$$y_k(s) = \left(\frac{a}{n\pi}\right)^k C_n\left\{\cos\left(\frac{n\pi s}{a} - a_{nk}\right) + R_k\right\},$$

then for $k$ sufficiently large $R_k$ is uniformly small on the interval $(-a, a)$. Now we may write

$$R_k = \frac{C_{n+1}}{C_n}\left(\frac{n}{n+1}\right)^k \cos\left(\frac{(n+1)\pi s}{a} - \alpha_{n+1k}\right) + \frac{C_{n+2}}{C_n}\left(\frac{n}{n+2}\right)^k \cos\left(\frac{(n+2)\pi s}{a} - \alpha_{n+2k}\right) + \cdots .$$

Since the Fourier series (1) is convergent, the $C$'s have a maximum absolute value $C$, and hence throughout the interval $(-a, a)$

$$|R_k| \leqq \frac{Cn^k}{|C_n|} T,$$

where

$$T = \frac{1}{(n+1)^k} + \frac{1}{(n+2)^k} + \cdots \leqq \int_0^\infty \frac{dx}{(n+x)^k} = \frac{1}{(k-1)n^{k-1}} \cdot$$

It follows that uniformly on the interval $(-a, a)$

$$\lim_{k\to\infty} |R_k| \leqq \lim_{k\to\infty} \frac{Cn}{|C_n|(k-1)} = 0,$$

and our theorem is a consequence.

---

## FRACTIONS*

L. R. FORD, Armour Institute of Technology

Perhaps the author owes an apology to the reader for asking him to lend his attention to so elementary a subject, for the fractions to be discussed in this paper are, for the most part, the halves, quarters, and thirds of arithmetic. But the fact is that the writer has, for some years, been looking on these entities in a somewhat new way. Here will be found a geometric picturization which will be novel to the reader and which will supply a visual representation of arithmetical results of diverse kinds.

The idea of representing a fraction by a circle is one at which the author arrived by an exceedingly circuitous journey. It began with the Group of Picard. In the geometric treatment of this group as carried out by Bianchi in accordance with the general ideas of Poincaré certain invariant families of spheres appear. These spheres, which are found at the complex rational fractions, are mentioned later in this paper. They suggest analogous known invariant families of circles at real rational points in the theory of the Elliptic Modular Group in the com-

---

* Some of the material of this article was presented in an address before the American Mathematical Society at Lawrence, Kansas, November 28, 1936. Other parts have been given in lectures at the Rice Institute, the University of Texas, Northwestern University, and the Armour Institute.

plex plane. Finally, it became apparent that this intricate scaffolding of group theory could be dispensed with and the whole subject be built up in a completely elementary fashion. It is this treatment that is undertaken here.

**1. The geometric representation.** We begin with real fractions. These are usually represented, along with real irrationals, by points on a line. Let this line be the $x$-axis in an $xy$-plane of rectangular coördinates.

Through each rational point $x=p/q$, where $p$ and $q$ are integers and the fraction is in its lowest terms, we construct a circle of radius $1/(2q^2)$ tangent to the $x$-axis and lying in the upper half-plane. It is this circle, which touches the $x$-axis at the point usually taken to represent the fraction, which will be the geometric picture of the fraction. The integers are represented by circles of radius $1/2$; the fractions $1/3$, $2/3$, $4/3$, etc., by circles of radius $1/18$; and so on. Every small interval of the $x$-axis contains points of tangency of infinitely many of these circles.*

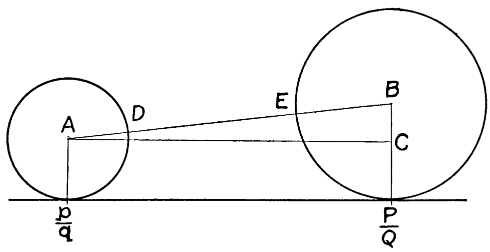


FIG. 1

Let $p/q$ and $P/Q$ be two different fractions in their lowest terms. Consider the distance between the centers of their representative circles. In the figure the horizontal distance $AC$ is $|(P/Q)-(p/q)|$ and the vertical distance $CB$ is the difference of the radii, $|(1/2Q^2)-(1/2q^2)|$. We have

$$\begin{aligned} AB^2 &= \left(\frac{P}{Q}-\frac{p}{q}\right)^2+\left(\frac{1}{2Q^2}-\frac{1}{2q^2}\right)^2 \\ &= \left(\frac{1}{2Q^2}+\frac{1}{2q^2}\right)^2+\frac{(Pq-pQ)^2-1}{Q^2q^2} \\ &= (AD+EB)^2+\frac{(Pq-pQ)^2-1}{Q^2q^2}. \end{aligned}$$

---

* Families of circles tangent to the real axis at the rational points are intimately involved in the geometry of the modular group and in the theory of quadratic forms. In these connections they have been used for some time. Circles defined as in the text except for the more general radius $1/(2hq^2)$ were used, with various values of $h$, by the present author in papers in the Proceedings of the Edinburgh Mathematical Society in 1917. They were probably used earlier by others.

These circles are called "Speiser circles" by some writers. This name appears to be due to a note of a dozen lines by A. Speiser in the Actes de la Société Helvétique des Sciences Naturelles in 1923. Mention should also be made of a very interesting ninety-page booklet by Züllig, Geometrische Deutung unendlicher Kettenbrüche (1928).

