A probabilistic proof of non-explosion of a non-linear PDE system

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Abstract

Using a representation in terms of a two-type branching particle system, we prove that positive solutions of the system $\dot{u} = Au + uv$, $\dot{v} = Bv + uv$ remain bounded for suitable bounded initial conditions, provided A and B generate processes with independent increments and one of the processes is transient with a uniform power decay of its semigroup. For the case of symmetric stable processes on $\mathbb{R}^1$, this answers a question raised in [LM-W].

1 Introduction and result

Consider the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta_{\alpha_1} u + uv, \quad u_0(x) = \varphi_1(x) \geq 0, \quad x \in \mathbb{R}^d, \\ \frac{\partial v}{\partial t} &= \Delta_{\alpha_2} v + uv, \quad v_0(x) = \varphi_2(x) \geq 0, \quad x \in \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $\Delta_\alpha := -(-\Delta^{\alpha/2})$, $0 < \alpha \leq 2$, stands for the α-Laplacian. In [LM-W] we showed that, for $d = 1$, (1.1) exhibits blow-up if $\min(\alpha_1, \alpha_2) > 1$, and we interpreted this fact in terms of the probabilistic representation of (1.1) by means of a two type branching particle system (which we will recall below): if both motions generated by Δ_{α_1} and Δ_{α_2} are "lazy enough," then the solution of (1.1) grows to infinity in a finite time (provided $\varphi_i \geq c1_D$ for some $c > 0$ and some nonempty interval D).

In [LM] it was shown that, for suitably bounded φ_1, φ_2, (1.1) admits a uniformly bounded solution if $\max(\alpha_1/d, \alpha_2/d) < 1$, i.e. if both motions are "mobile enough." It

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remained an open question what happens if, for $d = 1$, $\min(\alpha_1, \alpha_2) < 1 < \max(\alpha_1, \alpha_2)$. The result of the present note answers this question in a somewhat more general framework, revealing that it is the "most mobile type" only which is responsible for blow-up resp. stability of the system.

Instead of (1.1) we will consider the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= Au + uv, \quad u_0(x) = \varphi_1(x) \geq 0, \quad x \in \mathbb{R}^d, \\ \frac{\partial v}{\partial t} &= Bv + uv, \quad v_0(x) = \varphi_2(x) \geq 0, \quad x \in \mathbb{R}^d, \end{aligned} \tag{1.2}$$

where A and B are the generators of two Markov processes (W_t^A) and (W_t^B) on $\mathbb{R}^d$, having the semigroups (S_t) and (T_t), respectively.

Theorem 1.1 *Assume there exists some $\gamma > 0$ such that for all bounded $D \subset \mathbb{R}^d$,*

$$S_t T_s 1_D(x) \leq c_D t^{-(1+\gamma)}, \quad x \in \mathbb{R}^d, \quad t > 0, \quad s \geq 0, \tag{1.3}$$

where $c_D > 0$ may depend on D but not on x, s and t. Then (1.2) admits a bounded solution, provided

$$\varphi_i \leq c S_1 1_D, \quad i = 1, 2 \tag{1.4}$$

for some bounded $D \subset \mathbb{R}^d$ and some sufficiently small $c > 0$.

Remark 1.2 Condition (1.3) obviously is valid if

$$S_t 1_D(x) \leq c_D t^{-(1+\gamma)}, \quad x \in \mathbb{R}^d, \quad t > 0, \tag{1.5}$$

and if (W_t^A) and (W_t^B) both have independent increments (since then the two semigroups commute).

Corollary 1.3 *The system (1.1) admits a bounded solution if $\alpha_1/d < 1$ and if (1.4) is satisfied.*

Indeed, by the well-known scaling and unimodality properties of the symmetric stable densities, (1.5) holds with $\gamma := \frac{d}{\alpha_1} - 1$. 2

Remark 1.4 For $A = B$, (1.2) renders a special case of the so called Fujita equation

$$\frac{\partial u}{\partial t} = Au + u^\beta, \tag{1.6}$$

which, for the case $A = \Delta$, was studied in [Fu]. For integer $\beta \geq 2$, Nagasawa and Sirao [N-S] obtained a probabilistic representation of the solution of (1.6), which was further developed in [LM] into the form we are going to use here (cf. [LM-W], and (2.1) below). It is instructive to compare the representation obtained in [LM] with H.P. McKean's representation of the Kolmogorov-Petrovskii-Piskunov equation

$$\frac{\partial u}{\partial t} = Au + u^\beta - u \tag{1.7}$$

(cf. [McK]). Both are expectations of a functional of one and the same branching particle system, where the functionals differ by a factor "exponential of the tree length," which can be interpreted as a Feynman-Kac term correcting for the difference u between (1.6) and (1.7). (We owe this observation to A. Etheridge (personal communication.))

