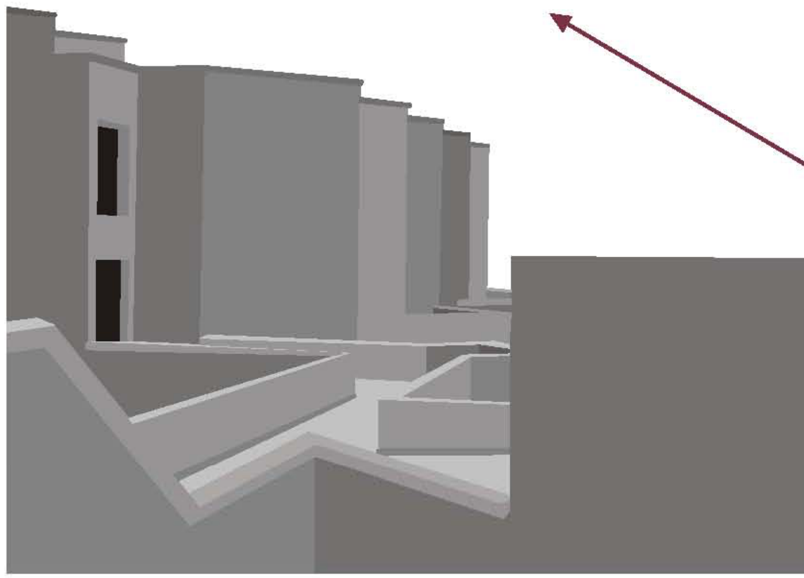

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Exponential Functionals of Brownian Motion and Explosion Times of a System of Semilinear SPDEs

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Abstract

We investigate lower and upper bounds for the blowup times of a system of semilinear SPDEs. Under certain conditions on the system parameters, we obtain explicit solutions of a related system of random PDEs, which allows us to use a formula due to Yor to obtain the ditribution functions of several explosion times. We also give the Laplace transforms at independent exponential times of related exponential functionals of Brownian motion.

1 Introduction

Existence and nonexistence of global solution for semilinear parabolic equations was investigated initially by Fujita [4], who proved that for a bounded smooth domain $D \subset \mathbb{R}^d$, the equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u^{1+\beta}(t,x), \quad x \in D,$$

with Dirichlet boundary condition, where $\beta > 0$ is a constant, explodes in finite time for all nonnegative initial values $u(0,x) \in L^2(D)$ satisfying $\int_D u(0,x)\psi(x)\,dx > \Lambda^{1/\beta}$. Here $\Lambda > 0$ is the first eigenvalue of the Laplacian on D and ψ the corresponding eigenfunction normalized so that $\|\psi\|_{L^1} = 1$.

In a previous work [2] we investigated blow-up times of semilinear SPDEs of the prototype

$$du(t,x) = \Big(\Delta u(t,x) + u^{1+\beta}(t,x)\Big)\,dt + \kappa u(t,x)\,dW_t, \quad x \in D, \tag{1.1}$$

with Dirichlet boundary conditions, where $\{W_t\}$ is a standard one-dimensional Brownian motion. We obtained bounds both for the probability of finite-time blow-up of $u(t,x)$, and for the probability of nonexplosion of $u(t,x)$ in finite time. In case of $\kappa = 0$, the bounds we found give the result of Fujita quoted above. We refer to [2] for definitions of blow-up times, and for types of solutions of SPDEs. In [2] it is also shown that the asymptotic behavior of (1.1) is determined to a great extent by the distribution of the exponential functional

$$\int_0^t \exp\{\kappa\beta W_r - \beta(\Lambda + \kappa^2/2)r\}\,dr.$$

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Functionals of the above form arise in many applications, specially in Financial and Actuarial Mathematics [3, 6, 14] and have been investigated by different methods and by various authors [1, 3, 7, 8, 11, 12, 13, 14].

From the results of [2] it follows that for initial values of the form $u(0,x) = k\psi(x)$, $x \in D$, where $k > 0$ is a parameter, the explosion time T of (1.1) satisfies $\sigma_* \le T \le \sigma^*$, where

$$\begin{aligned}
\sigma_* &= \inf\left\{ t \ge 0 : \int_0^t \exp\{\kappa\beta W_r - \beta(\Lambda + \kappa^2/2)r\}\, dr \ge 1/(\beta k^\beta \|\psi\|_\infty^\beta) \right\}, \\
\sigma^* &= \inf\left\{ t \ge 0 : \int_0^t \exp\{\kappa\beta W_r - \beta(\Lambda + \kappa^2/2)r\}\, dr \ge 1/\left(\beta k^\beta \left(\int \psi^2(x)\, dx\right)^\beta\right) \right\}.
\end{aligned}$$

Let us mention that the density of $\int_0^t \exp\{\kappa\beta W_r - \beta(\Lambda + \kappa^2/2)r\}\, dr$, $t > 0$, can be obtained from Yor's formula ([14], Ch. 4 and [13]). In particular, when $\int \psi^2(x)\, dx = \|\psi\|_\infty$, the above inequalities together with Yor's formula yield the exact distribution of the explosion time T. The upper bound σ^* of T was achieved in [2] by determining the blow-up time of a subsolution of Eq. (1.1), while for the lower bound σ_*, an scheme of successive approximations for the mild solution of (1.1) was used.

Our aim in this paper is to obtain lower and upper bounds for the blow-up times of the system of semilinear SPDEs

$$\begin{aligned}
du_1(t,x) &= [(\Delta + V_1)u_1(t,x) + u_2^p(t,x)]\, dt + \kappa_1 u_1(t,x)\, dW_t \\
du_2(t,x) &= [(\Delta + V_2)u_2(t,x) + u_1^q(t,x)]\, dt + \kappa_2 u_2(t,x)\, dW_t, \quad x \in D,
\end{aligned} \tag{1.2}$$

with Dirichlet boundary conditions

$$u_i(0,x) = f_i(x) \ge 0,\ x \in D \text{ and } u_i(t,x) = 0,\ t \ge 0,\ x \in \partial D, \quad i = 1,2. \tag{1.3}$$

Here $p \ge q > 1$ are constants, $D \subset \mathbb{R}^d$ is a bounded smooth domain, $V_i > 0$ and $\kappa_i \ne 0$ are constants, $i = 1,2$, and $\{W_t\}$, is a standard one-dimensional Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As in [2], we are going to consider weak solutions of (1.2)-(1.3).

