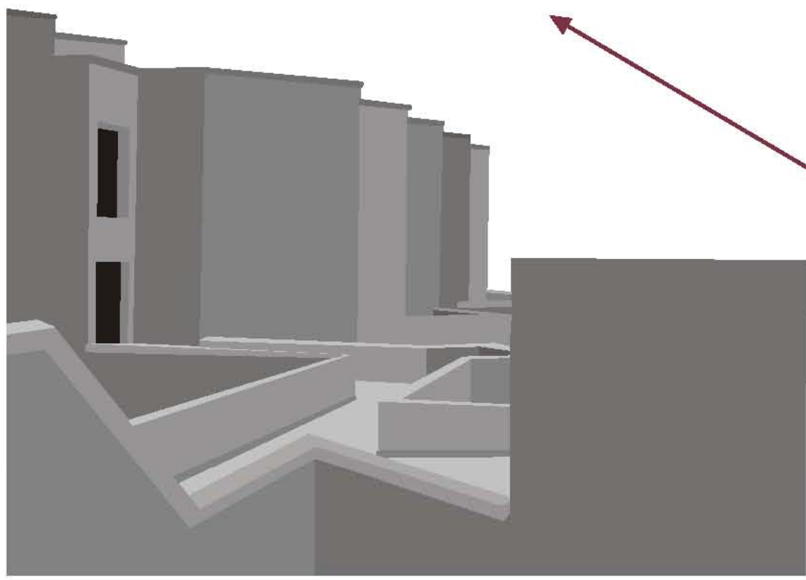

Comunicaciones del CIMAT

A BRIEF FOUNDATION OF THE LEFT-SYMMETRIC DIALGEBRAS

Raúl Felipe

Comunicación del CIMAT No I-11-02/18-03-2011
(MB/CIMAT)

CIMAT

A brief foundation of the Left-symmetric dialgebras

Raúl Felipe[a]

[a]CIMAT
Callejón Jalisco s/n Mineral de Valenciana
Guanajuato, Gto, México.
raulf@cimat.mx

Abstract

For introducing the concept of left-symmetric dialgebras, which includes as a particular case the notion of dialgebras, we can give a new impulse to the construction of Leibniz algebras and the geometric structures defined on them.

2000 Mathematics Subject Classification (MSC2000): Primary 17A32; Secondary 17D25.

Key words: Leibniz algebras, Left-symmetric dialgebras.

Introduction

In 1993, J. L. Loday introduced the notion of Leibniz algebra (see [11]), which is a generalization of the Lie algebras where the skew-symmetric of the bracket is dropped and the Jacobi identity is changed by the Leibniz identity. Loday also showed that the relationship between Lie algebras and associative algebras translates into an analogous relationship between Leibniz algebras and the so-called diassociative algebras or associative dialgebras (see [11]) which are a generalization of associative algebras possessing two products. In particular Loday showed, among other things, that any dialgebra $(D, \dashv, \vdash)$ becomes a Leibniz algebra D_{Leib} under the Leibniz bracket $[x, y] := x \dashv y - y \vdash x$.

Until now it is commonly accepted the exclusive role of the dialgebras in the construction of the Leibniz algebras, and as such these have received an intense attention. In this work we will show that there is an algebraic structure with two products slightly more complicated in its definition that a dialgebra which contains it as a particular case.

One of the concepts that in this paper we would like to introduce is the analogue of a left-symmetric algebra for a Leibniz algebra and it should be called left di-symmetric algebra or left-symmetric dialgebra. As we shall see later any of these new algebras generates a Leibniz algebra . It is then natural that in the course of our research we try use the tools already established for the study of dialgebras.

Next, we introduce some definitions and results of the classic theory of left-symmetric algebras. An algebra $(A, \cdot)$ over K with product $x \cdot y$ is called a left-symmetric algebra (LSA) or a left pre-Lie algebra, if the product is left-symmetric, i.e., if the identity

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z, \tag{1}$$

is satisfied for all $x, y, z \in A$.

The reader can consult [1] and [4] for an introduction about left-symmetric algebras. These left-symmetric algebras arise both in physics as in mathematics. Moreover, we must say that they were initially introduced by Cayley in 1896, and quickly forgotten until the 1960s of the past century, moment in which reappeared in the works of Vinberg and Koszul in the context of convex homogeneous cones (see for instance [14]) and independently it were introduced also by Gerstenhaber.

The associator (x, y, z) of three elements $x, y, z \in A$ is defined by $(x, y, z) = x \cdot (y \cdot z) - (x \cdot y) \cdot z$. Thus $(A, \cdot)$ is a (LSA) if $(x, y, z) = (y, x, z)$. An algebra is called RSA, if the identity $(x, y, z) = (x, z, y)$ is satisfied. It is clear that the opposite algebra of an LSA is an RSA. An associative product is right- and left-symmetric. The converse is not true in general. Notice that LSA and RSA are examples of Lie-admissible algebras, i.e., the commuter

$$[x, y] = x \cdot y - y \cdot x, \tag{2}$$

define a Lie bracket. We denote the Lie algebra by $\mathfrak{g}_A$.

There are many examples of left-symmetric algebras appearing in geometry and mathematical physics. Now we present some of them.