If $|Pq-pQ|>1$, then $AB>AD+EB$, and the two circles are wholly external to one another.

If $|Pq-pQ|=1$, then $AB=AD+EB$, and the two circles are tangent.

If $|Pq-pQ|<1$ we have, since $Pq-pQ$ is an integer, that $Pq-pQ=0$. From this $P/Q=p/q$, which is contrary to the assumption that the fractions are different. So this last inequality is not possible. We may state the following result:

THEOREM 1. *The representative circles of two distinct fractions are either tangent or wholly external to one another.*

**2. Adjacent fractions.** We shall call two fractions $p/q$ and $P/Q$ *adjacent* if their representative circles are tangent. The condition for this is that $|Pq-pQ|=1$. (We assume here and henceforth that fractions appear in their lowest terms.) We shall prove

THEOREM 2. *Each fraction $p/q$ possesses an adjacent fraction.*

That integers $P$ and $Q$ exist satisfying the equation $|Pq-pQ|=1$ (or, arranging signs suitably, satisfying $Pq-pQ=1$) is a fundamental proposition of the theory of numbers. A great many proofs have been given. For the sake of completeness we prove the theorem here. The proof given is so arranged as to cover the case of complex fractions which will appear subsequently in this paper.

The proof is by induction. Clearly the theorem holds for $|q|=1$, since $p/1$ then has the adjacent fraction $(p+1)/1$. To prove the theorem in general we assume that all fractions whose denominators are less in absolute value than $|q|$ possess adjacent fractions and prove it for the fraction $p/q$.

Let $n$ be the integer nearest to $p/q$, whence we can write, with $m$ integral,

$$\frac{p}{q} = n + \frac{m}{q} = \frac{nq+m}{q}, \qquad 0 < |m| < |q|.$$

Since $|m|<|q|$, then $q/m$ has an adjacent fraction $r/s$, so $|sq-rm|=1$. Then the fraction

$$\frac{P}{Q} = n + \frac{s}{r} = \frac{nr+s}{r}$$

is adjacent to $p/q$; for

$$|Pq - pQ| = |(nr+s)q - (nq+m)r| = |sq - rm| = 1.$$

This establishes the theorem.

We can now give a formula for all fractions adjacent to $p/q$.

THEOREM 3. *If $P/Q$ is adjacent to $p/q$ then all fractions adjacent to $p/q$ are*

$$\frac{P_n}{Q_n} = \frac{P+np}{Q+nq},$$

*where $n$ takes on all integral values.*

We find readily that the fractions given are adjacent to $p/q$, for

$$|(P+np)q-p(Q+nq)|=|Pq-pQ|=1.$$

We find that $P_n/Q_n$ and $P_{n+1}/Q_{n+1}$ are adjacent to one another, since

$$|[P+np][Q+(n+1)q]-[P+(n+1)p][Q+nq]|=|Pq-pQ|=1.$$

The circles corresponding to these fractions form a ring around the circle of $p/q$, all tangent to the circle of $p/q$, and each tangent to the circles which precede and follow it in the sequence (Figure 2). That this ring completely surrounds the circle of $p/q$ we see from

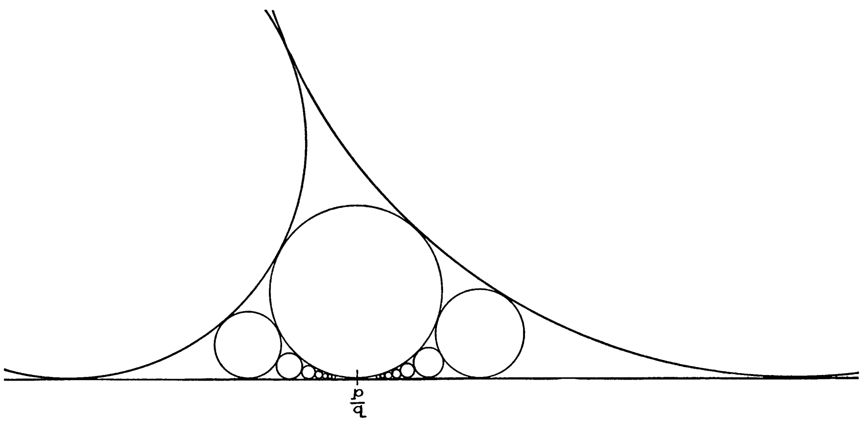


FIG. 2

$$(a)\qquad \frac{P_n}{Q_n}=\frac{p}{q}+\frac{Pq-pQ}{q(Q+nq)}=\frac{p}{q}\pm\frac{1}{q^2(n+Q/q)}\cdot$$

When $n\rightarrow+\infty$, $P_n/Q_n$ approaches $p/q$ from one side; when $n\rightarrow-\infty$, $P_n/Q_n$ approaches $p/q$ from the other side.

It is obvious from the geometric picture that it is not possible to draw a circle lying in the upper half-plane, touching the $x$-axis, and tangent to the circle of $p/q$ but not intersecting the circles of the ring surrounding the circle of $p/q$. It follows that there are no further fractions adjacent to $p/q$.

THEOREM 4. *Of the fractions adjacent to* $p/q$ ($|q|>1$) *exactly two have denominators numerically smaller than* $q$.

That two circles of the preceding ring about the circle of $p/q$ are larger than that circle is fairly evident from the geometrical picture. We see also that

$$|Q+nq|<|q|,\qquad\text{or}\qquad |n+Q/q|<1$$

for exactly two values of $n$; namely, those integers between which $-Q/q$ lies.