Below we are going to take the initial values of the form $f_i = k_i\psi$ for some constants $k_i > 0$, $i = 1,2$. Also, we are going to set $V_i = \Lambda + \kappa_i^2/2$, $i = 1,2$. These choices make it possible to obtain explicit solutions of a related system of random PDEs, and to use Yor's formula and some extensions of it. Moreover, in contrast with the case treated in [2], the blow-up times of (1.2) are finite with probability one; see e.g. [7], Prop. 6.4 and [8], Section 2. For general constants κ_1, κ_2 and $p \ge q > 1$, we obtain random times τ_{**}, τ^{**} such that $\tau_{**} \le T \le \tau^{**}$, where T is the blow-up time of (1.2). Moreover, we show that the distributions of τ_{**} and τ^{**} are given in terms of exponential functionals of the form

$$\mathcal{A}^{**}(t) = \int_0^t e^{aW_r} \wedge e^{bW_r}\, dr \quad \text{and} \quad \mathcal{A}_{**}(t) = \int_0^t e^{aW_r} \vee e^{bW_r}\, dr.$$

By adapting a method of Jeanblanc, Pitman and Yor [5] to our setting, we obtain the Laplace transforms of $\mathcal{A}^{**}(\theta_\lambda)$ and $\mathcal{A}_{**}(\theta_\lambda)$ at random times θ_λ which are exponentially distributed with parameter λ, for any value of $\lambda > 0$. From this we can derive upper bounds for probabilities of the form $\mathbb{P}\{T \ge \theta_\lambda\}$ for each $\lambda > 0$. In the particular case of $p\kappa_2 - \kappa_1 = q\kappa_1 - \kappa_2 =: \rho$, the relevant exponential functional is of the form $\int_0^t e^{\rho W_r} dr$, and the distributions of τ_{**} and τ^{**} can be obtained explicitly from Yor's formula. For a system of the form (1.2) with $V_1 = V_2 = 0$, similar results can be obtained for τ^{**}. However, in this case τ_{**} is given in terms of the solution of a non-homogeneous random Bernoulli equation for which we were not able to find an explicit solution.

2 A system of random PDEs

Let us define for $i = 1, 2$,

$$v_i(t,x) = \exp\{-\kappa_i W_t\} u_i(t,x), \quad t \geq 0, \; x \in D. \tag{2.1}$$

By Ito's formula,

$$e^{-\kappa_i W_t} = 1 - \kappa_i \int_0^t e^{-\kappa_i W_s} dW_s + \frac{\kappa_i^2}{2} \int_0^t e^{-\kappa_i W_s} ds.$$

Putting $u_i(t,\varphi_i) = \int_D u_i(t,x)\varphi_i(x)\,dx$, $i = 1, 2$, where φ_i is any smooth function with compact support, we get that a weak solution of (1.2) is given by

$$u_i(t,\varphi_i) = u_i(0,\varphi_i) + \int_0^t u_i(s,(\Delta + V_i)\varphi_i)\,ds + \int_0^t u_j^{\beta_i}(s,\varphi_i)\,ds + \kappa_i \int_0^t u_i(s,\varphi_i)\,dW_s, \tag{2.2}$$

where $\{j\} = \{1,2\}\backslash\{i\}$ and $\beta_1 = p, \; \beta_2 = q$. By applying the integration by parts formula we get that

$$\begin{aligned} v_i(t,\varphi_i) &:= \int_D v_i(t,x)\varphi_i(x)\,dx \\ &= v_i(0,\varphi_i) + \int_0^t e^{-\kappa_i W_s}\,du_i(s,\varphi_i) \\ &+ \int_0^t u_i(s,\varphi_i)\left(-\kappa_i e^{-\kappa_i W_s} dW_s + \frac{\kappa_i^2}{2} e^{-\kappa_i W_s}\,ds\right) + \left[e^{-\kappa_i W_\cdot}, u_i(\cdot,\varphi_i)\right](t), \end{aligned}$$

where the quadratic variation is given by $\left[e^{-\kappa_i W_\cdot}, u_i(\cdot,\varphi_i)\right](t) = -\kappa_i^2 \int_0^t e^{-\kappa_i W_s} u_i(s,\varphi_i)\,ds$, $t \geq 0$. Therefore,

$$\begin{aligned} v_i(t,\varphi_i) &= v_i(0,\varphi_i) + \int_0^t v_i(s,(\Delta + V_i)\varphi_i)\,ds \\ &\quad + \int_0^t e^{-\kappa_i W_s}\left(e^{\kappa_j W_s} v_j\right)^{\beta_i}\varphi_i(x)\,ds - \frac{\kappa_i^2}{2}\int_0^t e^{-\kappa_i W_s} u_i(s,\varphi_i)\,ds \\ &= v_i(0,\varphi_i) + \int_0^t \left[v_i(s,(\Delta + V_i)\varphi_i) - \frac{\kappa_i^2}{2} v_i(s,\varphi_i)\right] ds \\ &\quad + \int_0^t e^{-\kappa_i W_s}\left(e^{\kappa_j W_s} v_j\right)^{\beta_i}(s,\varphi_i)\,ds. \end{aligned} \tag{2.3}$$

Hence, the vector $(v_1(t,x), v_2(t,x))$ is a weak solution of the system

$$\frac{\partial v_i(t,x)}{\partial t} = \left(\Delta + V_i - \kappa_i^2/2\right) v_i(t,x) + e^{-\kappa_i W_t}\left(e^{\kappa_j W_t} v_j(t,x)\right)^{\beta_i}, \quad v_i(0,x) = f_i(x), \; i = 1, 2,$$

whose integral form is

$$v_i(t,x) = e^{t(V_i - \kappa_i^2/2)} T_t f_i(x) + \int_0^t e^{r(V_i - \kappa_i^2/2)} T_r \left[e^{-\kappa_i W_{t-r}}\left(e^{\kappa_j W_{t-r}} v_j(t-r,\cdot)\right)^{\beta_i}\right](x)\,dr \tag{2.4}$$

for $t \geq 0$, and $i = 1, 2$. Here $\{T_t\}$ is the semigroup of bounded linear operators given by

$$T_t f(x) = \mathbb{E}\left[f(X_t),\ t < \tau_D \middle| X_0 = x \right],\ x \in D,$$

for all bounded and measurable $f : D \to \mathbb{R}$, where $\{X_t\}_{t\geq 0}$ is the d-dimensional Brownian motion with variance parameter 2, killed at the time τ_D at which it hits ∂D. As above, we denote by $\Lambda > 0$ the first eigenvalue of the Laplacian on D, which satisfies

$$\Delta\psi(x) = -\Lambda\psi(x),\ x \in D, \tag{2.5}$$

ψ being the corresponding eigenfunction, which is strictly positive on D and $\psi|_{\partial D} = 0$. Recall that $T_t\psi = e^{-\Lambda t}\psi$, $t \geq 0$. We are going to assume that ψ is normalized so that $\int_D \psi(x)\,dx = 1$.

We write T for the blow up time of system (2.4) when the initial values are of the form $f_1 = L(1)\psi$ and $f_2 = L(2)\psi$ for some positive constants $L(1)$ and $L(2)$. Due to (2.1) and to the a.s. continuity of Brownian paths, T is also the explosion time of system (1.2) with initial values of the above form.