Let $\mathfrak{g}$ a $n-$dimensional Lie algebra over $\mathbb{R}$. An affine structure on $\mathfrak{g}$ is given by a bilinear mapping

$$\bigtriangledown : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g},$$

satisfying

$$\bigtriangledown(x, y) - \bigtriangledown(y, x) = [x, y],$$
$$\bigtriangledown(x, \bigtriangledown(y, z)) - \bigtriangledown(y, \bigtriangledown(x, z)) = \bigtriangledown([x, y], z),$$

for all $x, y, z \in \mathfrak{g}$. Then the product $x \cdot y = \bigtriangledown(x, y)$ endows the vector space $\mathfrak{g}$ with a structure the left-symmetric algebra.

The following example is due to S. Gelfand. Let $(A, \cdot)$ be a commutative associative algebra, and let D be a derivation of A. Then the new product

$$a * b = a \cdot Db, \quad \forall a, b \in A,$$

makes $(A, *)$ become a left-symmetric algebra.

Let V be a vector space over the complex field $\mathbb{C}$ with the usual inner product $(.,.)$ and let a be a fixed vector in A, then

$$u * v = (u, v)a + (u, a)v,$$

for all $u, v \in V$ defines a left-symmetric algebra on V.

Let $(A,\cdot)$ be an associative algebra and let $R: A \longrightarrow A$ be a linear map satisfying

$$R(x)\cdot R(y) + R(x\cdot y) = R(R(x)\cdot y + x\cdot R(y)),$$

for any $x, y \in A$. Then

$$x * y = R(x)\cdot y - y\cdot R(x) - x\cdot y, \quad \forall x, y \in A,$$

defines a left-symmetric algebra on A. The linear map R is called Rota-Baxter map of weight 1.

Also it is well known that left-symmetric structures appear of natural way in the theory of integrable systems of hydrodynamic type and the integrable generalized Burgers equation.

We would like comment that recently C. Bai, L. Liu and X. Ni have introduced the notion of $L-$dendriform algebra which constitutes a vector space with two bilinear products from which one can construct a left-symmetric algebra, that is, it is the Lie algebraic analogous of a dendriform dialgebra (see [2] and [3]). It is easy to show that the concept of left-symmetric dialgebra defined by us here is not a $L-$dendriform algebra neither coincides with the concept of dendriform dialgebra such as was introduced by Loday in [11] and [12]. For conclude this part and continuing the connection with the work of other authors we must inform that in [10], D. Liu introduced the notion of alternative dialgebra.

Thus, the main objective of this paper is to introduce the concept of left-symmetric dialgebra and to show its relationship with the Leibniz algebras. At the end of this work the emphasis is made in the study of affine transformations over this new structure that allows us give new classes of Leibniz algebras. On other hand, in opinion of the author all the work shows the strong geometric flavor of the left-symmetric dialgebras.

A review of Leibniz algebras and dialgebras

Around 1990, J. L. Loday introduces a non-antisymmetric version of Lie algebras, whose bracket satisfies the Leibniz relation (see (3)) and hence it is called Leibniz algebra. The Leibniz relation, combined with antisymmetry, leads to the Jacobi identity. Therefore Lie algebras are anti-symmetric Leibniz algebras. In [12], Loday also introduces an associative version of Leibniz algebras, called diassociative algebras or dialgebras, equipped with two bilinear and associative operations, $\vdash$ and $\dashv$, which satisfy three axioms all of them being variations of the associative law.

In this part we briefly surveying the theory of Leibniz algebras and dialgebras.

Definition 1 *A Leibniz algebra over a field K is a K-vector space L equipped with a binary operation, called Loday bracket, $[\cdot,\cdot] : L\times L \to L$ which satisfies the Leibniz identity*

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]\,, \qquad \text{for all } \; x,y,z\in L. \tag{3}$$

If the bracket is skew-symmetric, then L is a Lie algebra. Therefore Lie algebras are particular cases of Leibniz algebras.

Example 1 *Let L be a Lie algebra and let M be a L-module with action $M \times L \to M$, $(m,x) \mapsto mx$. Let $f : M \to L$ be a L-equivariant linear map, this is*

$$f(mx) = [f(m), x]\,, \quad \text{for all } \ m \in M \text{ and } x \in L,$$

then one can put a Leibniz structure on M as follows

$$[m,n]' := mf(n)\,, \quad \text{for all } \ m,n \in M.$$

Additionally, the map f defines a homomorphism between Leibniz algebras, since

$$f([m,n]') = f(mf(n)) = [f(m), f(n)]$$

Example 2 *Let (A,d) be a differential associative algebra. So, by hypothesis, $d(ab) = da\,b + a\,db$ and $d^2 = 0$. We define the bracket on A by the formula*

$$[a,b] := a\,db - db\,a,$$

then the vector space A equipped with this bracket is a Leibniz algebra.

It follows from the Leibniz identity (3) that in any Leibniz algebra we have

$$[x,[y,y]] = 0, \quad [x,[y,z]] + [x,[z,y]] = 0.$$

The Leibniz algebras are in fact right Leibniz algebras. For the opposite structure (left Leibniz algebras), that is $[x,y]' = [y,x]$, the left Leibniz identity is

$$[[x,y]',z]' = [y,[x,z]']' - [x,[y,z]']'. \tag{4}$$

The notion of dialgebra is a generalization of an associative algebra with two operations which gives rise to a Leibniz algebra instead of Lie algebra.

