METRIC SAMPLE SPACES OF CONTINUOUS GEOMETRIC CURVES AND ESTIMATION OF THEIR CENTROIDS

R. J. Biscay and C. M. Mora

Cuban Neuroscience Center and Havana University

Abstract. The metric sample space of Fréchet curves (Fréchet, 1934, 1951, 1961) is based on a generalization of regular curves that covers continuous curves in full generality. This makes it possible to deal with both smooth and non-smooth, even non-rectifiable geometric curves in statistical analysis. In the present paper this sample space is further extended in two directions that are relevant in practice: to incorporate information on landmark points in the curves and to impose invariance with respect to an arbitrary group of isometric spatial transformations. Properties of the introduced sample spaces of curves are studied, specially those concerning to the generation and representation of random curves by random functions. Also, centroids and restricted centroids of random curves are defined in a general metric framework, and methods for their consistent estimation are derived.

Introduction. In recent years many applied problems have led to deal with geometric curves as an important, distinguished type of data for statistical analysis. Typical examples arise from hand-written characters (Persoon and Fu, 1977), contours of particles in geostatistics (Serra, 1982; Stoyan and Stoyan, 1994), borders between patches in computer image segmentation (Kass *et al.*, 1988), and outlines of biological structures, specially in anatomical imaging (Bookstein, 1978, 1991; Valdés and Biscay, 1990; Grenander *et al.,* 1990; Grenander and Miller, 1994, 1998), to mention but a few.

However, to date, a general treatment of sample spaces of geometric curves is lacking. Instead, some partial approaches have been followed for the representation and statistical analysis of curves (see also comments in Stoyan, 1994, ch. 6). These can be summarized as follows:

1) Curves that are borders of objects in the plane are replaced by such objects as compact sets. In this way the sample space becomes the space $\mathcal{K}$ of compact sets in the plane endowed with the Hausdorff metric or the Matheron's topology of closed sets (Serra, 1982; Stoyan and Stoyan, 1994, Sec. 8.3). This approach is applicable only to simple closed curves in the plane, and it does not allow for the representation of information concerning the orientation of curves. Furthermore, notice that $\mathcal{K}$ contains as elements many plane sets whose borders are not curves.

2) Curves (typically, contours of biological structures) are replaced by finite dimensional vectors of landmark points (Bookstein, 1978, 1991), and the corresponding finite dimensional Euclidean space is then taken as the sample space. The main drawback in this approach is that information on the whole curves as geometric objects is lost. Also, there are situations in which the identification of landmarks is controversial or does not make much practical sense (see discussions in Grenander and Miller, 1994).

3) Curves are replaced by functions, and then the sample space is set as certain space of continuous functions. Some of these representations are only approximations by certain simple curves, such as linearly connected straight arcs (Grenander *et al.,* 1990; Grenander and Miller, 1994). Other representations are exact but only valid for curves of particular shapes, e.g. the radius-vector function for star-shaped curves and the support function for convex figures (Stoyan and Stoyan, 1994, ch. 7). Other ones are more general but assume rectifiability of the curves (as the arc length parametrization in Persoon and Fu, 1977) or even high order of smoothness (as the representation by curvatures, see, e.g., Klingenberg, 1978). Thus, they do not apply to continuous but non-rectifiable curves. For example, they can not be used to represent fractal curves (Barnsley, 1978; Stoyan and Stoyan, 1994, Part I) and noisy curves generated by non-rectifiable disturbance mechanisms (e.g., isotropized Brownian bridges, as in Stoyan and Stoyan, 1994, Sec. 7.6.4).

The limitations just mentioned pose the need for a sufficiently general concept of sample space of curves, suitable for the wide variety of situations faced in

practice. For curves with distinguished starting and ending points, it seems natural to require such a concept to satisfy the following conditions:

i) Each curve is a *subset* of the set of continuous functions x of $[0,1]$ into the space in which the curves live, say E (the starting and ending points are $x(0)$ and $x(1)$, respectively).

ii) To *any* (maybe non-rectifiable) continuous function x of $[0,1]$ into E corresponds a curve $[x]$.

iii) The set of curves, say $\mathcal{C}_2(E)$, is endowed with certain *distance* d_2, and the mapping $[\cdot] : C([0,1], E) \to \mathcal{C}_2(E), x \rightsquigarrow [x]$, is measurable.

Condition (i) is fulfilled by the standard definition of regular curves in Differential Geometry (see, e.g., Klingenberg, 1978). It expresses invariance of a curve w.r.t. parametrizations. Condition (ii) arises from the need to deal with non-rectifiable curves in a number of examples, as was pointed out above. It is not satisfied by regular curves, hence the need for a more general concept. Condition (iii) makes it possible to endow the set of curves with the Borel σ-algebra $\mathcal{B}(\mathcal{C}_2(E))$ induced by the distance, and therefore the sample space of curves can be simply taken as the measurable space $(\mathcal{C}_2(E), \mathcal{B}(\mathcal{C}_2(E)))$. In addition, measurability of the mapping $[\cdot]$ ensures that any random function X generates a random curve $[X]$, which is a convenient property for statistical modelling of random curves.

In fact, a concept of curves and a distance between curves that satisfy conditions (i)-(iii) was introduced by Fréchet long ago (Fréchet, 1934, 1951, see historical comments in Pastor *et al.,* 1963). This was the basis of the approach followed in Mora and Biscay (1995) to deal with random continuous curves. In the present paper this approach is further developed with two main purposes. First, to generalize the Fréchet sample space of curves to incorporate information on any number of landmarks (not only the starting and ending points) and invariance w.r.t. an arbitrary group of isometric spatial transformations. Second, to provide methods for estimating centroids of probability measures defined on the introduced sample spaces of curves.

More specifically, we define curves with two or more landmarks as vectors of linearly connected Fréchet curves or, equivalently, as certain classes of functions in which the correspondence between homologous landmarks is kept. We introduce Fréchet curves without landmarks by imposing not only invariance w.r.t. parametrizations but also w.r.t. their starting points. Under mild conditions, the sample spaces of Fréchet curves with arbitrary numbers of landmarks are Polish (i.e., separable and complete). This property is useful for various purposes, e. g., for the definition of curve-valued random processes and the construction of versions of conditional probabilities that are probability measures (Neveu, 1965).

It is also shown that these general spaces include standard sets of curves, such as rectifiable curves, regular curves and curves with Frenet-frames (Klingenberg, 1978), as Borel subsets. In turn, these distinguished subsets of curves are Borel isomorphic to sets of functions via arc length parametrization. Thus, both non-rectifiable and rectifiable curves with landmarks are covered by this approach, and for the latter the arc length parametrization can be used as an alternative, Borel isomorphic representation.

Also, we introduce metric sample spaces of Euclidean Fréchet curves, which generalize the space of Fréchet curves in $\mathbb{R}^p$ $(p \geq 1)$ to take into account not only invariance w.r.t. parametrization but also w.r.t. a group of isometric spatial transformations. Such a group can be any closed subgroup of Euclidean transformations in $\mathbb{R}^p$, e.g., the group of translations, the orthogonal group or the special Euclidean group (Birkhoff and MacLane, 1977). In this way, ideas from multivariate morphometrics (Bookstein, 1978, 1991; Kendall, 1984) are extended to continuous curves.

An important question for statistical modelling and analysis of random curves is wether they can be generated and represented by random functions. We say that a random curve C is represented by a random function X if there is a measurable and injective mapping φ such that $X = \varphi(C)$. Using results on measurable selections and universal completions (Parthasarathy, 1967; Dellacherie and Meyer, 1978) we show that, in principle, all random Fréchet curves (with any number of landmarks, and also Euclidean Fréchet curves) can be represented by random functions. As a consequence, for the *synthesis* of random curves it is sufficient to simulate random functions. However, in general, the results obtained on such representations are in the form of existence theorems, so the representations are unknown in practice. Thus, they do not allow for the reduction of the *analysis* of random Fréchet curves to that of random functions in all situations. Known representations are available only for particular types of curves (e.g., the arc length parametrization for rectifiable curves).

Finally, we deal with the estimation of centroids of *r.v.* in the introduced sample spaces of curves. Centroids of a *r.v.* X with probability law P on an abstract metric space $(\mathcal{X}, d)$ were defined by Fréchet (1948) by minimizing $Ed(X, z)$ over $z \in \mathcal{X}$. Results on consistent estimation of centroids and restricted centroids (i.e., centroids within the support of the measure P) have been obtained under the assumption of compactness of the space $\mathcal{X}$ (see, e.g., Sverdrup-Thygeson, 1981). But such results have poor applicability to sample spaces of curves because of these are typically non-compact. With this motivation, we define centroids and restricted centroids without said assumption. We also allow for functions of discrepancy between X and z more general than the metric d. Results on consistent estimation of centroids are proved, similar to that given in Sverdrup-Thygeson (1981).

The paper is organized as follows. For clarity, Section 2 presents some nota-

tions that are used through the paper. Section 3 and 4 introduce metric sample spaces of Fréchet curves with landmarks and Euclidean Fréchet curves, respectively. In Section 5 the generation and representation of random curves by random functions are discussed. Section 6 provides results on consistent estimation of centroids and restricted centroids that are applicable to the spaces of curves introduced in Sections 3 and 4. All proofs are deferred to Section 7.

2. Preliminary notations. The following notations will be used through this paper.

Unless otherwise specified, (E,d) is an arbitrary metric space. $(C(E),d)$ is the space of continuous functions of $[0,1]$ into E endowed with the supremum metric, $d(x,y)=\sup\{d(x(t),y(t)):t\in[0,1]\}$ for $x,\ y\in C(E)$.

$(H(E),h)$ is the set of non-null compact subsets of E endowed with the Hausdorff metric,

$$h(A,B)=\max\left(\sup\{d(x,B):x\in A\},\sup\{d(x,A):x\in B\}\right),$$

where $d(x,A)=\inf\{d(x,y):y\in A\}$ for $x\in E$, $A\in H(E)$.

$\mathbb{R}^p$ $(p\geq 1)$will be endowed with the distance induced by the Euclidean norm.

For integers $p\geq 1, k\geq 0$ and for $E=\mathbb{R}^p, C^k(E)=C^k([0,1],E)$ is the set of functions of $[0,1]$ into E with continuous derivatives up to order k. For $x\in C^k(E)$ and $j\leq k, x^{(j)}$ is the derivative function of x of order j. The notation $\dot{x}$ will be also used for $x^{(1)}$. The usual norm on $C^k(E)$ is denoted by $\|x-y\|_k=\sup\left\{d\left(x^{(j)}(t),y^{(j)}(t)\right):t\in[0,1],j=0,...,k\right\}$.

The set of regular parametrized curves in $E=\mathbb{R}^p$ (see, e.g., Klingenberg, 1978) is denoted by $R(E)=\left\{x\in C^1(E):\dot{x}(t)\neq 0 \text{ for all } t\in[0,1]\right\}$. We write $R^k(E)=R(E)\cap C^k(E)$ for $k\geq 1$.

The set of nondecreasing (resp., strictly increasing) functions of $[0,1]$ *onto* $[0,1]$ is denoted by U (resp., V). We write $V^k=V\cap C^k([0,1])$ for $k\geq 1$, and $V^\infty=\bigcap_{k=1}^{\infty}V^k$.

The symbol $\circ$ denotes composition of functions.

Unless otherwise specified, all metric spaces will be endowed with their Borel σ-algebras to become measurable spaces.

3. Sample spaces of Fréchet curves with any number of landmarks.

3.1. *Fréchet curves.* We will adopt the concept of Fréchet curves from Fréchet (1934, 1951, see also historical comments in Pastor *et al.,* 1963).

DEFINITION 3.1.1 (Fréchet, 1934, 1951). Two functions $x,y\in C(E)$ are said to be equivalent (in the Fréchet sense), written $x\sim_2 y$, if $\inf\{d(x,y\circ v):v\in V\}=0$. The space of *Fréchet curves* is the quotient space $\mathcal{C}_2(E)=C(E)/\sim_2$. Its elements will be denoted by $[x]_2$, $x\in C(E)$. Each $y\in[x]_2$ will be said a parametric

representation (for short, a representation) of the curve $[x]_2$. The *Fréchet distance* on $\mathcal{C}_2(E)$ is

$$d_2([x]_2, [y]_2) = \inf\{d(x, y \circ v) : v \in V\}.$$

REMARK 3.1.1. The subindex "2" in the symbols of this definition makes explicit that here the curves have two distinguished (landmark) points: the starting point $x(0)$ and the ending point $x(1)$, which are the same no matter the representation x of the curve. Curves with an arbitrary number of landmarks will be considered in Section 3.2.

The following known result makes it possible to characterize the Fréchet curves as equivalence classes of functions in $C(E)$ by means of parameter transformations (change of variables) in U.

LEMMA 3.1.1. (Kolmogorov and Fomin, 1978). *For any $x, y \in C(E)$, there are $u, v \in U$ such that*

$$\inf\{d(x, y \circ v) : v \in V\} = d(x \circ u, y \circ v).$$

In particular, $\inf\{d(x, y \circ v) : v \in V\} = 0$ *if and only if there are $u, v \in U$ such that $x \circ u = y \circ v$.*

To make explicit the relation of this result with the concept of Fréchet curve consider the following binary relation on $C(E)$: $x \sim y$ if and only there are $u, v \in U$ such that $x \circ u = y \circ v$. As a consequence of Lemma 3.1.1 and the fact that $\rho(x, y) = \inf\{d(x, y \circ v) : v \in V\}$ is a seudometric on $C(E)$ (see Lemma 7.1), $\sim$ is an equivalence relation and $d([x], [y]) = \rho(x, y)$ defines a metric on the quotient space $C(E)/\sim$. Kolmogorov and Fomin (1978) called $(C(E)/\sim, d)$ the metric space of continuous curves in E. But one can see from Lemma 3.1.1 and Definition 3.1.1 that $(C(E)/\sim, d) = (\mathcal{C}_2(E), d_2)$. Thus, the equivalence relations $\sim_2$ and $\sim$ are the same and thence a Fréchet curve $[x]_2$ can be characterized as

$$[x]_2 = \{y \in C(E) : x \circ u = y \circ v \text{ for some } u, v \in U\}.$$

For completeness, since we have not found in the literature a proof of Lemma 3.1.1, we prove this useful and far from evident result in Section 7.

REMARK 3.1.2. $d_2([x]_2, [y]_2)$ is defined as the best matching (in the sense of the supremum metric) between representations of the curves $[x]_2$ and $[y]_2$.

REMARK 3.1.3. It is worth to note some important differences between Fréchet curves and (oriented) regular curves (Klingenberg, 1978). In standard Differential Geometry, the space in which curves live is specifically $E = \mathbb{R}^p$; parametric representations (parametrized curves) are restricted to be regular, i.e., $x \in R(E)$; the admissible parameter transformations are restricted to be (increasing) smooth homeomorphisms, $v \in V^1$; and two regular representations x, y are said to be equivalent if they are equal except for an admissible parameter

transformation, i.e. $x = y \circ v$ for some $v \in V^1$. An (unparameterized) curve is then defined as an equivalent class of regular parametrized curves. In contrast, Fréchet curves can live in an arbitrary metric space E, their representations can be non-smooth (even non-rectifiable) functions, and the equivalence between representations is expressed in terms of continuous but possibly non-smooth and non-invertible parameter transformations, $u \in U$. This extension of the concept of curves makes it possible to associate a curve to *any* continuous function $x \in C(E)$ and to endow the resulting set of general curves with a distance. Regular curves can be thought of as a particular subspace of Fréchet curves, as will be shown in Section 3.4.

REMARK 3.1.4. Due to its flexibility in matching, the Fréchet distance may have some use as a measure of discrepancy for pattern recognition, curve pattern matching, cluster analysis, multidimensional scaling and the seek of centroids of samples of curves (for the latter, see Section 6). But it can be also used for more basic probabilistic purposes that are emphasized in the present paper. Namely, the definition of metric sample spaces of general (continuous) curves, and of random variables and probability distributions on such spaces. A further issue is to formulate and solve statistical problems in such a probability setting. For example, to discriminate from which of two given probability distributions P_1, P_2 on $\mathcal{C}_2(E)$ comes an observed curve c. The choice of suitable statistics for such problems depends on the probability model assumed in each particular situation. E.g., the optimal statistic to discriminate between two populations of curves is not, in general, the Fréchet distance but it is determined by the assumed distributions P_1 and P_2. For sufficiently smooth regular curves, some metrics oriented to several matching problems are being developed (Trouvé, 1995; Younes, 1998; Grenander and Miller, 1998). They lack the generality of the Fréchet metric but perform well in a number of matching problems involving smooth objects in computational anatomy and other applied fields.

REMARK 3.1.5. Non-oriented Fréchet curves can be defined in a similar fashion. Namely, by replacing in Definition 3.1.1 the set V by the set of homeomorphisms on the interval $[0, 1]$. In fact, this concept was also introduced by Fréchet (1934), as cited by Pastor *et al.*(1963). Similar concepts for surfaces were also defined in said paper.

A quotient metric space, such as $\mathcal{C}_2(E)$, inherits some properties of the underlying space. Specifically, the following lemma holds.