For one of these two values of $n$, $n+Q/q$ is positive, for the other, negative. We see then from $(a)$ that one of the fractions of Theorem 4 is greater than $p/q$, the other is less.

We shall add one further circle to our collection. The fraction $1/0$ is formally adjacent to the integers $p/1$, since $1\cdot 1-p\cdot 0=1$. We take as its circle the line $y=1$, which touches the circles of all the integers. We consider as the interior of the circle that part of the plane above the line.

**3. The mesh triangles.** The parts of the upper half plane exterior to all the circles of the system consist of an infinite number of circular arc triangles to which the name of "mesh triangles" will be given. Any two sides of a mesh triangle lie on circles belonging to adjacent fractions. Its angles are zero.

A further study of the mesh triangle will be made in a later section, where its properties will be used in the theory ot approximation.

4. **Farey's series.** Let a curve be drawn across the set of circles that we have just defined. We start with a point $A_0$ of the upper half-plane and trace a continuous curve $L$ which remains in the upper half plane except that its terminal point, if any, may possibly lie on the $x$-axis. We consider the fractions whose circles are passed through in succession by $L$. Each circle $K$ is surrounded by mesh triangles. If $L$ issues from $K$ into one of these triangles and if it does not return to $K$ it will on leaving the triangle pass in general into a circle tangent to $K$. We agree that $K$ shall not be counted twice if $L$ passes out of $K$ and then returns to $K$ again without entering another circle. We also make the convention that if $L$ touches two circles at their point of tangency without entering either then one or the other shall be considered as crossed by $L$ (the larger circle, say, or the one on the left, or the one whose fraction is the greater). We can then state as an established proposition the following:

PRINCIPLE. *If two circles of the system are penetrated in succsssion by L then the corresponding two fractions are adjacent.*

As a first illustration let $L$ be a line, $y=k$, parallel to the $x$-axis, starting say at a point on the positive $y$-axis and stopping at $x=1$. The points of tangency with the $x$-axis of the circles through which $L$ passes are arranged in order from left to right; that is, the corresponding fractions are arranged in numerical order. If $1/(n+1)^2<k<1/n^2$, $L$ intersects the circles of all fractions in the interval $0\leqq x\leqq 1$ whose denominators do not exceed $n$ and the circles of no other fractions. These fractions arranged in numerical order constitute what is known as a Farey's series of order $n$ for the interval.

Certain theorems concerning Farey's series are now readily established.* Let $p/q<p'/q'$ be successive fractions of the series, all the numbers being positive.

* See, for example, Landau, Vorlesungen über Zahlentheorie, 1927, Band I, pp. 98–100. Farey discussed these fractions more than a century ago.

(1) $p'q-pq'=1$. This results from the Principle stated earlier in this section.

(2) $q+q'\geqq n+1$. For $(p+p')/(q+q')$ is a fraction between $p/q$ and $p'/q'$; since it does not belong to the series its denominator is greater than $n$.

(3) For any number $\omega$ of the interval there is a fraction $p/q$ of Farey's series of order $n$ such that

$$\left|\frac{p}{q}-\omega\right|\leqq\frac{1}{(n+1)q}.$$

The number $\omega$ lies in an interval formed by a Farey's fraction $p/q$ and an adjacent fraction $(p+p_1)/(q+q_1)$ not of the series ($p_1/q_1$ being the Farey's fraction preceding or following $p/q$). The length of this interval is $1/(q+q_1)q$, where $q+q_1\geqq n+1$.

If we take other simple forms for the curve $L$ we get series analogous to Farey's. Thus if we take a line with a positive slope starting above $y=1$ and terminating at the origin we have all non-negative fractions such that

$$pq \leqq n,$$

where $n$ is suitably chosen. If these be arranged in order of numerical magnitude then successive fractions satisfy (1) above; (2) is replaced by $(p+p')(q+q')\geqq n+1$.

The series such that $0<p\leqq n$ may be got by taking for $L$ the arc of a suitable circle tangent to the $x$-axis at the origin.

**5. The problem of approximation.** Dirichlet showed by elementary means that if $\omega$ is a real irrational number then the inequality

$$\left|\frac{p}{q}-\omega\right|<\frac{k}{q^2} \tag{1}$$

is satisfied by infinitely many fractions $p/q$ if $k=1$. If, however, $\omega=r/s$, a rational, then the inequality is satisfied by only a finite number of rationals $p/q$ however large $k$ be chosen. For

$$\left|\frac{r}{s}-\frac{p}{q}\right|=\left|\frac{rq-ps}{sq}\right|\geqq\frac{1}{|sq|}=\frac{\left|\frac{q}{s}\right|}{q^2}>\frac{k}{q^2}$$

except for a finite number of values of $q$ for which $|q|\leqq|ks|$; also for each $q$ there is clearly only a finite number of fractions satisfying the inequality.

Various suites of fractions approximating to $\omega$ satisfy (1) with suitable $k$. The convergents in the ordinary continued fraction for $\omega$ satisfy the inequality with $k=1$. A suite of Hermite admits the smaller value $k=1/\sqrt{3}$.

The determination of the best value of $k$ is due to Hurwitz,* who proved the following theorem.

* Mathematische Annalen, vol. 39, 1891, pp. 279–284. See also Borel, Journal de Mathématiques, 5th ser., vol. 9, 1903, pp. 329 ff; L. R. Ford, Proceedings of the Edinburgh Mathematical Society, vol. 35, 1917.

