Optimal random designs for estimating relative exceedances of random fields

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1 Introduction

The objective of this work is to obtain optimal random designs to predict integrals of Level-Crossing random fields. We extend the seminal work of [2], where the problem of estimating the weighted average of second-order random fields was considered.

Let $Z = \{Z(x) : x \in A\}$, $A \subset R^d$ and $|A| < \infty$ be a random field defined over the probability space $(\Omega, \mathcal{A}, P)$ and let $u \in R$ be a constant. The constant u corresponds to a level. Consider the level-crossing random field generated by Z :

$$I_u(x) = \begin{cases} 1 \text{ if } Z(x) > u, \\ \\ 0 \text{ otherwise,} \end{cases} \tag{1}$$

referred to as I_u random field. Observe that

$$E(I_u(x)) = P(Z(x) > u)$$

and

$$Cov(I_u(x), I_u(y)) = P(Z(x) > u, Z(y) > u) - P(Z(x) > u) P(Z(y) > u).$$

In particular,

$$Cov(I_u(x), I_u(x)) = P(Z(x) > u)[1 - P(Z(x) > u)].$$

In order that the integral of level-crossing random field be well defined we need continuity of I_u in quadratic mean. The following proposition establishes a sufficient condition for that property.

Proposition 1.1. *If* $\{Z(x) : x \in A\}$ *is continuous in quadratic mean then* $\{I_u(x) : x \in A\}$ *is continuous in quadratic mean too.*

Proof. Let $x \in A$ and $\{y_n\}_{n=1}^{\infty} \subset A$ be a sequence converging to x. First observe that

$$E\left(I_u(y_n) - I(x)\right)^2 = P\left(Z(y_n) > u\right) + P\left(Z(x) > u\right) - 2P\left(Z(x) > u, Z(y_n) > u\right).$$

Clearly, the continuity of Z implies that $Z(y_n) \xrightarrow{D} Z(x)$. In particular,

$$P\left(Z(y_n) > u\right) \to P\left(Z(x) > u\right).$$

Moreover the vector $(Z(x), Z(y_n))$ converges in L^2 to the vector $(Z(x), Z(x))$, that is

$$E\left[(Z(x), Z(y_n)) - (Z(x), Z(x))\right]^2 = E\left[(0, Z(y_n) - Z(x))\right]^2 \to 0.$$

In the above notation, taking a square of a vector corresponds to multiplying it by its transpose. Therefore,

$$P\left(Z(x) > u, Z(y_n) > u\right) \to P\left(Z(x) > u\right).$$

It follows that

$$E\left(I_u(y_n) - I(x)\right)^2 \to 0,$$

that is, I_u is continuous in quadratic mean. ■

Following the last proposition, if Z is continuous in quadratic mean then we can well define the relative exceedance of Z with respect to level u over the set A as

$$q_u = \frac{1}{|A|} \int_A I_u(x)\, dx. \tag{2}$$

Observe that

$$E\left(q_u^2\right) = \frac{1}{|A|^2}\int_A\int_A P\left(Z\left(x\right) > u, Z\left(y\right) > u\right) dxdy \leq 1.$$

Following [2], a linear predictor of q_u can be defined as:

$$\widehat{q}_u = \frac{1}{n\left|A\right|}\sum_{k=1}^{n} c_n\left(X_{kn}\right) I_u\left(X_{kn}\right), \tag{3}$$

where $X_n = \{X_{1n}, \ldots, X_{nn}\}$ is a set of n random variables or a sample of size n over A. Moreover, X_{kn} has probability distribution function G_{kn}, $k = 1, \ldots, n$, and X_n is independent of Z. X_n or his joint probability distribution function are know as a design of size n.

Observe that bias of $\widehat{q}_u$ is:

$$B\left(\widehat{q}_u\right) = \frac{1}{|A|}\left\{\int_A c_n\left(x\right) P\left(Z\left(x\right) > u\right) dG_n\left(x\right) - \int_A P\left(Z\left(x\right) > u\right) dx\right\},$$

where $G_n = \dfrac{1}{n}\sum\limits_{k=1}^{n} G_{kn}$. A sufficient condition for zero bias is that

$$c_n\left(x\right) dG_n\left(x\right) = dx. \tag{4}$$

It follows that

$$E_{G_n}\left[c_n\left(X\right)\right] = \left|A\right|.$$

In others words, the expected value of predictor weights with respect to G_n distribution will be $|A|$.