LEMMA 3.1.2. *Suppose (X, d) is a metric space, $\mathcal{X}$ is a quotient space on X, with elements denoted by $[x]$ for $x \in X$, and ρ is a metric on $\mathcal{X}$ such that*

$$\rho([x], [y]) = \inf\{d(x', y') : x' \in [x], y' \in [y]\}.$$

Then, the following statements hold:

a) If (X, d) *is separable then* $(\mathcal{X}, \rho)$ *is separable*
b) If (X, d) *is complete then* $(\mathcal{X}, \rho)$ *is complete.*

In particular, the space of Fréchet curves is Polish under mild conditions, as the following shows.

THEOREM 3.1.1.
a) If (E, d) *is separable then* $(\mathcal{C}_2(E), d_2)$ *is separable.*
b) If (E, d) *is complete then* $(\mathcal{C}_2(E), d_2)$ *is complete.*

EXAMPLE 3.1.1. For any integer $p \geq 1$, $\left(\mathcal{C}_2(\mathbb{R}^P), d_2\right)$ is a Polish space.

REMARK 3.1.6. Fréchet curves are defined as classes of E-valued functions. Alternatively, curves have been also thought of as certain sets of points in E ("intrinsic " concepts of curves). Two well-known examples are the following: a) An Uryshon curve in E is any continuum (i.e., compact and connected) subset of E with topological dimension 1 (Kuratowski, 1968). b) Curves defined as images of the unit interval by continuous functions, $x\left([0,1]\right)$ with $x \in C\left(E\right)$ (also called Jordan curves in some measure-theoretic contexts). These concepts of curves differ w.r.t. their intensions and extensions. From a set-theoretic perspective, some relations between them have been studied in Kuratowski (1968). E.g., every continuum consisting exclusively of points of order 2 is a simple closed curve in the Jordan sense (b) above (Kuratowski, 1968, Vol. II, p. 294). From a metric viewpoint, the set of Uryshon curves, $\mathcal{UC}\left(E\right)$, and the set of Jordan curves, $\mathcal{IC}\left(E\right)$, are metric spaces w.r.t. the Hausdorff distance. The comparison of these paces with that of Fréchet curves shows a number of differences. First, only the Fréchet curves and metric reflect information about the oriented paths of the curves. Second, $\left(\mathcal{C}_2(\mathbb{R}^P), d_2\right)$ is Polish, as Example 3.1.1 shows. This is a convenient property on a sample space $\mathcal{X}$ for several purposes, e.g., for defining $\mathcal{X}$-valued stochastic processes (Neveu, 1965). Since $(H(\mathbb{R}^P), h)$ is separable (see, e.g., Barnsley, 1978) and subsets of a separable metric space are also separable, $\left(\mathcal{UC}\left(\mathbb{R}^P\right), h\right)$ and $\left(\mathcal{IC}\left(\mathbb{R}^P\right), h\right)$ are separable. However, they are not complete, as can be shown by simple examples. Hence they are not Polish. Third, information about landmarks and smoothness of the curves can be incorporated into the Fréchet metric in a natural way, as will be done in Sections 3.2 and 3.4. Fourth, as will be shown in Section 5, random Fréchet curves can be represented by random functions, which is a useful property for statistical modelling and analysis of random curves.

3.2. *Fréchet curves with more than two landmarks.* Specification of landmark points (usually by experts) is a common device to incorporate relevant constraints in comparing contours of the same kind of objects, specially in biological structures (see, e.g., Bookstein, 1978, 1991). A set of $m \geq 2$ landmarks divides a curve into $m - 1$ consecutive segments. The main idea is that the comparison of two

curves with landmarks has to be restricted by matching homologous segments. This constraint can be incorporated into the Fréchet distance as follows.

DEFINITION 3.2.1. For $m \geq 2$, the space of *Fréchet curves with m landmark points* is

$$\mathcal{C}_m(E) = \left\{([x_1]_2, ..., [x_{m-1}]_2) \in (\mathcal{C}_2(E))^{m-1} : x_j(1) = x_{j+1}(0) \text{ for } j = 1, ..., m-2\right\},$$

endowed with the distance

$$d_m\left(([x_1]_2, ..., [x_{m-1}]_2), ([y_1]_2, ..., [y_{m-1}]_2)\right) = \max\left\{d_2\left([x_k], [y_k]\right) : k = 1, ..., m-1\right\}.$$

Definition 3.2.1 regards Fréchet curves with landmarks as vectors of consecutive Fréchet curves. Alternatively, they can be defined as certain classes of parametric representations. Specifically, let $\backsim_m$be the binary relation on $C(E)$ defined by $x \backsim_m y$ if and only if $x \circ u = y \circ v$ for some $u, v \in U_m$, where

$$U_m = \left\{u \in U : u\left(\frac{i}{m-1}\right) = \frac{i}{m-1} \text{ for } i = 1, ..., m-1\right\}.$$

Notice that, for any $u \in U_m$, the parametrization $x \circ u$ leaves invariant the points $x \circ u\left(\frac{i}{m-1}\right) = x\left(\frac{i}{m-1}\right)$, $i = 1, ..., m-1$, and it does an arbitrary parameter transformation in each segment between these points. Without loss of generality, we can consider that these points are the landmarks. Thus, $x \backsim_m y$ means that, in the parametric representations x and y, the points corresponding to landmarks are the same and homologous segments coincide except for parameter transformations. This suggests that Fréchet curves with landmarks could also be defined as equivalence classes $[x]_m$ with respect to $\backsim_m$. The following theorem confirms this, and so both approaches will be regarded as equivalent henceforth.

THEOREM 3.2.1. *For $m \geq 2$, $\backsim_m$ is an equivalence relation,*

$$r_m\left([x]_m, [y]_m\right) = \inf\left\{d\left(x, y \circ u\right) : u \in U_m\right\} \text{ for } [x]_m, [y]_m \in C(E)/\backsim_m$$

is a metric on $C(E)/\backsim_m$, and the mapping $g : (C(E)/\backsim_m, r_m) \to (\mathcal{C}_m(E), d_m)$ defined by $g\left([x]_m\right) = \left(([x_1]_2, ..., [x_{m-1}]_2)\right), x_i(t) = x\left(\frac{i-1}{m-1} + \frac{1}{m-1}t\right)$ for $i = 1, ..., m-1, t \in [0, 1]$, is an isometry.

3.3. *Fréchet curves without landmarks.* In practice, curves without landmark points, i.e., closed curves for which no specific point is distinguished, can also be of interest (see, e.g., Grenander and Miller, 1994).

In order to introduce Fréchet curves without landmarks some notations are convenient. Let $C_c(E)$ denotes the set of closed continuous functions of $[0, 1]$ into E,

$$C_c(E) = \left\{x \in C(E) : x\left(0\right) = x\left(1\right)\right\},$$

and let

$$U_0 = \{w_{u,s} : [0,1] \to [0,1],\, w_{u,s}(t) = (u(t) + s) \bmod (1),\, u \in U, s, t \in [0,1]\}.$$

DEFINITION 3.3.1. For $x, y \in C_c(E)$, write $x \backsim_0 y$ if and only if there are $u, v \in U_0$ such that $x \circ u = y \circ v$. The equivalence classes $[x]_0 = \{y \in C_c : x \backsim_0 y\}$, $x \in C_c(E)$, are called *Fréchet curves without landmarks*, and we write $\mathcal{C}_0(E) = C_c(E)/\backsim_0$. The Fréchet distance on $\mathcal{C}_0(E)$ is

$$d_0(b,c) = \inf\{d(x, y \circ w) : w \in U_0\},$$

for $b = [x]_0$, $c = [y]_0 \in \mathcal{C}_0(E)$.

REMARK 3.3.1. Notice that, according to the notations introduced above, the number of landmarks of a curve is $m = 0$ or $m \geq 2$. If a curve have a unique distinguished point then it is a closed curve and we adopt the convention that it has two coincident landmarks, $x(0) = x(1)$.

THEOREM 3.3.1. *$\backsim_0$ is an equivalence relation on $C_c(E)$, and $(\mathcal{C}_0(E), d_0)$ is a metric space.*

The following generalizes Theorem 3.1.1 to any number of landmarks.

THEOREM 3.3.2. *Suppose $m = 0$ or $m \geq 2$.*
a) If (E,d) is separable then $(\mathcal{C}_m(E), d_m)$ is separable.
b) If (E,d) is complete then $(\mathcal{C}_m(E), d_m)$ is complete.

3.4. *Some distinguished subspaces of Fréchet curves.* In this Section we will show that some types of Fréchet curves satisfying different regularity conditions form Borel sets, and these are Borel isomorphic to sets of functions. We recall that a Borel isomorphism is a measurable mapping that has measurable inverse (Parthasarathy, 1967).

Rectifiable curves. We follow Kolmogorov and Fomin (1978) in defining rectifiable continuous curves and their representations by arc length parametrization except we deal with curves with any number of landmarks.

For $x \in C(E)$ and $s, t \in [0,1]$, let

$$L(x; s, t) = \sup\left\{\sum_{k=1}^{n} d(x(t_k), x(t_{k-1})) :\ s = t_0 \leq \cdots \leq t_n = t, n = 2, 3...\right\}.$$

DEFINITION 3.4.1. Let $m \geq 2$. The length of a curve $[x]_m \in \mathcal{C}_m(E)$ is $L_{[x]_m} = L(x, 0, 1)$; $[x]_m$ is said to be *rectifiable* if $L(x, 0, 1) < \infty$. $\mu \in C(E)$ is called a *natural parametric representation* (or natural representation, for short) of a rectifiable curve $c \in \mathcal{C}_m(E)$ if $c = [\mu]_m$ and $L(\mu_i; 0, t) = tL_{[\mu_i]}$ for $t \in [0,1]$, $i = 1, ..., m-1$, where $\mu_i(t) = \mu\left(\frac{i-1}{m-1} + \frac{1}{m-1}t\right)$. A natural representation of $[x]_m$

will be denoted by $\widetilde{x}$. The subset of rectifiable curves in $\mathcal{C}_m(E)$ will be denoted by $\mathcal{C}_m^R(E)$, and the set of natural representations by

$$N_m(E) = \{x \in C(E) : x \text{ is a natural representation of } [x]_m\}.$$

REMARK 3.4.1. Notice that a natural representation $\widetilde{x}$ of $[x]_2 \in \mathcal{C}_2(E)$ is essentially an arc length parametrization except it does not satisfy $L(\widetilde{x}; 0, t) = t$ ("unit speed") but $L(\widetilde{x}; 0, t) = t\, L_{[x]_2}$. This convention is adopted to maintain the interval $[0, 1]$ as the range of the parameter t no matter the length of the curve. A natural representation $\widetilde{x}$ of a curve with $m \geq 2$ landmarks, $[x]_m \in \mathcal{C}_m(E)$, is a piece-wise natural representation between consecutive landmarks.

LEMMA 3.4.1. *For $m \geq 2$, each rectifiable curve $[x]_m \in \mathcal{C}_m^R(E)$ has an unique natural representation $\widetilde{x}$, and $[x]_m = \{\widetilde{x} \circ u : u \in U_m\}$.*

THEOREM 3.4.1. *Let E be a Polish space and $m \geq 2$. Then, $\mathcal{C}_m^R(E)$ is a Borel subset of $(\mathcal{C}_m(E), d_m)$ and the mapping $\varphi : \left(\mathcal{C}_m^R(E), d_m\right) \to (C(E), d)$, $[x]_m \rightsquigarrow \widetilde{x}$, is a Borel isomorphism between $\mathcal{C}_m^R(E)$ and $N_m(E)$.*

REMARK 3.4.2. The fact that $\mathcal{C}_m^R(E)$ is a Borel set allows one to restrict any probability measure P defined on $\mathcal{C}_m(E)$ to the subset $\mathcal{C}_m^R(E)$. (Recall that all metric spaces are endowed with their Borel σ-algebras, as was pointed out in Section 2). This is convenient because of a number of applications involve only rectifiable curves. In such cases, according to Theorem 3.4.1 it is equivalent (Borel isomorphic) to deal with a probability measure on the space of curves $\mathcal{C}_m^R(E)$ and with the induced measure on the space of natural representations $N_m(E)$. Any of these spaces can be chosen as most convenient for probability modelling and analysis in each concrete situation. Models on $\mathcal{C}_m^R(E)$ require of general tools from the probability theory on metric spaces. In contrast, $N_m(E)$ is contained in the normed space $C(E)$. But notice that functions $\widetilde{x}$ in $N_m(E)$ have to satisfy the constraints $L(\widetilde{x}_i; 0, t) = tL_{[\widetilde{x}_i]}$ for $t \in [0, 1]$, $i = 1, ..., m-1$, where $\widetilde{x}_i(t) = \widetilde{x}\left(\frac{i-1}{m-1} + \frac{1}{m-1}t\right)$. This causes that probability models on $N_m(E)$ are not so simple to specify as those on $C(E)$. To illustrate, consider a quite simple model on $C(\mathbb{R}^2)$, say $X = \mu + \varepsilon$, where $\mu \in C^1(\mathbb{R}^2)$ is the natural representation of a smooth non-random function and ε is a stationary Gaussian process with smooth realizations *a.s.* Since $\widetilde{X} = X \circ l_X^{-1}$, where $l_X(t) = L(X; 0, t)/L(X; 0, 1)$ and $L(X; 0, t) = \int_0^t \left|\dot{X}(s)\right| ds$, the process $\widetilde{X}$ has a quite complex probability structure, neither Gaussian nor stationary. This is a common situation in random deformation models of a deterministic curve: even the most simple deformation mechanisms lead to quite complicated random natural representations. Notice that in statistical analysis of rectifiable curves, the representation X associated with some assumed deformation mechanism is unknown; the data is only the curve $[X]_2$ or, equivalently, the natural representation $\widetilde{X}$.

Regular curves. To avoid cumbersome details, in what follows in this Section the number of landmarks will be set $m = 2$ though the results to be presented can be formulated for any m. Unless otherwise specified, $p \geq 1$ and $k \geq 1$ will be integers, and $E = \mathbb{R}^p$.

Denote by $N_2^k(E)$ the set of natural representations with continuous derivatives up to order k, i.e. $N_2^k(E) = N_2(E) \cap C^k(E)$. The set of Fréchet curves corresponding to smooth regular representations $x \in R^k(E)$ will be denoted by $\mathcal{C}_2^k(E) = \left\{[x]_2 : x \in R^k(E)\right\}$.

THEOREM 3.4.2. *For all $k \geq 1, \mathcal{C}_2^k(E)$ is a Borel subset of $\left(\mathcal{C}_2^R(E), d_2\right)$, and the mapping $\varphi : \left(\mathcal{C}_2^k(E), d_2\right) \to \left(C(E), \|.\|_0\right), [x]_2 \rightsquigarrow \widetilde{x}$, is a Borel isomorphism between $\mathcal{C}_2^k(E)$ and $N_2^k(E)$.*

Let $\mathcal{R}_2^k(E)$ be the set of regular curves in the ordinary sense in Differential Geometry, i.e., $\mathcal{R}_2^k(E) = \left\{(x)_2 : x \in R^k(E)\right\}$, where

$$(x)_2 = \left\{y \in R^k(E) : x = y \circ v \text{ for some } v \in V^k\right\}$$

(Klingenberg, 1978, see also Remark 3.1.3). For $x, y \in R^k(E)$, define

$$\rho_2\left((x)_2, (y)_2\right) = \inf\left\{d\left(x, y \circ v\right) : v \in V\right\}.$$

The following theorem shows that regular curves with the distance ρ_2 and regular Fréchet curves are isometric.

THEOREM 3.4.3. *For $k \geq 1$, the mapping*

$$\left(\mathcal{R}^k(E), \rho_2\right) \to \left(\mathcal{C}_2^k(E), d_2\right), (x)_2 \rightsquigarrow [x]_2 \ \ for\ x \in R^k(E),$$

is isometric.

REMARK 3.4.3. For arbitrary continuous representations $x, y \in C(E)$, $d_2\left([x]_2, [y]_2\right) = 0$ (i.e., $[x]_2 = [y]_2$) does not imply that $x = y \circ v$ for some smooth parameter transformation $v \in V^k$, $k \geq 1$ (i.e., $(x)_2 = (y)_2$) but only that $x \circ u = y \circ v$ for some $u, v \in U$. This motivated the generalization of the ordinary concept of equivalent parametric representations that led to the introduction of Fréchet curves (see Remark 3.1.2). But in case of regular representations $x, y \in R^k(E)$ $(k \geq 1)$, Theorem 3.4.3 shows that $[x]_2 = [y]_2$ if and only if $(x)_2 = (y)_2$. Therefore, ordinary regular curves can be endowed with the Fréchet distance defined as ρ_2 without need for any modification of the concept of such curves. Furthermore, in the definition of this distance one can restrict the infimum to be only over regular representations, i.e., for $x, y \in R^k(E)$,

$$\rho_2\left((x)_2, (y)_2\right) = \inf\left\{d\left(x, y \circ v\right) : v \in V^k\right\}.$$

This is an immediate consequence of the following.