THEOREM 5. *If $k=1/\sqrt{5}$, then for each irrational $\omega$ there are infinitely many fractions $p/q$ satisfying the inequality* (1).

*If $k<1/\sqrt{5}$, then there are irrationals $\omega$ such that only a finite number of fractions $p/q$ satisfy* (1).

Let the curve $L$ of the previous section be a vertical line which terminates at the irrational $\omega$. Then $L$ intersects infinitely many circles of the system we are considering and is tangent to none. If $L$ cuts the circle of $p/q$ then the distance from $\omega$ to $p/q$ is less than the radius:

$$\left| \frac{p}{q} - \omega \right| < \frac{1}{2q^2} \cdot$$

The curve $L$ thus provides us with an infinite suite of all those fractions satisfying (1) when $k=1/2$. If $p/q$ and $p'/q'$ are successive fractions of this suite then, from the Principle of the preceding section, we have $p'q-pq'=\pm 1$. Also, we see geometrically that $|q'|>|q|$. The convergence of the suite to $\omega$ is then immediately proved.

We propose now to prove Hurwitz' theorem by a study of the circles of our system. This elementary proof will be free of continued fractions on the one hand and of the theory of the Modular Group on the other.

The line $L$ cuts across infinitely many mesh triangles. The first part of the theorem of Hurwitz is a consequence of the following result which will now be proved:

THEOREM 6. *Of the three fractions whose circles form the boundary of a mesh triangle which $L$ crosses, at least one satisfies* (1) *with $k=1/\sqrt{5}$.*

We consider the mesh triangle in some detail. Let $P/Q$, $p/q$, $p_1/q_1$ be the fractions whose circles bound the area, where

$$0 < Q \leqq q < q_1 = q+Q, \quad p_1 = p + P.$$

We suppose, to fix the picture, that the largest of the three circles is on the right; that is, $P/Q>p/q$, whence $Pq-pQ=+1$. If not, we could make suitable changes of sign in the following; or, more simply, we could reflect the entire figure in the $y$-axis by changing the signs of the numerators and of $\omega$ and at the conclusion of our analysis we could reflect again.

The vertex $A$ (see Figure 3) is the point of contact of the circles of $p/q$ and $P/Q$. It divides the line of centers of these circles in the ratio of their radii, $1/2q^2:1/2Q^2$, or $Q^2:q^2$. The abscissa of this point is found by an early formula of analytic geometry, or by the methods of high school plane geometry, to be

$$a = \frac{q^2 \cdot \dfrac{p}{q} + Q^2 \cdot \dfrac{P}{Q}}{q^2 + Q^2} = \frac{pq + PQ}{q^2 + Q^2} \cdot$$

The abscissas of $B$ and $C$, the other vertices, may be written down by an interchange of letters,

$$b = \frac{pq + p_1q_1}{q^2 + q_1^2}, \qquad c = \frac{PQ + p_1q_1}{Q^2 + q_1^2}.$$

In order that $L$ cross the mesh triangle under consideration it is necessary and sufficient that $\omega$ lie in the interval whose right end is $c$ and whose left end

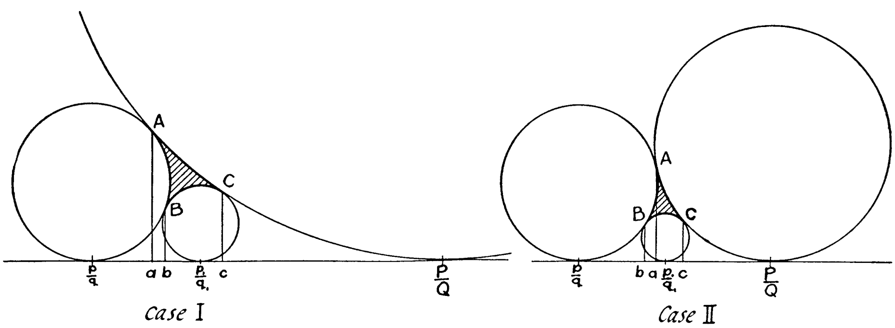


FIG. 3

is the lesser of the two quantities $a$ and $b$. We find

$$a - \frac{p}{q} = \frac{Q}{q(q^2 + Q^2)}, \qquad b - \frac{p}{q} = \frac{q_1}{q(q^2 + q_1^2)},$$

whence subtracting and remembering that $q_1 = q + Q$, we have

$$b - a = \frac{q^2 - qQ - Q^2}{(q^2 + Q^2)(q^2 + q_1^2)}.$$

We put $q/Q = s \geqq 1$. The sign of $b - a$ is (dividing the preceding numerator by $Q^2$) the same as the sign of $s^2 - s - 1$. Now, factoring,

$$s^2 - s - 1 = \left(s + \frac{\sqrt{5} - 1}{2}\right)\left(s - \frac{\sqrt{5} + 1}{2}\right),$$

and the square root of 5 first enters the picture. Here the first factor is positive and the sign is determined by the second factor. We have two cases to consider.

Case I. $a < b$, and * $s > \frac{1}{2}(\sqrt{5} + 1)$. We shall show that $P/Q$ satisfies the inequality (1) with $k = 1/\sqrt{5}$. We have

$$\left|\frac{P}{Q} - \omega\right| < \frac{P}{Q} - a = \frac{q}{Q(q^2 + Q^2)} = \frac{s}{s^2 + 1} \cdot \frac{1}{Q^2}.$$

* Here and in many subsequent relations the sign of equality is not possible, for one member is rational and the other is irrational.