2 Simple random sampling

We define simple random sampling as a set of n random variables X_n that are independent and identically distributed as G. Note that the marginal

distribution function G does not change with n. Under this design the mean square error of predictor (3) is, after some algebraic manipulation equals to:

$$MSE\left(\widehat{q}_u^{srs}\right) =$$

$$\frac{1}{n\left|A\right|^2}\left\{\int_A c_n^2\left(x\right)P\left(Z\left(x\right)>u\right)dG\left(x\right)-\int_A\int_A P\left(Z\left(x\right)>u,Z\left(y\right)>u\right)dxdy\right\} \tag{5}$$

where $\widehat{q}_u^{srs}$ is the predictor (3) under simple random sampling.

Observe that MSE depends on marginal and bivariate distributions of the random field Z. If we now want to minimize (2) with respect to design, then we need to minimize the first part of the right hand size of (2) . In order to accomplish that, suppose that G has a density g with support A. Using the condition of zero bias (4) we get that

$$\int_A c_n^2\left(x\right)P\left(Z\left(x\right)>u\right)dG\left(x\right)=\int_A\frac{P\left(Z\left(x\right)>u\right)}{g\left(x\right)}dx. \tag{6}$$

Observe that $c_n\left(x\right)=1/g\left(x\right)$ and c_n does not depend on n.

To minimize the integral expression (6) with respect to g, with respect to design, we have used techniques of Gâteaux Variations (see Appendix A and section 2.4 of [3]) and we have obtained the unique optimal design:

$$g\left(x\right)=\frac{\sqrt{P\left(Z\left(x\right)>u\right)}}{\int_A\sqrt{P\left(Z\left(y\right)>u\right)}dy}. \tag{7}$$

Observe that under this optimal design the predictor weights c_n are

$$c_n\left(x\right)=\frac{\int_A\sqrt{P\left(Z\left(y\right)>u\right)}}{\sqrt{P\left(Z\left(x\right)>u\right)}}dy,$$

and do not depend on n.

With the previous results we can prove the following Proposition.

Proposition 2.1. *For simple random sampling, the error* (2) *is minimized if and only if the probability distribution function* G *has density of the form* (7)*. Under this optimal design*

$$MSE\left(\widehat{q}_u^{*srs}\right) =$$

$$\frac{1}{n\left|A\right|^2}\left\{\left[\int_A \sqrt{P\left(Z\left(x\right)>u\right)}dx\right]^2 - \int_A\int_A P\left(Z\left(x\right)>u, Z\left(y\right)>u\right)dxdy\right\},$$

where

$$\widehat{q}_u^{*srs} = \frac{\int_A \sqrt{P\left(Z\left(y\right)>u\right)}dy}{n\left|A\right|}\sum_{k=1}^{n}\frac{I_u\left(X_{kn}\right)}{\sqrt{P\left(Z\left(X_{kn}\right)>u\right)}}. \tag{8}$$

Remark 2.1. *Observe that the support of the optimal design* (7) *has the form* $\{x\in A : P\left(Z\left(x\right)>u\right)>0\}$*. Such selection is intuitively clear as the values of the field* I_u *where* $P\left(Z\left(x\right)>u\right)=0$ *are not informative.*

Remark 2.2. *The predictor* (8) *is a Horvitz-Thompson predictor type for the proposed optimal design (see for example [1]).*

To prove the remark, define the inclusion density function

$$\pi\left(x\right) = \sum_{k=1}^{n} g_{kn}\left(x\right),$$

where g_{kn} is the density function of sampling point $X_{kn}, k=1,\ldots,n$. Thus the Horvitz-Thompson predictor of (2) is

$$q_u^{HT} = \frac{1}{|A|}\sum_{k=1}^{n}\frac{I_u\left(X_{kn}\right)}{\pi\left(X_{kn}\right)}. \tag{9}$$

Under the simple random design, $g_1 = \cdots = g_n = g$, so $\pi = ng$ and the predictor (9) is reduced to

$$q_u^{HT,srs} = \frac{1}{n|A|} \sum_{k=1}^{n} \frac{I_u(X_{kn})}{g(X_{kn})}.$$

In the case of the optimal design (7) , we obtain the optimal Horvitz-Thompson predictor.