LEMMA 3.4.2. *For all* $k \geq 1$ *and* $x, y \in C(E)$,

$$\inf\{d(x, y \circ v) : v \in V\} = \inf\left\{d(x, y \circ v) : v \in V^k\right\}.$$

For regular curves, it is possible to incorporate information about higher order derivatives into the Fréchet distance in a natural way.

DEFINITION 3.4.2. The *Fréchet distance with higher order derivatives* for regular curves $[x]_2, [y]_2 \in \mathcal{C}_2^k(E)$ is

$$d_{2,k}([x]_2, [y]_2) = d_2\left(\left[\left(\widetilde{x}^{(0)}, ..., \widetilde{x}^{(k)}\right)\right]_2, \left[\left(\widetilde{y}^{(0)}, ..., \widetilde{y}^{(k)}\right)\right]_2\right),$$

where $\left[\left(\widetilde{x}^{(0)}, ..., \widetilde{x}^{(k)}\right)\right]_2, \left[\left(\widetilde{y}^{(0)}, ..., \widetilde{y}^{(k)}\right)\right]_2 \in \mathcal{C}_2(E^k)$ denote the Fréchet curves in E^k corresponding to the functions $\left(\widetilde{x}^{(0)}, ..., \widetilde{x}^{(k)}\right), \left(\widetilde{y}^{(0)}, ..., \widetilde{y}^{(k)}\right) \in C(E^k)$, respectively. Here $\widetilde{x}^{(j)}, \widetilde{y}^{(j)}$ denote the derivatives of order j of the natural representations $\widetilde{x}$ and $\widetilde{y}$, respectively. Furthermore, on $E^k = (\mathbb{R}^p)^k$ we consider the metric induced by the norm $\|(a_1, ..., a_k)\| = \max\left(\|a_1\|_{\mathbb{R}^p}, ..., \|a_k\|_{\mathbb{R}^p}\right)$ for $(a_1, ..., a_k) \in (\mathbb{R}^p)^k$, where $\|\cdot\|_{\mathbb{R}^p}$ is the euclidean norm in $\mathbb{R}^p$.

REMARK 3.4.3. The fact that $\left(\mathcal{C}_2^k(E), d_{2,k}\right)$ is a metric space straightforwardly follows from the same property for $\left(\mathcal{C}_2(E^k), d_2\right)$ and Lemma 3.3.1. Also notice that $d_{2,k}([x]_2, [y]_2) \leq \|\widetilde{x} - \widetilde{y}\|_k$.

REMARK 3.4.4. Since $\widetilde{x}^{(0)}, ..., \widetilde{x}^{(k)}$ are invariant with respect to the representation $x \in C^k(E)$, $d_{2,k}$ is based on matching (with respect to the supremum norm) intrinsic features of the curves. More generally, let $D(E)$ be any subset of $C(E)$ and γ be a mapping $D(E) \subset C(E) \to (C(E))^q$, where $q \geq 1$. Suppose γ is an intrinsic feature in the sense that $\gamma(x \circ v) = \gamma(x) \circ v$ for all $x \in D(E)$ and $v \in V_D$, where V_D is a dense subset of V (e.g., $V_D = V^\infty = \bigcap_{k=1}^{\infty} V^k$, see Proof of Lemma 3.4.2). Then,

$$\rho([x]_2, [y]_2) = d_2([\gamma(x)]_2, [\gamma(y)]_2),$$

where d_2 denotes the Fréchet distance on $\mathcal{C}_2(E^q)$, is a seudometric on $\mathcal{D}_2(E) = \{[x]_2 : x \in D(E)\}$. Thus, Fréchet-like distances can be defined on the basis of matching any desirable intrinsic features of the curves.

A version of Theorem 3.4.2 with the stronger metric $d_{2,k}$ is the following.

THEOREM 3.4.4. *The mapping* $\left(\mathcal{C}_2^k(E), d_{2,k}\right) \to \left(C^k(E), \|.\|_k\right), [x]_2 \rightsquigarrow \widetilde{x}$, *is a Borel isomorphism between* $\mathcal{C}_2^k(E)$ *and* $N_2^k(E)$.

Curves with Frenet representations. Let $E = \mathbb{R}^p, p \geq 2$. The sufficient condition usually invoked to guarantee the existence of a distinguished Frenet-frame

for a regular parametrized curve $x \in R^p(E)$ is that $x^{(1)}(t), ..., x^{(p-1)}(t)$ be linearly independent (*l.i.*) vectors for all $t \in [0,1]$ (Klingenberg, 1978, Proposition 1.12). Let

$$F(E) = \left\{x \in C^p(E) : x^{(1)}(t), ..., x^{(p-1)}(t) \text{ are } l.i. \text{ for all } t \in [0,1]\right\},$$

and denote the corresponding set of Fréchet curves with Frenet-frames as $\mathcal{F}_2(E) = \{[x]_2 : x \in F(E)\}$.

THEOREM 3.4.5. *$\mathcal{F}_2(E)$ is a Borel subset of $\mathcal{C}_2^R(E)$, and the mapping $\varphi : (\mathcal{F}_2(E), d_2) \to (C(E), \|.\|_0)$, $[x]_2 \rightsquigarrow \widetilde{x}$, is a Borel isomorphism between $\mathcal{F}_2(E)$ and $F(E) \cap N_2(E)$.*

Simple rectifiable curves. Let $S(E)$ be the set of simple parametric representations in E, $S(E) = \{x \in C(E) : x(t_1) \neq x(t_2) \text{ for all } t_1 \neq t_2\}$, and $\mathcal{S}(E)$ be the corresponding set of simple Fréchet curves, $\mathcal{S}_2(E) = \{[x]_2 : x \in S(E)\}$. Furthermore, denote by $\mathcal{S}_2^R(E)$ the set of simple rectifiable Fréchet curves, $\mathcal{S}_2^R(E) = \mathcal{S}_2(E) \cap \mathcal{C}_2^R(E)$.

THEOREM 3.4.6. *Let E be an arbitrary Polish metric space. Then, $\mathcal{S}_2^R(E)$ is a Borel subset of $\left(\mathcal{C}_2^R(E), d_2\right)$, and the mapping $\left(\mathcal{S}_2^R(E), d_2\right) \to (C(E), \|.\|_0)$, $[x]_2 \rightsquigarrow \widetilde{x}$, is a Borel isomorphism between $\mathcal{S}_2^R(E)$ and $S(E) \cap N_2(E)$.*

4. Sample spaces of euclidean Fréchet curves. In this Section, $E = \mathbb{R}^p$ and Λ is the Euclidean group on $\mathbb{R}^p$, i.e., the group of isometric transformations on $\mathbb{R}^p$ (Birkhoff and MacLane, 1977). It is known that each $\lambda \in \Lambda$ can be expressed in the form $\lambda = (T, q), \lambda a = Ta + q$ for all $a \in E$, where $q \in E$ and T is an element of the orthogonal group, $T \in O(p)$. Λ is a group w.r.t. the operation defined by $\lambda_1 \circ \lambda_2 = (T_2T_1, q_2 + T_2q_1)$ for $\lambda_1 = (T_1, q_1)$ and $\lambda_2 = (T_2, q_2)$. On Λ we will consider the metric induced by the norm $\|\lambda\| = \|(T, q)\| = \|T\| + \|q\|$, where $\|q\|$ is the Euclidean norm of the vector $q \in E$ and $\|T\|$ is the operator norm of the matrix T induced by the Euclidean norm of E. By Λ_S we will denote an arbitrary closed subgroup of Λ.

DEFINITION 4.1. Let $m = 0$ or $m \geq 2$. Two curves $c, c' \in \mathcal{C}_m(E)$ are Λ_S-equivalent, denoted by $c \backsim_{\Lambda_S} c'$, if there is $g \in G(\Lambda_S)$ such that $g(c) = c'$, where

$$G(\Lambda_S) = \{g_\lambda : \mathcal{C}_m(E) \longrightarrow \mathcal{C}_m(E) : g_\lambda([x]_m) = [\lambda x]_m, \lambda \in \Lambda_S\}.$$

Here $(\lambda x)(t) = \lambda x(t)$ for $\lambda \in \Lambda_S$, $x \in C(E)$. The class corresponding to a Fréchet curve $c \in \mathcal{C}_m(E)$ by the equivalence relation $\backsim_{\Lambda_S}$ will be denoted by $< c >$, and the quotient space of Euclidean curves by $\mathcal{C}_m(E)/_{\Lambda_S}$.

For $< c >, < c' > \in \mathcal{C}_m(E)/_{\Lambda_S}$, we define

$$\rho_m(< c >, < c' >) = \inf\{d_m(c, \lambda(c')) ; \lambda \in \Lambda_S\}.$$

EXAMPLE 4.1. Examples of the closed subgroup Λ_S are the Euclidean group Λ, the orthogonal group $O(p)$, the group of translations, the special orthogonal group $SO(p)$ (rotations), and the special Euclidean group, also called group of congruencies or rigid displacements (rotations and translations). If Λ_S consists of only the identity transformation then $\mathcal{C}_m(E)/_{\Lambda_S} = \mathcal{C}_m(E)$.

THEOREM 4.1. *$\mathcal{C}_m(E)/_{\Lambda_S}$ is well-defined, and $(\mathcal{C}_m(E)/_{\Lambda_S}, \rho_m)$ is a Polish metric space.*

REMARK 4.2. If Λ_S is the special Euclidean group then $\rho_m(< c >, < c' >)$ is invariant w.r.t. translations and rotations of the curves, and each equivalence class $< c >$ consists of congruent Fréchet curves with landmarks. Notice that this a basic step for the extension to continuous curves of morphometric concepts (invariance w.r.t. some isometric transformations and dilations, Procustean metrics, shapes, etc.) that have been developed for finite dimensional vectors of points (configurations) in $\mathbb{R}^p$ (Bookstein, 1978, 1991; Kendall, 1984; Stoyan and Stoyan, 1994).

REMARK 4.3. For Euclidean regular curves $< c >, < c' > \in \mathcal{C}_m^k(E)/_{\Lambda_S} = \left\{< c >: c \in \mathcal{C}_m^k(E)\right\}, k \geq 1$, a distance with higher order derivatives can be defined on the basis of $d_{m,k}$ as

$$\rho_{m,_k}(< c >, < c' >) = \inf\left\{d_{m,k}(c, \lambda(c'));\ \lambda \in \Lambda_S\right\}.$$

REMARK 4.4. To each parametrized curve with Frenet-frame, $x \in F(E)$, one can associate its arc length function L_x and its curvature functions $\kappa_1^x, ..., \kappa_{p-1}^x$, and these are uniquely determined modulo translations and rotations (Klingenberg, 1978, Sec. 1.3). Furthermore, they are intrinsic features in the sense that $L_{x\circ v} = L_x \circ v, \kappa_1^{x\circ v} = \kappa_1^x \circ v, ..., \kappa_{p-1}^{x\circ v} = \kappa_{p-1}^x \circ v$ for all $v \in V^\infty$. Thus,

$$\tau_2(< [x]_2 >, < [y]_2 >) = d_2\left(\left[\left(L_x, \kappa_1^x, ..., \kappa_{p-1}^x\right)\right]_2, \left[\left(L_y, \kappa_1^y, ..., \kappa_{p-1}^y\right)\right]_2\right)$$

for $< [x]_2 >, < [y]_2 > \in \mathcal{F}_2(E)/_{\Lambda_S} = \{< [x]_2 >: x \in F(E)\}$ defines a distance between Euclidean curves with Frenet-frames. Here d_2 denotes the Fréchet distance on $\mathcal{C}_2(E^p)$. A similar definition can be introduced for any number of landmarks m.

5. Random Fréchet curves.

5.1.Generation of random curves by random functions. To define random curves we follow the general concept of random elements in a metric space (Fréchet, 1948, Parthasarathy, 1967; Stoyan and Stoyan, 1994). This concept has been previously specialized to Fréchet curves (Fréchet, 1961) but herein, in addition, we deal with curves with any number of landmarks and Euclidean curves.

DEFINITION 5.1.1. A random curve (resp. random Euclidean curve) is a random variable (*r.v.*) with values in $\mathcal{C}_m(E)$ (resp. $\mathcal{C}_m(E)/_{\Lambda_S}$), i.e., a measurable mapping of a probability space $(\Omega, \mathcal{F}, P)$ into $\mathcal{C}_m(E)$ (resp. $\mathcal{C}_m(E)/_{\Lambda_S}$).

The following theorem shows that random continuous functions generate random curves. This makes it possible to use statistical models of stochastic processes in modelling random curves.

THEOREM 5.1.1.

a) If X is a $C_c(E)$-valued r.v. then $[X]_0$ is a $\mathcal{C}_0(E)$-valued r.v.

b) If X is a $C(E)$-valued r.v. and $m \geq 2$ then $[(X_1, ..., X_{m-1})]_m$ is a $\mathcal{C}_m(E)$-valued r.v, where $X_i(t) = X\left(\frac{i-1}{m-1} + \frac{1}{m-1}t\right)$ for $i = 1, ..., m-1, t \in [0,1]$.

c) Let $E = \mathbb{R}^p, p \geq 1, m = 0$ or $m \geq 2$, and Λ_S be a closed subgroup of the Euclidean group of transformations on E. If C is a $\mathcal{C}_m(E)$-valued r.v then $\langle C \rangle$ is a $\mathcal{C}_m(E)/_{\Lambda_S}$-valued r.v.

EXAMPLE 5.1.1. A number of probability models for generating random parametrized curves in $\mathbb{R}^2$ or $\mathbb{R}^3$ have been introduced in the last years. All of them generate continuous functions with probability 1, so they induce random curves in the Fréchet sense, as a consequence of Theorem 5.1.1. To illustrate, we mention some examples: models of random closed polygonal functions, such as the Circular Autoregressive (CAR) model for edges (Mardia *et al.*, 1991) and the model of linearly connected arcs subject to circulant random scale-rotation transformations (Grenander *et al.,* 1990; Grenander and Miller, 1994); models based on spline interpolation of a random vector of landmark points (Mardia *et al.*, 1991); models of random curves with Frenet representations based on probability laws for their curvature functions (Grenander and Miller, 1998); and models of contours radially disturbed by isotropized Brownian bridges (Stoyan and Stoyan, 1994). The latter class of models generates parametrized curves that are non-rectifiable with probability 1.

5.2. Representation of random curves by random functions. Let $\mathcal{X}, \mathcal{Y}$ be arbitrary measurable spaces. We will say that random elements in $\mathcal{X}$ can be represented by random elements in $\mathcal{Y}$ if there is some measurable and injective mapping φ of $\mathcal{X}$ into $\mathcal{Y}$. In such a case, any $\mathcal{X}$-valued *r.v.* X induces a unique $\mathcal{Y}$-valued *r.v.* $Y = \varphi(X)$ through said mapping.

>From the results in Section 3.3 (Theorems 3.3.1, 3.3.2, 3.3.4 and 3.3.5) we know that random rectifiable Fréchet curves (and more specific types of curves, such as regular, with Frenet representations and simple rectifiable curves) can be represented by random functions via natural parameterization. For general random Fréchet curves, a representation by functions can be obtained on the basis

of the following lemma. For this, the Borel σ-algebras of the metric spaces involved will be slightly increased. Namely, they will be replaced by their universal completions. This will not imply any essential modification from the viewpoint of probability measures, see (Remark 5.2.2 below).

For a metric space Y, the universal completion of its Borel σ-algebra $\mathcal{B}(Y)$ is the intersection of all the completions $\mathcal{B}^P(Y)$ of $\mathcal{B}(Y)$ associated with all the probability measures P on $\mathcal{B}(Y)$ (see, e.g., Dellacherie and Meyer, 1978).

LEMMA 5.2.1. *Let (X,d) and $(\mathcal{X},\rho)$ be Polish metric spaces, and π be a continuous mapping of X onto $\mathcal{X}$. Then, there exist a $\mathcal{B}_u(\mathcal{X})-\mathcal{B}_u(X)$ measurable mapping h of $\mathcal{X}$ into X such that $\pi\circ h(x)=x$ for any $x\in\mathcal{X}$.*

THEOREM 5.2.1. *Let (E,d) be a Polish metric space, $m=0$ or $m\geq 2$, $(X,d)=(C(E),d)$ and $(\mathcal{X},\rho)=(\mathcal{C}_m(E),d_m)$. Then, there is a $\mathcal{B}_u(\mathcal{X})-\mathcal{B}_u(X)$ measurable mapping h of $\mathcal{X}$ into X such that $[h(c)]_m=c$ for all $c\in\mathcal{X}$.*

REMARK 5.2.1. $[h(c)]_m=c$ for all $c\in\mathcal{X}$ implies that h in Theorem 5.2.1 is an injective mapping.