If now

$$\frac{s}{s^2+1} > \frac{1}{\sqrt{5}}$$

we have

$$s^2 - \sqrt{5}\,s + 1 < 0, \qquad \left(s - \frac{\sqrt{5}+1}{2}\right)\left(s - \frac{\sqrt{5}-1}{2}\right) < 0,$$

which is impossible since both factors in the first member are positive. It follows that $s/(s^2+1) < 1/\sqrt{5}$, and the inequality is satisfied.

Case II. $b < a$, and $s < \frac{1}{2}(\sqrt{5}+1)$. We shall show that $p_1/q_1$ satisfies the inequality (1) with $k = 1/\sqrt{5}$. It is clear that $c$ is nearer $p_1/q_1$ than $b$ is, since $C$ is higher on the circle than $B$. Then

$$\left|\frac{p_1}{q_1} - \omega\right| < \frac{p_1}{q_1} - b = \frac{q}{q_1(q^2+q_1^2)} = \frac{s(s+1)}{s^2+(s+1)^2}\cdot\frac{1}{q_1^2}\,.$$

If

$$\frac{s(s+1)}{s^2+(s+1)^2} > \frac{1}{\sqrt{5}},$$

then

$$(\sqrt{5}-2)s^2 + (\sqrt{5}-2)s - 1 > 0,$$

or, dividing,

$$s^2 + s - \sqrt{5} - 2 > 0.$$

Factoring,

$$\left(s - \frac{\sqrt{5}+1}{2}\right)\left(s + \frac{\sqrt{5}+3}{2}\right) > 0,$$

which is impossible since the first factor is negative and the second is positive.

We have proved that one, at least, of the three fractions whose circles bound the mesh triangle satisfies (1) with $k = 1/\sqrt{5}$. Since $L$ passes through infinitely many mesh triangles it follows that infinitely many rational fractions satisfy the inequality with $k = 1/\sqrt{5}$.

It remains to exhibit an irrational $\omega$ for which (1) holds for only a finite number of fractions for $k < 1/\sqrt{5}$. We take, in fact, $\omega = \frac{1}{2}(\sqrt{5}+1)$ and any $h < 1$ and show that there is a finite number of fractions for which

$$\left|\frac{p}{q} - \frac{\sqrt{5}+1}{2}\right| < \frac{h}{\sqrt{5}q^2}\,. \tag{2}$$

For a fraction satisfying this we may write

$$\frac{p}{q} - \frac{\sqrt{5}+1}{2} = \frac{\theta}{\sqrt{5}q^2},$$

where $|\theta| < h < 1$. Writing this

$$\frac{p}{q} - \frac{1}{2} = \frac{\sqrt{5}}{2} + \frac{\theta}{\sqrt{5}q^2},$$

squaring, and rearranging, we get

$$5q^2[(p^2 - pq - q^2) - \theta] = \theta^2.$$

The expression in square brackets must be positive, whence the integer $p^2 - pq - q^2$, which cannot be zero (since then $p/q$ would turn out irrational) must equal or exceed 1. We have

$$q^2 = \frac{\theta^2}{5[(p^2 - pq - q^2) - \theta]} < \frac{h^2}{5(1 - h)}.$$

This limits $q$ to a finite number of values. For each $q$, the inequality (2) then limits $p$; and the number of fractions is finite.

Other irrationals might have been used here; for example, $\omega = (r\sqrt{5}+s)/t$, where $r$, $s$, $t$ are integers. These last irrational numbers are found in every interval of the $x$-axis.

**6. Continued fractions.** One of the most successful interpretations supplied by our system of circles is the geometric picture of the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}; \tag{3}$$

or, as often written,

$$a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots$$

Here the quantities $a_n$ are integers. If all the integers are positive, except possibly $a_0$, the continued fraction is *simple*.*

The $n$th *convergent*, $p_n/q_n$, is the quantity that remains when all that part of the expression following $a_{n-1}$ is erased. We have

$$\frac{p_1}{q_1} = \frac{a_0}{1}, \qquad \frac{p_2}{q_2} = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1},$$

$$\frac{p_3}{q_3} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = \frac{(a_0 a_1 + 1)a_2 + a_0}{a_1 a_2 + 1} = \frac{a_2 p_2 + p_1}{a_2 q_2 + q_1}.$$

---

* So called by Chrystal in whose Algebra, Part II, pp. 396ff, will be found one of the best elementary treatments of continued fractions in English.

Here and in the following we are using the $p$'s and $q$'s to represent the actual numerators and denominators of the fractions in the second members and not merely numbers proportional thereto.

The last equation exhibits for $n=2$ the recurrence formula from which we calculate the convergents step by step,

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a_n p_n + p_{n-1}}{a_n q_n + q_{n-1}}. \tag{4}$$

This formula is readily established by induction. It holds, we have just seen, for $n=2$; assuming it up to any later $n$ we prove it for $n+1$. We get $p_{n+2}/q_{n+2}$ from $p_{n+1}/q_{n+1}$ by replacing $a_n$ by $a_n+1/a_{n+1}$;

$$\frac{p_{n+2}}{q_{n+2}} = \frac{(a_n + 1/a_{n+1})p_n + p_{n-1}}{(a_n + 1/a_{n+1})q_n + q_{n-1}} = \frac{a_{n+1}(a_n p_n + p_{n-1}) + p_n}{a_{n+1}(a_n q_n + q_{n-1}) + q_n} = \frac{a_{n+1}p_{n+1} + p_n}{a_{n+1}q_{n+1} + q_n}.$$

This is the required formula with $n$ replaced by $n+1$ and (4) holds for $n \geqq 2$. It holds also for $n=1$ if we take $p_0/q_0=1/0$.