Definition 2 *A dialgebra over a field K is a K-vector space D equipped with two associative products*

$$\dashv : D \times D \to D$$
$$\vdash : D \times D \to D$$

satisfying the identities:

$$x \dashv (y \dashv z) = x \dashv (y \vdash z), \tag{5}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{6}$$

$$(x \vdash y) \vdash z = (x \dashv y) \vdash z, \tag{7}$$

Observe that the analogue of (6), with the product symbols pointing outward, is not valid in general in dialgebras: $(x \dashv y) \vdash z \neq x \dashv (y \vdash z)$.

Example 3 *If A is an associative algebra, then the formula $x \dashv y = xy = x \vdash y$ defines a structure of dialgebra on A.*

Example 4 *If (A,d) is a differential associative algebra, then the formulas $x \dashv y = x\,dy$ and $x \vdash y = dx\,y$ define a structure of dialgebra on A.*

Example 5 *Let V be a vector space and fix $\varphi \in V'$ (the algebraic dual), then one can define a dialgebra structure on V by setting $x \dashv y = \varphi(y)x$ and $x \vdash y = \varphi(x)y$, denoted by V_φ. If $\varphi \neq 0$, then V_φ is a dialgebra with non-trivial bar-units. Moreover, its halo is an affine space modeled after the subspace $Ker\varphi$.*

If D is a dialgebra and we define the bracket $[\cdot,\cdot] : D \times D \to D$ by

$$[x, y] := x \dashv y - y \vdash x\,, \quad \text{for all } x, y \in D,$$

then $(D, [\cdot,\cdot])$ is a Leibniz algebra. Moreover, Loday showed that the following diagram is commutative

$$\begin{array}{ccc} Dias & \xrightarrow{-} & Leib \\ \uparrow & & \uparrow \\ As & \xrightarrow{-} & Lie \end{array}$$

where *Dias*, *As*, *Lie* and *Leib* are denoted, respectively, as the categories of dialgebras, associative, Lie and Leibniz algebras ([12]).

Left-symmetric dialgebras

In this section we introduce the notion of left-symmetric dialgebra.

Definition 3 *Let S be a vector space over a field K. Let us assume that S is equipped with two bilinear products, not necessarily associative*

$$\dashv : S \times S \to S$$

$$\vdash : S \times S \to S$$

satisfying the identities:

$$x \dashv (y \dashv z) = x \dashv (y \vdash z), \tag{8}$$

$$(x \vdash y) \vdash z = (x \dashv y) \vdash z, \tag{9}$$

$$x \dashv (y \dashv z) - (x \dashv y) \dashv z = y \vdash (x \dashv z) - (y \vdash x) \dashv z, \tag{10}$$

$$x \vdash (y \vdash z) - (x \vdash y) \vdash z = y \vdash (x \vdash z) - (y \vdash x) \vdash z, \tag{11}$$

then we say that S is a ***left-symmetric dialgebra*** *(LSDA) or a* ***left di-symmetric algebra****.*

For us is a bit surprising how this definition escaped to the attention of specialists in the subject until now.

Let us denote by $\mathfrak{S}$ the set of all left-symmetric dialgebra and let us reserve the notation $\mathfrak{D}$ for the set of all dialgebras.

Example 6 *All dialgebra D is a left-symmetric dialgebra. Hence, $\mathfrak{D} \subset \mathfrak{S}$.*

Example 7 *Any left-symmetric algebra is a left-symmetric dialgebra in which $\vdash = \dashv$. On the other hand a nonassociative left symmetric algebra is not a dialgebra. It confirms that $\mathfrak{D} \neq \mathfrak{S}$.*

Example 8 *Let $(S, \vdash, \dashv)$ be a left-symmetric dialgebra and $(A, .)$ a left-symmetric algebra, then $S \times A$ is a left-symmetric dialgebra with the products $\vdash$ and $\dashv$ defined in the following form:*

$$(x, \alpha) \vdash (y, \beta) = (x \vdash y, \alpha\beta),$$

$$(x, \alpha) \dashv (y, \beta) = (x \dashv y, \alpha\beta).$$

Example 9 *If $((A, \cdot), d)$ is a differential left-symmetric algebra, then the formulas $x \dashv y = x \cdot dy$ and $x \vdash y = dx \cdot y$ define a structure of left-symmetric dialgebra on A. It is clear that it constitutes an analogous for the Gelfand example. Even more, let $(D, \vdash, \dashv)$ be a dialgebra and $\delta \in Der(D)$ such that $\delta^2 = 0$, then the products $x \succ y = \delta x \vdash y$ and $x \prec y = x \dashv \delta y$ convert D into a left-symmetric dialgebra.*

Proposition 4 *A left-symmetric dialgebras S is a dialgebra if and only if both products of S are associative.*

Proof. Let the left-symmetric dialgebra S be a dialgebra, then justly both products $\vdash$ and $\dashv$ defined over S are associative. Let us assume now that for a left-symmetric dialgebra S each product is associative, then from (10)

$$x \vdash (y \dashv z) - (x \vdash y) \dashv z = 0,$$

for every $x, y, z \in S$. Hence S is a dialgebra. ■

A morphism of left-symmetric dialgebras from S to S' is a linear map $f : S \to S'$ such that

$$f(x \dashv y) = f(x) \dashv f(y) \quad \text{and} \quad f(x \vdash y) = f(x) \vdash f(y),$$

for all x, y in S. In what follows if there exists a morphism of left-symmetric dialgebras between two of these structures then the products defined on these will be denoted by the same symbols.

