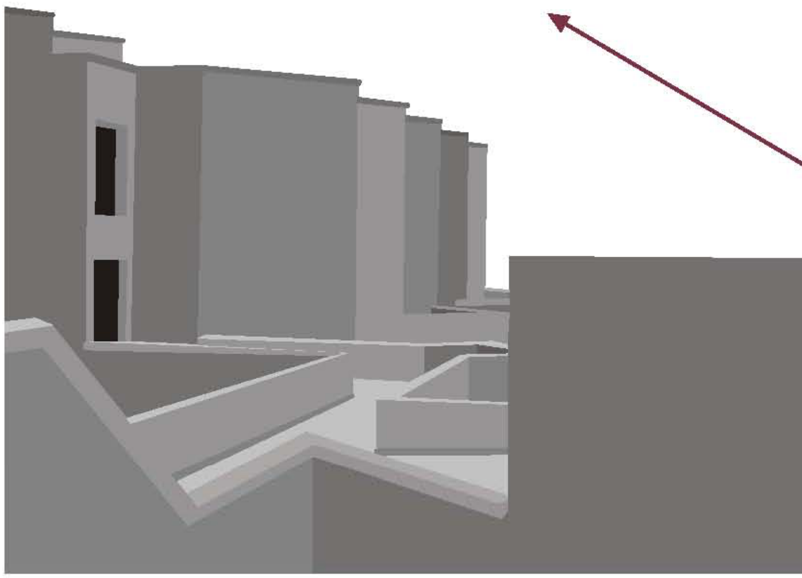

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GERBER-SHIU FUNCTIONAL FOR THE CLASSICAL RISK PROCESSES PERTURBED BY AN α-STABLE MOTION

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1 Introduction

In the classical risk model, the surplus of the insurance company at time t is given by

$$X(t) = u + ct - \sum_{i=1}^{N(t)} Y_i := u + ct - S(t), \tag{1}$$

where $u \geq 0$ is the company's initial capital, c is a premium per unit time, $S(t) = \sum_{i=1}^{N(t)} Y_i$ is a compound Poisson process and $Y_1, Y_2, \ldots$ is a sequence of non-negative, independent and identically distributed random variables with common distribution F such that $F(0) = 0$. For the process X, the discounted Gerber-Shiu penalty function is defined by

$$\phi_X(u) = \mathrm{E}[e^{-\delta\tau_0}\omega(|X(\tau_0)|, X(\tau_0-))I_{\{\tau_0<\infty\}}|X(0) = u], \tag{2}$$

where $\tau_0 = \inf\{t > 0 : X(t) < 0\}$, is the time of ruin, $\delta \geq 0$ is the discounted force of interest and $\omega(x, y) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a given non-negative function called penalty function. It was introduced by Gerber and Shiu (1998) in the context of the classical risk model and represents the joint distribution of the time of ruin, the surplus immediately before ruin and the deficit at time of ruin. As a particular case, when $\delta = 0$ and $\omega(x, y) = 1$, the ruin probability $\psi(u) = \mathbb{P}[\tau_0 < \infty | X(0) = u]$ is obtained.

Gerber (1970) extended the classical risk model (1), by adding an independent Brownian motion B:

$$X_B(t) = u + ct - \sum_{i=1}^{N(t)} Y_i - \eta B(t), \eta \geq 0,\ t \geq 0, \tag{3}$$

and Dufresne and Gerber(1991) obtained an explicit expression for the probability of ruin for this model, as a series of convolutions. Based on ideas of Dufresne and Gerber, Tsai and Willmot (2002) studied the Gerber-Shiu penalty function for (3) and proved that, under some conditions on ω, it satisfies a defective renewal equation. Then Sarkar and Sen (2005) proved the result of Tsai and Willmot under more general conditions.