3 Simple stratified sampling

Stratified sampling consists of independently choosing a sample of size n_i from each stratum A_{im}, where $i = 1, \ldots, m$. Moreover, $\sum_{i=1}^{m} n_i = n$, where $\{A_{1m}, \ldots, A_{mm}\}$ is a partition of A. Thus in stratified sampling the sample point X_{kim}, the kth sample point of stratum i, has the probability distribution function G_{im} with support A_{im}, $k = 1, \ldots, n_i$, $i = 1, \ldots, m$. For simple stratified sampling the set of random variables $\{X_{1im}, \ldots, X_{n_i im}\}$ is a simple random sample over A_{im}, $i = 1, \ldots, m$.

Under simple stratified sampling design the predictor (3) has the form

$$\widehat{q}_u^{sts} = \frac{1}{n|A|} \sum_{i=1}^{m} \sum_{k=1}^{n_i} c_n(X_{kim}) I_u(X_{kim}).$$

The mean square error of the above estimator has the form

$$\frac{1}{|A|^2} \sum_{i=1}^{m} \frac{1}{n_i} \left\{ \int_{A_{im}} \frac{P(Z(x) > u)}{g_{im}(x)} dx - \int_{A_{im}} \int_{A_{im}} P(Z(x) > u, Z(y) > u)\, dxdy \right\}, \tag{10}$$

where g_{im} is the density function of G_{im}. In this expression we have used the condition for zero bias (4), where $G_n = \sum_{i=1}^{m} (n_i/n) G_{im}$.

Using the results of section (2) we are able to prove the following.

Proposition 3.1. *For simple stratified sampling, the MSE* (10) *is minimized if and only if the probability distribution function* $G_{im}, i = 1, \ldots, m$ *has the density*

$$g_{im}(x) = \frac{\sqrt{P(Z(x) > u)}}{\int_{A_{im}} \sqrt{P(Z(y) > u)} dy}.$$

Under this optimal design the MSE has the form

$$\frac{1}{|A|^2} \sum_{i=1}^{m} \frac{1}{n_i} \left\{ \left[\int_{A_{im}} \sqrt{P(Z(x) > u)} dx \right]^2 - \int_{A_{im}} \int_{A_{im}} P(Z(x) > u, Z(y) > u) \, dxdy \right\},$$

where

$$\widehat{q}_u^{*sts} = \frac{1}{n|A|} \sum_{i=1}^{m} \left[\int_{A_{im}} \sqrt{P(Z(y) > u)} dy \right] \sum_{k=1}^{n_i} \frac{I_u(X_{kim})}{\sqrt{P(Z(X_{kim}) > u)}}.$$

This is the equivalent of the Proposition (4.1) of [2].

An advantage of simple stratified sampling over simple random sampling is in getting smaller mean square error. The following Proposition establishes this property.

Proposition 3.2. *For every simple random design of size* n *there exists one simple stratified design of the same size* n *such that* $MSE(\widehat{q}_u^{sts}) \leq MSE(\widehat{q}_u^{srs})$.

Proof. Let G be the sampling distribution, with density g, for simple random sampling. Consider the simple stratified sampling with $m = n$, $n_i = 1$ and sampling distribution $G_{in} = nG$, with density g_{in}, over $A_{in}, i = 1, \ldots, n$,where $\{A_{1n}, \ldots, A_{nn}\}$ is a partition of A such that $\int_{A_{in}} dG(x) = 1/n$. Thus, under this simple stratified sampling, $G_n = G$.