REMARK 5.2.2. Every measure P on a measurable space $(\Omega,\mathcal{F})$ can be uniquely extended to a measure $\widehat{P}$ on the universal completion $\left(\Omega,\widehat{\mathcal{F}}\right)$ of $(\Omega,\mathcal{F})$, and the mapping $\psi: P\longrightarrow\widehat{P}$ is a bijection between the sets of measures on $\mathcal{F}$ and $\widehat{\mathcal{F}}$ (Dellacherie and Meyer, 1978, Section II-32, Remark (c)). Thus, from the viewpoint of probability measures, a *r.v.* C in $(\mathcal{C}_m(E),\mathcal{B}(\mathcal{C}_m(E)))$ with law P can be replaced by a *r.v.* $\widehat{C}$ in $(\mathcal{C}_m(E),\mathcal{B}_u(\mathcal{C}_m(E)))$ with corresponding law $\widehat{P}$ via the bijection ψ. By Theorem 5.2.1, this induces the *r.v* $h\left(\widehat{C}\right)$ in $(C(E),\mathcal{B}_u(C(E)))$, which in turn, again via the bijection of measures, can be replaced by a *r.v.* in $(C(E),\mathcal{B}(C(E)))$. From this and Theorem 5.1.1 it follows that there is a bijection between the sets of probability laws on $(\mathcal{C}_m(E),\mathcal{B}(\mathcal{C}_m(E)))$ and $(C(E),\mathcal{B}(C(E)))$.

REMARK 5.2.3. In general, the representation stated in Theorem 5.2.1 can not be used in practice to reduce random curves to random functions because the mapping h is unknown. On the other hand, as was noticed in Remark 5.2.2, the existence of this representation ensures that all probability models of curves can be induced from probability models of functions through the mapping $x\in C(E)\rightarrow[x]_m\in\mathcal{C}_m(E)$. Thus, probability *modelling* of Fréchet curves can be reduced to modelling random functions, but the *statistical analysis* of the induced models on the basis of data curves $[x]_m$ can not be reduced to the analysis of sample functions $h([x]_m)$, except for particular types of curves. This is an important difference between *synthesis* and *analysis* of random curves.

In principle, random Euclidean Fréchet curves can be represented by random Fréchet curves, as the next Theorem shows.

THEOREM 5.2.2. *Let $E=\mathbb{R}^p, p\geq 1, \Lambda_S$ be a closed subgroup of the Euclidean group on $\mathbb{R}^p$, and $m=0$ or $m\geq 2$. Then, there is a measurable mapping $\varphi: \mathcal{C}_m(E)/_{\Lambda_S}\to \mathcal{C}_m(E)$ such that $\langle\varphi(\eta)\rangle=\eta$ for all $\eta\in\mathcal{C}_m(E)/_{\Lambda_S}$.*

As an immediate consequence of Theorems 5.2.1 and 5.2.2 we have the following.

COROLLARY 5.2.1. *Let $E=\mathbb{R}^p$, $p\geq 1$, $m=0$ or $m\geq 2$, Λ_S be a closed subgroup of the euclidean group Λ on $\mathbb{R}^p$, $(X,d)=(C(E),d)$ and $(\mathcal{X},\rho)=(\mathcal{C}_m(E)/\Lambda_S,\rho_m)$. Then, there is a $\mathcal{B}_u(\mathcal{X})-\mathcal{B}_u(X)$ measurable mapping φ of $\mathcal{X}$ into X such that $\langle\varphi(\eta)\rangle=\eta$ for all $\eta\in\mathcal{X}$.*

In general, the representation of Euclidean Fréchet curves provided by Corollary 5.2.1 is unknown. If in addition the curves are regular, an explicit representation based on the natural parametrization can be constructed, as in the following Theorem. For notational simplicity, we set $m=2$ though similar results hold for any m.

THEOREM 5.2.3. *Let $E=\mathbb{R}^p, p\geq 1$, and Λ_S be the special Euclidean group Λ on $\mathbb{R}^p$. Define the mapping $\varphi:(\mathcal{R}_2^1(E)/\Lambda_S,\rho_2)\to(C(E),d),\langle[x]_2\rangle\rightsquigarrow\widetilde{x}$ such that $\widetilde{x}$ is a natural parametrization of x, $\widetilde{x}(0)=0$ and $\dot{\widetilde{x}}(0)=L_xe_1$, where e_1 is the unit vector $(1,...,0)\in\mathbb{R}^p$. Then, φ is a Borel isomorphism between $\mathcal{R}_2^1(E)/\Lambda_S$ and $\overline{N}_2^1=\{x\in N_2^1(E): x(0)=0, \dot{x}(0)=L_xe_1\}$.*

In case of Euclidean curves with Frenet-frames, an alternative representation can be constructed in terms of curvatures. Specifically, to each parametrized curve $x\in F(E)$ in $E=\mathbb{R}^p$ $(p\geq 2)$ with Frenet-frame we can associate its length $L>0$ and its curvatures $\kappa_1,...,\kappa_{p-1}\in C(\mathbb{R})$ (with $\kappa_1(t)>0,...,\kappa_{p-2}(t)>0$ for all $t\in[0,1]$) corresponding to its natural parametrization (Klingenberg, 1978, Sec. 1.3). Denote this mapping by $\gamma: x\rightsquigarrow(\kappa_1,...,\kappa_{p-1},L)$. Reciprocally, if $L>0$ and $(\kappa_1,...,\kappa_{p-1})\in K^p$, where

$$K^p=\left\{(\kappa_1,...,\kappa_{p-1})\in C^1\left(\mathbb{R}^{p-1}\right): \kappa_1(t)>0,...,\kappa_{p-2}(t)>0 \text{ for all } t\in[0,1]\right\},$$

we can assign to $(\kappa_1,...,\kappa_{p-1},L)$ a parametrized curve $x\in F(E)$ with constant speed L, length L and curvatures $\kappa_1,...,\kappa_{p-1}$. This assignment is unique modulo the special Euclidean group (Klingenberg, 1978, Sec. 1.3). Thus, we can set $x(0)=0$ and $\dot{x}(0)=Le_1$, where e_1 is the unit vector $(1,...,0)\in\mathbb{R}^p$, to obtain a mapping $x=\phi((\kappa_1,...,\kappa_{p-1},L)), K^p\times\mathbb{R}_+\to C(E)$. Next theorem shows that this mapping allows for an explicit representation of random Euclidean Fréchet curves with smooth curvatures, i.e. curves in $\mathcal{K}_2(E)/\Lambda_S=\{\langle[\phi(\kappa,L)]_2\rangle:(\kappa,L)\in K^p\times\mathbb{R}_+\}$.

THEOREM 5.2.4. *Let $E=\mathbb{R}^p, p\geq 2$ and Λ_S be the special Euclidean group of transformations on $\mathbb{R}^p$. On $C(\mathbb{R}^{p-1})\times\mathbb{R}_+$ consider the metric d induced by the norm $\|\kappa\|_0+L$ for $(\kappa,L)\in C(\mathbb{R}^{p-1})\times\mathbb{R}_+$. Then, the mapping $\varphi:(\mathcal{K}_2(E)/\Lambda_S,\rho_2)\to(C(\mathbb{R}^{p-1})\times\mathbb{R}_+,d),\langle[x]_2\rangle\rightsquigarrow\varphi(\langle[x]_2\rangle)=\gamma(x)=(\kappa_1,...,\kappa_{p-1},L)$, is a Borel isomorphism between $\mathcal{K}_2(E)/\Lambda_S$ and $K^p\times\mathbb{R}_+$.*

6. Centroids of random curves. Through this Section, $\mathcal{X}$ is a metric space, $(\Omega, \mathcal{F}, P)$ is a probability space, and $X_1, ..., X_n$ are *i.i.d.* random variables $(\Omega, \mathcal{F}) \to (\mathcal{X}, \mathcal{B}(\mathcal{X}))$ with probability distribution P. The empirical probability measure of the sample is denoted by $P_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}$.

Fréchet (1948) defined a centroid of a *r.v.* X with values in $(\mathcal{X}, d)$ as any $z \in \mathcal{X}$ that minimizes $Ed(X, z)$. Consistent estimation of centroids has been studied by Sverdrup-Thygeson (1981) under the constraint that $\mathcal{X}$ is compact. In view of applications to random curves, we generalize the concept of centroid in two aspects: we allow for non-compact sample spaces $\mathcal{X}$ and discrepancy functions different from the metric d of $\mathcal{X}$.

6.1. Non-restricted centroids.

DEFINITION 6.1.1. Let $\mathcal{Z}$ be a metric space and $f : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}_+$ be a functional such that $f(., z)$ is measurable for each $z \in \mathcal{Z}$. It is said that $z \in \mathcal{Z}$ is a (*non-restricted*) *centroid* of $(\mathcal{X}, \mathcal{B}(\mathcal{X}), P)$ w.r.t. $(\mathcal{Z}, f)$ if

$$V_{P,f}(z) = \inf \{V_{P,f}(y) : y \in \mathcal{Z}\},$$

where $V_{P,f}(y) = \int_{\mathcal{X}} f(x, y) P(dx)$. A *sample centroid* of $(\mathcal{X}, \mathcal{B}(\mathcal{X}), P)$ w.r.t. $(\mathcal{Z}, f)$ is any $z \in \mathcal{Z}$ such that

$$V_{P_n,f}(z) = \inf \{V_{P_n,f}(y) : y \in \mathcal{Z}\},$$

where

$$V_{P_n,f}(y) = \int_{\mathcal{X}} f(x, y) P_n(dx) = \frac{1}{n}\sum_{i=1}^{n} f(x_i, y).$$

REMARK 6.1.1. If $\mathcal{Z} = \mathcal{X}$, $f = d^q$ $(q > 0)$ and $\mathcal{X}$ is compact, Definition 6.1.1 reduced to that given in Sverdrup-Thygeson (1981).

REMARK 6.1.2. $f(x, z)$ can be thought as a measure of discrepancy between x and z, and $V_{P,f}(z)$ as the mean variation (as reflected by f) of the population around the point z. Typically, $\mathcal{Z} \subset \mathcal{X}$. The flexibility in defining $\mathcal{Z}$ allows one to impose desired properties to the centroids (see examples below).

In what follows, the notations and conditions in Definition 6.1.1 are assumed. Furthermore, the set of centroids (resp., sample centroids) will be denoted by Q (resp., $Q_n = Q_n(\omega), \omega \in \Omega$). $\widehat{Q}$ will denote the set of all cluster points of sequences $(z_n) \in \prod_{n=1}^{\infty} Q_n$. As in Sverdrup-Thygeson (1981) we will deal with arbitrary, non-measurable selections $z_n = z_n(\omega) \in Q_n(\omega)$. We will express probability statements about such selections in terms of the outer measure P^* corresponding to P.

The following conditions will be referred to in the theorems below:

C1) $\mathcal{X}$ is separable.

C2) $\{f(.,z) : z \in \mathcal{Z}\}$ is an equicontinuous family of functions at each point of $\mathcal{X}$.

C3) There is a continuous function $\phi : \mathcal{X} \to \mathbb{R}_+$ such that $\int \phi(x)P(dx) < \infty$

and, for all $x \in \mathcal{X}$ and $z \in \mathcal{Z}, |f(x.,z)| \leq \phi(x)$.

THEOREM 6.1.1.

a) Suppose that $\mathcal{Z}$ is compact and $f(x,.)$ is lower semicontinuous for each $x \in \mathcal{X}$. Then, $Q \neq \emptyset$ and $Q_n \neq \emptyset$.

b) In addition, assume that conditions C1)-C3) hold. Then,

i) For any sequence $z_n \in Q_n$ and any $z^* \in Q, V_{P_n,f}(z_n) \underset{n}{\to} V_{P,f}(z^*)$ $P^*-a.s.$

ii) $\widehat{Q} \neq \emptyset$ and $\widehat{Q} \subset Q$ $P^*-a.s.$

REMARK 6.1.3. Let $\Phi : \mathcal{X} \to \mathbb{R}_+$ be a lower semicontinuous functional and $0 < \gamma < \infty$. Consider the following alternative definition of a centroid z of $(\mathcal{X}, \mathcal{B}(\mathcal{X}), P)$, now w.r.t. the functional Φ:

$$D_{P,f}(z) = \inf \{D_{P,f}(y) : y \in \mathcal{X}\},$$

where

$$D_{P,g}(y) = \int f(x,y) P(dx) + \gamma\Phi(y).$$

It can be shown (in a way completely similar to the proof of part *(a)* of Theorem 6.1.1) that the set Q_γ of centroids so defined is non-null. Let $z \in Q_\gamma$. Define $\beta = \Phi(z)$ and $\mathcal{Z} = \{z \in \mathcal{X} : \Phi(z) \leq \beta\}$. Since Φ is a lower semicontinuous, $\mathcal{Z}$ is compact. Denote by Q_β the set of centroids of $(\mathcal{X}, \mathcal{B}(\mathcal{X}), P)$ w.r.t. $(\mathcal{Z}, f)$ in the sense of Definition 6.1.1. Then, it can be shown that $Q_\gamma = Q_\beta$. Thus, the alternative definition of centroids in terms of Φ is a particular case of Definition 6.1.1. Notwithstanding, the former may be preferred for numerical computations because it has the form of an unconstrained optimization problem. Notice that Φ plays here the role of a regularizing functional to solve the problem of minimizing $\int f(x,y) P(dx)$ over $\mathcal{X}$ (Tijonov *et al.*, 1995).

In order to apply Theorem 6.1.1 it is necessary to specify the compact set $\mathcal{Z}$ and the functional f. For spaces of curves introduced in previous sections, some examples of $\mathcal{Z}$ are the following.

EXAMPLE 6.1.1. (Centroids within a compact parametric family of curves). Let $m = 0$ or $m \geq 2, \Theta \subset \mathbb{R}^q$ be a compact q-dimensional set, and $A =$

$\{\mu(.;\theta) \in C(E) : \theta \in \Theta\}$ be a family of continuous functions such that the mapping $\pi : \Theta \to (C(E), d), \theta \to \mu(.;\theta)$, is continuous. From $d_m \leq d$ it follows that the mapping $[.]_m : (C(E), d) \to (\mathcal{C}_m(E), d_m)$ is continuous. Define

$$\mathcal{Z} = [\pi(\Theta)]_m = \{[\mu(.;\theta)]_m : \theta \in \Theta\}.$$

Since it is the image of a compact set by a continuous mapping, $\mathcal{Z}$ is compact in $\mathcal{X} = \mathcal{C}_m(E)$. In the case of $E = \mathbb{R}^p$, a sufficient condition for the continuity of π is that $\frac{\partial \mu}{\partial \theta}(t;\theta)$ be bounded as a function of $(t;\theta) \in [0,1] \times \Theta$.

EXAMPLE 6.1.2. (Centroids with bounded length). Let $m = 0$ or $m \geq 2$. If the metric space E is compact and $\alpha \geq 0$ then

$$\mathcal{Z} = \{c \in \mathcal{C}_m(E) : L_c \leq \alpha\}$$

is a compact set in $\mathcal{X} = \mathcal{C}_m(E)$ (see Lemma 7.7 in Section 7). Notice that compactness of E is a realistic assumption in many applications, e.g., in computational anatomy the curves usually are contained in a compact subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ determined by the spatial range of some sensor equipment.

The following illustrates the specification of functionals f that satisfy the conditions of Theorem 6.1.1 on spaces of curves.

EXAMPLE 6.1.3. Let $(\mathcal{X}, d) = (\mathcal{C}_m(E), d_m), \mathcal{Z} \subset \mathcal{X}, q > 0$, and ρ be any seudometric on $\mathcal{C}_m(E)$ such that $\rho \leq d_m$. For example, $\rho(c_1, c_2) = d_m(c_1, c_2)$ or $\rho(c_1, c_2) = h(x_1([0,1]), x_2([0,1]))$, where $c_1, c_2 \in \mathcal{C}_m(E), x_1 \in c_1, x_2 \in c_2$. Define $f = \rho^q$. Then, f satisfies the conditions of Theorem 6.1.1 in either of the following two situations: 1)E is bounded, or 2)$q \in \mathbb{N}$ and $\int \rho^q(c_1, c_2) P(dc_1) < \infty$ for some $c_2 \in \mathcal{C}_m(E)$.

6.2. Restricted centroids. The main idea in defining restricted centroids is to impose the constraint that centroids belong to the support of the probability measure P assumed on the sample space $\mathcal{X}$. To guarantee the existence of restricted centroids, compactness of $\mathcal{X}$ is usually assumed (see, e.g., Sverdrup-Thygeson, 1981). However, in general this condition is not satisfied in spaces of curves. For example, $\mathcal{C}_m(E)$ is not compact even if E is compact. To overcome this, we will introduce a mapping r from the original sample space $\mathcal{X}$ to other space $\mathcal{Y}$ in such a way that the induced measure P^r be of compact support, and we define restricted centroids w.r.t. P^r. In practice, $\mathcal{Y}$ will typically be close to $\mathcal{X}$ but more regular in some sense (see examples below).

Specifically, let $\mathcal{Y}$ be a subspace of $\mathcal{X}$, $r : \mathcal{X} \to \mathcal{Y}$ be a measurable mapping, $\mathcal{X}^r = \overline{r(\mathcal{X})}$ be the closure of $r(\mathcal{X})$, and $f : \mathcal{X} \times \mathcal{S}^r \to \mathbb{R}_+$ be a non-negative functional. Here $\mathcal{S}^r$ is the support of the probability measure P^r induced by r on $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$, i.e., $P^r(\cdot) = P(r^{-1}(\cdot))$. Suppose that $f(., y)$ is a measurable function for each $y \in \mathcal{S}^r$.