Furrer(1998) generalized the model (3) by introducing the classical risk process perturbed by an α-stable Levy process:

$$V(t) = u + ct - \sum_{i=1}^{N(t)} Y_i - \eta W_\alpha(t), \tag{4}$$

where W_α is a standard α-stable Levy process with parameters $1 < \alpha \leq 2$, $\beta = 1$, and obtained a formula for the probability of ruin in terms of series of convolutions. Furrer, Michna and Weron (1997) proved that the α-stable process with drift, for $\alpha \in (1, 2)$ arises as a weak limit of risk processes with heavy-tailed claims, hence the process V is important in insurance modelling in the presence of big oscillations in the data.

The processes (1), (3) and (4) belong to the general class of processes with stationary independent increments and no positive jumps, called spectrally negative Levy processes, and a generalized Gerber-Shiu function for these processes has been investigated recently by Biffis and Kyprianou (2010) and Biffis and Morales (2010). In the first paper the authors give an expression for a path-dependent version of the Gerber-Shiu function in terms of integrals of asociated scale functions for the Levy process, and present some examples in which the scale functions are known. In Biffis and Morales (2010), Theorem 4.1, the authors obtain an expression for the Generalized Gerber-Shiu function of in terms of an infinite series of convolution of integral functions. However, these results are difficult to use in numerical calculations since the integral functions involved are not easy to solve in general, and the formulae contain integration with respect to pure jump measures and require Laplace transform inversion techniques.

In this communication we consider the Gerber-Shiu discounted penalty function for V, defined by (2), and present formulae for its the Laplace transform for two particular penalty functions ω. They allow us to obtain results for important risk measures for the process, such as the ruin probability, the Laplace transform of the ruin time, the joint tail distribution of the severity of ruin and surplus prior to ruin, and the bivariate Laplace transform of the severity of ruin and surplus prior to ruin.

This formulae allow us further to obtain a defective renewal equation and asymptotic expressions for $\phi(u)$ when the initial capital u tends to infinity. When $\alpha = 2$, the results of Tsai and Willmot (2002) for the classical risk process perturbed by independent Brownian motion are recovered, and for $\delta = 0$ and $\omega(x, y) = 1$ for all $x, y \geq 0$, we obtain the result of Furrer (1998) for the ruin probability of the classical risk process perturbed by an α-stable process.

The main difficulty in working with this model lies in the lack of a closed expression for the α-stable density, and the fact that we can not use the standart tool of a first step analysis in order to obtain a renewal equation for ϕ, because of the infinite number of jumps of the α-stable process in each time interval.

Our results are obtained by constructing a weak approximation sequences of classical risk processes to our process, and then proving the convergence of the corresponding Gerber-Shiu penalty functions. Finally, we obtain the limits of the Laplace transforms of the Gerber-Shiu functions of the approximating processes, and a corresponding defective renewal equation for the Gerber-Shiu penalty function of the process (4). Weak approximations in risk theory have been used in Iglehart (1969), Grandell (1977), and recently in Sarkar and Sen (2005) to obtain the Gerber-Shiu function for (3) in the case $\alpha = 2$, and in Furrer et al. (1997) to estimate ruin probabilities within a finite time horizon.

2 Preliminaries and main results

In this section we give some definitions and preliminary results, and present the main results. The proofs will be given in the forthcomming Ph.D. thesis of Ehyter M. González.

We recall that for the classical risk process (1) the probability of ruin $\psi(u)$ has closed expression only for some particular cases of the claim distribution F. For example, when F is the exponential distribution with mean μ, and the intensity of the claim arrival process $N(t)$ is λ, the probability of ruin is given by

$$\psi(u) = \frac{\lambda\mu}{c} e^{-(\frac{1}{\mu}-\frac{\lambda}{c})u}. \tag{5}$$

In general, only a formula for the Laplace transform of ruin probability is available (see e.g. Rolski et al. (1999) for more details).

Definition 1 *For any nonnegative function f we define its Laplace transform by*

$$\widehat{f}(r) = \int_{-\infty}^{\infty} e^{-rx} f(x) dx,$$

for each $r \geq 0$, for which the integral above exists and is finite.