The main result in this part which shows the importance of the notion of left-symmetric dialgebra can be stated as follows

Theorem 5 *Let $(S, \vdash, \dashv)$ be a left-symmetric dialgebra. Then the Loday commutator defined before as*

$$[x, y]_S = x \dashv y - y \vdash x,$$

defines a structure of Leibniz algebra on S. In others words, $(S, [.,.]_S)$ is a Leibniz algebra denoted by S_{Leib}.

Proof. The proof is a straightforward checking in which the identities (8) and (9) are used once. In fact, if we denote $\Sigma_{Lieb} = [x, [y, z]_S]_S - [[x, y]_S, z]_S + [[x, z]_S, y]_S$ then

$$\begin{aligned}\Sigma_{Lieb} = &(x \dashv (y \dashv z) - (x \dashv y) \dashv z) - (y \vdash (x \dashv z) - (y \vdash x) \dashv z)\\ &- (x \dashv (z \dashv y) - (x \dashv z) \dashv y) + (z \vdash (x \dashv y) - (z \vdash x) \dashv y)\\ &+ (y \vdash (z \vdash x) - (y \vdash z) \vdash x) - (z \vdash (y \vdash x) - (z \vdash y) \vdash x).\end{aligned}$$

Now, by (10) and (11) it follows that $\Sigma_{Leib} = 0$. ■

Now equipped with the previous result we can write (10) and (11) in the way

$$L_x^{\dashv}L_y^{\vdash} - L_y^{\vdash}L_x^{\dashv} = L_{[x,y]_S}^{\dashv}, \tag{12}$$

and

$$L_x^{\vdash}L_y^{\vdash} - L_y^{\vdash}L_x^{\vdash} = L_{[x,y]_S}^{\vdash}, \tag{13}$$

respectively for all $x, y \in S$, where $L_x^{\vdash}y = x \vdash y$ and $L_x^{\dashv}y = x \dashv y$. Thus we have the following Theorem

Proposition 6 *Let us define* $Leib(S) = \{L_x^{\vdash} | x \in S\}$. *Then* $Leib(S)$ *is a Leibniz algebra with the commutator defined as* $[L_x^{\vdash}, L_y^{\vdash}]_{Leib(S)} = L_x^{\vdash}L_y^{\vdash} - L_y^{\vdash}L_x^{\vdash}$.

Proof. It follows of (13). ■

Let $f : S \longrightarrow S'$ a morphism of left-symmetric dialgebras between S and S' then for all $x, y \in S$ we have

$$\begin{aligned} [f(x), f(y)]_{S'} &= f(x) \dashv f(y) - f(y) \vdash f(x) \\ &= f(x \dashv y) - f(y \vdash x) \\ &= f(x \dashv y - y \vdash x) = f([x, y]_S). \end{aligned}$$

Next, we adopt the same concept of bar-unit known for dialgebras for left-symmetric dialgebras S. Thus a bar-unit in S is an element e in S such that

$$x \dashv e = x = e \vdash x\,, \quad \text{for all} \;\; x \in S.$$

A bar-unit needs not to be unique (because, for instance, this fact holds in dialgebras). The subset of bar-units of S is called its **halo**. A unital left-symmetric dialgebra is a left-symmetric dialgebra with a specified bar-unit e. The problem of adding a unit-bar to left-symmetric dialgebras remains open.

Observe that if a left-symmetric dialgebra has a unit ϵ, which satisfies $\epsilon \dashv x = x = x \vdash \epsilon$ for any $x \in S$, then from (10), we deduce $\dashv = \vdash$ and S is a left-symmetric algebra with unit ϵ.

Let $(S, \vdash, \dashv)$ be a left-symmetric dialgebra. Let S^{ann} be the subspace of S spanned by the elements of the form $x \dashv y - x \vdash y$, for $x, y \in S$. From (8) and (9) we conclude

$$x \dashv z = 0 = z \vdash x, \tag{14}$$

for all $x \in S$ and $z \in S^{ann}$. One can see that S^{ann} is an ideal of S with respect to both products. On other hand S is a left-symmetric algebra if and only if $S^{ann} = \{0\}$.

Let S be a left-symmetric dialgebra, a derivation of S is an element of $End(S)$ such that $\delta(x \vdash y) = \delta x \vdash y + x \vdash \delta y$ and $\delta(x \dashv y) = \delta x \dashv y + x \dashv \delta y$ for every $x, y \in S$. $Der(S)$ denotes the set consists of all derivations of S.

