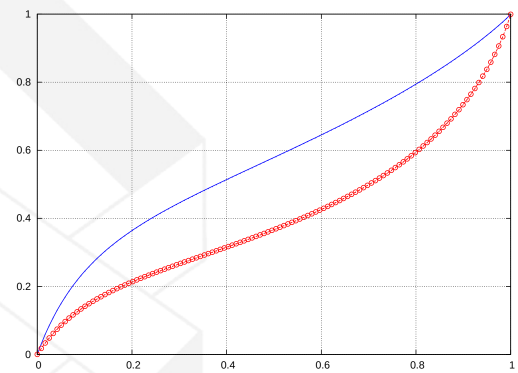
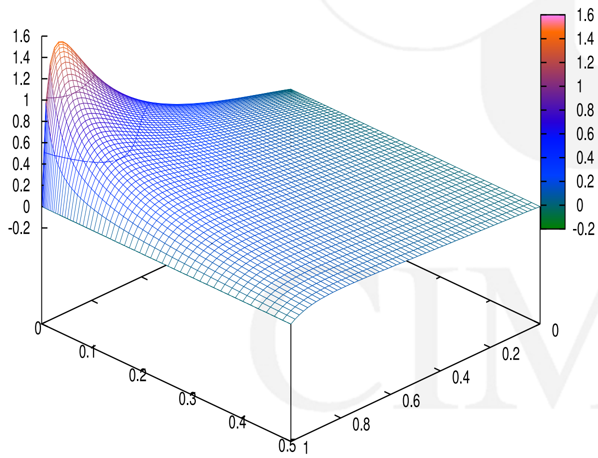


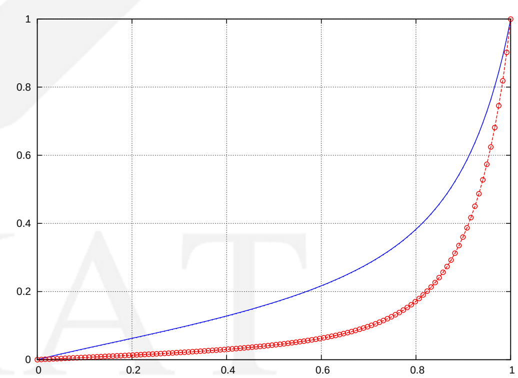
(a) Set 1: $f_1(x, y)$



(b) Set 1: $f_2(x)$ (solid) and $f_3(x)$ (circles)

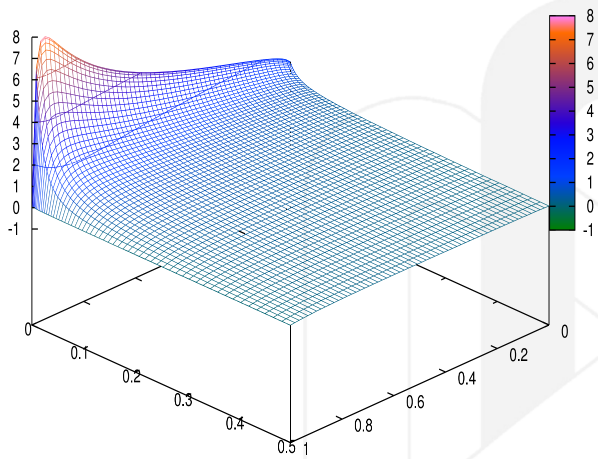


(c) Set 2: $f_1(x, y)$

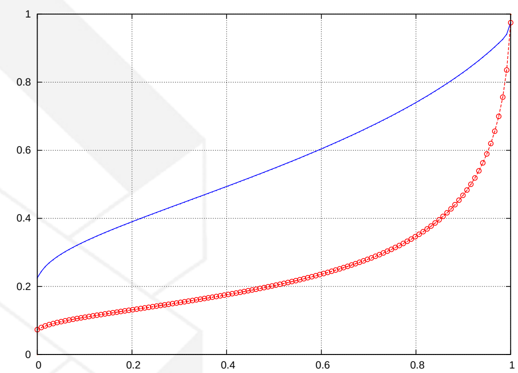


(d) Set 2: $f_2(x)$ (solid) and $f_3(x)$ (circles)

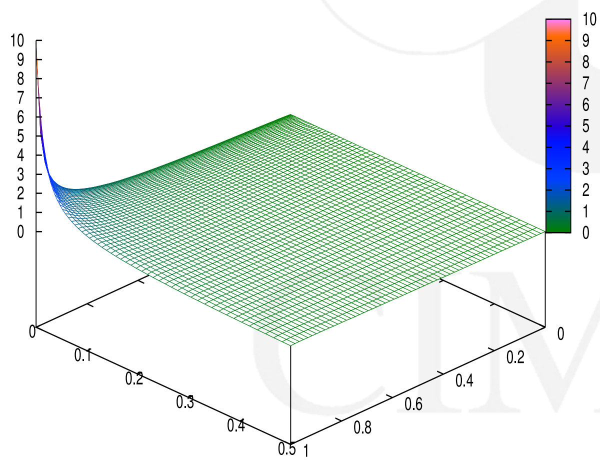
Figure 1: Unbounded variation case ($\sigma = 0.5$): computing the density of the overshoot $f_1(x, y)$ ($x \in (0, 1)$, $y \in (0, 0.5)$), probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$ for parameter Set 1 (positive drift $\mu = 1$) and Set 2 (negative drift $\mu = -1$).