DEFINITION 6.2.1. A *restricted centroid* of $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ with respect to (r, f) is any centroid of $(\mathcal{X}^r, \mathcal{B}(\mathcal{X}^r), P^r)$ w.r.t. $(\mathcal{S}^r, f)$ (in the sense of Definition 6.1.1), i.e., any $z \in \mathcal{S}^r$ such that

$$V_{P,f}(z) = \inf \{V_{P^r,f}(y) : y \in \mathcal{S}^r\},$$

where

$$V_{P^r,f}(y) = \int_{\mathcal{X}^r} f(x,y) P^r(dx).$$

A *sample restricted centroid* is any $z \in \mathcal{S}_n^r$ such that

$$V_{P_n^r,f}(z) = \inf \left\{V_{P_n^r,f}(y) : y \in \mathcal{S}_n^r\right\},$$

where $S_n^r = \{r(X_1), ..., r(X_n)\}$ and

$$V_{P_n^r,f}(y) = \int_{\mathcal{X}^r} f(x,y) P_n^r(dx) = \frac{1}{n} \sum_{i=1}^{n} f(r(x_i), y).$$

In what follows, notations and conditions stated in Definition 6.2.1 are assumed. The sets of restricted centroids and sample restricted centroids will be denoted by R and R_n, respectively. $\widehat{R}$ will be the set of all cluster points of sequences $(z_n) \in \prod_{n=1}^{\infty} R_n$.

THEOREM 6.2.1. *Suppose that $\mathcal{X}^r$ is compact and $f(y,.)$ is lower semicontinuous for each $y \in \mathcal{X}^r$. Then,*

a) $R \neq \emptyset$.

b) In addition, assume that conditions C1)-C3) of Section 6.1 hold for $\mathcal{X}$, f and $\mathcal{Z} = \mathcal{X}^r$. Then,

i) For any sequence $z_n \in R_n$ and any $z^* \in R$, $V_{P_n,g}(z_n) \underset{n}{\to} V_{P,g}(z^*) \quad P^* - a.s.$

ii) $\widehat{R} \neq \emptyset$ and $\widehat{R} \subset R \quad P^* - a.s.$

COROLLARY 6.2.1. *All the theses of Theorem 6.2.1 hold if $\mathcal{X}$ is separable, $\mathcal{X}^r$ is compact and f is continuous.*

REMARK 6.2.1. We are interested here in applications of Theorems 6.1.1 and 6.2.1 to random curves. But notice that they may be also applied to *r.v.* in other metric sample spaces, e.g., to random compact sets in $(H(E), h)$ that arise from probability modelling in mathematical morphology (Matheron, 1975; Serra, 1982; Cressie, 1991).

EXAMPLE 6.2.1. Let $\mathcal{X} = \mathcal{C}_2^R(E), E$ be a bounded set in $\mathbb{R}^p, p \geq 1$, and $\mathcal{Y} = \mathcal{C}_2^R(E)$. For $c \in \mathcal{C}_2^R(E)$, define $r : \mathcal{X} \to \mathcal{Y}$ by $r(c) = [R(\widetilde{c})]_2$, where

$$R(\widetilde{c})(t) = \int_{\mathbb{R}} \widetilde{c}(s) J_\varepsilon(t-s) ds,$$

$J_\varepsilon \in C^k(\mathbb{R}, \mathbb{R})$ with support $[-\varepsilon, \varepsilon]$, $0 < \varepsilon < 1$ is a small constant and $k \geq 1$ (for examples of such functions J_ε, see, e.g., Adams, 1978, p. 29). Here $\widetilde{c}(t), t \in [0,1]$, is the natural parametrization of c and, in the integral that defines R, we set $\widetilde{c}(t) = 0$ for $t \notin [0,1]$. Also, define $f(c_1, c_2) = d_2^q(c_1, c_2)$ or $f(c_1, c_2) = h(x_1([0,1]), x_2([0,1]))$ for $c_1 = [x_1]_2, c_2 = [x_2]_2 \in \mathcal{C}_2^R(E)$, where $q > 0$. Then, the hypotheses of Corollary 6.2.1 are satisfied.

Indeed, f is a continuous mapping. By Theorem 3.2.3, $\mathcal{X}$ is separable. Furthermore, r is the composition of the mappings:

$$\mathcal{C}_2^R(E) \overset{\varphi_1}{\to} C(E) \overset{\varphi_2}{\to} C(E) \overset{\varphi_3}{\to} \mathcal{C}_2^R(E) : c \rightsquigarrow \widetilde{c} \rightsquigarrow R(\widetilde{c}) \rightsquigarrow [R(\widetilde{c})]_2 .$$

It is straightforward to verify that φ_2 and φ_3 are continuous, and Theorem 3.3.1 states that φ_1 is measurable. Therefore, r is measurable. Moreover, taking into consideration that E is bounded and J_ε has bounded derivatives up to order k, it can be verified that there is a closed ball $B \subset \left(C^k(E), \|\cdot\|_k\right)$ such that $\varphi_2(C(E)) \subset B$. Since the imbedding $\left(C^k(E), \|\cdot\|_k\right) \to (C(E), \|\cdot\|_0)$ is compact (Adams, 1978, Theorem 1.31), B has compact closure $\overline{B}$ in $C(E)$. From this and continuity of φ_3 it follows that $\varphi_3\left(\overline{B}\right)$ is compact. Thus, $\mathcal{X}^r = \overline{r(\mathcal{C}_2^R(E))} \subset \varphi_3\left(\overline{B}\right)$ is a closed subset of a compact set, hence $\mathcal{X}^r$ is compact. This completes the verification of the hypotheses of Corollary 6.2.1. Notice that the parameter $\varepsilon > 0$ can be selected arbitrarily small, so $r(c)$ can be as close to c as desired.

EXAMPLE 6.2.2. Let $E \subset \mathbb{R}^p, p \geq 1, \mathcal{X} = \mathcal{C}_2^R(E), \mathcal{Y} = \mathcal{C}_2^R(E)$, and $k \geq 1$. Denote by $H_2^k([0,1])$ the Sobolev space $H_2^k([0,1])$ =the completion of $C^k(\mathbb{R})$ w.r.t. the norm

$$\|g\|_{k,2} = \left\{ \sum_{0 \leq j \leq k} \int_0^1 \left| g^{(j)}(t) \right|^2 dt \right\}^{1/2},$$

(see, e.g., Adams, 1978, ch. III) and B be a closed ball in $H_2^k([0,1])$. For $c \in \mathcal{C}_2^R(E)$, define $r(c) = [\Pi\widetilde{c}]_2$, where $\widetilde{c}$ is the natural parametrization of c and $\Pi\widetilde{c}$ is the projection of $\widetilde{c}$ on B w.r.t. the norm $\|\cdot\|_{k,2}$. This projection is well-defined because B is a closed set in a Hilbert space $\left(H_2^k([0,1]), \|\cdot\|_{k,2}\right)$ (Adams, 1978, Th. 3.2). Furthermore, define $f(c_1, c_2) = d_2^q(c_1, c_2)$ or $f(c_1, c_2) = h(c_1([0,1]), c_2([0,1]))$ for $c_1, c_2 \in \mathcal{C}_2^R(E)$, where $q > 0$. Then, all the hypotheses of Corollary 6.2.1 are satisfied.

Indeed, f is a continuous mapping and, by Theorem 3.2.3, $\mathcal{X}$ is separable. Furthermore, r is the composition of the mappings:

$$\mathcal{C}_2^R(E) \overset{\varphi_1}{\to} C(E) \overset{\varphi_2}{\to} C(E) \overset{\varphi_3}{\to} \mathcal{C}_2^R(E) : c \rightsquigarrow \widetilde{c} \rightsquigarrow \Pi(\widetilde{c}) \rightsquigarrow [\Pi(\widetilde{c})]_2 .$$

Here φ_2 and φ_3 are continuous mappings, and, by Theorem 3.3.1, φ_1 is measurable. Therefore, r is measurable. Since the imbedding $\left(H_2^k([0,1]), \|\cdot\|_{k,2}\right) \to (C(E), \|\cdot\|_0)$ is compact (Adams, 1978, Th. 6.2, part III), B has a compact closure $\overline{B}$ in $C(E)$. This and continuity of φ_3 imply that $\varphi_3\left(\overline{B}\right)$ is compact. Therefore, $\mathcal{X}^r = \overline{r(\mathcal{C}_2^R(E))} \subset \varphi_3\left(\overline{B}\right)$ is a closed subset of a compact set, hence $\mathcal{X}^r$ is compact. This completes the verification of the hypotheses of Lemma 6.2.1.

7. Proofs.

LEMMA 7.1 *Let (X,d) be a seudometric space and G be a group of transformations of X into X such that $d(gx,gy) = d(x,y)$ for all $x,y \in X$, $g \in G$. Then,*

$$\rho(x,y) = \inf\{d(x,gy) : g \in G\}$$

is a seudometric on X.

PROOF. Let $x,y,z \in X$. Then,
i) $\rho(x,x) \le d(x,x) = 0$, and therefore $\rho(x,x) = 0$.
ii) $\rho(x,y) = \inf\{d(g^{-1}x,y) : g \in G\} = \inf\{d(gx,y) : g \in G\} = \rho(y,x)$.
iii) For all $g_1, g_2 \in G$,

$$d(x,g_1y) \le d(x,g_2z) + d(g_2z,g_1y) = d(x,g_2z) + d\left(z, g_2^{-1} \circ g_1y\right).$$

Taking here infimum w.r.t. g_1, g_2 one obtains $\rho(x,y) \le \rho(x,z) + \rho(z,y)$. □

PROOF OF LEMMA 3.1.1. Write $d_0 = \inf\limits_{w \in V} d(x, y \circ w)$. Then, there is a sequence $(u_n) \subset U$ such that

$$d(x, y \circ u_n) \underset{n}{\to} d_0.$$

Since (u_n) is a sequence of nondecreasing functions, second Helly's theorem (Kolmogorov and Fomin, 1978) ensures the existence of a subsequence (u_{n_k}) of (u_n) such that, for each $t \in [0,1]$, $(u_{n_k}(t))$ converges. Write $u(t) = \lim_k u_{n_k}(t)$.

Since u is nondecreasing, the set of its discontinuity points is at most countable, say $D = \{t_n\}$, and u has finite lateral limits at such points. Define

$$g(t) = \left(1 - \sum_{n=1}^{N} 3^{-n}\right) t + \sum_{t_n \le t} 3^{-n}$$

and

$$\tau(s) = \inf\{t : g(t) \ge s\},$$

where $N = M$ if D is a set of finite size M, and $N = \infty$ in other case.

Since g is strictly increasing, τ is so too and $\tau(g(t)) = t$. Thus, $\tau \in V$.

Define

$$v(s) = \begin{cases} u(\tau(s)) & \text{if } s \notin \bigcup_n [g(t_n-), g(t_n)] \\ 3^n (u_n^+ - u_n^-)(s - g(t_n-)) + u_n^- & \text{if } s \in \bigcup_n [g(t_n-), g(t_n)] \end{cases}$$

where $g(0-) = 0$, $g(t-) = \lim_{s \to t-} g(t)$ if $t \neq 0$,

$$u_n^- = \begin{cases} \lim_{t \to t_n-} u(t) & \text{si } t_n \neq 0 \\ 0 & \text{si } t_n = 0 \end{cases} \text{ and } u_n^+ = \begin{cases} \lim_{t \to t_n+} u(t) & \text{si } t_n \neq 1 \\ 1 & \text{si } t_n = 1 \end{cases}.$$

Since g is strictly increasing, $g(s) \in [g(t_n-), g(t_n)]$ implies $s = t_n$. Thus, if $s \in [0,1] \setminus D$ then $g(s) \notin \bigcup_n [g(t_n-), g(t_n)]$ and thence $v(g(s)) = u(\tau(g(s))) = u(s)$. From this and the facts that u is nondecreasing, u is continuous on $[0,1] \setminus D$ and $v([g(t_n-), g(t_n)]) = [u_n^-, u_n^+]$ it follows that v is surjective. In addition, it can be verified that v is nondecreasing, $v(0) = 0$ and $v(1) = 1$. Therefore, $v \in U$.

Let us verify that $d(x \circ \tau(s), y \circ v(s)) \leq d_0$ for all $s \in [0,1]$, which would complete the proof.

First, consider the case in which $s \in [g(t_n-), g(t_n)]$ for some n.

Then, $\tau(s) = t_n$ and $v(s) \in [u_n^-, u_n^+]$. Thus, it suffices to prove that

$$d(x(t_n), y(r)) \leq d_0$$

for all $r \in [u_n^-, u_n^+]$. Let $r \in]u_n^-, u_n^+[$. Since u_{n_k} are surjective functions, there is a numeric sequence (l_k) in $[0,1]$ such that $u_{n_k}(l_k) = r$ for all k. Since $[0,1]$ is compact, (l_k) has a convergent subsequence, which will be denoted by the same symbol. Write $l = \lim_k l_k$. If $l > t_n$ then there is some t such that $l > t > t_n$, and hence

$$u_n^+ \leq u(t) = \lim u_{n_k}(t) \leq \lim u_{n_k}(l_k) = r,$$

which contradicts the assumption $r \in]u_n^-, u_n^+[$. Thus, $l \leq t_n$. Similarly, it can be proved that $l \geq t_n$. Therefore, $l = t_n$ and

$$d(x(t_n), y(r)) = \lim_k d(x(l_k), y \circ u_{n_k}(l_k)) \leq \lim_k d(x, y \circ u_{n_k}) = d_0.$$

Now consider the case $s \notin \bigcup_n [g^-(t_n), g(t_n)]$.

In this case there is t such that $g(t) = s$. Thus,

$$\begin{aligned} d(x \circ \tau(s), y \circ v(s)) &= d(x \circ \tau \circ g(t), y \circ v \circ g(t)) \\ &= d(x(t), y \circ u(t)) = \lim_k d(x(t), y \circ u_{n_k}(t)) = d_0. \end{aligned}$$

This completes the proof. □

PROOF OF LEMMA 3.1.2.

a)Let D be a countable dense set in X. For all $x, y \in X, \rho([x],[y]) \leq d(x,y)$. Thus, $[D] = \{[x] : x \in D\}$ is dense in $\mathcal{X}$, which implies that $\mathcal{X}$ is separable.

b)Let $([x_n])$ be a Cauchy sequence in $\mathcal{X}$. Then, there is some subsequence $([x_n])$ such that $\rho\left([x_{n_k}], \left[x_{n_{k+1}}\right]\right) < 2^{-k}$. By definition of ρ, there is a sequence (y_k) in X such that $y_k \in [x_{n_k}]$ and $d(y_k, y_{k+1}) \leq 2^{-k}$. Thus, (y_k) is a Cauchy sequence in X. Completeness of X implies that (y_k) converges to some $y \in X$. From $\rho([x_{n_k}],[y]) = \rho([y_k],[y]) \leq d(y_k, y) \underset{k}{\to} 0$ it follows that $[x_{n_k}]$ converges to $[y]$. This and the fact that $([x_n])$ is a Cauchy sequence imply that $([x_n])$ is convergent, which completes the proof. □

LEMMA 7.2.

a) If E is a separable metric space then $C(E)$ is separable.

b) If E is a complete metric space then $C(E)$ is complete.

PROOF.

a) If A is a compact metric space and E is a separable metric space then the space $C(A,E)$ of continuous mappings of A into E is separable w.r.t. the supremum metric (Kuratowski, 1968, Vol. I, paragraph 22, p. 120). Thus, $C(E) = C([0,1],E)$ is separable.

b) If A is compact and E is a complete metric space then $C(A,E)$ is complete w.r.t. the supremum metric (Kuratowski, 1968, Vol. II, paragraph 44, p. 90). Thus, $C(E)$ is complete. □

PROOF OF THEOREM 3.1.1.

a) By part (a) of Lemma 7.2, $C(E)$ is separable. Thus, separability of $(\mathcal{C}_2(E), d_2)$ follows from Lemma 3.1.2, part (a).

b) By part (b) of Lemma 7.2, $C(E)$ is complete. Completeness of $(\mathcal{C}_2(E), d_2)$ then follows from Lemma 3.1.2, part (b). □

PROOF OF THEOREM 3.2.1. For $x, y \in C(E)$, it is obvious that $x \backsim_m y$ if and only if $x_i \backsim_2 y_i$ for $i = 1, ..., m-1$, where $x_i(t) = x\left(\frac{i-1}{m-1} + \frac{1}{m-1}t\right)$ and $y_i(t) = y\left(\frac{i-1}{m-1} + \frac{1}{m-1}t\right)$ for all $t \in [0,1]$. Thus, $\backsim_m$ is an equivalence relation.