In this moment it is convenient to introduce the set $Z(S) = Z_{\vdash}(S) \cap Z_{\dashv}(S)$ where by definition $Z_{\vdash}(S) = \{z \in S | z \vdash x = 0, \forall x \in S\}$ and $Z_{\dashv}(S) = \{z \in S | x \dashv z = 0, \forall x \in S\}$. It is clear that $S^{ann} \subset Z(S)$. Also $Z(S)$ is an ideal of S with respect to $\vdash$ and $\dashv$. Observe that if $e \in S$ is a bar unit of S, then $\delta e \in Z(S)$. On other hand, let us suppose that the left-symmetric dialgebra S has a bar unit then $S^{ann} = Z(S)$. In fact, let $e \in S$ be a bar unit then for any $z \in Z(S)$ we have $z = z \dashv e - z \vdash e \in S^{ann}$, thus $Z(S) \subset S^{ann}$.

Lemma 7 *Let S be a left-symmetric dialgebra and $\delta \in Der(S)$ arbitrary. Then S^{ann} and $Z(S)$ are invariant subspaces of δ.*

Proof. For any $x, y \in S$ we have

$$\begin{aligned}\delta(x \dashv y - x \vdash y) &= \delta(x \dashv y) - \delta(x \vdash y)\\ &= (\delta x \dashv y + x \dashv \delta y) - (\delta x \vdash y + x \vdash \delta y)\\ &= (\delta x \dashv y - \delta x \vdash y) + (x \dashv \delta y - x \vdash \delta y),\end{aligned} \tag{15}$$

thus $\delta z \in S^{ann}$ for all $z \in S^{ann}$. Let us suppose that $z \in Z(B)$ then for any $x \in S$ we have $\delta z \vdash x = \delta z \vdash x + z \vdash \delta x = \delta(z \vdash x) = \delta 0 = 0$. In a similar form is possible to show that $x \dashv \delta z = 0$. It follows that $\delta z \in Z(S)$. The Lemma has been proved. ∎

Definition 8 *Let $(S, \vdash, \dashv)$ be a left-symmetric dialgebra. Suppose that there exists an ideal I such that $S^{ann} \subset I \subset Z(S)$ for which $S = I \oplus A$ as subspaces, where A is a left-symmetric subdialgebra of S. Then S is called a* ***split*** *left-symmetric dialgebra.*

Note that if $S = S^{ann} \oplus A$ is split then A is a left-symmetric algebra. In fact if $x, y \in A$ are arbitrary, then $x \dashv y - x \vdash y \in S^{ann} \cap A$, hence $x \dashv y - x \vdash y = 0$.

Let us assume now that the split left-symmetric dialgebra $S = S^{ann} \oplus A$ has a bar unit e. Consider the decomposition $e = e^{ann} + e_A$ holds, then $\epsilon = e_A$ is a unit of A. More even ϵ is a unit of S and any unit of S can be written as $i + \epsilon$.

Theorem 9 *Let $(A, \cdot)$ be a left-symmetric algebra and let I be a bimodule over A. We define two products on $S = I \oplus A$ as follows*

$$(i_1 + a_1) \vdash (i_2 + a_2) = a_1 i_2 + a_1 a_2, \;\; (i_1 + a_1) \dashv (i_2 + a_2) = i_1 a_2 + a_1 a_2. \tag{16}$$

Assume that the following equalities hold

$$i(ab) - (ia)b = a(ib) - (ai)b, \;\; c(dj) - (cd)j = d(cj) - (dc)j, \tag{17}$$

for every $i, j \in I$ and all $a, b, c, d \in A$. Then $(S, \vdash, \dashv)$ is a left-symmetric dialgebra. On other hand $S^{ann} = \{ia - bj |\; i, j \in I \text{ and } a, b \in A\}$.

Proof. the proof is a simple calculation. ∎

It is clear that the left-symmetric dialgebra which was constructed in the previous Theorem is split.

We know that if $(S, \vdash, \dashv)$ is a lef-symmetric dialgebra then $Leib(S)$ is a Leibniz algebra. Assume now that $S = S^{ann} \oplus A$ is split. Let $x \in S$ be given then we denote by $a(x)$ its projection on A. We say that x and y are $Leib(S)$-equivalent if $a(x) = a(y)$ in whose case we write $x \sim y$. It is clear that $\sim$ constitutes a relation of equivalence for which the classes of equivalence have the form $\Im_x = \{i + a(x) | i \in S^{ann}\}$. Observe that for all $y \in \Im_x$ we have $L_y^{\vdash} = L_{a(x)}^{\vdash}$ where $D(L_{a(x)}^{\vdash}) = S$. Since the map $a(x) : S \longrightarrow A$ is linear then it is easy to see that $\Re = \{\Im_x | x \in S\}$ is vector space by to define $\Im_x + \Im_y = \Im_{x+y}$ and $\lambda \Im_x = \Im_{\lambda x}$ for all$x, y \in S$ and λ an arbitrary scalar.