Remark 1.5 For $A = B = \Delta$, and $u^{p_i} v^{q_i}$ $(i = 1, 2)$ instead of uv in lines 1 and 2 of (1.2), respectively, Escobedo and Levine [E-L] showed, under the assumptions $p_1 > 1$, $p_1 q_2 > 0$, and $p_1 + q_2 \leq p_2 + q_2$, that the system admits global solutions if $2/d < p_1 + q_1 - 1$, and blows up otherwise. In this note we are focussing on another case, namely $p_1 = p_2 = q_1 = q_2 = 1$, and possibly *different* operators A and B.

2 The probabilistic framework

In order to recall the probabilistic solution of (1.2), let us introduce some concepts and notations.

Let $\mathcal{T}_t$ be a *Yule tree* (i.e. a continuous time Galton-Watson tree with offspring distribution δ_2) with branching rate 1, growing from one ancestor at time 0 up to time t. For our purpose, it is convenient to think of $\mathcal{T}_t$ being generated as follows: The "original" branch gives, in between times 0 and t, rise to offspring branches at rate ds, each of which, when born at time s, gives again, between times s and t, rise to offspring branches at rate dr, and so on.

For each realization τ of $\mathcal{T}_t$, we denote by $L(\tau)$ the *length* of τ, i.e. the sum of the branch lengths of τ. In addition, for each realization of $\mathcal{T}_t$, we perform a colouring of each of the branches of τ by the "colours" a and b, in such a way that an offspring branch always gets a colour different from that of its parent branch. The *coloured tree* $\tau^{(i)}$ (where $i = a$ or $i = b$) is thus determined by the colour i of the original branch.

For such a coloured tree $\tau^{(i)}$, and $x \in \mathbb{R}^d$, let $(X_s^{x,\tau^{(i)}})_{0 \leq s \leq t}$ be a two-type process indexed by $\tau^{(i)}$ which evolves as follows. An original particle starts in x and moves up to time t with A-motion if $i = a$ and with B-motion if $i = b$. This particle generates offspring particles at its respective position according to the branching points of $\tau^{(i)}$, which then move on independently according to the colouring of $\tau^{(i)}$, and so on. For every time $s \in [0, t]$, this gives rise to a random population of coloured particles on $\mathbb{R}^d$, which we denote by $X_s^{x,\tau^{(i)}}$. Let

$$X_s^{x,\tau^{(i)}} = X_{s,a}^{x,\tau^{(i)}} + X_{s,b}^{x,\tau^{(i)}}$$

be the decomposition of $X_s^{x,\tau^{(i)}}$ into its subpopulations of colours a and b.

Finally, for every counting measure $\nu = \sum_n \delta_{y_n}$ and $\varphi : \mathbb{R}^d \to \mathbb{R}_+$, we write

$$\widehat{\varphi}(\nu) := \prod_n \varphi(y_n).$$

We now recall the probabilistic representation of the solution of (1.2), of which we include a proof for the sake of self-containedness.

Proposition 2.1. **([LM])** *The solution of (1.2) is given by* $u_t(x) = w_t(x,a)$, $v_t(x) = w_t(x,b)$, *where*

$$w_t(x,i) = \mathbb{E}\left[\widehat{\varphi}_1\left(X_{t,a}^{x,\mathcal{T}_t^{(i)}}\right)\widehat{\varphi}_2\left(X_{t,b}^{x,\mathcal{T}_t^{(i)}}\right)e^{L(\mathcal{T}_t)}\right], \quad i = 1,\, 2. \tag{2.1}$$

Proof Conditioning on the length l of the original branch of $\mathcal{T}_t$ renders

$$\begin{aligned} w_t(x,i) &= e^{-t}\,\mathbb{E}\left[\widehat{\varphi}_1\left(X_{t,a}^{x,\mathcal{T}_t^{(i)}}\right)\widehat{\varphi}_2\left(X_{t,b}^{x,\mathcal{T}_t^{(i)}}\right)e^{L(\mathcal{T}_t)}\middle|\, l \geq t\right] \\ &\quad + \int_0^t dr\, e^{-r}\,\mathbb{E}\left[\widehat{\varphi}_1\left(X_{t,a}^{x,\mathcal{T}_t^{(i)}}\right)\widehat{\varphi}_2\left(X_{t,b}^{x,\mathcal{T}_t^{(i)}}\right)e^{L(\mathcal{T}_t)}\middle|\, l = r\right]. \end{aligned}$$