For $u \in U_m$, write

$$u_i(t) = \left(u\left(\frac{i-1}{m-1} + \frac{1}{m-1}t\right) - \frac{i-1}{m-1}\right)(m-1), i = 1, ..., m-1, t \in [0,1].$$

Then, for $[x]_m, [y]_m \in C(E)/\backsim_m$,

$$\begin{aligned} r_m([x]_m, [y]_m) &= \inf\{d(x, y \circ u) : u \in U_m\} \\ &= \inf\left\{\max_{1 \leq i \leq m-1} d(x_i, y_i \circ u_i) : u_1, ..., u_{m-1} \in U\right\} \\ &\geq d(x_i, y_i \circ u_i) \end{aligned}$$

for all $u_i \in U, 1 \leq i \leq m-1$. Thus,

$$\begin{aligned} r_m\left([x]_m, [y]_m\right) &\geq \max_{1\leq i\leq m-1} \inf\left\{d\left(x_i, y_i \circ u_i\right) : u_i \in U\right\} = \max_{1\leq i\leq m-1} d_2\left([x_i]_2, [y_i]_2\right) \\ &= d_m\left(g\left(x\right), g\left(y\right)\right) \end{aligned}$$

By Lemma 3.1.1 there are $\overline{u}_1, \overline{v}_1, ..., \overline{u}_{m-1}, \overline{v}_{m-1} \in U$ such that

$$d_2\left([x_i]_2, [y_i]_2\right) = d\left(x_i \circ \overline{v}_i, y_i \circ \overline{u}_i\right) \qquad i = 1, ..., m-1.$$

Therefore

$$\begin{aligned} d_m\left(f\left(x\right), f\left(y\right)\right) &= \max_{1\leq i\leq m-1} d\left(x_i \circ \overline{v}_i, y_i \circ \overline{u}_i\right) \\ &\geq \inf\left\{\max_{1\leq i\leq m-1} d\left(x_i \circ v_i, y_i \circ u_i\right) : u_1, v_1, ..., u_{m-1}, v_{m-1} \in U\right\} \\ &= \inf\left\{\max_{1\leq i\leq m-1} d\left(x_i =, y_i \circ u_i\right) : u_1, v_1, ..., u_{m-1}, v_{m-1} \in U\right\} \\ &= r_m\left([x]_m, [y]_m\right). \end{aligned}$$

Thus, $d_m\left(g\left(x\right), g\left(y\right)\right) = r_m\left([x]_m, [y]_m\right)$. This and the fact that g is bijective complete the proof. □

PROOF OF THEOREM 3.3.1. Write $V_0 = \{w_{u,s} \in U_0 : u \in V\}$ and

$$G_0 = \left\{g_w : w \in V_0, g_w\left(x\right) = x \circ w \text{ for all } x \in C_c(E)\right\}.$$

It can be verified that, for all $w_{u,r}$, $w_{v,s} \in V_0$,

$$w_{u,r} \circ w_{v,s} = w_{\tilde{u}\circ v, \tilde{t}},$$

where $\tilde{t} = \left(u\left(s\right) + r\right)$ and

$$\tilde{u}\left(t\right) = \begin{cases} u\left(t+s\right) - u\left(s\right) & if\ t \in [0, 1-s) \\ u\left(t+s-1\right) + 1 - u\left(s\right) & if\ t \in [1-s, 1] \end{cases}$$

It is straightforward to show that $\tilde{u} \in V$ if $u \in V$. Thus, the set of transformations G_0 is closed w.r.t. composition. Furthermore, each $g_{w_{v,s}} \in G_0$ has an inverse $g^{-1}_{w_{u,r}} = g_{w_{v,s}}$, where $s = u^{-1}\left(1-r\right)$, $v = \tilde{u}^{-1}$, and $\tilde{u}$ is defined as above. Therefore, $(G_0, \circ)$ is a group.

For $x, y \in C_c(E)$, define

$$\rho\left(x, y\right) = \inf\left\{d(x, g(y)) : g \in G_0\right\}.$$

Since the metric d is invariant w.r.t. the group of transformations G_0, Lemma 7.1 implies that ρ is a seudometric on $C_c(E)$.

Furthermore, since V_0 is dense in U_0,

$$d_0\left([x]_0,[y]_0\right)=\rho\left(x,y\right).$$

Thus, in order to prove that $\sim_0$ is an equivalence relation and d_0 is a metric on $\mathcal{C}_0$ it suffices to prove that $[x]_0=[y]_0$ if and only if $\rho\left(x,y\right)=0$.

Since V_0 is dense in U_0 and d is invariant by transformations in G_0,

$$\rho\left(x,y\right)=\inf\left\{d(x,y\circ w):w\in V_0\right\}=\inf\left\{d(x\circ w_1,y\circ w_2):w_1,w_2\in U_0\right\}.$$

Therefore, $[x]_0=[y]_0$ implies $\rho\left(x,y\right)=0$. To prove the opposite implication, let $x,y\in C_c(E)$ and write $\beta=\rho\left(x,y\right)$. From the definition of infimum and compactness of $[0,1]$ it follows that there is a convergent sequence in $[0,1]$, say $t_n\to t$, such that $\lim_n\inf_{u\in U}d\left(x,g_{u,t_n}(y)\right)=\beta$. Since y is uniformly continuous, $d\left(x,g_{u,s}(y)\right)$ is uniformly continuous as a function of (u,s) on $U\times[0,1]$. Therefore, $\inf_{u\in U}d\left(x,g_{u,s}(y)\right)$ is a continuous function on $[0,1]$, and thence

$$\beta=\inf_{u\in U}d\left(x,g_{u,t}(y)\right)=\inf_{u\in U}d\left(x,y\left(\left(u+t\right)\bmod\left(1\right)\right)\right)=d_2\left([x]_2,[y_t]_2\right),$$

where $y_t\in C(E)$ is defined by $y_t(s)=y\left(\left(s+t\right)\bmod\left(1\right)\right)$ for $s\in[0,1]$. Suppose $\beta=0$. Then, the last equation and Lemma 3.1.1 imply that there are $u,v\in U$ such that $x\circ u=y_t\circ v$. Thus, $x\circ w_{u,0}=y\circ w_{v,t}$, and thence $[x]_0=[y]_0$. This completes the proof. □

PROOF OF THEOREM 3.3.2.

a)Case $m\geq2$. Separability of $(\mathcal{C}_m(E),d_m)$ follows from the facts that $(\mathcal{C}_2(E),d_2)$ is separable (Theorem 3.1.1), $\mathcal{C}_m(E)\subset\mathcal{C}_2(E))^{m-1}$ and the definition of d_m.

Case $m=0$. By Lemma 7.2, if E is separable then $C(E)$ is separable. Hence, the subset $C_c(E)$ of $C(E)$ is separable. This and Lemma 3.1.2 imply that $(\mathcal{C}_0(E),d_0)$ is separable.

b)Case $m\geq2$. By Theorem 3.1.1, $(\mathcal{C}_2(E),d_2)$ is complete. Then,

$$\left((\mathcal{C}_2(E))^{m-1},\max\left(d_2,...,d_2\right)\right)=\left((\mathcal{C}_2(E))^{m-1},d_m\right)$$

is complete. This and the fact that $(\mathcal{C}_m(E),d_m)$ is a closed subset of $((\mathcal{C}_2(E))^{m-1},d_m)$ imply that $(\mathcal{C}_m(E),d_m)$ is complete.

Case $m=0$. If E is complete then $C(E)$ is complete (by Lemma 7.2), hence the closed subset $C_c(E)$ of $C(E)$ is complete too. Completeness of $(\mathcal{C}_0(E),d_0)$ then follows from Lemma 3.1.2. □

PROOF OF LEMMA 3.4.1. First consider $m=2$.

If $L_{[x]_2}=0$ then $L\left(x,0,t\right)=0$ for all $t\in[0,1]$, and then x is a constant mapping, $\widetilde{x}=x$ is the natural parametrization of x and $[x]_2=\{\widetilde{x}\circ u:u\in U\}=\{\widetilde{x}\}$.

Assume $L_{[x]_2} > 0$. It is known that $L(x,0,\cdot)$ is a continuous function (Kolmogorov and Fomin, 1978). Thus, the set

$$K_t = \left\{ s \in \left[0, L_{[x]_2}\right] : L(x,0,s) = t\ L_{[x]} \right\}$$

is not empty for each $t \in [0,1]$. By Zorn's lemma (see, e.g., Dudley, 1989) there is a mapping $\tau : [0,1] \to \left[0, L_{[x]_2}\right]$ such that $\tau(t) \in K_t$ for each t. Define $\widetilde{x} = x \circ \tau$. From

$$\begin{aligned} d(\widetilde{x}(t), \widetilde{x}(s)) &= d(x \circ \tau(t), x \circ \tau(s)) \leq L(x, \tau(t), \tau(s)) = \\ &= |L(x,0,\tau(t)) - L(x,0,\tau(s))| = L_x |t-s| \end{aligned}$$

it follows that $\widetilde{x} \in C(E)$.

Since

$$0 \leq d(x(t), x(s)) \leq L(x,s,t) = |L(x,0,t) - L(x,0,s)|,$$

x is constant on K_t. This and $t \in K_{L(x,0,t)/L_{[x]_2}}$ imply

$$x(t) = x \circ \tau\left(\frac{L(x,0,t)}{L_{[x]_2}}\right) = \widetilde{x} \circ u(t),$$

where $u(t) = \tau\left(L(x,0,t)/L_{[x]_2}\right)$. Thus, $[x]_2 = [\widetilde{x}]_2$, which together with

$$L(\widetilde{x},0,t) = L(x,0,\tau(t)) = tL_{[\widetilde{x}]_2}$$

and $\widetilde{x} \in C(E)$ prove that $\widetilde{x}$ is a natural parametrization of x.

Suppose μ and γ are natural representations of $[x]_2$. Then, there are $u, v \in U$ such that $\gamma \circ v = \mu \circ u$. Thus,

$$\begin{aligned} u(t) L_{[x]} &= L(\mu, 0, u(t)) = L(\mu \circ u, 0, t) = L(\gamma \circ v, 0, t) = L(\gamma, 0, v(t)) \\ &= v(t) L_{[x]} \end{aligned}$$

for all $t \in [0,1]$. Therefore, $u(t) = v(t)$ for all t, and hence $\gamma = \mu$. This proves the uniqueness of the natural representation.

The equation $x(t) = \widetilde{x} \circ u(t)$ obtained above implies that $[x]_2 = \{\widetilde{x} \circ u : u \in U\}$. This completes the proof of Lemma 3.4.1 for $m = 2$.

For $m > 2$, this lemma follows from the fact that if $\widetilde{x}_i$ are natural representations of curves $[x_i]_2 \in \mathcal{C}_2(E)$ $(i = 1, ..., m-1)$ with $x_i(1) = x_{i+1}(0) (i = 1, ..., m-2)$ then the natural representation $\widetilde{x}$ of $([x_1]_2, ..., [x_{m-1}]_2) \in \mathcal{C}_m(E)$ is $\widetilde{x}(t) = \widetilde{x}_i(\left(t - \frac{i-1}{m-1}\right)(m-1)), \frac{i-1}{m-1} \leq t \leq \frac{i}{m-1}, i = 1, ..., m-1$. □

The following result (Parthasarathy, 1967, p. 22) will be frequently used in the sequel, in particular in the proof of Theorem 3.4.1.

LEMMA 7.3. (Kuratowski). *Let X_1 be a Polish metric space and X_2 be a separable metric space. If $E_1 \subset X_1$ is a Borel set and $\varphi : E_1 \to X_2$ is a measurable and injective mapping then $\varphi(E_1)$ is a Borel set and φ is a Borel isomorphism of E_1 onto $\varphi(E_1)$.*

The following is a consequence of Lemma 7.3.

LEMMA 7.4. *Let (X, d) be a Polish metric space, $\mathcal{X}$ be a quotient space on X with equivalence classes denoted by $[x]$ for $x \in X$, and d be a distance in X such that, for all $[x], [y] \in \mathcal{X}$,*

$$d([x],[y]) = \inf\{d(x', y') : x' \in [x], y' \in [y]\}.$$

If $\mathcal{A}$ is a subset of $\mathcal{X}$, $\psi : \mathcal{A} \to X$ is a mapping such that $\psi([x]) \in [x]$ for all $[x] \in \mathcal{X}$ and $\psi(\mathcal{A})$ is a Borel set then $\mathcal{A}$ is a Borel set and ψ is a Borel isomorphism of $\mathcal{A}$ onto $\psi(\mathcal{A})$.

PROOF. $\psi([x]) \in [x]$ for all $[x] \in \mathcal{X}$ implies that ψ is injective. Define $\varphi = \psi^{-1} : \psi(\mathcal{A}) \to \mathcal{A}$, $x \rightsquigarrow [x]$.

Since

$$d(\varphi(x), \varphi(y)) = d([x],[y]) \leq d(x, y),$$

φ is continuous, and thence measurable. Furthermore, φ is injective. Also, $\mathcal{X}$ is separable, as a consequence of the separability of X.

Then, by setting $X_1 = X, E_1 = \psi(\mathcal{A})$ and $X_2 = \mathcal{X}$ in Lemma 7.3, it follows that $\varphi(\psi(\mathcal{A})) = \mathcal{A}$ is a Borel set and φ (or equivalently, ψ) is a Borel isomorphism between $\psi(\mathcal{A})$ and $\mathcal{A}$. □

LEMMA 7.5. *For any $m \geq 2, N_m(E)$ is a Borel set of $C(E)$.*

PROOF. For each $t \in [0,1]$, the functional $C(E) \to \overline{\mathbb{R}}, x \rightsquigarrow L(x, 0, t)$, is lower semicontinuous (Kolmogorov and Fomin, 1978).

Therefore, $L(.) = L(., 0, 1) : C(E) \to \overline{\mathbb{R}}$ is measurable. Thus, $C^R(E) = \overset{\infty}{\underset{n=1}{\cup}} L^{-1}([0, n])$ is a Borel set in $C(E)$.

Define the mapping

$$g : C^R(E) \to C(E), g(x)(t) = L(x, 0, t) - tL(x, 0, 1)$$

for $x \in C^R(E), t \in [0,1]$.

For each t, $g(\cdot)(t)$ is lower semicontinuous, thence measurable. This and the fact that the topology of $C(E)$ is generated by the cylindric sets

$$\{y \in C(E) : y(t_i) \in B_i, i = 1, ..., n\}, n \in \mathbb{N},$$

imply that g is measurable. Therefore, $N_2(E) = g^{-1}(\{0\})$ is a Borel set in the Borel $\sigma-$algebra of $C(E)$ relative to $C^R(E)$. From this and the fact that $C^R(E)$ is a Borel subset of $C(E)$ it follows that $N_2(E)$ is a Borel subset of $C(E)$.

For any $m > 2$. Define

$$A = \left\{(x_1, ..., x_{m-1}) \in (C(E))^{m-1} : x_j(1) = x_{j+1}(0), j = 1, ..., m-2\right\}$$

and the mapping

$$\pi : C(E) \to (C(E))^{m-1}$$
$$x \rightsquigarrow (x_1, ..., x_{m-1}), x_j(t) = x\left(\frac{j-1}{m-1} + \frac{1}{m-1}t\right), 1 \leq, j \leq m-2.$$

>From the facts that $N_2(E)$ is a Borel set, A is closed and π is continuous it follows that $N_m(E) = \pi^{-1}\left(A \cap (N_2(E))^{m-1}\right)$ is a Borel set.□

PROOF OF THEOREM 3.4.1. Since (E, d) is Polish, (by Lemma 7.2) $X = C(E)$ is Polish and (by Theorem 3.2.2) $\mathcal{X} = \mathcal{C}_m^R(E)$ too. According to Theorem 3.2.1, $\mathcal{X} = \mathcal{C}_m(E)$ is isometric to the quotient space $X = (C(E)/\sim_m, r_m)$. Furthermore, by Lemma 7.5, $\varphi\left(\mathcal{C}_m^R(E)\right) = N_m(E)$ is a Borel set. The thesis of Theorem 3.3.1 then follows from Lemma 7.3 applied to the mapping φ and the set $\mathcal{A} = \mathcal{C}_m^R(E)$.□

PROOF OF THEOREM 3.4.2. Let ψ be the mapping

$$\psi : N_2^k(E) \to \mathcal{C}_2(E), x \rightsquigarrow [x]_2.$$

Then, the following statements hold: a) Since $d([x]_2, [y]_2) \leq \|x - y\|_0$ for all $x, y \in C(E)$, ψ is continuous, hence measurable. b) By uniqueness of the natural representation (Lemma 3.3.1), ψ is injective. c) Both $N_2(E)$ (by Lemma 7.4) and $C^k(E)$ are Borel subsets of $C(E)$. Thus, $N_2^k(E) = N_2(E) \cap C^k(E)$ is a Borel subset of $C(E)$. d) Since E is a Polish metric space, Lemma 7.2 and Theorem 3.1.1 imply that $C(E)$ and $\mathcal{C}_2(E)$ are Polish.

Under these conditions, from Lemma 7.5 it follows that $\psi\left(N_2^k(E)\right)$ is a Borel subset of $\mathcal{C}_2(E)$ and ψ is a Borel isomorphism between $N_2^k(E)$ and $\psi\left(N_2^k(E)\right)$.

Furthermore, notice that, for $k \geq 1$, if $x \in R^k(E)$ then $\widetilde{x} \in R^k(E)$, where $\widetilde{x}$ denotes the natural parametrization of x. Thus, $\mathcal{C}_2^k(E) = \left\{[x]_2 \in R^k(E)\right\} = \left\{[x]_2 \in N_2^k(E)\right\}$, and so $\mathcal{C}_2^k(E) = \psi\left(N_2^k(E)\right)$.