If the continued fraction terminates, its value is the last convergent. If there are infinitely many $a$'s the value to be assigned to the fraction is $\lim p_n/q_n$ if this limit exists.

We investigate the suite of convergents

$$\frac{p_0}{q_0}\left(=\frac{1}{0}\right), \quad \frac{p_1}{q_1}\left(=\frac{a_0}{1}\right), \quad \frac{p_2}{q_2}, \quad \frac{p_3}{q_3}, \quad \cdots.$$

THEOREM 7. *Successive convergents, $p_n/q_n$ and $p_{n+1}/q_{n+1}$, are adjacent fractions.*

From (4)

$$p_{n+1}q_n - q_{n+1}p_n = (a_n p_n + p_{n-1})q_n - (a_n q_n + q_{n-1})p_n = -(p_n q_{n-1} - q_n p_{n-1}).$$

Since $p_1q_0 - p_0q_1 = a_0 \cdot 0 - 1 \cdot 1 = -1$ we have that

$$p_{n+1}q_n - q_{n+1}p_n = (-1)^{n+1},$$

which establishes the theorem.

Given the tangent circles of $p_{n-1}/q_{n-1}$ and $p_n/q_n$ and the integer $a_n$, how do we find the circle of $p_{n+1}/q_{n+1}$? For all integral $a_n$ we get from (4) (see Theorem 3, Section 2) the ring of tangent circles around the circle of $p_n/q_n$. For $a_n=0$ we get the circle of $p_{n-1}/q_{n-1}$; for $a_n=1$ we get a circle next to this; for $a_n=2$ we get the next around the ring; and so on. For $a_n=-1$, $-2$, *etc.* we pass around in the opposite direction.

To determine the direction consider

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^{n+1}}{q_{n+1}q_n} = \frac{(-1)^{n+1}}{(a_n q_n + q_{n-1})q_n} = \frac{(-1)^{n+1}}{q_n^2\left(a_n + \dfrac{q_{n-1}}{q_n}\right)}.$$

For $a_n$ large and positive the last denominator is positive. If $n+1$ is even then $p_{n+1}/q_{n+1}$ is to the right of $p_n/q_n$ and we have counted around the circle of $p_n/q_n$ in a clockwise direction; if $n+1$ is odd $p_{n+1}/q_{n+1}$ is less than $p_n/q_n$ and we have counted in a counter-clockwise direction.

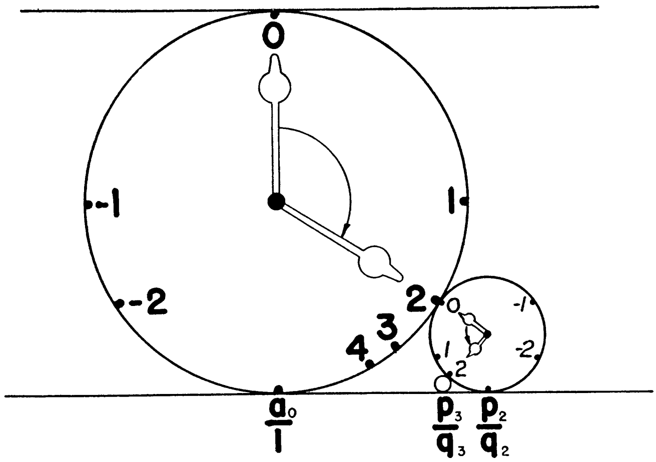


FIG. 4

We may liken the circle of $p_n/q_n$ to the face of a clock (Figure 4). Set the hand to point toward the center of the circle of $p_{n-1}/q_{n-1}$. It is now in the zero position. If we seek an even convergent turn clockwise to the $a_n$th point of tangency; if we seek an odd convergent turn counter-clockwise to the $a_n$th point of tangency. If $a_n$ is negative the turning is in the opposite direction in each case. The hand now points to the center of $p_{n+1}/q_{n+1}$.

Initially ($n=1$) we set the hand to point vertically upward to the point of tangency of $y=1$ with the circle of $a_0/1$, as in the figure. The positive direction is clockwise. On the next clock the positive direction is counter-clockwise; on the next, clockwise; and so on. In the figure $a_1=2$, $a_2=2$, and if $a_0=0$ the first four convergents, starting with $p_0/q_0$, are

$$\frac{1}{0}, \quad \frac{0}{1}, \quad \frac{1}{2}, \quad \frac{2}{5}.$$

We see then that *the circles of the convergents, in order, of a continued fraction form a sequence, or chain, of circles. The chain begins with the circle* $y=1$ *and each circle is tangent to the circles preceding and following it in the sequence.*

Conversely, any such chain has a unique corresponding continued fraction. We determine $a_0$ as the integer whose circle touches the first circle, $y=1$, in the chain; and $a_2$, $a_3$, etc., are got uniquely by counting points of tangency, as already explained.

