Chapter 2

Vectors in $\mathbf{R}^n$ and $\mathbf{C}^n$, Spatial Vectors

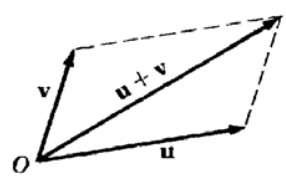
2.1 INTRODUCTION

In various physical applications there appear certain quantities, such as temperature and speed, which possess only "magnitude." These can be represented by real numbers and are called *scalars*. On the other hand, there are also quantities, such as force and velocity, which possess both "magnitude" and "direction." These quantities can be represented by arrows (having appropriate lengths and directions and emanating from some given reference point O) and are called *vectors*.

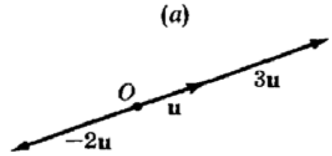
We begin by considering the following operations on vectors.

(i) ***Addition:*** The resultant $\mathbf{u} + \mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$ is obtained by the so-called parallelogram law, i.e., $\mathbf{u} + \mathbf{v}$ is the diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ as shown in Fig. 2-1(*a*).

(ii) ***Scalar multiplication:*** The product $k\mathbf{u}$ of a vector $\mathbf{u}$ by a real number k is obtained by multiplying the magnitude of $\mathbf{u}$ by k and retaining the same direction if $k > 0$ or the opposite direction if $k < 0$, as shown in Fig. 2-1(*b*).



(*a*)



(*b*)

Fig. 2-1

Now we assume the reader is familiar with the representation of the points in the plane by ordered pairs of real numbers. If the origin of the axes is chosen at the reference point O above, then every vector is uniquely determined by the coordinates of its endpoint. The relationship between the above operations and endpoints follows.

(i) ***Addition:*** If (a, b) and (c, d) are the endpoints of the vectors $\mathbf{u}$ and $\mathbf{v}$, then $(a + c, b + d)$ will be the endpoint of $\mathbf{u} + \mathbf{v}$, as shown in Fig. 2-2(*a*).

(ii) ***Scalar multiplication:*** If (a, b) is the endpoint of the vector $\mathbf{u}$, then (ka, kb) will be the endpoint of the vector $k\mathbf{u}$, as shown in Fig. 2-2(*b*).

Mathematically, we identify the vector $\mathbf{u}$ with its endpoint (a, b) and write $\mathbf{u} = (a, b)$. In addition we call the ordered pair (a, b) of real numbers a point or vector depending upon its interpretation. We generalize this notion and call an n-tuple $(a_1, a_2, \ldots, a_n)$ of real numbers a vector. However, special notation may be used for spatial vectors in $\mathbf{R}^3$ (Section 2.8).

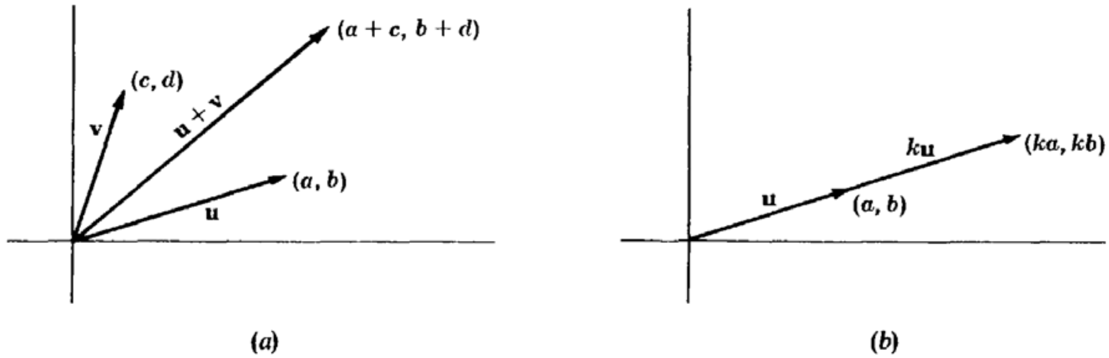


Fig. 2-2

We assume the reader is familiar with the elementary properties of the real number field which we denote by $\mathbf{R}$.