The proof is completed by noting that φ is the inverse of the isomorphism ψ.□

PROOF OF THEOREM 3.4.3. By definition, $\rho_2((x)_2, (y)_2) = d_2([x]_2, [y]_2)$ for all $x, y \in R^k(E)$. Thus, it suffices to prove that the mapping $\varphi : \mathcal{R}^k(E) \to \mathcal{C}_2^k(E), (x)_2 \rightsquigarrow [x]_2$ for $x \in R^k(E)$, is bijective. This mapping is well-defined because $(x)_2$ and $[x]_2$ do not depend on the particular representation x of $(x)_2$.

Let $x, y \in R^k(E)$. By definition, $(x)_2 = (y)_2$ implies $[x]_2 = [y]_2$.

Suppose $[x]_2 = [y]_2$. Then, by uniqueness of the natural representation of a Fréchet curve (Lemma 3.3.1), $\widetilde{x} = \widetilde{y}$. But $\widetilde{x} = x \circ l_x^{-1}$ and $\widetilde{y} = y \circ l_y^{-1}$, where l_x^{-1}

and l_y^{-1} denote the inverses of the normalized arc length functions of x and y, respectively. These inverses exist for regular representations $x, y \in R^k(E)$, and they are functions with derivatives up to order k. From $x \circ l_x^{-1} = y \circ l_y^{-1}$ we have $x = y \circ l_y^{-1} \circ l_x$. Hence, $(x)_2 = (y)_2$.

Thus, φ is bijective. □

PROOF OF LEMMA 3.4.2. It straightforwardly follows from the fact that V^k is a dense subset of V for any $k \geq 1$.□

PROOF OF THEOREM 3.4.4.

i) Since $\left(C^k(E), \|\cdot\|_k\right)$ is separable and $N_2^k(E) \subset C^k(E)$, $\left(N_2^k(E), \|\cdot\|_k\right)$ is separable. This together with $\mathcal{C}_2^k(E) = \left\{[\widetilde{x}]_2 : \widetilde{x} \in N_2^k(E)\right\}$ and $d_{2,k}\left([\widetilde{x}]_2, [\widetilde{y}]_2\right) \leq \|\widetilde{x} - \widetilde{y}\|_k$ for all $[\widetilde{x}]_2, [\widetilde{y}]_2 \in \mathcal{C}_2^k(E)$ imply that $\mathcal{C}_2^k(E)$ is separable.

ii) Define the mapping $\phi : \left(N_2^k(E), \|.\|_k\right) \to \left(\mathcal{C}_2^k(E), d_{2,k}\right), x \rightsquigarrow [x]_2$.

Then, the following statements hold: a) Since $d_{2,k}\left([\widetilde{x}]_2, [\widetilde{y}]_2\right) \leq \|\widetilde{x} - \widetilde{y}\|_k$, ϕ is continuous and thence measurable. b) Since, by Lemma 3.4.1, the natural representation is unique, ϕ is injective. c) $N_2^k(E)$ is a Borel subset of $\left(C^k(E), \|.\|_k\right)$ (this can be proved similarly to Lemma 7.5). d) $\left(C^k(E), \|.\|_k\right) = \left(C^k(\mathbb{R}^p), \|.\|_k\right)$ is a Polish space and (by the step (i) above) $\mathcal{C}_2^k(E)$ is separable.

>From these conditions and Lemma 7.3 the thesis of Theorem 3.5.1 follows. □

PROOF OF THEOREM 3.4.5. Define the mapping

$$\psi : F(E) \cap N_2(E) \subset (C(E), \|.\|_0) \to (\mathcal{C}_2(E), d_2), x \rightsquigarrow [x]_2.$$

Then, the following statements hold: a) ψ is measurable (since it is a continuous mapping) and injective (due to the uniqueness of the natural representation stated in Lemma 3.3.1). b) $F(E) \cap N_2(E)$ is a Borel subset of $C(E)$ (as a consequence of the fact that $F(E)$ is a Borel subset of $C(E)$ and Lemma 7.5). c) The space $C(E) = C(\mathbb{R}^p)$ is Polish and (by Theorem 3.2.2) $\mathcal{C}_2(E)$ is Polish.

Under these conditions, from Lemma 7.3 it follows that $\psi(F(E) \cap N_2(E))$ is a Borel set in $\mathcal{C}_2(E)$ and ψ is a Borel isomorphism between $F(E) \cap N_2(E)$ and $\psi(F(E) \cap N_2(E))$. It can be shown that $\mathcal{F}_2(E) = \psi(F(E) \cap N_2(E))$. Furthermore, $\mathcal{F}_2(E) \subset \mathcal{C}_2^R(E)$ and (by Theorem 3.3.1) $\mathcal{C}_2^R(E)$ is a Borel subset of $\mathcal{C}_2(E)$. Thus, $\mathcal{F}_2(E)$ is a Borel set in $\mathcal{C}_2^R(E)$. The proof is completed by noting that φ is the inverse of the isomorphism ψ. □

PROOF OF THEOREM 3.4.6. Define the mapping $\phi : S(E) \cap N_2(E) \to \mathcal{C}_2(E), x \rightsquigarrow [x]_2$. Then, the following statements hold: a) ϕ is measurable (since it is continuous) and injective (by uniqueness of the natural representation stated in Lemma 3.3.1). b) $S(E) \cap N_2(E)$ is a Borel subset of $C(E)$ (as a consequence of Lemma 7.5 and the fact that $S(E)$ is a Borel subset of $C(E)$). c) $C(E)$ is Polish (by Lemma 7.2) and $\mathcal{C}_2(E)$ too (by Theorem 3.2.2).

>From these facts and Lemma 7.3 it follows that $\phi\left(S\left(E\right)\cap N_2(E)\right)=\mathcal{S}_2^R(E)$ is a Borel set in $\mathcal{C}_2(E)$ (and hence, by Theorem 3.3.1, $\mathcal{S}_2^R(E)$ is also a Borel set in $\mathcal{C}_2^R(E)$) and ϕ is a Borel isomorphism between $S\left(E\right)\cap N_2(E)$ and $\mathcal{S}_2^R(E)$. $\square$

PROOF OF THEOREM 4.1. The proof will be organized in several steps.

i) Let $\lambda\in\Lambda_S$ and $[x]_m=[y]_m\in\mathcal{C}_m(E)$. By Lemma 3.1.1, there are $u,v\in U_m$ such that $x\circ u=y\circ v$. Thus, $(\lambda x)\circ u=\lambda\left(x\circ u\right)=\lambda\left(y\circ u\right)=(\lambda y)\circ u$, and hence $g_\lambda\left([x]_m\right)=[\lambda x]_m=[\lambda y]_m=g_\lambda\left([y]_m\right)$. This proves that $g_\lambda\left([x]_m\right)$ is well-defined, i.e., it is the same no matter the representation x of the curve $c=[x]_m$.

ii) $G\left(\Lambda_S\right)$ is a group w.r.t. the composition of functions. Indeed, since $(\Lambda_S,\circ)$ is a subgroup of Λ, each $g_\lambda\in G\left(\Lambda_S\right)$ has inverse $\left(g_\lambda\right)^{-1}=g_{\lambda^{-1}}$. Therefore $\sim_{\Lambda_S}$ is an equivalence relation and thence $\mathcal{C}_m(E)/\Lambda_S$ is well-defined.

Since Λ keeps the metric d of E, $G\left(\Lambda_S\right)$ keeps the metric d_m. This together with Lemma 7.1 imply that ρ_m is a seudometric. In order to prove that ρ_m is a metric we then only need to verify that $\rho_m\left(\langle b\rangle,\langle c\rangle\right)=0$ implies $\langle b\rangle=\langle c\rangle$.

Let $b=[x]_m,c=[y]_m\in\mathcal{C}_m(E)$, and write $\beta=\rho_m\left(\langle b\rangle,\langle c\rangle\right)$. It is known that the orthogonal group $O\left(p\right)$ is a compact subspace of the general linear group of transformations on E (w.r.t. the operator norm induced by the Euclidean norm on E). This and the definition of infimum imply that there are sequences $(T_j)\subset O\left(p\right)$ and $(q_j)\subset E$ such that

$$d_m\left(\left[T_jx+q_j\right]_m,[y]_m\right)\underset{j}{\to}\beta$$

and $T_j\underset{j}{\to}T$ for some $T\in O\left(p\right)$.

Let j_0 be such that, for $j>j_0$,

$$d_m\left(\left[T_jx+q_j\right]_m,[y]_m\right)<\beta+1.$$

This and Definition 3.2.1 ensure that there is a sequence $(u_j)\subset U_m$ such that, for $j>j_0$,

$$d\left(T_jx+q_j,y\circ u_j\right)<\beta+1.$$

The points q_j in E can also be thought of as constant functions, i.e., elements of $(C(E),d)$, where $d=\|\cdot\|_0$, and it holds that

$$\begin{aligned}\left\|q_j\right\|_E&=d\left(q_j,0\right)\leq d\left(q_j,y\circ u_j-T_jx\right)+d\left(y\circ u_j-T_jx,0\right)\\&=d\left(y\circ u_j,T_jx+q_j\right)+d\left(y\circ u_j,T_jx\right)\\&\leq d\left(y\circ u_j,T_jx+q_j\right)+d\left(y\circ u_j,0\right)++d\left(0,T_jx\right)\\&=d\left(y\circ u_j,T_jx+q_j\right)+d\left(y,0\right)++d\left(0,x\right)\\&\leq\beta+1+d\left(y,0\right)++d\left(0,x\right)\ \text{ for } j>j_0.\end{aligned}$$

Thus, the sequence $(q_j)\subset E=\mathbb{R}^p$ is bounded, hence it has some convergent subsequence, say $q_{n_j}\underset{j}{\to}q$. This implies that $d\left(T_{n_j}x+q_j,Tx+q\right)\underset{j}{\to}0$, and

thence

$$d_m\left(\left[T_{n_j}x+q_j\right]_m,[Tx+q]_m\right)\underset{j}{\to}0.$$

>From this and $d_m\left(\left[T_jx+q_j\right]_m,[y]_m\right)\underset{j}{\to}\beta$ it follows that

$$\beta=d_m\left([Tx+q]_m,[y]_m\right).$$

Since Λ_S is closed, $(T,q)\in\Lambda_S$. Therefore, $\beta=0$ implies $[Tx+q]_m=[y]_m$ and thence $\langle[x]_m\rangle=\langle[y]_m\rangle$. This completes the proof of the fact that $(\mathcal{C}_m(E)/_{\Lambda_S},\rho_m)$ is a metric space.

iii) Since $E=\mathbb{R}^p$ is a Polish space, Theorem 3.2.3 ensures that $(\mathcal{C}_m(E),d_m)$ is Polish. Then, by Lemma 3.1.2, $(\mathcal{C}_m(E)/_{\Lambda_S},\rho_m)$ is Polish. □

PROOF OF LEMMA 5.2.1. Let $A=\{(x,\pi(x))\ ;\ x\in X\}$. Since π is a continuous mapping, A is a closed subset of $(X\times\mathcal{X},d+\rho)$. Since $(X\times\mathcal{X},d+\rho)$ is a Polish space, all its Borel subsets are Lusin sets (Dellacherie and Meyer, 1978, Sec. III-17), and thence Souslin sets (Dellacherie and Meyer, 1978, Sec. III-16). Thus, A is a Souslin set.

Let T be the projection of $X\times\mathcal{X}$ onto $\mathcal{X}$ and $\mathcal{S}$ be the σ-algebra on $\mathcal{X}$ generated by the Souslin subsets of $\mathcal{X}$. Since π is surjective, $T(A)=\mathcal{X}$. Then, the first theorem of Sec. III-81 in Dellacherie and Meyer (1978) implies that there is a $\mathcal{S}-B(X)$ measurable mapping h of $\mathcal{X}$ into X such that, for all $x\in X$, $(h(x),x)\in A$. This implies $\pi\circ h(x)=x$ for all $x\in\mathcal{X}$.

Since Souslin subsets of a metric space are universally measurable (Dellacherie and Meyer, 1978, Sec. III-81, Remark (a)), $\mathcal{S}\subset B_u(\mathcal{X})$. Thus, h is $B_u(\mathcal{X})-B(X)$ measurable. This and Remark (c) of Sec. II-32 in Dellacherie and Meyer (1978) imply that h is $B_u(\mathcal{X})-B_u(X)$ measurable. □

PROOF OF THEOREM 5.2.1. Lemma 7.2 and Theorem 3.2.2 ensure that (X,d) and $(\mathcal{X},\rho)$ are Polish spaces. Then, Theorem 5.2.1 follows from Lemma 5.2.1 applied to $\pi(x)=[x]_m, x\in C(E)$. □

The following Kuratowski's lemma will be used in proving Theorem 5.2.2.

LEMMA 7.6. (Kuratowski). If $\mathcal{X}$ is a compact metric space, Ω is a measurable space and $g:\mathcal{X}\times\Omega\to\mathbb{R}$ is a functional such that $g(x,.)$ is measurable for each $x\in\mathcal{X}$ and $g(.,\omega)$ is continuous for each $\omega\in\Omega$ then there is a measurable mapping $X:\Omega\to\mathcal{X}$ such that $g(X(\omega),\omega)=\sup\{g(x,\omega):x\in\mathcal{X}\}$.

LEMMA 7.7. *Let $m=0$ or $m\geq2$, and $0\leq\beta<\infty$. If E is a compact metric space then $\mathcal{K}_\beta=\{c\in\mathcal{C}_m(E):L_c\leq\beta\}$ is compact in $(\mathcal{C}_m(E),d_m)$.*

PROOF. By the generalized Arzela-Ascoli theorem (Kolmogorov and Fomin, 1978),

$$K_\beta=\{x\in C(E):d(x(t),x(s))\leq\beta|t-s|\ \text{ for all } t,s\in[0,1]\}$$

is compact in $C(E)$. Since $d_m([x]_m,[y]_m) \leq d(x,y)$ for all $x,y \in C(E)$, the mapping $C(E) \to \mathcal{C}_m(E), x \rightsquigarrow [x]_m$, is continuous. Since $[K_\beta]_m$ is the image of a compact set by a continuous function, it is compact. The proof is completed by noting that, as a consequence of Lemma 3.3.1, $[K_\beta]_m = \mathcal{K}_\beta$. □

PROOF OF THEOREM 5.2.2. For $n = 1, 2, ...$, write

$$k_n = \{c \in \mathcal{C}_m(E) : d(c,0) + L_c \leq n\} \text{ and } K_n = \{\langle c \rangle : c \in k_n\},$$

Define the mapping $g : \mathcal{C}_m(E) \times \mathcal{C}_m(E)/_{\Lambda_S} \to \mathbb{R}$ by $g(c,\eta) = -d_m(\langle c \rangle, \eta)$.

By Lemma 7.7, k_n is compact. On the other hand, since $\rho_m(\langle c_1 \rangle, \langle c_2 \rangle) \leq d_m(c_1,c_2)$ for all $c_1, c_2 \in \mathcal{C}_m(E)$, the mapping $\langle . \rangle : \mathcal{C}_m(E) \to \mathcal{C}_m(E)/_{\Lambda_S}$ is continuous. Thus, K_n is compact.

>From Lemma 7.6 it follows that there is a measurable mapping $\varphi_n : K_n \to k_n$ such that $\langle \varphi_n(\eta) \rangle = \eta$ for all $\eta \in K_n$.

Define $A_1 = k_1, B_1 = K_1, A_n = k_n \backslash k_{n-1}, B_n = K_n \backslash K_{n-1}$ for $n = 2, 3, ...$Since B_n are Borel sets, the restriction of φ_n to B_n, $\varphi_n \mid_{B_n}$: $B_n \to A_n$, is measurable. This and the fact that A_n are disjoint sets imply that the mapping $\varphi : \mathcal{C}_m(E)/_{\Lambda_S} \to \mathcal{C}_m(E)$ defined by

$$\varphi(\eta) = \varphi_n(\eta) \text{ if } \eta \in B_n$$

is measurable. □

PROOF OF THEOREM 5.2.3. Define the mapping

$$\phi : \left(\overline{N}_2^1, d\right) \to \left(\mathcal{R}_2^1(E)/\Lambda_S, \rho_2\right), x \rightsquigarrow \langle [x]_2 \rangle .$$

Then, the following statements hold: a) Since $\rho_2 \langle [x]_2, [y]_2 \rangle \leq d(x,y)$ for all $x, y \in C(E)$, ϕ is continuous and thence measurable. b) ϕ is injective. c) From Lemma 7.5, part (i) of the Proof of Theorem 3.3.2 and the fact that $L(\cdot, 0, 1) : C(E) \to \overline{\mathbb{R}}$ is lower semicontinuous (Kolmogorov and Fomin, 1978) it follows that $\overline{N}_2^1 = N_2 \cap C^1(E) \cap \left\{ x \in C(E) : x(0) = 0 \text{ and } \dot{\widetilde{x}}(0) = L_x e_1 \right\}$ is a Borel subset of $(C(E), d)$. d) By Lemma 7.2 and Theorem 4.1, $\overline{N}_2^1$ and $\mathcal{R}_2^1(E)/\Lambda_S$ are contained in the Polish spaces $(C(E), d)$ and $(\mathcal{C}_2(E)/\Lambda_S, \rho_2)$, respectively.