Writing $W_s^{(x,i)}$ for the position of the original particle at time $s \leq l$, it follows that

$$\begin{aligned} w_t(x,i) &= e^{-t}e^{t}\,\mathbb{E}\left[\varphi_i\left(W_t^{(x,i)}\right)\right] + \int_0^t dr\, e^{-r}e^{r}\,\mathbb{E}\left[w_{t-r}\left(W_r^{(x,i)},a\right)\, w_{t-r}\left(W_r^{(x,i)},b\right)\right] \\ &= \begin{cases} S_t\varphi_1(x) + \displaystyle\int_0^t dr\, S_r\left(w_{t-r}(\cdot,a)\; w_{t-r}(\cdot,b)\right)(x) & \text{for } i = a, \\[2ex] T_t\varphi_2(x) + \displaystyle\int_0^t dr\, T_r\left(w_{t-r}(\cdot,a)\; w_{t-r}(\cdot,b)\right)(x) & \text{for } i = b. \end{cases} \end{aligned}$$

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3 Proof of Theorem 1.1

Let $\mathbf{T}_t$ be the set of trees which arise as realizations of $\mathcal{T}_t$ (as described in Section 2). On $\mathbf{T}_t$ we define the measure μ_t which arises by reweighing the distribution of $\mathcal{T}_t$ by $e^{L(\tau)}$:

$$\mu_t(d\tau) := \mathbb{P}\left[\mathcal{T}_t \in d\tau\right] e^{L(\tau)}. \tag{3.1}$$

In view of (2.1) and (3.1), we are going to analyse $u_t(x) = w_t(x,a)$ and $v_t(x) = w_t(x,b)$, where

$$w_t(x,i) := \int \mathbb{E}\left[\widehat{\varphi}_1\left(X_{t,a}^{x,\tau_t^{(i)}}\right)\widehat{\varphi}_2\left(X_{t,b}^{x,\tau_t^{(i)}}\right)\right]\,\mu_t(d\tau).$$

For $\tau \in \mathbf{T}_t$, let $K(\tau)$ denote the number of inner nodes of τ. Let us write

$$\mu_t^{(\geq 1)}(d\tau) := \mu_t(d\tau)1_{[K(\tau)\geq 1]} \quad \text{and} \quad \mu_t^k(d\tau) := \mu_t(d\tau)1_{[K(\tau)=k]}.$$

For example, if ρ denotes the tree in $\mathbf{T}_t$ which consists of one single branch, then $K(\rho) = 0$, and $\mu^{(0)}(\{\rho\}) = e^{-t}e^t = 1$. If $\sigma(r)$, $0 \leq r \leq t$, denotes the tree with one single branching point at time r, then $\mu^{(1)}(d(\sigma(r))) = 1_{[0,t]}(r)e^{-r}e^r e^{-2(t-r)}e^{2(t-r)}\,dr = 1_{[0,t]}(r)\,dr$.

Definition 3.1 For $\tau \in \mathbf{T}_t$ with $K(\tau) \geq 1$, we denote by $r = r(\tau)$ the time of its first branching, and by τ' and τ'' ($\in \mathbf{T}_{t-r}$) its two subtrees originating from there. Let us also introduce the notation

$$\begin{aligned}
M_t(d\tau) &:= \mathbb{P}[\mathcal{T}_t \in d\tau]\,,\\
M_t^{(\geq 1)}(d\tau) &:= \mathbb{P}[\mathcal{T}_t \in d\tau]\,1_{[K(\tau)\geq 1]},\\
M_t^{(k)}(d\tau) &:= \mathbb{P}[\mathcal{T}_t \in d\tau]\,1_{[K(\tau)=k]}.
\end{aligned}$$

Lemma 3.2 *(a)* $M_t^{(\geq 1)}(d\tau) = 1_{[0,t]}(r)e^{-r}M_{t-r}(d\tau')\,M_{t-r}(d\tau'')\,dr.$

(b) For $k \geq 1$, $M_t^{(k)}(d\tau) = 1_{[0,t]}(r)e^{-r}\,dr\,\dfrac{1}{k}\displaystyle\sum_{j=0}^{k-1} M_{t-r}^{(j)}(d\tau')\,M_{t-r}^{(k-1-j)}(d\tau'').$