Now, combining the expressions (2) and (10), we have

$$MSE\left(\widehat{q}_u^{srs}\right) - MSE\left(\widehat{q}_u^{sts}\right) =$$

$$\frac{1}{n\left|A\right|^2}\left\{\int_A \frac{P\left(Z\left(x\right)>u\right)}{g\left(x\right)}dx - \int_A\int_A P\left(Z\left(x\right)>u, Z\left(y\right)>u\right)dxdy\right\}$$

$$-\frac{1}{\left|A\right|^2}\sum_{i=1}^{n}\left\{\int_{A_{in}} \frac{P\left(Z\left(x\right)>u\right)}{g_{in}\left(x\right)}dx - \int_{A_{in}}\int_{A_{in}} P\left(Z\left(x\right)>u, Z\left(y\right)>u\right)dxdy\right\} \tag{11}$$

Observe that $g_{in} = ng$ over $A_{in}, i = 1, \ldots, n$. So

$$\begin{aligned}\sum_{i=1}^{n}\int_{A_{in}} \frac{P\left(Z\left(x\right)>u\right)}{g_{in}\left(x\right)}dx &= \frac{1}{n}\sum_{i=1}^{n}\int_{A_{in}} \frac{P\left(Z\left(x\right)>u\right)}{g\left(x\right)}dx \\ &= \frac{1}{n}\int_A \frac{P\left(Z\left(x\right)>u\right)}{g\left(x\right)}dx.\end{aligned}$$

Thus

$$\frac{1}{n}\int_A \frac{P\left(Z\left(x\right)>u\right)}{g\left(x\right)}dx - \sum_{i=1}^{n}\int_{A_{in}} \frac{P\left(Z\left(x\right)>u\right)}{g_{in}\left(x\right)}dx = 0.$$

In this form, the above difference of MSE can be reduced to

$$
\begin{aligned}
& MSE\left(\widehat{q}_u^{srs}\right) - MSE\left(\widehat{q}_u^{sts}\right) \\
= & \frac{1}{|A|^2}\left\{\sum_{i=1}^{n}\int_{A_{in}}\int_{A_{in}} P\left(Z\left(x\right)>u, Z\left(y\right)>u\right)dxdy\right. \\
& \left.-\frac{1}{n}\int_{A}\int_{A} P\left(Z\left(x\right)>u, Z\left(y\right)>u\right)dxdy\right\} \\
= & \frac{1}{|A|^2}\left\{\sum_{i=1}^{n}\int_{A_{in}}\int_{A_{in}} P\left(Z\left(x\right)>u, Z\left(y\right)>u\right)dxdy\right. \\
& \left.-\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{A_{in}}\int_{A_{jn}} P\left(Z\left(x\right)>u, Z\left(y\right)>u\right)dxdy\right\} \\
= & \frac{1}{2n\left|A\right|^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{A}\int_{A} P\left(Z\left(x\right)>u, Z\left(y\right)>u\right)\left[I_{A_{in}}\left(x\right)-I_{A_{jn}}\left(x\right)\right]\left[I_{A_{in}}\left(y\right)-I_{A_{jn}}\left(y\right)\right]dxdy \\
= & \frac{1}{2n\left|A\right|^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left\{\int_{A} I_u\left(x\right)\left[I_{A_{in}}\left(x\right)-I_{A_{jn}}\left(x\right)\right]dx\right\}^2 \geq 0,
\end{aligned}
$$

where $I_{A_{in}}(x)$ is the indicator function of the set A_{in}, see expression (1). From the above calculation it easily follows that the mean square error of $\widehat{q}_u^{sts}$ is not greater than the mean square error of $\widehat{q}_u^{srs}$. ■

4 Systematic sampling

Let $\{A_{1n}, \ldots, A_{nn}\}$ be a partition of A, where A_{in} represents a stratum, $i = 1, \ldots, n$. Let $T_{ijn} : A_{jn} \to A_{in}$ be a 1-1 transformation and onto, such that $T_{ijn} = T_{ikn}T_{kjn}$ for $i, j, k = 1, \ldots, n$. Systematic sampling consists of selecting a sample from the first stratum and then choosing the samples on each remaining stratum by applying the specified transformation on the sample from the first stratum. We consider only the case where a sample

of size one is taken from each stratum. Thus the sample point X_{1n} has distribution G_{1n} with support on A_{1n} and the remaining sample points are defined by $X_{in} = T_{i1n}X_{1n}$ with X_{in} having distribution G_{in} on A_{in}, $i = 2, \ldots, n$.