3 The case $p\kappa_2 - \kappa_1 = q\kappa_1 - \kappa_2$

First we consider Eq. (1.2) with parameters

$$p \geq q > 1, \quad V_i = \Lambda + \frac{\kappa_i^2}{2},\ i = 1, 2, \quad \text{and} \quad p\kappa_2 - \kappa_1 = q\kappa_1 - \kappa_2 =: \rho. \tag{3.1}$$

We are going to find random times τ_* and τ^* which are given in terms of the exponential functional $\mathcal{A}_t := \int_0^t e^{\rho W_s}\,ds$, $t \geq 0$, and such that $0 \leq \tau_* \leq T \leq \tau^*$. The density function of $\mathcal{A}_t$ is given explicitly in [14], Section 4.

3.1 A lower bound for T

We are going to obtain a random time τ_* which satisfies $\tau_* \leq T$.

Theorem 1 *Assume conditions (3.1), and let the initial values be of the form*

$$f_1 = L(1)\psi \text{ and } f_2 = L(2)\psi \tag{3.2}$$

for some positive constants $L(1)$ and $L(2)$. Let τ_* be given by

$$\tau_* = \inf\left\{ t \geq 0 : \int_0^t \exp\{\rho W_r\}\,dr \ \geq\ \min\left\{ \frac{1}{(p-1)L^{p-1}(1)\|\psi\|_\infty^{p-1}}, \frac{1}{(q-1)L^{q-1}(2)\|\psi\|_\infty^{q-1}} \right\} \right\}.$$

Then $\tau_* \leq T$.

Proof. Let v_1 and v_2 solve (2.4). Then we have

$$\begin{aligned} v_1(t,x) &= e^{\Lambda t} T_t f_1(x) + \int_0^t e^{\Lambda(t-r)} T_{t-r}\left[e^{\rho W_r} v_2^p(r,x) \right] dr \\ v_2(t,x) &= e^{\Lambda t} T_t f_2(x) + \int_0^t e^{\Lambda(t-r)} T_{t-r}\left[e^{\rho W_r} v_1^q(r,x) \right] dr, \quad x \in D, \quad t \geq 0. \end{aligned}$$

We define the operators $\mathcal{R}_1$, $\mathcal{R}_2$ by

$$\begin{aligned}
\mathcal{R}_1 v(t,x) &= e^{\Lambda t} T_t f_1(x) + \int_0^t e^{\rho W_r} e^{\Lambda(t-r)} \left(T_{t-r} v\right)^p dr, \\
\mathcal{R}_2 v(t,x) &= e^{\Lambda t} T_t f_2(x) + \int_0^t e^{\rho W_r} e^{\Lambda(t-r)} \left(T_{t-r} v\right)^q dr, \quad x \in D, \quad t \geq 0,
\end{aligned}$$

where v is any nonnegative, bounded and measurable function. Moreover, on the set $t \leq \tau_*$ we put

$$\mathcal{B}_1(t) = \left[1 - (p-1) \int_0^t e^{\rho W_r} \left\| e^{\Lambda r} T_r f_1 \right\|_\infty^{p-1} dr\right]^{-\frac{1}{p-1}},$$

$$\mathcal{B}_2(t) = \left[1 - (q-1) \int_0^t e^{\rho W_r} \left\| e^{\Lambda r} T_r f_2 \right\|_\infty^{q-1} dr\right]^{-\frac{1}{q-1}}.$$

Then we have

$$\frac{d\mathcal{B}_1(t)}{dt} = e^{\rho W_t + (p-1)\Lambda t} \left\| T_t f_1 \right\|_\infty^{p-1} \mathcal{B}_1^p(t), \quad \mathcal{B}_1(0) = 1,$$

hence

$$\mathcal{B}_1(t) = 1 + \int_0^t e^{\rho W_r + (p-1)\Lambda r} \left\| T_r f_1 \right\|_\infty^{p-1} \mathcal{B}_1^p(r)\, dr,$$

and similarly,

$$\mathcal{B}_2(t) = 1 + \int_0^t e^{\rho W_r + (q-1)\Lambda r} \left\| T_r f_2 \right\|_\infty^{q-1} \mathcal{B}_1^q(r)\, dr.$$

Let us choose $v \geq 0$ such that $v(t,x) \leq e^{\Lambda t} T_t f_1(x) \mathcal{B}_1(t)$ for $x \in D$ and $t < \tau_*$. Then $e^{\Lambda t} T_t f_1(x) \leq \mathcal{R}_1 v(t,x)$ and

$$\begin{aligned}
\mathcal{R}_1 v(t,x) &= e^{\Lambda t} T_t f_1(x) + \int_0^t e^{\rho W_r + \Lambda(t-r)} \left(T_{t-r} v(r,x)\right)^p \, dr \\
&\leq e^{\Lambda t} T_t f_1(x) + \int_0^t e^{\rho W_r + \Lambda(t-r)} \mathcal{B}_1(r)^{p-1} e^{\Lambda r(p-1)} \left\| T_r f_1 \right\|_\infty^{p-1} \mathcal{B}_1(r) e^{\Lambda r} T_{t-r} \left(T_r f_1\right)(x)\, dr \\
&= e\Lambda t T_t f_1(x) \left[1 + \int_0^t e^{\rho W_r} \left\| e^{\Lambda r} T_r f_1 \right\|_\infty^{p-1} B_1^p(r)\, dr\right] \\
&= e^{\Lambda t} T_t f_1(x) \mathcal{B}_1(t),
\end{aligned}$$

and similarly,

$$e^{\Lambda t} T_t f_2(x) \leq \mathcal{R}_2 u(t,x) \leq e^{\Lambda t} T_t f_2(x) \mathcal{B}_2(t)$$

for all u such that $0 \leq u(t,x) \leq e^{\Lambda t} T_t f_2(x) \mathcal{B}_2(t)$. Let us take, for $x \in D$ and $0 \leq t < \tau_*$,

$$u_1^{(0)}(t,x) = e^{\Lambda t} T_t f_1(x), \qquad u_2^{(0)}(t,x) = e^{\Lambda t} T_t f_2(x)$$

and

$$u_1^{(n)}(t,x) = \mathcal{R}_1 u_2^{(n-1)}(t,x), \qquad u_2^{(n)}(t,x) = \mathcal{R}_2 u_1^{(n-1)}(t,x), \quad n \geq 1.$$

We are going to show that the function sequences $\{u_1^{(n)}\}$, $\{u_2^{(n)}\}$ are increasing. As a matter of fact,

$$u_1^{(0)}(t,x) \leq e^{\Lambda t} T_t f_1(x) + \int_0^t e^{\rho W_r + \Lambda(t-r)} \left(T_{t-r} u_2^{(0)}(r,x)\right)^p \, dr = \mathcal{R}_1 u_2^{(0)}(t,x) = u_1^{(1)}(t,x).$$