Definition 2 *A random variable X has a stable distribution if for any positive numbers a, b there exist a positive number c and a real number d such that*

$$aX_1 + bX_2 \stackrel{d}{=} cX + d,$$

where X_1 and X_2 are independent copies of X and $\stackrel{d}{=}$ denotes equality in distribution. In this case there exist parameters α, σ, β and μ such that $0 < \alpha \leq 2$, $\sigma > 0$, $-1 \leq \beta \leq 1$ and $-\infty < \mu < \infty$, respectively called index of stability, scale, skewness and shift parameter, and such that the characteristic function of X is given by

$$\mathrm{E}[e^{i\theta X}] = \begin{cases} e^{\sigma(i\mu\theta - |\theta|^{\alpha} exp\{-i(\pi/2)\beta K(\alpha) sgn\theta\})} & \text{for } \alpha \neq 1, \\ e^{\sigma(i\mu\theta - |\theta|(\pi/2 + i\beta log|\theta| sgn\theta))} & \text{for } \alpha = 1, \end{cases} \tag{6}$$

where $K(\alpha) = \alpha - 1 + sgn(1-\alpha)$ and $sgn\theta = 1_{\{\theta>0\}} + \theta 1_{\{\theta=0\}} - 1_{\{\theta<0\}}$. If X an α-stable distribution, we write $X \sim S_{\alpha}(\sigma, \beta, \mu)$.

Definition 3 *The process $X = \{X(t), t \geq 0\}$. X is a Levy process if it satisfies the following conditions:*

- *$X(0) = 0$ a.s.*
- *X has $\mathbb{P}$-a.s. right-continuous paths with left limits (càdlàg trajectories),*

- *For $0 \leq s \leq t$, $X(t)-X(s) \stackrel{d}{=} X(t-s)$ and $X(t)-X(s)$ is independent of $\{X(u), u \leq s\}$.*

*When $W_\alpha = \{W_\alpha(t), t \geq 0\}$ is a Levy process such that $W_\alpha(t) - W_\alpha(s) \sim S_\alpha\left[(t-s)^{1/\alpha}, \beta, \mu\right]$, $0 \leq s < t < \infty$, then W_α is called **α-stable Levy motion**. It is **standard α-stable motion** when $\sigma = 1, \mu = 0$.*

If $1 < \alpha < 2$, the moments of W_α of order less than α are finite, and when $\beta = 1$, only positive jumps of W_α are possible. For $\alpha = 2$, the process $\{\frac{1}{\sqrt{2}}B(t), t \geq 0\}$ is obtained, where $\{B(t), t \geq 0\}$ is the standard Brownian motion. We refer the reader to Bertoin (1996), Kyprianou (2006), or Sato (1999) for other properties of Levy processes.

We consider the classical risk process perturbed by an α-stable motion, given by (4), where the Poisson arrival process $\{N(t),\, t \geq 0\}$, has intensity parameter $\lambda > 0$, and the claim distributions $\{Y_i, i = 1, 2 \ldots\}$ have a common distribution F, which has density f and finite first moment μ. Furthermore, the process $W_\alpha = \{W_\alpha(t), t \geq 0\}$ is a standard α-stable motion with $1 < \alpha < 2$ and $\beta = 1$. Hence, when $1 < \alpha < 2$, the first moment of $V = \{V(t), t \geq 0\}$ is finite and only negative jumps are allowed. We investigate the Gerber-Shiu function for V, for two particular penalty functions ω:

$$
\begin{aligned}
&\text{Case 1:} \quad \omega(x, y) = 1_{\{x > a, y > b\}}, \\
&\text{Case 2:} \quad \omega(x, y) = e^{-sx-ty},
\end{aligned} \tag{7}
$$

where $a, b > 0$ and $s, t \geq 0$ are fixed real numbers. The penalty function in case 2 allows us to obtain, as particular cases, expressions for the moments of the severity and the surplus prior to ruin, and the probability of ruin. The penalty function in case 1 is used to obtain expressions for the tails of the joint distribution of the severity of ruin and the surplus prior to ruin. We observe that the distribution that the joint tail distribution of the severity and surplus prior to ruin can also be obtained from case 2, but while this requires inversion of the Laplace transform (which might not be easy to do analytically), the penalty function in case 1 gives us directly the analytical expression of the desired joint tail distribution.