2.2 VECTORS IN $\mathbf{R}^n$

The set of all n-tuples of real numbers, denoted by $\mathbf{R}^n$, is called *n-space*. A particular n-tuple in $\mathbf{R}^n$, say

$$u = (u_1, u_2, \ldots, u_n)$$

is called a *point* or *vector*; the real numbers u_i are called the *components* (or *coordinates*) of the vector u. Moreover, when discussing the space $\mathbf{R}^n$ we use the term *scalar* for the elements of $\mathbf{R}$.

Two vectors u and v are *equal*, written $u = v$, if they have the same number of components, i.e., belong to the same space, and if corresponding components are equal. The vectors (1, 2, 3) and (2, 3, 1) are not equal, since corresponding elements are not equal.

Example 2.1

(*a*) Consider the following vectors

$$(0, 1) \qquad (1, -3) \qquad (1, 2, \sqrt{3}, 4) \qquad (-5, \tfrac{1}{2}, 0, \pi)$$

The first two vectors have two components and so are points in $\mathbf{R}^2$; the last two vectors have four components and so are points in $\mathbf{R}^4$.

(*b*) Suppose $(x - y, x + y, z - 1) = (4, 2, 3)$. Then, by definition of equality of vectors,

$$\begin{aligned} x - y &= 4 \\ x + y &= 2 \\ z - 1 &= 3 \end{aligned}$$

Solving the above system of equations gives $x = 3$, $y = -1$, and $z = 4$.

Sometimes vectors in n-space are written vertically as columns rather than horizontally as rows, as above. Such vectors are called *column vectors*. For example,

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 7 \\ -8 \end{pmatrix} \quad \begin{pmatrix} 1.2 \\ -35 \\ 28 \end{pmatrix}$$

are column vectors with 2, 2, 3, and 3 components, respectively.

2.3 VECTOR ADDITION AND SCALAR MULTIPLICATION

Let u and v be vectors in $\mathbf{R}^n$:

$$u = (u_1, u_2, \ldots, u_n) \qquad \text{and} \qquad v = (v_1, v_2, \ldots, v_n)$$

The *sum* of u and v, written $u + v$, is the vector obtained by adding corresponding components:

$$u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

The *product* of the vector u by a real number k, written ku, is the vector obtained by multiplying each component of u by k:

$$ku = (ku_1, ku_2, \ldots, ku_n)$$

Observe that $u + v$ and ku are also vectors in $\mathbf{R}^n$. We also define

$$-u = -1u \qquad \text{and} \qquad u - v = u + (-v)$$

The sum of vectors with different numbers of components is not defined.

Basic properties of the vectors in $\mathbf{R}^n$ under vector addition and scalar multiplication are described in the following theorem (proved in Problem 2.4). In the theorem, $0 \equiv (0, 0, \ldots, 0)$, the *zero vector* of $\mathbf{R}^n$.

Theorem 2.1: For any vectors $u, v, w \in \mathbf{R}^n$ and any scalars $k, k' \in \mathbf{R}$,

(i) $(u + v) + w = u + (v + w)$ (v) $k(u + v) = ku + kv$

(ii) $u + 0 = u$ (vi) $(k + k')u = ku + k'u$

(iii) $u + (-u) = 0$ (vii) $(kk')u = k(k'u)$

(iv) $u + v = v + u$ (viii) $1u = u$

Suppose u and v are vectors in $\mathbf{R}^n$ for which $u = kv$ for some nonzero scalar $k \in \mathbf{R}$. Then u is called a *multiple* of v; and u is said to be in the *same direction* as v if $k > 0$, and in the *opposite direction* if $k < 0$.

2.4 VECTORS AND LINEAR EQUATIONS

Two important concepts, linear combinations and linear dependence, are closely related to systems of linear equations as follows.