A chain can be defined in many ways and with much arbitrariness. A simple way is to use a curve $L$, starting above $y=1$, as explained in Section 4. We shall not object to $a_n=0$, which gives $p_{n+1}/q_{n+1}=p_{n-1}/q_{n-1}$, nor to $q_n=0$, which means that $y=1$ reappears as a circle of the chain.

The preceding picture answers various questions about convergence. We see that it is possible so to choose the chain that $p_n/q_n$ approaches a prescribed rational or irrational limit in a great variety of ways, or so that $p_n/q_n$ becomes positively or negatively infinite, or so that $p_n/q_n$ wanders more or less arbitrarily over the $x$-axis. We may even choose the chain so that each interval of the $x$-axis contains infinitely many convergents. Such a chain is the set of circles passed over by the broken line $L$ formed by joining in succession the points $A_1, A_2, \cdots$ where $A_n$ has the coördinates $[(-1)^n n, 1/n]$. In this example each real rational number appears infinitely often among the convergents.

**7. Simple continued fractions.*** The situation is quite different if all $a_n (n>0)$ are positive integers. The first clock runs right (see Figure 4) and makes $p_2/q_2$ greater than $p_1/q_1$; the next runs left and places $p_3/q_3$ between $p_1/q_1$ and $p_2/q_2$. Each convergent falls between the two immediately preceding it. Certain facts then appear:

(*a*) The fraction converges to a value $\omega$.

(*b*) The odd convergents increase monotonically to $\omega$, the even convergents decrease monotonically to $\omega$.

(*c*) $\omega$ lies between each pair of successive convergents.

(*d*) $q_{n+1} > q_n \qquad (n > 1).$

(*e*) $$\left| \frac{p_n}{q_n} - \omega \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{a_n q_n^2} \leqq \frac{1}{q_n^2}.$$

Here the second expression is the distance between the convergents $p_n/q_n$ and $p_{n+1}/q_{n+1}$, and $q_{n+1}=a_n q_n+q_{n-1}>a_n q_n$ $(n>1)$.

(*f*) $$\left| \frac{p_n}{q_n} - \omega \right| > \frac{a_{n+1}}{q_n q_{n+2}},$$

for the second member is the distance between $p_n/q_n$ and $p_{n+2}/q_{n+2}$.

We see geometrically how to find the simple continued fraction for a given $\omega$. We select for $a_0$ the integer next below $\omega$ (see Figure 4). We then turn the clock to the right until $p_2/q_2$ lies as far to the left as possible without passing $\omega$, thus finding $a_1$. We turn the second clock to the left until $p_3/q_3$ is as far to the right as possible without passing $\omega$, whence we have $a_2$; and so on.

That the development is unique is apparent. The first $a_n$'s that differ in two developments throw the values of the two continued fractions into different intervals thereafter. There is an unimportant exception in the case of a fraction which terminates. A last $a_n>1$ may be written $(a_n-1)+1/1$, thus inserting an additional convergent.

* Züllig, *loc. cit.*, has the chain of circles given here with figures for many numerical cases.

It is not difficult to identify this process of forming the continued fraction for $\omega$ with the usual arithmetic process: (1) taking $a_0$ as the largest integer in $\omega$; (2) forming $\omega_1$, the reciprocal of the remainder $\omega - a_0$, and taking $a_1$ as the largest integer in $\omega_1$; (3) taking $a_2$ as the largest integer in $\omega_2$, the reciprocal of $\omega_1 - a_1$; and so on.

We see geometrically that $p_n/q_n$ is a good approximation to $\omega$ if $a_n$ is very large, for our clock has then turned far toward the bottom. This is clear, of course, from (*e*) above. Thus in the continued fraction

$$\pi = 3 + \frac{1}{7 +} \frac{1}{15 +} \frac{1}{1 +} \frac{1}{292 +} \frac{1}{1 +} \frac{1}{1 + \cdots},$$

the second convergent 22/7 and the fourth 355/113 (here $a_4 = 292$) are especially good and are, in fact, much used.

Suppose we take $a_n = 1$ for all $n$ in order to avoid a good approximation as far as we can. We have

$$\omega = 1 + \frac{1}{1 +} \frac{1}{1 + \cdots} = 1 + \frac{1}{\omega},$$

or $\omega^2 - \omega - 1 = 0$. From this, since $\omega > 1$,

$$\omega = \tfrac{1}{2}(\sqrt{5} + 1).$$

This is the irrational quantity we used in the latter part of the proof of Hurwitz's theorem.

**8. Complex fractions.*** Complex numbers $x + iy$, where $x$ and $y$ are real and $i = \sqrt{-1}$, are commonly represented by points $(x, y)$ in an $xy$-plane using rectangular coördinates. The complex integers, $n = n' + in''$, where $n'$ and $n''$ are real integers, form a square lattice in the plane. A fraction is the quotient of two integers

$$\frac{p}{q} = \frac{p' + ip''}{q' + iq''} = \frac{p' + ip''}{q' + iq''} \cdot \frac{q' - iq''}{q' - iq''} = \frac{p'q' + p''q''}{q'^2 + q''^2} + i\,\frac{p''q' - p'q''}{q'^2 + q''^2}$$

and is represented by the point with the coördinates shown in the last member. The complex fractions thus consist of the numbers $x + iy$, where $x$ and $y$ are real fractions. We think of the $xy$-plane as being horizontal, and we introduce a third or $z$-axis perpendicular to the $x$- and $y$-axes.