Now assume that $u_1^{(n)} \geq u_1^{(n-1)}$ and $u_2^{(n)} \geq u_2^{(n-1)}$ for some $n \geq 1$. Then

$$u_1^{(n+1)} = \mathcal{R}_1 u_2^{(n)} \geq \mathcal{R}_1 u_2^{(n-1)} = u_1^{(n)},$$

where we have used the monotonicity of $\mathcal{R}_1$ to obtain the above inequality. In the same way is proved the monotonicity of the other sequence. Therefore the limits

$$v_1(t,x) = \lim_{n\to\infty} u_1^{(n)}(t,x), \quad v_2(t,x) = \lim_{n\to\infty} u_2^{(n)}(t,x)$$

exist for $x \in D$ and $0 \leq t < \tau_*$. In virtue of the monotone convergence theorem we obtain that

$$v_1(t,x) = \mathcal{R}_1 v_2(t,x), \quad v_2(t,x) = \mathcal{R}_2 v_1(t,x), \quad x \in D, \quad 0 \leq t < \tau_*,$$

and moreover,

$$\begin{aligned} v_1(t,x) &\leq \frac{e^{\Lambda t} T_t f_1(x)}{\left[1-(p-1)\int_0^t e^{\rho W_r} \|e^{\Lambda r} T_r f_1\|_\infty^{p-1}\, dr\right]^{\frac{1}{p-1}}}, \\ v_2(t,x) &\leq \frac{e^{\Lambda t} T_t f_2(x)}{\left[1-(q-1)\int_0^t e^{\rho W_r} \|e^{\Lambda r} T_r f_2\|_\infty^{q-1}\, dr\right]^{\frac{1}{q-1}}}. \end{aligned}$$

The assertion follows by choosing the initial values according to (3.2). ■

Remark Notice that for general bounded, measurable and positive f_i, $i = 1, 2$, the blow-up time of (1.2) is lower-bounded by the random time

$$\inf\left\{t \geq 0 : \int_0^t e^{\rho W_r} \|e^{\Lambda r} T_r f_1\|_\infty^{p-1}\, dr \geq (p-1)^{-1} \quad \text{or} \quad \int_0^t e^{\rho W_r} \|e^{\Lambda r} T_r f_2\|_\infty^{q-1}\, dr \geq (q-1)^{-1}\right\},$$

which coincides with τ_* when the initial values satisfy (3.2).

3.2 An upper bound for T

We now set $\varphi_i = \psi$ and $V_i = \Lambda + \kappa_i^2/2$ in (2.3), $i = 1, 2$, thus obtaining the system

$$v_i(t,\psi) = v_i(0,\psi) + \int_0^t e^{-\kappa_i W_s} \left(e^{\kappa_j W_s} v_j\right)^{\beta_j}(s,\psi)\, ds, \quad i = 1, 2, \tag{3.3}$$

where $\left(e^{\kappa_j W_s} v_j\right)^{\beta_i}(s,\psi) = \int_D \left[e^{\kappa_j W_s} v_j(s,x)\right]^{\beta_i} \psi(x)\, dx$ and $\beta_1 = p$, $\beta_2 = q$. By Jensen's inequality,

$$\left(e^{\kappa_j W_s} v_j\right)^{\beta_i}(s,\psi) \geq \left[\int_D \left(e^{\kappa_j W_s} v_j(s,x)\right)\psi(x)\, dx\right]^{\beta_i} = e^{\beta_i \kappa_j W_s} v_j(s,\psi)^{\beta_i},$$

which gives

$$\frac{\partial v_i(t,\psi)}{\partial t} \geq e^{-\kappa_i W_t}\left(e^{\beta_i \kappa_j W_t} v_j(t,\psi)^{\beta_i}\right) = e^{-\kappa_i W_t + \beta_i \kappa_j W_t} v_j(t,\psi)^{\beta_i}, \quad i = 1, 2, \quad j \in \{1,2\}\backslash\{i\}.$$

In this way, $v_i(t,\psi) \geq h_i(t)$, $i = 1, 2$, where

$$\frac{dh_1(t)}{dt} = e^{\rho W_t} h_2(t)^p, \quad \frac{dh_2(t)}{dt} = e^{\rho W_t} h_1(t)^q, \quad h_i(0) = v_i(0,\psi), \quad i = 1, 2, \tag{3.4}$$

and $\rho = p\kappa_2 - \kappa_1 = q\kappa_1 - \kappa_2$. We define $E(t) = h_1(t) + h_2(t)$, $t \geq 0$.

Theorem 2

1. *Assume that $p = q > 1$. Then $T \le \tau^*$, where*

$$\tau^* = \inf\left\{ t \ge 0 : \int_0^t e^{\rho W_s}\, ds \ge 2^p (p-1)^{-1} E^{1-p}(0) \right\}.$$

2. *Let $p > q > 1$, and let $A_0 = \frac{p-q}{p}\left(\frac{p}{q}\right)^{\frac{q}{p-q}}$ and $\epsilon_0 = 1 \wedge (h_2(0)/A_0^{1/q})^{p-q} \wedge (2^{-q}E(0)^q/A_0)^{(p-q)/p}$. Assume that*

$$2^{-q}\epsilon_0 E^q(0) \ge \epsilon_0^{\frac{p}{p-q}} A_0. \tag{3.5}$$

Then $T \le \tau^*$, where

$$\tau^* = \inf\left\{ t \ge 0 : \int_0^t e^{\rho W_s}\, ds \ge \frac{1}{(q-1)E^{q-1}(0)[2^{-q}\epsilon_0 - \epsilon_0^{p/(p-q)} A_0 E^{-q}(0)]} \right\}.$$

Remark Notice that (3.5) follows from the condition $\int_D f_i(x)\psi(x)\, dx > A_0^{1/q},\ i = 1, 2.$

Proof. Suppose first that $p = q > 1$. We get from (3.4) that

$$\frac{dE(t)}{dt} = e^{\rho W_t}\left(h_1^p(t) + h_2^p(t)\right). \tag{3.6}$$

By substituting $y \equiv h_1/h_2$ into the inequality $1 + y^p \ge 2^{-p}(1+y)^p$, which is valid for $y \ge 0$, we obtain

$$1 + \left(\frac{h_1(t)}{h_2(t)}\right)^p \ge 2^{-p}\left(1 + \frac{h_1(t)}{h_2(t)}\right)^p,$$

hence $h_1^p(t) + h_2^p(t) \ge 2^{-p}\left(h_1(t) + h_2(t)\right)^p$. Plugging this into (3.6) leads to the differential inequality

$$\frac{dE(t)}{dt} \ge e^{\rho W_t} 2^{-p} E^p(t).$$

Thus, $E(t)$ blows up no latter than the solution $I(t)$ of the equation

$$\frac{dI(t)}{dt} = e^{\rho W_t} 2^{-p} I^p(t), \quad I(0) = E(0),$$

whose explosion time is given by

$$\tau = \inf\left\{ t \ge 0 : \int_0^t e^{\rho W_s}\, ds \ge 2^{-p}(p-1)^{-1} E^{1-p}(0) \right\}. \tag{3.7}$$

Now we assume that $p > q > 1$. We want to lower-bound the solution components of system (3.4). Our approach to achieve this is an adaptation of a technique used in [9].