>From these conditions and Lemma 7.3 it follows that $\phi\left(\overline{N}_2^1\right) = \mathcal{R}_2^1(E)/\Lambda_S$ is a Borel set and ϕ is a Borel isomorphism between $\overline{N}_2^1$ and $\mathcal{R}_2^1(E)/\Lambda_S$. □

PROOF OF THEOREM 5.2.4. Define the mapping

$$\Psi : K^p \times \mathbb{R}_+ \to \mathcal{C}_2(E)/\Lambda_S, (\kappa_1, ..., \kappa_{p-1}, L) \rightsquigarrow \left\langle [\phi(\kappa_1, ..., \kappa_{p-1}, L)]_2 \right\rangle,$$

where $x = \phi((\kappa_1, ..., \kappa_{p-1}, L))$ is the parametrized curve with constant speed L, length L, curvatures $\kappa_1, ..., \kappa_{p-1}$, $x(0) = 0$ and $\dot{x}(0) = Le_1$ (Klingenberg, 1978, Sec. 1.3).

Then, the following statements hold: a) Ψ is continuous, and thence measurable. b) Ψ is injective. c) $K^p \times \mathbb{R}_+$ is a Borel subset of the Polish space $(C(\mathbb{R}^{p-1}) \times \mathbb{R}_+, d)$. d) By Theorem 3.2.3, $\mathcal{C}_2(E)/\Lambda_S$ is separable.

These conditions and Lemma 7.3 imply that $\Psi(K^p \times \mathbb{R}_+) = \mathcal{K}_2(E)/\Lambda_S$ is a Borel set and Ψ is a Borel isomorphism between $K^p \times \mathbb{R}_+$ and $\mathcal{K}_2(E)/\Lambda_S$. □

PROOF OF THEOREM 6.1.1. For simplicity, denote $V_{P_n}(.) = V_{P_n,f}(.)$ and $V_P(.) = V_{P,f}(.)$.

a) It suffices to prove that $V_{P_n}(.)$ and $V_P(.)$ are lower semicontinuous functionals, for then they achieve their infimum values on the compact set $\mathcal{Z}$.

Let $z_j, z \in \mathcal{Z}, z_j \underset{j}{\to} z$. Since $f(x, .)$ is lower semicontinuous for each $x \in \mathcal{X}$,

$$f(x,z) \leq \liminf_j f(x, z_j).$$

Thus,

$$\begin{aligned} V_P(z) &= \int_{\mathcal{X}} f(x,z) P(dx) \leq \int_{\mathcal{X}} \liminf_j f(x,z_j) P(dx) \leq \liminf_j \int_{\mathcal{X}} f(x,z_j) P(dx) \\ &= \liminf_j V_P(z_j). \end{aligned}$$

This proves that $V_P(.)$ is lower semicontinuous. A similar proof holds for $V_P(.)$.

b) i) Let $z^* \in Q$. For $z^* \in Z$, define $T_n^*(z) = V_{P_n}(z) - V_P(z^*)$ and $T_n(z) = V_{P_n}(z) - V_P(z)$.

The strong law of large numbers for i.i.d. real-valued *r.v.* (Dudley, 1989) ensures that the following ergodic property is satisfied: For any function $\varphi : \mathcal{X} \to \mathbb{R}$ with $\int_{\mathcal{X}} |\varphi(x)| dP(dx) < \infty$ it holds that $\int_{\mathcal{X}} \varphi(x) dP_n(dx) \underset{n}{\to} \int_{\mathcal{X}} \varphi(x) dP(dx)$ with probability 1. By Theorem 6.2 in Ranga-Rao (1962), this ergodic property and conditions C1)-C3) imply that $\sup_{z \in Z} |T_n(z)| \underset{n}{\to} 0$ with probability 1.

Therefore, $T_n(z^*) \underset{n}{\to} 0$ $P - a.s.$ and $T_n(z_n) \underset{n}{\to} 0$ $P^* - a.s.$ This and the inequality

$$T_n(z_n) \leq T_n^*(z_n) \leq T_n(z^*)$$

imply that $T_n^*(z_n) \underset{n}{\to} 0 \quad P^* - a.s.$

ii) Since Z is a compact subset of a metric space, each sequence $(z_n(\omega)) \in \prod_{n=1}^{\infty} Q_n(\omega) \subset Z$ has a convergent subsequence. This and $Q_n(\omega) \neq \emptyset$ for any $\omega \in \Omega$ imply that $\widehat{Q}(\omega) \neq \emptyset$.

Let $z^* \in Q$, $\widehat{z} \in \widehat{Q}, \widehat{z} = \lim_k z_{n_k}$, $z_{n_k} \in Q_{n_k}$. From

$$\begin{aligned} |V_P(z_n) - V_P(z^*)| &= |T_n(z_n) - T_n^*(z_n)| \leq |T_n(z_n)| + |T_n^*(z_n)| \\ &\leq |T_n(z_n)| + \max(|T_n(z_n)|, T_n(z^*)) \\ &\leq 2 \sup_{z \in Z} |T_n(z)| \underset{n}{\to} 0 \quad P - a.s. \end{aligned}$$

we have $P^* \left\{ \lim_n V_P(z_n) = V_P(z^*) \text{ for all sequences } (z_n) \in \prod_{n=1}^{\infty} Q_n \right\} = 1$.

On the other hand, since $V_P(.)$ is lower semicontinuous (see proof of part (a) above), $V_P(\widehat{z}) \leq \liminf_k V_P(z_{n_k})$ for all $\widehat{z}, z_{n_k}$ such that $\widehat{z} = \lim_k z_{n_k}$.

Therefore, $P^* \left\{ V_P(\widehat{z}) = V_P(z^*) \text{ for all } \widehat{z} \in \widehat{Q} \right\} = 1$, .i.e. $\widehat{Q} \subset Q \quad P^* - a.s.$ □

PROOF OF THEOREM 6.2.1. For simplicity, denote $V_{P_n^r}(.) = V_{P_n^r, f}(.)$ and $V_{P^r}(.) = V_{P^r, f}(.)$.

a) $\overline{r(S)}$ is a closed set with $P^r\left(\overline{r(S)}\right) \geq P^r(r(S)) = P(S) = 1$. This and definition of the support of a measure implies $S^r \subset \overline{r(S)}$. On the other hand, $\overline{r(S)} \subset \overline{r(\mathcal{X})} = \mathcal{X}^r$. Thus, S^r is a closed subset of a compact set $\mathcal{X}^r$ in a metric space, which implies that S^r is compact.

In a completely similar way to the proof of part (a) of Theorem 6.1.1 it can be shown that $V_{P^r}(.)$ and $V_{P_n^r}(.)$ are lower semicontinuous.

Thus, $V_{P^r}(.)$ achieves their minimum value on the compact set S^r, i.e. $R \neq \emptyset$.

b) i) Let $Y_i = r(X_i), i = 1, ..., n$. For $z \in S^r$, define $T_n^*(z) = V_{P_n^r}(z) - V_{P^r}(z)$ and $T_n(z) = V_{P_n^r}(.) - V_{P^r}(.)$.

By definition of z_n,

$$T_n^*(z_n) = T_n^*(z^*) + T_n^*(z_n) - T_n^*(z^*) \leq T_n^*(z^*) + \min_{1 \leq i \leq n} (T_n^*(y_i) - T_n^*(z^*)).$$

On the other hand, by definition of $z^*, T_n^*(z_n) \geq T_n(z_n)$. Thus, taking also into account that $T_n^*(z^*) = T_n(z^*)$, it holds that

$$(7.1) \qquad |T_n^*(z_n)| \leq \max\left(|T_n(z_n)|, T_n(z^*) + \min_{1 \leq i \leq n} (T_n^*(y_i) - T_n(z^*)) \right).$$

Similarly to the proof of part b)i) of Theorem 6.1.1**,** from conditions C1)-C3) it follows that

$$(7.2) \qquad T_n(z^*) \underset{n}{\to} 0 \text{ and } T_n(z_n) \underset{n}{\to} 0 \; P^{r*} - a.s.,$$

where P^{r*} is the outer measure corresponding to P^r.

Furthermore, for $i = 1, ..., n$,

$$\begin{aligned} |T_n^*(y_i) - T_n(z^*)| &= \left| \frac{1}{n} \sum_{j=1}^{n} g(r(x_j), y_i) - \frac{1}{n} \sum_{j=1}^{n} g(r(x_j), z^*) \right| \\ &\leq \frac{1}{n} \sum_{j=1}^{n} |g(r(x_j), y_i) - g(r(x_j), z^*)|, \end{aligned}$$

and thence

$$\min_{1\leq i\leq n}\left|T_n^*\left(y_i\right)-T_n\left(z^*\right)\right|\leq\sup_{y\in\mathcal{X}^r}\ \sup\left\{\left|g\left(y,z\right)-g\left(y,z^*\right)\right|:z\in S^r,d\left(z,z^*\right)\leq\delta_n\right\},$$

where $\delta_n=\min_{1\leq i\leq n}\ d\left(y_i,z^*\right)$.

It can be shown that $S^r=\overline{S_\infty}\ P-a.s.$, where $S_\infty=\left\{Y_1,...,Y_n,...\right\}$. Hence $\delta_n\to 0\ P-a.s.$, which together with condition C2) implies that, with P^{r*} measure 1, the right member of the last inequality above tends to 0 as n increases. This, (7.1) and (7.2) complete the proof of b)i).

ii) Let $z^*\in R$ and $\widehat{z}\in\widehat{R}$. By definition of $\widehat{R}$, there is a sequence $z_n\in R_n$ such that $z_n\to\widehat{z}$. From

$$\left|V_{P^r}(z_n)-V_{P^r}(z^*)\right|=\left|T_n\left(z_n\right)-T_n^*\left(z_n\right)\right|\leq\left|T_n\left(z_n\right)\right|+\left|T_n^*\left(z_n\right)\right|,$$

$T_n\left(z_n\right)\to 0$ and $T_n^*\left(z_n\right)\to 0\ P^{r*}-a.s.$ (as was proved in part b)i) above) it follows that $V_{P^r}(z_n)\to V_{P^r}(z^*)$ with P^{r*} measure 1. From this and lower semicontinuity of $V_{P^r}(.)$ we have that, with measure P^{r*} equal to 1,

$$V_{P^r}(\widehat{z})\leq\liminf V_{P^r}(z_n)=V_{P^r}(z^*).$$

On the other hand, $z_n\in r\left(S_n\right)$ for all n implies that $\widehat{z}\in\overline{S_\infty}=S^r$, and hence, by definition of z^*,

$$V_{P^r}(\widehat{z})\geq V_{P^r}(z^*).$$

The last two inequalities imply that $V_{P^r}(\widehat{z})=V_{P^r}(z^*)$, and thence $\widehat{z}\in R$ $P^{r*}-a.s.$ □

PROOF OF COROLLARY 6.2.1. Condition (C1) is satisfied by hypothesis, so it suffices to verify conditions (C2) and (C3). If $\mathcal{X}^r$ is compact then $\mathcal{S}^r\subset\mathcal{X}^r$ is compact. Since f is continuous, it is uniformly continuous on the compact set $\mathcal{X}^r\times\mathcal{S}^r$. This implies (C2) and (C3). □

Acknowledgments. We are much indebted to Dr. P. Valdés, from the Cuban Neuroscience Center, for his emphasis on the need for a general framework for the statistical analysis of curves. Part of this work has been carried out while one of the authors was an invited researcher at the Centro de Investigación en Matemáticas, Guanajuato, México. We are grateful to the staff of the Department of Probability and Statistics of this Center for its support. We also thank the financial support by the Pontificia Universidad Católica de Chile to the work of one of the authors. This paper is partly based on a lecture given at the Workshop on Statistics of Brain Mapping, Montreal, June 13-14, 1998. We thank Professor K. Worsley for facilitating our participation in this event, and we also express our thanks to participants that provided valuable comments on our work, in particular to M. I. Miller.

REFERENCES

Adams, R.A. (1978). *Sobolev Spaces.* Academic Press, Boston.

Barnsley, M. (1978). *Fractals Everywhere.* Academic Press, Boston.

Birkhoff, G. and MacLane, S. (1977, 4th ed.). *A Survey of Modern Algebra.* MacMillan Publishing, New York.

Bookstein, F. L.. (1978). *The Measurement of Biological Shape and Shape Change.* Springer-Verlag, New York.

Bookstein, F. L. (1991). *Morphometric Tools for Landmark Data: Geometry and Biology.* Cambridge University Press, Cambridge.

Cressie, N. (1991). *Statistics for Spatial Data.* J. Wiley & Sons, New York.

Dellacherie, C. and Meyer. P. A. (1978) *Probabilities and Potential.* North-Holland, Amsterdam.

Dudley, R. M. (1989). *Real Analysis and Probability.* Warsworth &Brooks, California.

Fréchet, M. (1934). Sur la distance de deux surfaces. *Ann. Soc. Polonaise Math.* **3**, 4-19.

Fréchet, M. (1948). Les éléments aléatories de nature quelconque dans un espace distancié. *Ann. Inst. H. Poincaré* **10**, 215-310.

Fréchet, M. (1951) *Les Espaces Abstraits.* Gauthier-Villars, Paris.

Fréchet, M. (1961) Les courbes aléatoires. *Bull. Inst. Internat. Statist* **38**, 499-504.

Grenander, U., Chow, Y., and Keenan, D. (1990). *HANDS: a Pattern Theoretic Study of Biological Shapes.* Springer, New York.

Grenander, U. and Miller, M. I.. (1994). Representations of knowledge in complex systems. *Journal of The Royal Statistical Society* **B** 56**,** 549-603.

Grenander, U. and Miller, M. I.. (1998). *Computational Anatomy: an Emerging Discipline. CIS.*

Kass, M., Witkin, A. and Terzopolous, D. (1988). Snakes: Active contour models. *International Journal of Computer Vision* **1**, 321-331.

Kendall, D. G.. (1984). Shape manifolds, procrustean metrics and complex projected spaces. *Bull. London Math. Soc.* 16, 81-121.

Klingenberg, W. (1978). *A Course in Differential Geometry.* Springer, New York.

Kolmogorov, A. N. and Fomin, S. V.(1978) *Elementos de la Teoría de Funciones y del Análisis Funcional.* MIR, Moscow.

Kuratowsky, K. (1968) *Topology.* 2 Vols. Academic Press, New York.

Mardia, K. V., Kent, J. T., and Walder, A. N. (1991). Statistical shape models in image analysis. In: *Computer Science and Statistics: Proc. 23rd Symp. Interface*, E. M. Keramidas (ed.), pp. 550-557, Fairfax.

Matheron, G. (1975). *Random Sets and Integral Geometry.* J. Wiley & Sons, New York.

Mora, C. and Biscay, R. J. (1995). Espacios métricos probabilísticos de contornos. *Investigación Operacional* **16**, 77-123.

Neveu, J. (1965). *Mathematical Foundations of the Calculus of Probability.* Holden-Day, San Francisco.

Parthasarathy, K. R. (1967). *Probability Measures on Metric Spaces.* Academic Press, New York.

Persoon, E. and Fu, K. S. (1977). Shape discrimination using Fourier descriptors. *IEEE Trans. on Systems, Man and Cybernetics SCMC-***7**, 170-179.

Ranga Rao, R.(1962) Relations between weak and uniform convergence of measures with applications. *Ann. Math. Statist.* **33,** 659-680.

Serra, J. P. (1982). *Image Analysis and Mathematical Morphology.* Academic Press, London.

Stoyan, D. and Stoyan, H. (1994). *Fractals, Random Shapes and Point Fields.* John Wiley & Sons, Chichester.

Sverdrup-Thygeson, H.(1981). Strong law of large numbers for measures of central tendency and dispersion of random variables in compact metric spaces. *Ann. Statist.* **9**, 141-145.

Thijonov, A. N., Gonsharsky, A. V., Stepanov, V. V. and Yagola, A. G. (1995). *Numerical Methods for the Solution of Ill-Posed Problems.* Kluwer Academic Publishers, Dordrecht.

Trouvé, A. (1995). Action de groupe de dimension infinie et reconnaissanse de formes. *C. R. Acad. Sci. Paris, Série I,* **321**, 1031-1034.

Valdés, P. and Biscay, R. J. (1990). The statistical analysis of brain images. In: *Machinery of the Mind*, E. Roy John *et al.* (eds.), pp. 405-434. Birkhauser, Boston.

Younes, L. (1998). Computable elastic distances between shapes. *SIAM J. Applied Mathematics* (in press).

Centro de Neurociencias de Cuba
Apartado 6880
Cuba, Ciudad Habana
E-mail: na@cneuro.edu.cu

Departamento Matemáticas Aplicadas
Universidad de La Habana
Ciudad Habana, Cuba
E-mail: cmora@mat.puc.cl