Define now

$$\Im_x \vdash \Im_y = \Im_{x \vdash y}, \;\; \Im_x \dashv \Im_y = \Im_{x \dashv y}, \tag{18}$$

then $(\Re, \vdash, \dashv)$ is a left-symmetric dialgebra and the map $\Im : S \longrightarrow \Re$ defined as $\Im(x) = \Im_x$ is a morphism of left-symmetric dialgebras. Obviously

$$[\Im_x, \Im_y]_{\Re} = \Im_{[x,y]_S}. \tag{19}$$

Below we introduce affine Leibniz structures on Leibniz algebras

Definition 10 *Let $(L, [\cdot,\cdot])$ be a Leibniz algebra. A pair $(\bigtriangledown_1, \bigtriangledown_2)$ of bilinear mappings*

$$\bigtriangledown_1 : L \times L \longrightarrow L, \tag{20}$$

and

$$\bigtriangledown_2 : L \times L \longrightarrow L, \tag{21}$$

*is called an **affine Leibniz structure** if*

$$\bigtriangledown_2(x, y) - \bigtriangledown_1(y, x) = [x, y], \tag{22}$$

$$\bigtriangledown_1(\bigtriangledown_1(x, y), z) = \bigtriangledown_1(\bigtriangledown_2(x, y), z), \quad \bigtriangledown_2(x, \bigtriangledown_2(y, z)) = \bigtriangledown_2(x, \bigtriangledown_1(y, z)), \tag{23}$$

$$\bigtriangledown_2(x, \bigtriangledown_2(y, z)) - \bigtriangledown_1(y, \bigtriangledown_2(x, z)) = \bigtriangledown_2([x, y], z), \tag{24}$$

and

$$\bigtriangledown_1(x, \bigtriangledown_1(y, z)) - \bigtriangledown_1(y, \bigtriangledown_1(x, z)) = \bigtriangledown_1([x, y], z), \tag{25}$$

for all $x, y, z \in L$.

We have

Theorem 11 *Let $(L, [\cdot,\cdot])$ be a Leibniz algebra and let $(\bigtriangledown_1, \bigtriangledown_2)$ be an affine Leibniz structure. Then L is a left-symmetric dialgebra with $\vdash$ and $\dashv$ defined as*

$$x \vdash y = \bigtriangledown_1(x, y), \quad x \dashv y = \bigtriangledown_2(x, y). \tag{26}$$

Proof. (23) implies (8) and (9). On other hand, (10) and (11) follow of (24) and (25) respectively. ■

Affine transformations defined over left-symmetric dialgebras

In this section we shall study affine transformations over left-symmetric dialgebras.

Let $(S, \vdash, \dashv)$ be a left-symmetric dialgebra given. We choose a point $s_0 \in S$ and then we associate to each $x \in S$ two special affine transformations as follows

$$D_x^{\vdash}(s) = x \vdash (s - s_0) + x = L_x^{\vdash}(s - s_0) + x, \tag{27}$$

and

$$D_x^{\dashv}(s) = x \dashv (s - s_0) + x = L_x^{\dashv}(s - s_0) + x. \tag{28}$$

It is clear that $D(S)^{\vdash} = \{D_x^{\vdash} | x \in S\}$ and $D(S)^{\dashv} = \{D_x^{\dashv} | x \in S\}$ can be considered vector spaces.

Theorem 12 *Let us define in $D(S)^{\vdash}$ the following bracket*

$$[D_x^{\vdash}, D_y^{\vdash}]_l(s) = L_x^{\vdash} D_y^{\vdash}(s) - L_y^{\vdash} D_x^{\vdash}(s), \tag{29}$$

for all $s \in S$ and any $x, y \in S$. Then

$$[D_x^{\vdash}, D_y^{\vdash}]_l(s) = D_{[x,y]_S}^{\vdash}(s), \tag{30}$$

and $(D(S)^{\vdash}, [.,.]_l)$ is a Leibniz algebra.

Proof. In order to simplify the notation in the calculations of the proof of this Theorem where appropriate we write $[x, y]$ instead of $[x, y]_S$. A simple calculation shows that

$$[D_x^{\vdash}, D_y^{\vdash}]_l(s) = x \vdash (y \vdash (s - s_0)) - y \vdash (x \vdash (s - s_0)) + [x, y]_S.$$

From this last equality, (9) and (11) follow at once (30). Now, we must verify the Leibniz identity for $[.,.]_l$. In fact, using (30) we have

$$\begin{aligned}
[D_x^{\vdash}, [D_y^{\vdash}, D_z^{\vdash}]_l]_l(s) &= [D_x^{\vdash}, D_{[y,z]}^{\vdash}]_l(s) = D_{[x,[y,z]]}^{\vdash}(s) \\
&= D_{[[x,y],z]-[[x,z],y]}^{\vdash}(s) = D_{[[x,y],z]}^{\vdash}(s) - D_{[[x,z],y]}^{\vdash}(s) \\
&= [D_{[x,y]}^{\vdash}, D_z^{\vdash}]_l(s) - [D_{[x,z]}^{\vdash}, D_y^{\vdash}]_l(s) \\
&= [[D_x^{\vdash}, D_y^{\vdash}]_l, D_z^{\vdash}]_l(s) - [[D_x^{\vdash}, D_z^{\vdash}]_l, D_y^{\vdash}]_l(s),
\end{aligned}$$

thus $(D(S)^{\vdash}, [.,.]_l)$ is a Leibniz algebra. ■

This result acquires special significance by the fact that both $L_x^{\vdash} D_y^{\vdash}$ and $L_y^{\vdash} D_x^{\vdash}(s)$ do not belong to $D(S)^{\vdash}$, hence they could not be used to define a structure of left-symmetric dialgebra on $D(S)^{\vdash}$.