Let us recall Young's inequality (see e.g. [?] or [10]). Let $1 < b < \infty$ and $\delta > 0$, and let $a = b/(b-1)$. Then

$$xy \le \frac{\delta^a x^a}{a} + \frac{\delta^{-b} y^b}{b}, \quad x, y \ge 0. \tag{3.8}$$

From Young's inequality (3.8) it follows that for any $\varepsilon > 0$,

$$h_2^p(t) \ge \varepsilon h_2^q(t) - \varepsilon^{\frac{p}{p-q}} A_0. \tag{3.9}$$

Indeed, (3.9) follows from (3.8) by setting $b = p/q$, $y = h_2^q(t)$, $x = \epsilon$, $\delta = (p/q)^{q/p}$, and using that $q < p$. We also have

$$\epsilon_0\, h_2(0)^q - \epsilon_0^{\frac{p}{p-q}} A_0 \;\geq\; 0. \tag{3.10}$$

In fact, if $\epsilon_0 = 1$ then $1 \leq (h_2(0)/A_0^{1/q})^{p-q}$, which immediately gives $A_0 \leq h_2^q(0)$. If $\epsilon_0 = (h_2(0)/A_0^{1/q})^{p-q}$ then $\epsilon_0 h_2^q(0) - \epsilon_0^{p/(p-q)} A_0 = (h_2^p(0)/A_0^{p/q}) \cdot 0$, and if $\epsilon_0 = (2^{-q}E^q(0)/A_0)^{(p-q)/p}$ we obtain

$$\epsilon_0 h_2^q(0) - \epsilon_0^{p/(p-q)} A_0 = \frac{2^{-q}E^q(0)}{A_0}\left[\left(\frac{A_0}{2^{-q}E^q(0)}\right)^{q/p} h_2^q(0) - A_0\right],$$

which together with the inequality $(A_0/(2^{-q}E^q(0)))^{q/p} \geq A_0/h_2^q(0)$, render that the expression in brackets is lower-bounded by zero.

It follows from (3.4) and (3.9) that

$$\frac{dh_1(t)}{dt} \geq e^{\rho W_t}\left[\epsilon_0 h_2^q(t) - \epsilon_0^{\frac{p}{p-q}} A_0\right],$$

and therefore

$$\frac{dE(t)}{dt} \geq e^{\rho W_t}\left[h_1^q(t) + \epsilon_0 h_2^q(t) - \epsilon_0^{\frac{p}{p-q}} A_0\right]. \tag{3.11}$$

Setting $y \equiv \epsilon_0^{1/q} h_2(t)/h_1(t)$ and using again the inequality $1 + y^q \geq 2^{-q}(1+y)^q$, we see that

$$h_1^q(t) + \epsilon_0 h_2^q(t) = h_1^q(t)\left[1 + \left(\epsilon_0^{1/q}\frac{h_2(t)}{h_1(t)}\right)^q\right] \geq 2^{-q}\left(h_1(t) + \epsilon_0^{1/q} h_2(t)\right)^q \geq 2^{-q}\epsilon_0 E^q(t)$$

because $\epsilon_0 \leq 1$. Plugging this into (3.11) renders

$$\frac{dE(t)}{dt} \geq e^{\rho W_t}\left[2^{-q}\epsilon_0 E^q(t) - \epsilon_0^{\frac{p}{p-q}} A_0\right]. \tag{3.12}$$

Due to assumption (3.5) we get $E(t) \geq E(0) > 0$ for any $t \geq 0$. Moreover, (3.12) transforms into

$$\frac{dE(t)}{E^q(t)} \;\geq\; e^{\rho W_t}\left[2^{-q}\epsilon_0 - \frac{\epsilon_0^{p/(p-q)} A_0}{E(0)^q}\right] dt \tag{3.13}$$

which gives

$$E^{q-1}(t) \;\geq\; \frac{E^{q-1}(0)}{1 - (q-1)E^{q-1}(0)\left[2^{-q}\epsilon_0 - \epsilon_0^{p/(p-q)} A_0 E^{-q}(0)\right]\int_0^t e^{\rho W_s} ds}. \tag{3.14}$$

In this way $E(t)$, and therefore $(h_1(t), h_2(t))$, is going to blow-up earlier than the random time τ^* given by

$$\tau^* = \inf\left\{t \geq 0 : \int_0^t e^{\rho W_s}\, ds \;\geq\; \frac{1}{(q-1)E^{q-1}(0)\left[2^{-q}\epsilon_0 - \epsilon_0^{p/(p-q)} A_0 E^{-q}(0)\right]}\right\}.$$

■

4 A more general case

In this section we consider system (1.2) under the assumptions $p \geq q > 1$ and $V_i = \Lambda + \kappa_i^2/2$, $i = 1, 2$. Let us again write $\beta_1 = p$, $\beta_2 = q$ and $j \in \{1,2\}\backslash\{i\}$. We have from (2.4)

$$v_i(t,x) = e^{\Lambda t} T_t f_i(x) + \int_0^t e^{(\beta_i \kappa_j - \kappa_i)W_r + \Lambda(t-r)} T_{t-r}\left(v_j(r,\cdot)^{\beta_j}\right)(x)\, dr, \quad i = 1, 2.$$

Theorem 3 *Assume that $p \geq q > 1$ and $V_i = \Lambda + \kappa_i^2/2$, $i = 1, 2$, and let $f_1 = L(1)\psi$ and $f_2 = L(2)\psi$ for some positive constants $L(1)$ and $L(2)$. Let τ_{**} be given by*

$$\begin{aligned} \tau_{**} &= \inf\Bigg\{ t \geq 0 : \int_0^t e^{(p\kappa_2 - \kappa_1)W_r} dr \geq \frac{1}{(p-1)L^{p-1}(1)\|\psi\|_\infty^{p-1}} \\ &\qquad \text{or} \int_0^t e^{(q\kappa_1 - \kappa_2)W_r} \geq \frac{1}{(q-1)L^{q-1}(2)\|\psi\|_\infty^{q-1}} \Bigg\}. \end{aligned}$$

Then $\tau_{**} \leq T$.