Under systematic sampling design the predictor (3) has the form

$$\widehat{q}_u^{sys} = \frac{1}{n\,|A|} \sum_{k=1}^{n} c_n\left(X_{kn}\right) I_u\left(X_{kn}\right)$$

and the respective mean square error is

$$\frac{1}{|A|^2}\left\{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{A_{in}} \frac{P\left(Z\left(x\right)>u, Z\left(T_{jin}\left(x\right)\right)>u\right)}{g_n\left(T_{jin}\left(x\right)\right)}dx - \int_A\int_A P\left(Z\left(x\right)>u, Z\left(y\right)>u\right)dxdy\right\},$$

where g_n is the density function of $G_n = \sum_{i=1}^{n} G_{in}/n$. In this expression we have used the condition for zero bias (4).

Remark 4.1. *In general, the minimization of the above MSE seems to be quite difficult. However, the above MSE can serve as a benchmark with other sampling methods.*

Proposition 4.1. *Consider a simple stratified design with $n = m$, $n_i = 1$ and ith sample point has distribution G_{in} on A_{in}, $i = 1, \ldots, n$. Moreover, let the systematic design be constructed in such a way that the ith sample point again has distribution G_{in} on A_{in}, $i = 1, \ldots, n$. Let also the transformations*

$\{T_{ijk}\}$ *be consistent with this choice. Let* c_n *be the same for both designs and*

$$\begin{aligned} M_{jin} &= \int_{A_{in}} c_n(x)\, c_n(T_{jin}(x))\, P(Z(x) > u, Z(T_{jin}(x)) > u)\, dG_{in}(x) \\ &\quad - \int_{A_{in}} \int_{A_{jn}} c_n(x)\, c_n(y)\, P(Z(x) > u, Z(y) > u)\, dG_{jn}(y)\, dG_{in}(x). \end{aligned} \tag{12}$$

Under the above conditions the following statements are true:

i) If $M_{jin} \geq 0$ *for all* $i \neq j$*, then* $MSE(\widehat{q}_u^{sts}) \leq MSE(\widehat{q}_u^{sys})$.

ii) If $M_{jin} \leq 0$ *for all* $i \neq j$*, then* $MSE(\widehat{q}_u^{sts}) \geq MSE(\widehat{q}_u^{sys})$.

Proof. Observe first that for this simple stratified design we can express the respective mean square error in the form

$$\begin{aligned} MSE\left(\widehat{q}_u^{sts}\right) &= \frac{1}{|A|^2} E\left\{ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} c_n(X_{in})\, c_n(X_{jn})\, P(Z(X_{in}) > u, Z(X_{jn}) > u) \right. \\ &\quad + \int_A \int_A P(Z(x) > u, Z(y) > u)\, dydx \\ &\quad \left. - \frac{2}{n} \sum_{i=1}^{n} \int_A c_n(X_{in})\, P(Z(X_{in}) > u, Z(x) > u)\, dx \right\}. \end{aligned} \tag{13}$$

Secondly, for the same scenario we also can express the mean square error for respective systematic design in the form

$$\frac{1}{|A|^2}E\left\{\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}c_n\left(X_{in}\right)c_n\left(T_{ji}\left(X_{in}\right)\right)P\left(Z\left(X_{in}\right)>u,Z\left(T_{jin}\left(X_{in}\right)\right)>u\right)\right.$$
$$+\int_A\int_A P\left(Z\left(x\right)>u,Z\left(y\right)>u\right)dydx$$
$$\left.-\frac{2}{n}\sum_{i=1}^{n}\int_A c_n\left(X_{in}\right)P\left(Z\left(X_{in}\right)>u,Z\left(x\right)>u\right)dx\right\}. \tag{14}$$

Now, combining the expressions (13) and (14) we obtain

$$\begin{aligned}
& n^2\left|A\right|^2\left[MSE\left(\widehat{q}_u^{sys}\right)-MSE\left(\widehat{q}_u^{sts}\right)\right] \\
= \; & E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}c_n\left(X_{in}\right)c_n\left(T_{ji}\left(X_{in}\right)\right)P\left(Z\left(X_{in}\right)>u,Z\left(T_{jin}\left(X_{in}\right)\right)>u\right)\right. \\
& \left.-\sum_{i=1}^{n}\sum_{j=1}^{n}c_n\left(X_{in}\right)c_n\left(X_{jn}\right)P\left(Z\left(X_{in}\right)>u,Z\left(X_{jn}\right)>u\right)\right\} \\
= \; & \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\left\{\int_{A_{in}}c_n\left(x\right)c_n\left(T_{ji}\left(x\right)\right)P\left(Z\left(x\right)>u,Z\left(T_{jin}\left(x\right)\right)>u\right)dG_{in}\left(x\right)\right. \\
& \left.-\int_{A_{in}}\int_{A_{jn}}c_n\left(x\right)c_n\left(y\right)P\left(Z\left(x\right)>u,Z\left(y\right)>u\right)dG_{jn}\left(y\right)dG_{in}\left(x\right)\right\} \\
= \; & \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}M_{jin}.
\end{aligned}$$