The following result has the same features as above, thus we have

Theorem 13 *If in $D(S)^{\dashv}$ we introduce the bracket*

$$[D_x^{\dashv}, D_y^{\dashv}]_r(s) = L_x^{\dashv} D_y^{\dashv}(s) - L_y^{\vdash} D_x^{\dashv}(s), \tag{31}$$

for all $s \in S$ and any $x, y \in S$. Then the following equality

$$[D_x^{\dashv}, D_y^{\dashv}]_r(s) = D_{[x,y]_S}^{\dashv}(s), \tag{32}$$

holds and $(D(S)^{\dashv}, [.,.]_r)$ is a Leibniz algebra.

Proof. The proof of this Theorem is similar to the proof of the previous Theorem except that to prove (32) we use (10) instead of (11). ■

Let us suppose that $S = S^{ann} \oplus A$ is a split left-symmetric dialgebra. Set $x = i(x) + a(x)$, $s = i(s) + a(s)$ and $s_0 = i(s_0) + a(s_0)$ then

$$D_x^{\vdash} s = D_{a(x)}^{ann,\vdash} i(s) + D_{a(x)}^{A,\vdash} a(s), \tag{33}$$

where $D_{a(x)}^{ann,\vdash} i(s) = a(x) \vdash (i(s) - i(s_0)) + i(x)$ and $D_{a(x)}^{A,\vdash} a(s) = a(x)(a(s) - a(s_0)) + a(x)$. Obviously $D_{a(x)}^{ann,\vdash} i(s) \in S^{ann}$ and $D_{a(x)}^{A,\vdash} a(s) \in A$. Moreover $D_{a(x)}^{A,\vdash}$ can be considered as an affine transformations over A. We understand that the notation $D_x^{ann,\vdash}i(s)$ and $D_x^{A,\vdash}$ would be more appropriate to denote $D_{a(x)}^{ann,\vdash}$ and $D_{a(x)}^{A,\vdash}$ respectively but we maintain the last for convenience, even more, both will be used interchangeably.

On other hand, of (33) one concludes that

$$D^{\vdash}_{[x,y]_S}(s) = D^{ann,\vdash}_{[a(x),a(y)]}i(s) + D^{A,\vdash}_{[a(x),a(y)]}a(s), \tag{34}$$

Now a simple calculation leads to equality

$$\begin{aligned}[D^{\vdash}_x, D^{\vdash}_y]_l(s) =& (L^{\vdash}_{a(x)}D^{ann,\vdash}_{a(y)} - L^{\vdash}_{a(y)}D^{ann,\vdash}_{a(x)})(i(s)) \\ &+ (L^{\vdash}_{a(x)}D^{A,\vdash}_{a(y)} - L^{\vdash}_{a(y)}D^{A,\vdash}_{a(x)})(a(s)),\end{aligned} \tag{35}$$

hence, combining equations (30), (34) and (35) one obtains

$$(L^{\vdash}_{a(x)}D^{A,\vdash}_{a(y)} - L^{\vdash}_{a(y)}D^{A,\vdash}_{a(x)})(a(s)) = D^{A,\vdash}_{[a(x),a(y)]}a(s) = D^{A,\vdash}_{[x,y]}a(s), \tag{36}$$

and

$$(L^{\vdash}_{a(x)}D^{ann,\vdash}_{a(y)} - L^{\vdash}_{a(y)}D^{ann,\vdash}_{a(x)})(i(s)) = D^{ann,\vdash}_{[a(x),a(y)]}i(s) = D^{ann,\vdash}_{[x,y]}i(s). \tag{37}$$

The equality (36) implies that we can state the following result

Proposition 14 *Let us assume that $S = S^{ann} \oplus A$ is a split left-symmetric dialgebra. Then, the vector space $D^{A,\vdash}(S) = \{D^{A,\vdash}_{a(x)} = D^{A,\vdash}_x | x \in S\}$ is a Lie algebra with respect to the bracket defined in the form*

$$[D^{A,\vdash}_{a(x)}, D^{A,\vdash}_{a(y)}]a(s) = [D^{A,\vdash}_x, D^{A,\vdash}_y]a(s) = (L^{\vdash}_{a(x)}D^{A,\vdash}_{a(y)} - L^{\vdash}_{a(y)}D^{A,\vdash}_{a(x)})a(s), \tag{38}$$

for all $x, y \in S$.

Proof. It is clear that $(L^{\vdash}_{a(x)}D^{A,\vdash}_{a(y)} - L^{\vdash}_{a(y)}D^{A,\vdash}_{a(x)})(a(s)) \in A$. On other hand, the Leibniz identity follows of (36). ■

Observe that the expression $(L^{\vdash}_{a(x)}D^{ann,\vdash}_{a(y)} - L^{\vdash}_{a(y)}D^{ann,\vdash}_{a(x)})(i(s))$ does not define a Lie bracket on $D^{ann,\vdash}(S) = \{D^{ann,\vdash}_{a(x)} | x \in S\}$ despite (37). This is because in

$$D^{ann,\vdash}_{[a(x),a(y)]}i(s) = D^{ann,\vdash}_{[x,y]}i(s) = [a(x), a(y)] \vdash (i(s) - i(s_0)) + [i(x), a(y)]_S, \tag{39}$$

the term $[i(x), a(y)]_S$ does not allow us to obtain the Jacobi identity.