Proof. Let us define, for $i = 1, 2$,

$$\mathcal{R}_i v(t,x) = e^{\Lambda t} T_t f_i(x) + \int_0^t e^{(\beta_i \kappa_j - \kappa_i)W_r + \Lambda(t-r)} \left(T_{t-r} v(r,x)\right)^{\beta_j} dr,$$

and

$$\mathcal{B}_i(t) = \left[1 - (\beta_i - 1)\int_0^t e^{(\beta_i \kappa_j - \kappa_i)W_r + \Lambda r} \|T_r f_i\|_\infty^{\beta_i - 1} dr\right]^{-1/(\beta_i - 1)}.$$

Proceeding in the same way as in the proof of Theorem 1, we get that

$$v_1(t,x) = \mathcal{R}_1 v_2(t,x), \quad v_2(t,x) = \mathcal{R}_2 v_1(t,x)$$

as long as $t \leq \tau_{**}$ and $x \in D$. Moreover,

$$\begin{aligned} v_i(t,x) &\leq \frac{e^{\Lambda t} T_t f_i(x)}{\left[1 - (\beta_i - 1)\int_0^t e^{(\beta_i \kappa_j - \kappa_i)W_r} \|e^{\Lambda r} T_r f_i\|_\infty^{\beta_i - 1} dr\right]^{\frac{1}{\beta_i - 1}}} \\ &= \frac{L(i)\psi(x)}{\left[1 - (\beta_i - 1)L(i)^{\beta_i - 1}\|\psi\|_\infty^{\beta_i - 1}\int_0^t e^{(\beta_i \kappa_j - \kappa_i)W_r} dr\right]^{\frac{1}{\beta_i - 1}}} \end{aligned}$$

by our choice of f_1 and f_2. ■

Corollary 4 *Let the random time τ' be defined by*

$$\begin{aligned} \tau' &= \inf\Bigg\{ t \geq 0 : \int_0^t \max\left\{ e^{(p\kappa_2 - \kappa_1)W_r}, e^{(q\kappa_1 - \kappa_2)W_r} \right\} dr \\ &\qquad \geq \min\left\{ \frac{1}{(p-1)L^{p-1}(1)\|\psi\|_\infty^{p-1}}, \frac{1}{(q-1)L^{q-1}(2)\|\psi\|_\infty^{q-1}} \right\} \Bigg\}. \end{aligned}$$

Then $\tau' \leq \tau_{**}$

In order to obtain an upper bound for T when $p \geq q > 1$ and $V_i = \Lambda + \kappa_i^2/2$, $i = 1, 2$, we first notice that a sub-solution for Eq. (3.3) is given by the solution of

$$\frac{dh_1(t)}{dt} = e^{(p\kappa_2-\kappa_1)W_t} h_2(t)^p, \quad \frac{dh_2(t)}{dt} = e^{(q\kappa_1-\kappa_2)W_t} h_1(t)^q, \quad h_i(0) = v_i(0,\psi), \quad i = 1, 2. \tag{4.1}$$

Working with system (4.1) as we did with (3.4) in the proof of Theorem 2, we get that

$$\frac{dE(t)}{dt} \geq \min\left\{e^{(p\kappa_2-\kappa_1)W_t}, e^{(q\kappa_1-\kappa_2)W_t}\right\}\left[2^{-q}\epsilon_0 E^q(t) - \epsilon_0^{\frac{p}{p-q}} A_0\right],$$

and in case of $p = q$,

$$\frac{dE(t)}{dt} \geq \min\left\{e^{(p\kappa_2-\kappa_1)W_t}, e^{(p\kappa_1-\kappa_2)W_t}\right\} 2^{-p} E^p(t),$$

which can be handled using the same method as in the proof of Theorem 2.1. In this way we obtain the following result for the explosion time of system (1.2).

Theorem 5 *Let $p \geq q > 1$.*

1. *If $p = q$ then $T \leq \tau^{**}$, where*

$$\tau^{**} = \inf\left\{t \geq 0 : \int_0^t \min\left\{e^{(p\kappa_2-\kappa_1)W_r}, e^{(p\kappa_1-\kappa_2)W_r}\right\} ds \geq 2^p (p-1)^{-1} E^{1-p}(0)\right\}.$$

2. *If $p > q$ and (3.5) holds true, then $T \leq \tau^{**}$ where*

$$\begin{aligned}\tau^{**} &= \inf\left\{t \geq 0 : \int_0^t \min\left\{e^{(p\kappa_2-\kappa_1)W_r}, e^{(q\kappa_1-\kappa_2)W_r}\right\} dr\right.\\ &\qquad\left.\geq \frac{1}{(q-1)E^{q-1}(0)\left[2^{-q}\epsilon_0 - \epsilon_0^{p/(p-q)} A_0 E^{-q}(0)\right]}\right\}.\end{aligned}$$

Remark Setting $t \geq 0$ and

$$M := \left((q-1)E^{q-1}(0)[2^{-q}\epsilon_0 - \epsilon_0^{p/(p-q)} A_0 E^{-q}(0)]\right)^{-1}, \tag{4.2}$$

we get

$$\begin{aligned}\mathbb{P}\{\tau^{**} \geq t\} &= \mathbb{P}\left\{\int_0^t \min\left\{e^{(p\kappa_2-\kappa_1)W_r}, e^{(q\kappa_1-\kappa_2)W_r}\right\} dr \leq M\right\}\\ &= \mathbb{P}\left\{e^{-\int_0^t \min\left\{e^{(p\kappa_2-\kappa_1)W_r}, e^{(q\kappa_1-\kappa_2)W_r}\right\} dr} \geq e^{-M}\right\}\\ &\leq e^M \mathbb{E}\left[e^{-\int_0^t \min\left\{e^{(p\kappa_2-\kappa_1)W_r}, e^{(q\kappa_1-\kappa_2)W_r}\right\} dr}\right],\end{aligned}$$

where we used Chebyshev's inequality. A similar estimate for the distribution function of τ_{**}, but involving the functional $\int_0^t \max\left\{e^{(p\kappa_2-\kappa_1)W_r}, e^{(q\kappa_1-\kappa_2)W_r}\right\} dr$, can be obtained using that $\tau' \leq \tau_{**}$. The Laplace transforms of the random functionals $\int_0^\theta \max\left\{e^{aW_r}, e^{bW_r}\right\} dr$, $\int_0^\theta \min\left\{e^{aW_r}, e^{bW_r}\right\} dr$ where θ is an independent exponentially distributed time and $a > 0$, $b > 0$ are constants, are given in Theorem 7 below.