We suppose that the **net profit condition**, $\mathrm{E}[V(1)-u]>0$, is satisfied in order to avoid the trivial case $\psi(u)=1$ for all $u\geq 0$. Then the following results hold.

Theorem 1 *Suppose that $\omega(x,y)$ is one of the penalty functions given in (7). Then the Laplace transform of the Gerber-Shiu penalty function of the perturbed risk process V is given by*

$$\widehat{\phi}(r)=\frac{N(\rho)-N(r)+M_\alpha(\rho)-M_\alpha(r)}{L(r)}, \tag{8}$$

where $N(r)=\lambda\int\limits_0^\infty\int\limits_u^\infty e^{-ru}\omega(x-u,u)f(x)dxdu$,

$$M_\alpha(r)=\begin{cases}\eta^\alpha\frac{\alpha-1}{\Gamma(\alpha-2)}\int\limits_b^\infty e^{-ru}(u+a)^{-\alpha}du & if\ \omega(x,y)=1_{\{x>a,y>b\}},\\ \eta^\alpha\frac{(t+r)^\alpha-s^\alpha}{s-(t+r)} & if\ \omega(x,y)=e^{-sx-ty},\end{cases} \tag{9}$$

$$L(r)=\lambda\widehat{f}(r)+cr+(\eta r)^\alpha-(\lambda+\delta) \tag{10}$$

is the Generalized Lundberg function for V, and ρ is the unique nonnegative root of L.

In order to obtain a simpler expression for the formula (8), we introduce the translation operator T_r, defined in Dickson and Hipp (2001).

Definition 4 *For any nonnegative function f defined on $[0,\infty)$ and $r \geq 0$, let us define the translation operator $T_r f(x)$ by*

$$T_r f(x) = \int_x^\infty e^{-r(y-x)} f(y)dy, x \geq 0. \tag{11}$$

For $1 < \alpha < 2$, we also define the function

$$\ell_\alpha(x) = \frac{(\alpha-1)x^{-\alpha}}{\Gamma(2-\alpha)}, x > 0, \tag{12}$$

Theorem 2 *For the penalty functions $\omega(x,y)$ given in (7) and for all $r_1, r_2 \geq 0$, the functions $M_\alpha(r)$ and $N(r)$ given in Theorem 1 satisfy the following equalities in terms of the operator T:*

$$\frac{N(r_1)-N(r_2)}{r_2-r_1} = \begin{cases} \lambda \widehat{T}_{r_2} n_{1,a,b}(r_1) & if\ \omega(x,y) = 1_{\{x>a,y>b\}}, \\ \lambda \widehat{T}_{r_2} n_{2,s,t}(r_1) & if\ \omega(x,y) = e^{-sx-ty}, \end{cases} \tag{13}$$

where

$$n_{1,a,b}(x) = \overline{F}(x+a)1_{\{x>b\}},\ n_{2,s,t}(x) = e^{-tx}T_s f(x), \tag{14}$$

and

$$\frac{M_\alpha(r_1)-M_\alpha(r_2)}{r_2-r_1} = \begin{cases} \eta^\alpha \widehat{T}_\rho m_{1,a,b}(r) & if\ \omega(x,y) = 1_{\{x>a,y>b\}}, \\ \eta^\alpha \widehat{T}_\rho m_{2,s,t}(r) & if\ \omega(x,y) = e^{-sx-ty}, \end{cases} \tag{15}$$

with

$$m_{1,a,b}(x) = \frac{\alpha-1}{\Gamma(2-\alpha)}(x+a)^{-\alpha}1_{\{x>b\}},\ m_{2,s,t}(x) = e^{-tx}\ell_\alpha(x) - se^{-tx}T_s\ell_\alpha(x). \tag{16}$$

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