It is easy to see that if we define

$$\begin{aligned}[D^{ann,\vdash}_{a(x)}, D^{ann,\vdash}_{a(y)}](i(s)) &= [D^{ann,\vdash}_x, D^{ann,\vdash}_y](i(s)) \\ &= (L^{\vdash}_{a(x)}D^{ann,\vdash}_{a(y)} - L^{\vdash}_{a(y)}D^{ann,\vdash}_{a(x)})(i(s)),\end{aligned} \tag{40}$$

then

$$[D^{\vdash}_x, D^{\vdash}_y](s) = [D^{ann,\vdash}_{a(x)}, D^{ann,\vdash}_{a(y)}](i(s)) + [D^{A,\vdash}_{a(x)}, D^{A,\vdash}_{a(y)}](a(s)). \tag{41}$$

Thus, we have a less obvious result

Proposition 15 *For a split left-symmetric dialgebra $S = S^{ann} \oplus A$, the vector space $D^{ann,\vdash}(S) = \{D^{ann,\vdash}_{a(x)} = D^{ann,\vdash}_x | x \in S\}$ is a Leibniz algebra with respect to the bracket (40).*

Proof. We already know that for any $s \in S$

$$[D_x^{\vdash}, D_{[y,z]_S}^{\vdash}]_l s = [D_{[x,y]_S}^{\vdash}, D_z^{\vdash}]_l s - [D_{[x,z]_S}^{\vdash}, D_y^{\vdash}]_l s, \tag{42}$$

now, from (41) one obtains

$$[D_x^{\vdash}, D_{[y,z]_S}^{\vdash}]_l s = D_{[a(x),[a(y),a(z)]]}^{ann,\vdash} i(s) + D_{[a(x),[a(y),a(z)]]}^{A,\vdash} a(s),$$

$$[D_{[x,y]_S}^{\vdash}, D_z^{\vdash}]_l s = D_{[[a(x),a(y)],a(z)]}^{ann,\vdash} i(s) + D_{[[a(x),a(y)],a(z)]}^{A,\vdash} a(s),$$

and

$$[D_{[x,z]_S}^{\vdash}, D_y^{\vdash}]_l s = D_{[[a(x),a(z)],a(y)]}^{ann,\vdash} i(s) + D_{[[a(x),a(z)],a(y)]}^{A,\vdash} a(s).$$

But, since $D^{A,\vdash}(S)$ is a Lie algebra and $D^{A,\vdash}{-x} = D^{A,\vdash}{-a(x)} = -D^{A,\vdash}x = -D^{A,\vdash}a(x)$ then substituting the previous expressions in (42) we arrive to the following equality

$$D_{[a(x),[a(y),a(z)]]}^{ann,\vdash} i(s) = D_{[[a(x),a(y)],a(z)]}^{ann,\vdash} i(s) - D_{[[a(x),a(z)],a(y)]}^{ann,\vdash} i(s), \tag{43}$$

now, it follows of (37), (40) and (43) that the Leibniz identity holds for the bracket (40). ■

Acknowledgment

The author was supported under CONACYT grant 106923.

References

[1] C. Bai, *Left-symmetric algebras from linear functions.* Journal of Algebra, **281** (2004), 651-665.

[2] C. Bai, L. Liu and X. Ni, *Some results on $L-$dendriform algebras.* To appear in Journal of Geometry and Physics.

[3] C. Bai, D. Hou and Z. Chen, *On a class of Lie groups with a left invariant flat pseudo-metric.* To appear in Monatsh Math.

[4] D. Burdes, *Left-symmetric algebras, or pre-Lie algebras in geometry and physics.* Central Eur. J. Math., **4** (2006), 323-357.

[5] D. Burdes, K. Dekimpe, *Novikov structures on solvable Lie algebras.* J. Geom. Phys., **56** (2006), 1837-1855.

[6] D. Burdes, K. Dekimpe and K. Vercammen, *Novikov algebras and Novikov structures on Lie algebras.* Linear Algebras and its Aplications, **429** (2008), 31-41.

[7] R. Felipe, N. López and F. Ongay, *R-Matrices for Leibniz algebras.* Letters in Mathematical Physics, **63** (2003), 157-164.

[8] R. Felipe, N. López, R. Velasquez and F. Ongay, *Yang-Baxter equations on matrix dialgebras with bar unit.* Linear Algebra and its Applications, **403** (2005), 31-44.

[9] R. Felipe, *An analogue to functional analysis in dialgebras.* Intern. Math. Forum, **2** (2007), 1069-1091.

[10] D. Liu, *Steinberg-Leibniz algebras and superalgebras.* J. Algebra, **283** (2005), 199-221.

[11] J. L. Loday, *Une version non commutative des algèbres de Lie: les algèbres de Leibniz.* Ens. Math., **39** (1993), 269-293.

[12] J. L. Loday, *Dialgebras*, in: Dialgebras and related operads, Lectures Notes in Mathematics, vol. **1763**, Springer Verlag, 2001, pp. 7-66.

[13] R. Velasquez and R. Felipe, *Split dialgebras, split quasi-Jordan algebras and regular elements.* Journal of Algebra and its Applications, **8** (2009), 191-218.

[14] E. Vinberg, *The theory of convex homogeneous cones.* Trans. Moscow Math. Soc., **12** (1963), 340-403.