5 The Laplace transform of a Brownian motion functional

Let $a, b \in [0, \infty)$ be given constants, and let $F : \mathbb{R} \to (0, \infty)$ be the locally bounded function defined by

$$F(z) = e^{az}\mathbf{1}_{[0,\infty)}(z) + e^{bz}\mathbf{1}_{(-\infty,0)}(z).$$

Hence, for any $z \geq 0$,

$$F(z) = \begin{cases} \min\{e^{az}, e^{bz}\}, & \text{if } a \leq b, \\ \max\{e^{az}, e^{bz}\}, & \text{if } a \geq b. \end{cases} \tag{5.1}$$

We want to investigate the functional of Brownian motion $\mathcal{A}_t^F$ defined by

$$\mathcal{A}_t^F = \int_0^t F(W_s)\, ds = \int_0^t \left(e^{aW_s}\mathbf{1}_{[0,\infty)}(W_s) + e^{bW_s}\mathbf{1}_{(-\infty,0)}(W_s) \right) ds. \tag{5.2}$$

Let θ_k be an exponentially distributed random variable of parameter k, independent of $\{W_s\}$, and let $g_{\theta_k}(\omega)$ be the last zero of $W_\cdot(\omega)$ before $\theta_k(\omega)$. Our first aim is to calculate the Laplace transform of $\mathcal{A}_{\theta_k}^F$, which follows from a simple adaptation of the approaches in [5] and [14]. In fact, as in [5] we can write

$$\mathbb{E}\left[\exp\left\{-\mathcal{A}_{\theta_k}^F\right\}\right] = \mathbb{E}\left[\exp\left\{-\mathcal{A}_{g_{\theta_k}}^F\right\}\right] \mathbb{E}\left[\exp - \left\{\mathcal{A}_{\theta_k}^F - \mathcal{A}_{g_{\theta_k}}^F\right\}\right], \tag{5.3}$$

and substituting λF instead of F in the above equality, where $\lambda \geq 0$, we get an expression for the Laplace transform of $\mathcal{A}_{\theta_k}^F$ in terms of the two expectations in the RHS of (5.3). Let $\{\ell_t\}$ be the local time at 0 of $\{W_s\}$, and

$$\tau_s := \inf\{t \geq 0 :\ \ell_t > s\}$$

the inverse local time. Using formula (2.4) in [5] we obtain that

$$\mathbb{E}\left[\exp\left\{-\mathcal{A}_{g_{\theta_k}}^F\right\}\right] = k\int_0^\infty dl\, \mathbb{E}\left[\exp - \left\{\frac{k^2}{2}\tau_l + \mathcal{A}_{\tau_l}^F\right\}\right]. \tag{5.4}$$

From Corollary 3.4 and Theorem 3.1 in [5] we know that for general F,

$$\mathbb{E}\left[\exp - \left\{\frac{k^2}{2}\tau_l + \mathcal{A}_{\tau_l}^F\right\}\right] = \exp\left\{\frac{l}{2}\left(\Phi^{F_+}\right)_t'(k,0) + \frac{l}{2}\left(\Phi^{F_-}\right)_t'(k,0)\right\}, \tag{5.5}$$

where $\Phi^F(k,t) =: u(k,t)$ is the unique bounded solution of the Sturm-Liouville equation

$$\frac{1}{2}U'' = \left(\frac{k^2}{2} + F\right)U, \quad U(0) = 1,$$

and $F_+ := F|_{\mathbb{R}_+}$, $F_-(z) := F(-z)$, $z \geq 0$. In our case

$$F_+(z) = e^{az} \quad \text{and} \quad F_-(z) = e^{-bz}, \quad z \geq 0.$$

The remaining expectation in the RHS of (5.3) can be written as

$$\mathbb{E}\left[\exp\left\{-\mathcal{A}_{\theta_k}^F - \mathcal{A}_{g_{\theta_k}}^F\right\}\right] = \frac{k}{2}\int_0^\infty dt\ \left[\Phi^{F_+}(k,t) + \Phi^{F_-}(k,t)\right]; \tag{5.6}$$

see [5], Corollary 3.2. We are going to calculate the functions $\Phi^{\frac{\lambda^2}{2}F_+}(k,t)$ and $\Phi^{\frac{\lambda^2}{2}F_-}(k,t)$ in our special setting. The Sturm-Liouville equation corresponding to $\frac{\lambda^2}{2}F_+$, $\lambda \geq 0$, is given by

$$\frac{1}{2}U''(a,k,t) = \left(\frac{k^2}{2} + \frac{\lambda^2}{2}e^{at}\right)U(a,k,t), \quad U(a,k,0) = 1. \tag{5.7}$$

When $a = 2$ we can use Yor's solution of the previous equation, according to which

$$U(2,k,t) = \frac{K_k(\lambda e^t)}{K_k(\lambda)} =: \Phi^{\frac{\lambda^2}{2}F_+}(k,t), \tag{5.8}$$

where K_ν is the usual modified Bessel function with index ν. When $a \neq 2$ we solve (5.7) by means of a change of variables. We set $t = 2s/a$ and for simplicity denote $U(a,k,t)$ by $v(k,t)$. Then we have,

$$\frac{1}{2}v''_{tt}\left(k, \frac{2s}{a}\right) = \left(\frac{k^2}{2} + \frac{\lambda^2}{2}e^{2s}\right)v\left(k, \frac{2s}{a}\right), \quad v(k,0) = 1,$$

and

$$v'_t = v'_s\frac{ds}{dt} = \frac{a}{2}v'_s, \quad v''_{tt} = \left(\frac{a}{2}\right)^2 v''_{ss}.$$

Therefore,

$$\frac{1}{2}\frac{a^2}{4}v''_{ss}\left(k, \frac{2s}{a}\right) = \left(\frac{k^2}{2} + \frac{\lambda^2}{2}e^{2s}\right)v\left(k, \frac{2s}{a}\right),$$

or

$$\frac{1}{2}v''_{ss}\left(k, \frac{2s}{a}\right) = \left(\frac{2k^2}{a^2} + \frac{2\lambda^2}{a^2}e^{2s}\right)v\left(k, \frac{2s}{a}\right).$$

Putting $w(k,s) = v(k, 2s/a)$, we get $w'(k,s) = (2/a)v'(k,2s/a)$ and $(w'')^2(k,s) = \frac{4}{a^2}v''(k,2s/a)$. Hence, w satisfies

$$\frac{1}{2}\frac{a^2}{4}w''(k,s) = \left(\frac{2k^2}{a^2} + \frac{2\lambda^2}{a^2}e^{2s}\right)w(k,s),$$

and

$$\frac{1}{2}w''(k,s) = \left(\frac{8k^2}{a^4} + \frac{8\lambda^2}{a^4}e^{2s}\right)w(k,s),$$

which is the equation (5.7) with $a = 2$ and new constants k and λ. Thus, by (5.8), w has the form

$$w(k,t) = \frac{K_{k'}(\lambda' e^t)}{K_{k'}(\lambda')} \quad \text{with} \quad k' = \frac{4k}{a^2},\ \lambda' = \frac{4\lambda}{a^2},$$

and therefore

$$\Phi^{\frac{\lambda^2}{2}F_+}(k,t) = v(k,t) = w(k, at/2) = \frac{K_{4k/a^2}\left(4\lambda e^{at/2}/a^2\right)}{K_{4k/a^2}\left(4\lambda/a^2\right)}.$$