Proof (a) is immediate from the definition of Yule trees. (b) results from (a) and the fact that the genealogy of a Yule tree is identical in law with that of a Polya urn (starting with two balls after the first branching point). Consequently, given there are k inner nodes (and therefore $k+1$ leaves), the total number of leaves of one of the two subtrees, say $\mathcal{T}'_{t-r}$, is $1+J$, where J is uniformly distributed on $\{0,1,\ldots,k-1\}$. 2

Corollary 3.3 *For* $k = 1$,

$$\mu_t^{(k)}(d\tau) = dr\,\frac{1}{k}\sum_{j=0}^{k-1}\mu_{t-r}^{(j)}(d\tau')\,\mu_{t-r}^{(k-1-j)}(d\tau''). \tag{3.2}$$

Proof This is immediate from the previous lemma and the fact that $L(\tau) = r + L(\tau') + L(\tau'')$. 2

Definition and Remark 3.4 We write $u_t^{(k)}(x) := w_t^{(k)}(x,a)$, $v_t^{(k)}(x) := w_t^{(k)}(x,b)$, and

$$w_t^{(k)}(x,i) := \int \mathbb{E}\left[\widehat{\varphi}_1\left(X_{t,a}^{x,\tau_t^{(i)}}\right)\widehat{\varphi}_2\left(X_{t,b}^{x,\tau_t^{(i)}}\right)\right]\,\mu_t^{(k)}(d\tau). \tag{3.3}$$

We obviously have

$$u_t(x) = \sum_{k=0}^{\infty} u_t^{(k)}(x), \quad v_t(x) = \sum_{k=0}^{\infty} v_t^{(k)}(x), \tag{3.4}$$

and from Corollary 3.4 and (3.3) it is clear that for $k \geq 1$

$$u_t^{(k)}(x) = \frac{1}{k}\sum_{j=0}^{k-1}\int_0^t S_r\left(u_{t-r}^{(j)}\,u_{t-r}^{(k-1-j)}\right)(x)\,dr, \tag{3.5}$$

with the analogous formula being valid for $v_t^{(k)}$. In order to bound $u_t^{(k)}$ in a suitable manner, we are going to work with a decomposition of $u_t^{(k)}$ along the "second branch," splitting off successively all the offspring of newborn type a-individuals. To write this decomposition in a neat form, consider the following stickbreaking scheme:

Let J_1 be uniformly distributed on $\{0,1,\dots,k-1\}$; given J_1 let J_2 be uniformly distributed on $\{0,1,\dots,k-J_1-2\}$; given (J_1,J_2), let J_3 be uniformly distributed on $\{0,1,\dots,k-J_1-J_2-3\}$, and so on, till $k-J_1-J_2-\cdots-J_N-N=0$. Iterating (3.5), we arrive at the following decomposition of $u_t^{(k)}$:

$$\begin{aligned} u_t^{(k)}(x) \;=\; \mathbb{E}\Bigg[\int_0^t dr_1\,S_{r_1}\Bigg(u_{t-r_1}^{(J_1)}\int_0^{t-r_1} dr_2\,T_{r_2}\Bigg(u_{t-r_1-r_2}^{(J_2)} \\ \cdots\int_0^{t-r_1-\cdots-r_{N-1}} dr_N\,T_{r_N}\left(u_{t-r_1-\cdots-r_N}\,T_{t-r_1-\cdots-r_N}\varphi_2\right)\cdots\Bigg)\Bigg)(x)\Bigg]. \end{aligned} \tag{3.6}$$

With the probability weights

$$\mathbb{P}\left[N=n; J_1=j_1,\dots,J_n=j_n\right] =: \pi(n;j_1,\dots,j_n),$$

(3.6) can be rewritten as

$$\begin{aligned} u_t^{(k)}(x) \;=\; \sum_n \sum_{j_1+\cdots+j_n=0}^{k-n}\pi(n;j_1,\dots,j_n)\int_0^t dr_1\,S_{r_1}\Bigg(u_{t-r_1}^{(j_1)}\int_0^{t-r_1} dr_2\,T_{r_2}\Bigg(u_{t-r_1-r_2}^{(j_2)} \\ \cdots\int_0^{t-r_1-\cdots-r_{n-1}} dr_n\,T_{r_n}\left(u_{t-r_1-\cdots-r_n}\,T_{t-r_1-\cdots-r_n}\varphi_2\right)\cdots\Bigg)\Bigg)(x) \end{aligned}$$

$$=: \sum_{n} \sum_{j_1+\cdots+j_n=0}^{k-n} \pi(n; j_1, \ldots, j_n) A_t(n; j_1, \ldots, j_n). \tag{3.7}$$