The analogue of the circle of Section 1 is a sphere. Through the point in the plane which represents the fraction $p/q$ (in its lowest terms) we construct a sphere touching the complex plane there, lying in the upper half-space, and having the radius $1/(2q\bar{q})$. Here $\bar{q}$ is the conjugate of $q$, $\bar{q} = q' - iq''$; whence

* For the matters treated in this section see L. R. Ford, Transactions of the American Mathematical Society, vol. 19, 1918, pp. 1–42.

$q\bar{q}=q'^2+q''^2=|q|^2$. This sphere will be the geometric representation of $p/q$. Every small area in the complex plane contains points of tangency of infinitely many of these spheres.

Interpreting Figure 1 as a pair of spheres at complex fractions, we have

$$AB^2=\left|\frac{P}{Q}-\frac{p}{q}\right|^2+\left(\frac{1}{2Q\bar{Q}}-\frac{1}{2q\bar{q}}\right)^2=\left(\frac{1}{2Q\bar{Q}}+\frac{1}{2q\bar{q}}\right)+\frac{|Pq-pQ|^2-1}{Q\bar{Q}q\bar{q}}.$$

If $|Pq-pQ|>1$, then $AB>AD+EB$, and the spheres are wholly external to one another. If $|Pq-pQ|=1$ the spheres are tangent. In no case do the spheres intersect. We call the fractions *adjacent* if the spheres are tangent.

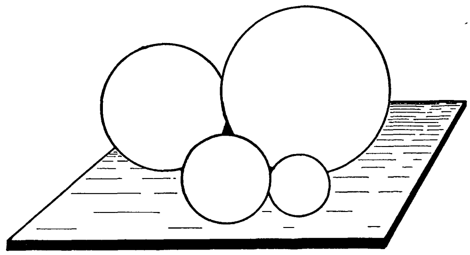

FIG. 5

That $p/q$ has an adjacent fraction $P/Q$ we already know, for we have designed the proof of Theorem 2 to cover this case. Then the fraction

$$\frac{P_n}{Q_n}=\frac{P+np}{Q+nq}$$

of Theorem 3, where $n$ is any complex integer, is adjacent to $p/q$. We note that the four fractions $P_m/Q_m$, where $m=n+1,\ n-1,\ n+i,\ n-i$, are adjacent to $P_n/Q_n$; for $|(P+np)(Q+mq)-(P+mp)(Q+nq)|=|m-n|\cdot|Pq-pQ|=1$. That is, each sphere tangent to a given sphere touches four other spheres tangent to the given sphere. It is not difficult to show that the fractions $P_n/Q_n$ constitute the complete set of fractions adjacent to $p/q$.

How many of the spheres touching the sphere of $p/q$ are larger than that sphere? We require the number of solutions of

$$|Q+nq|<|q|, \qquad \text{or} \qquad |n+Q/q|<1.$$

Now $|n+Q/q|$ is the distance from the integer $n$ to the point $-Q/q$. The problem takes the form: If a unit circle is drawn in the complex plane, how many complex integers does it contain? For various positions of the center $(-Q/q)$ the answer is 2, 3, or 4, as is readily seen. We put aside the case $|q|=1$, which puts the center at an integer, when clearly there are no larger tangent spheres. We introduce the plane $z=1$ to touch the spheres of the integers and be the sphere of their adjacent fraction $1/0$.

The use of complex integers in the continued fraction (3) offers no difficulty. We are led to the geometric picture of the continued fraction as a chain of spheres, beginning with the plane $z=1$, and proceeding thence from sphere to tangent sphere. We have here the same possibilities of convergence or divergence as before. At the worst we can set up continued fractions whose convergents are present in every small region of the complex plane. There is no immediate generalization of the simple continued fraction to the complex case. There are, however, schemes for developing a complex number in a continued fraction, the best processes being due to Hurwitz.*

The analogue of Hurwitz's theorem on rational approximations to an irrational number was first discovered by the present author.† The inequality (1) is replaced by

$$\left| \frac{p}{q} - \omega \right| < \frac{k}{q\bar{q}},$$

and the minimum $k$ for which infinitely many fractions always satisfy the inequality is $k=1/\sqrt{3}$. The proof of this, however, is not elementary.

---

## MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

*This Department of the* MONTHLY *has been created as an experiment to afford a place for the discussion of the place of mathematics in education. With this topic will naturally be associated other matters emphasizing the educational interests of those who teach mathematics. It is not intended to take up minute details of teaching technique. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. The success of this department obviously will depend upon the cooperation of the readers of the* MONTHLY. *Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

## MATHEMATICAL EDUCATION IN GERMANY BEFORE 1933

RICHARD COURANT, New York University

At the suggestion of the editors I shall try to give a brief account of trends in the teaching of mathematics in Germany, particularly at German universities, in the period from the World War until 1933.

The situation of mathematics in Germany, since the early part of the 19th century, has been pivoted around a definite connection between university and high school. In a sweeping reform following the French Revolution, the institution of the German "humanistisches Gymnasium" was established. Teachers at these institutions were required to undergo a very thorough academic preparation, and the task of preparing them was entrusted to the philosophical faculties of the universities. The teachers' training was in no way of an elementary char-

---

* Acta Mathematica, vol. 11, 1887, pp. 187–200.

† L. R. Ford, Transactions of the American Mathematical Society, vol. 27, 1925, pp. 146–154.