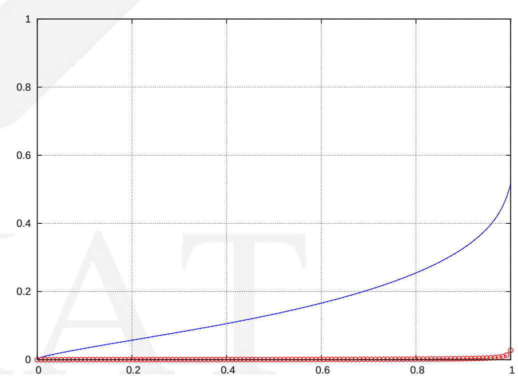
(a) Set 3: $f_1(x, y)$



(b) Set 3: $f_2(x)$ (solid) and $f_3(x)$ (circles)

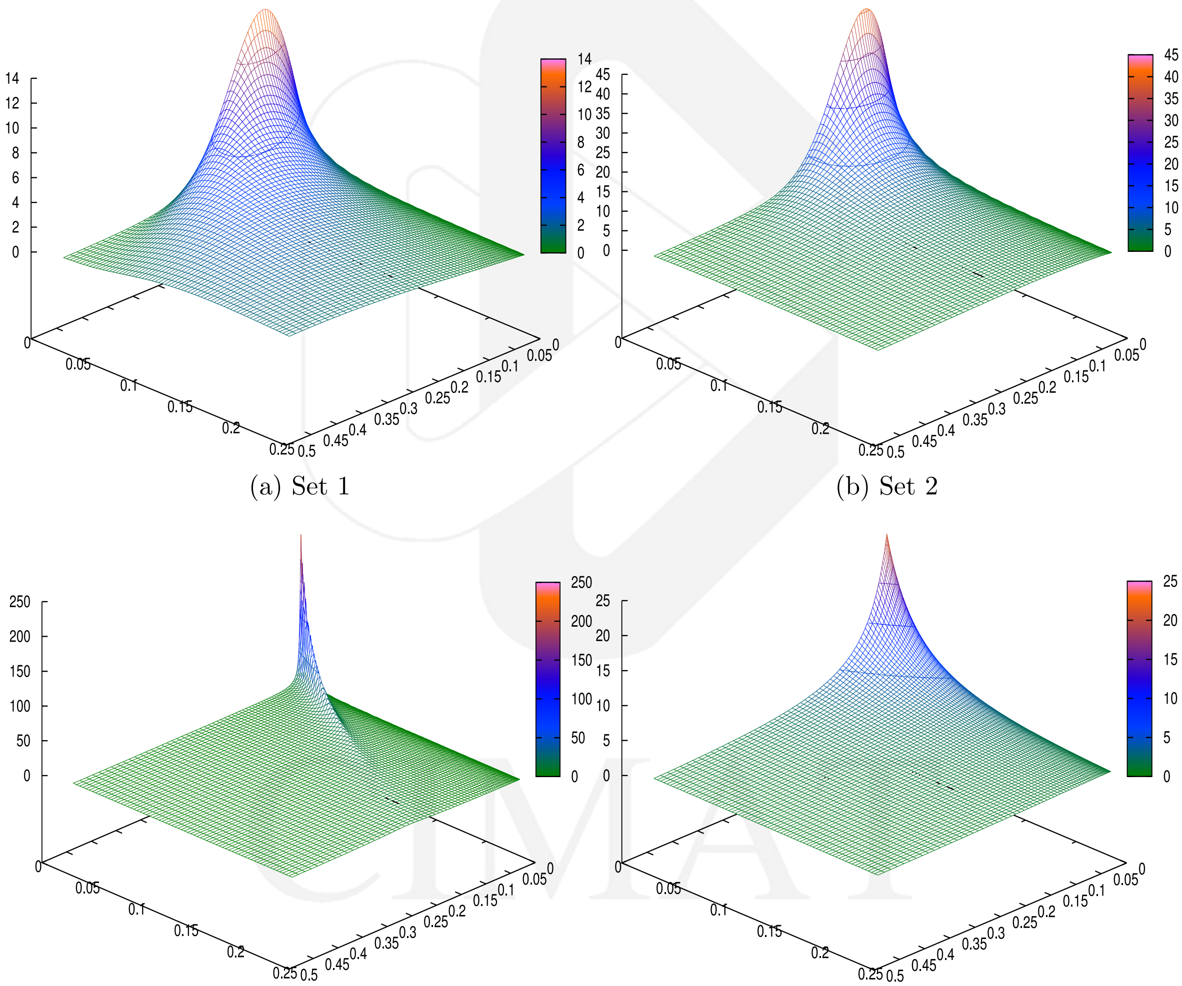


(c) Set 4: $f_1(x, y)$



(d) Set 4: $f_2(x)$ (solid) and $f_3(x)$ (circles)

Figure 2: Bounded variation case ($\sigma = 0$): computing the density of the overshoot $f_1(x, y)$ ($x \in (0, 1)$, $y \in (0, 0.5)$), probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$ for parameter Set 3 (positive drift $\mu = 1$) and Set 4 (negative drift $\mu = -1$).



(a) Set 1 (b) Set 2

(c) Set 3 (d) Set 4

Figure 3: Computing the density of the renewal measure $\mathcal{U}(\mathrm{d}s, \mathrm{d}x)$, $s \in (0, 0.25)$ and $x \in (0, 0.5)$.