Proposition 3.5 *Assume (S_t) and (T_t) satisfy (1.3), and assume*

$$\varphi_i \le S_1 1_D, \quad i = 1, 2, \tag{3.8}$$

for some bounded $D \subset \mathbb{R}^d$. Then

$$u_s^{(k)}(x) \le \left(2^{2+\gamma}\right)^k c_D^{k+1} \gamma^{-k} (s+1)^{-(1+\gamma)} \tag{3.9}$$

uniformly in $s \ge 0$ and $x \in \mathbb{R}^d$, and

$$v_s^{(k)}(x) \le \left(2^{2+\gamma}\right)^k c_D^{k+1} \gamma^{-k} T_s \varphi_2(x). \tag{3.10}$$

Proof We will use induction over k.
For $k = 0$, $u_s^{(0)}(x) = S_s \varphi_1(x) \le S_{s+1} 1_D(x) \le c_D (s+1)^{-(1+\gamma)}$ by (3.8) and (1.3).
Now assume (3.9) holds true for $l = 0, \ldots, k-1$. Since for all $s > 0$ there holds

$$\int_0^s (r+1)^{-(1+\gamma)} \, dr = \frac{1}{\gamma} \left(1 - (s+1)^{-\gamma}\right) \le \frac{1}{\gamma},$$

by the induction assumption the term $A_t(n; j_1, \ldots, j_n)$ in the decomposition (3.7) is bounded by

$$c_D^{j_1+\cdots+j_n+n} \left(2^{2+\gamma}\right)^{j_1+\cdots+j_n} \left(\frac{1}{\gamma}\right)^{j_1+\cdots+j_n+(n-1)} \int_0^t dr_1 \; (t+1-r_1)^{-(1+\gamma)} S_{r_1} T_{t-r_1} \varphi_2(x)$$

$$\le c_D^k \left(2^{2+\gamma} \frac{1}{\gamma}\right)^{k-1} c_D \int_0^t (t+1-r)^{-(1+\gamma)} (r+1)^{-(1+\gamma)} \, dr \tag{3.11}$$

where we again used (1.3) and (3.8) in the last inequality. The induction argument is completed by observing that

$$\begin{aligned} \int_0^t (t+1-r)^{-(1+\gamma)} (r+1)^{-(1+\gamma)} \, dr &\le 2 \left(\frac{t}{2} + 1\right)^{-(1+\gamma)} \int_0^{\frac{t}{2}} (r+1)^{-(1+\gamma)} \, dr \\ &\le 2^{2+\gamma} \frac{1}{\gamma} (t+1)^{-(1+\gamma)}. \end{aligned}$$

In order to prove (3.10) first observe that $v_t^{(k)}$ has the same representation as $u_t^{(k)}$ in (3.7), but with S_{r_1} replaced by T_{r_1}. Replacing S_{r_1} by T_{r_1} also in the LHS of (3.11) we obtain (3.10). 2

To conclude the proof of Theorem 1.1 it suffices to remark that, if the initial conditions φ_1 and φ_2 both are multiplied by a factor $c > 0$, then a factor c^{k+1} enters into both $u^{(k)}$ and $v^{(k)}$. Hence, due to (3.4), (3.9) and (3.10), $u_t(x)$ and $v_t(x)$ are majorized by convergent geometric series, provided (1.4) holds true with sufficiently small $c > 0$. 2

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References

[E-L] Escobedo, M. and Levine, H. (1995). Critical blowup and global existence numbers for a weakly coupled system of reaction-diffusion equations. *Arch. Rational Mech. Anal.* Vol. 129, 47-100.

[Fu] Fujita, H. (1966). On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I* Vol. 13, 109-124.

[LM] López-Mimbela, J.A. (1996). A probabilistic approach to existence of global solutions of a system of nonlinear differential equations. *Aportaciones Matemáticas Notas de Investigación* Vol. 12, 147-155.

[LM-W] López-Mimbela, J.A. and Wakolbinger, A. (1998). Length of Galton-Watson trees and blow-up of semilinear systems. *J. Appl. Prob.* Vol. 35, 802-811.

[McK] McKean, H.P. (1975). Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure and Applied Math.* Vol. 28, 323-331.

[N-S] Nagasawa, M. and Sirao, T. (1969). Probabilistic treatment of the blowing up of solutions for a nonlinear integral equation. *Trans. Amer. Math. Soc.* Vol. 139, 301-310.