To calculate $\Phi^{\frac{\lambda^2}{2}F_-}$ we proceed in the same fashion, but now using the fact that the corresponding Sturm-Liouville equation (for $a = 2$) has as solution

$$U(2,k,t) = \Phi^{F_-}(k,t) = \frac{I_k(\lambda e^{-t})}{I_k(\lambda)}, \quad t \geq 0,$$

where I_k is the other modified Bessel function. Hence,

$$\Phi^{\frac{\lambda^2}{2}F_-}(k,t) = \frac{I_{4k/b^2}\left(4\lambda e^{bt/2}/b^2\right)}{I_{4k/b^2}\left(4\lambda/b^2\right)}, \quad t \geq 0.$$

We now continue with the development of (5.5). We recall from [14] (pag. 131) that

$$\lambda K_{k+1}(\lambda) = -\lambda K_k'(\lambda) + kK_k(\lambda) \quad \text{and} \quad \lambda I_{k-1}(\lambda) = \lambda I_k'(\lambda) + kI_k(\lambda), \tag{5.9}$$

where the derivatives are with respect to λ. The derivatives of $\Phi^{\frac{\lambda^2}{2}F_+}(k,t)$ and $\Phi^{\frac{\lambda^2}{2}F_-}(k,t)$ at the point $t=0$ are given by

$$\begin{aligned}
\left(\Phi^{\frac{\lambda^2}{2}F_+}\right)_t'(k,0) &= \frac{K_{4k/a^2}'(4\lambda/a^2)(a/2)(4\lambda/a^2)}{K_{4k/a^2}(4\lambda/a^2)} \\
&= \frac{2\lambda}{a}\left(\frac{4k/(a^2)}{4\lambda/(a^2)} - \frac{K_{\frac{4k}{a^2}+1}(4\lambda/a^2)}{K_{4k/a^2}(4\lambda/a^2)}\right) \\
&= \frac{a}{2}\left(\frac{4k}{a^2} - \frac{(4\lambda/a^2)K_{4k/a^2+1}(4\lambda/a^2)}{K_{4k/a^2}(4\lambda/a^2)}\right),
\end{aligned}$$

where in the last equality we used (5.9). In a similar way,

$$\left(\Phi^{\frac{\lambda^2}{2}F_-}\right)_t'(k,0) = \frac{I_{4k/b^2}'(4\lambda/b^2)(b/2)(4\lambda/b^2)}{I_{4k/b^2}(4\lambda/b^2)} = \frac{b}{2}\left(\frac{(4\lambda/b^2)I_{4k/b^2-1}(4\lambda/b^2)}{I_{4k/b^2}(4\lambda/b^2)} - \frac{4k}{b^2}\right).$$

Substituting the two expressions above into (5.5) we obtain the following.

Lemma 6

$$\mathbb{E}\left[\exp - \left\{\frac{k^2}{2}\tau_l + \frac{\lambda^2}{2}\mathcal{A}_{\tau_l}^F\right\}\right] = \exp\left\{-\frac{l}{2}\left(R(a,\lambda,k) \; + \; S(b,\lambda,k)\right)\right\},$$

where

$$R(a,\lambda,k) = \frac{a}{2}\left(\frac{\frac{4\lambda}{a^2}K_{\frac{4k}{a^2}+1}\left(\frac{4\lambda}{a^2}\right)}{K_{\frac{4k}{a^2}}\left(\frac{4\lambda}{a^2}\right)} - \frac{4k}{a^2}\right) \quad \textit{and} \quad S(b,\lambda,k) = \frac{b}{2}\left(\frac{4k}{b^2} - \frac{\frac{4\lambda}{b^2}I_{\frac{4k}{b^2}-1}\left(\frac{4\lambda}{b^2}\right)}{I_{\frac{4k}{b^2}}\left(\frac{4\lambda}{b^2}\right)}\right). \tag{5.10}$$

Putting together (5.3), (5.4), (5.6) and Lemma 6 we get the following formulas.

Theorem 7 *The Laplace transforms of $\mathcal{A}_{g_{\theta_k}}^F$ and $\mathcal{A}_{\theta_k}^F$ are given by*

$$\mathbb{E}\left[e^{-\frac{\lambda^2}{2}\mathcal{A}_{g_{\theta_k}}^F}\right] = k\int_0^\infty \exp\left\{-\frac{l}{2}(R(a,\lambda,k)+S(b,\lambda,k))\right\}dl = \frac{2k}{R(a,\lambda,k)+S(b,\lambda,k)}, \tag{5.11}$$

and

$$\mathbb{E}\left[e^{-\frac{\lambda^2}{2}\mathcal{A}_{\theta_k}^F}\right] = \frac{k^2}{R(a,\lambda,k)+S(b,\lambda,k)}\int_0^\infty \left[\frac{K_{\frac{4k}{a^2}}\left(\frac{4\lambda}{a^2}e^{\frac{ax}{2}}\right)}{K_{\frac{4k}{a^2}}\left(\frac{4\lambda}{a^2}\right)} + \frac{I_{\frac{4k}{b^2}}\left(\frac{4\lambda}{b^2}e^{\frac{bx}{2}}\right)}{I_{\frac{4k}{b^2}}\left(\frac{4\lambda}{b^2}\right)}\right]dx, \tag{5.12}$$

where $R(a,\lambda,k)$ and $S(b,\lambda,k)$ are given by (5.10).

Corollary 8 *Let θ_k be an exponentially distributed random variable of parameter $k>0$, independent of $\{W_t\}$. For all $k>0$,*

$$\mathbb{P}\{T \geq \theta_k\} \leq \frac{k^2 M}{R(P,\sqrt{2},k)+S(Q,\sqrt{2},k)}\int_0^\infty \left[\frac{K_{\frac{4k}{P^2}}\left(\frac{4\lambda}{P^2}e^{\frac{Px}{2}}\right)}{K_{\frac{4k}{P^2}}\left(\frac{4\lambda}{P^2}\right)} + \frac{I_{\frac{4k}{Q^2}}\left(\frac{4\lambda}{Q^2}e^{\frac{Qx}{2}}\right)}{I_{\frac{4k}{Q^2}}\left(\frac{4\lambda}{Q^2}\right)}\right]dx,$$

where M is given by (4.2), $P=\min\{p\kappa_2-\kappa_1, q\kappa_1-\kappa 2\}$, $Q=\max\{p\kappa_2-\kappa_1, q\kappa_1-\kappa 2\}$, and R and S are given by (5.10).

Proof. From Theorem 5 we know that for any $s \geq 0$,

$$\mathbb{P}\{T \geq s\} \leq \mathbb{P}\{\tau^{**} \geq s\} = \mathbb{P}\{\mathcal{A}_s^F \leq M\},$$

where $\mathcal{A}_s^F$ is given by (5.2) with $a = p\kappa_2 - \kappa_1$ and $b = q\kappa_1 - \kappa_2$. Using Chebyshev's inequality as before we get $\mathbb{P}\{\mathcal{A}_s^F \leq M\} \leq e^M \mathbb{E}\left[\exp\{-\mathcal{A}_s^F\}\right]$. The result follows from (5.12), putting $\lambda = \sqrt{2}$ and $s = \theta_k$. ■

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