The result follows easily from the above calculations. ■

Remark 4.2. *If we use the condition for zero bias* (4)*, then we can express*

(12) *in the form*

$$M_{jin} = n^2 \left\{ \int_{A_{in}} \frac{P\left(Z\left(x\right) > u, Z\left(T_{jin}\left(x\right)\right) > u\right)}{g_{jn}\left(T_{jin}\left(x\right)\right)} dx - \int_{A_{in}} \int_{A_{jn}} P\left(Z\left(x\right) > u, Z\left(y\right) > u\right) dydx \right\},$$

where $j \neq i = 1, \ldots, n$. *The first term of the right size of the equality expresses the second moment function of the field* I_u *between the possible pairs of sampling points under systematic sampling and weighted by reciprocal of the design. The second term expresses the same function but between the possible pairs of sampling points under simple stratified sampling. That is, for a field* I_u *given, the value of* M_{jin} *depend of the behavior of* $g_{jn} \circ T_{jin}$ *only.*

Appendix A Calculation of the optimal random design

In this appendix we will find the optimal marginal density that minimized the expression (6) using tools of variational calculus like those in[3].

For that, let $D = \{g : g \text{ is a density function with support } A\}$ and let T be a real function over D given by

$$T\left(g\right) = \int_A \frac{P\left(Z\left(x\right) > u\right)}{g\left(x\right)} dx.$$

Observe that for each $g \in D$, $g + \varepsilon f \in D$ if $\int_A f(x)dx = 0$ and $g + \varepsilon f > 0$ on A and for all sufficiently small ε.

We will prove first that T is strictly convex and then we will find the design g that achieves the minimum. For that observe that the Gâteaux

variation of T at g in the direction f is

$$\frac{\partial}{\partial\varepsilon}T\left(g+\varepsilon f\right)\Big|_{\varepsilon=0}=-\int_{A}\frac{P\left(Z\left(x\right)>u\right)}{g^{2}\left(x\right)}f\left(x\right)dx, \tag{15}$$

for each $g\in D$ and for each f such that $\int_{A}f(x)dx=0$ and $g+\varepsilon f>0$ on A and for all sufficiently small ε. We have that

$$\begin{aligned}
T\left(g+f\right)-T\left(g\right) &= \int_{A}\frac{P\left(Z\left(x\right)>u\right)}{g\left(x\right)+f\left(x\right)}dx-\int_{A}\frac{P\left(Z\left(x\right)>u\right)}{g\left(x\right)}dx \\
&= -\int_{A}\frac{P\left(Z\left(x\right)>u\right)}{g^{2}\left(x\right)+g\left(x\right)f\left(x\right)}f\left(x\right)dx \\
&\geq -\int_{A}\frac{P\left(Z\left(x\right)>u\right)}{g^{2}\left(x\right)}f\left(x\right)dx \\
&= \frac{\partial}{\partial\varepsilon}T\left(g+\varepsilon f\right)\Big|_{\varepsilon=0},
\end{aligned}$$

with equality if $f\equiv 0$. This establishes the strict convexity of T on D provided that g and $g+f\in D$.

Now, T is minimized uniquely at g if $\dfrac{\partial}{\partial\varepsilon}T\left(g+\varepsilon f\right)\Big|_{\varepsilon=0}=0$. But this is possible if in (15), $P\left(Z\left(x\right)>u\right)/g^{2}\left(x\right)\equiv c$, where $c>0$ is a constant. Given this, the expression (15) is reduced to

$$c\int_{A}f(x)=0.$$

Thus,

$$g\left(x\right)=\frac{\sqrt{P\left(Z\left(x\right)>u\right)}}{\int_{A}\sqrt{P\left(Z\left(y\right)>u\right)}dy},$$

the expression given in (7).

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