Linear Combinations

Consider a nonhomogeneous system of m equations in n unknowns:

$$\begin{array}{c}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{array}$$

This system is equivalent to the following vector equation

$$x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

that is, the vector equation

$$x_1 u_1 + x_2 u_2 + \cdots + x_n u_n = v$$

where $u_1, u_2, \ldots, u_n, v$ are the above column vectors, respectively.

Now if the above system has a solution, then v is said to be a linear combination of the vectors u_i. We state this important concept formally.

Definition: A vector v is a *linear combination* of vectors $u_1, u_2, \ldots, u_n$ if there exist scalars $k_1, k_2, \ldots, k_n$ such that

$$v = k_1 u_1 + k_2 u_2 + \cdots + k_n u_n$$

that is, if the vector equation

$$v = x_1 u_1 + x_2 u_2 + \cdots + x_n u_n$$

has a solution where the x_i are unknown scalars.

The above definition applies to both column vectors and row vectors, although our illustration was in terms of column vectors.

Example 2.2. Suppose

$$v = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}, \qquad u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Then v is a linear combination of u_1, u_2, u_3 since the vector equation (or system)

$$\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{or} \qquad \begin{cases} 2 = x + y + z \\ 3 = x + y \\ -4 = x \end{cases}$$

has a solution $x = -4$, $y = 7$, $z = -1$. In other words,

$$v = -4u_1 + 7u_2 - u_3$$

Linear Dependence

Consider a homogeneous system of m equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

This system is equivalent to the following vector equation:

$$x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

that is, the vector equation

$$x_1 u_1 + x_2 u_2 + \cdots + x_n u_n = 0$$

where $u_1, u_2, \ldots, u_n$ are the above column vectors, respectively.

Now, if the above homogeneous system has a nonzero solution, the vectors $u_1, u_2, \ldots, u_n$ are said to be linearly dependent; on the other hand, if the equation has only the zero solution, then the vectors are said to be linearly independent. We state this important concept formally.

Definition: Vectors $u_1, u_2, \ldots, u_n$ in $\mathbf{R}^n$ are *linearly dependent* if there exist scalars $k_1, k_2, \ldots, k_n$, not all zero, such that

$$k_1 u_1 + k_2 u_2 + \cdots + k_n u_n = 0$$

that is, if the vector equation

$$x_1 u_1 + x_2 u_2 + \cdots + x_n u_n = 0$$

has a nonzero solution where the x_i are unknown scalars. Otherwise, the vectors are said to be *linearly independent*.

The above definition applies to both column vectors and row vectors, although our illustration was in terms of column vectors.

Example 2.3

(*a*) The only solution to

$$x\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}1\\1\\0\end{pmatrix} + z\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad \text{or} \quad \begin{cases} x + y + z = 0 \\ x + y \qquad = 0 \\ x \qquad\qquad = 0 \end{cases}$$

is the zero solution $x = 0, y = 0, z = 0$. Hence the three vectors are linearly independent.

(*b*) The vector equation (or system of linear equations)

$$x\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}2\\-1\\3\end{pmatrix} + z\begin{pmatrix}1\\-5\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad \text{or} \quad \begin{cases} x + 2y + z = 0 \\ x - y - 5z = 0 \\ x + 3y + 3z = 0 \end{cases}$$

has a nonzero solution $(3, -2, 1)$, i.e., $x = 3, y = -2, z = 1$. Thus the three vectors are linearly dependent.

2.5 DOT (SCALAR) PRODUCT

Let u and v be vectors in $\mathbf{R}^n$:

$$u = (u_1, u_2, \ldots, u_n) \qquad \text{and} \qquad v = (v_1, v_2, \ldots, v_n)$$

The *dot*, *scalar*, or *inner product* of u and v, denoted by $u \cdot v$, is the scalar obtained by multiplying corresponding components and adding the resulting products:

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

The vectors u and v are said to be *orthogonal* (or *perpendicular*) if their dot product is zero, that is, if $u \cdot v = 0$.

Example 2.4. Let $u = (1, -2, 3, -4)$, $v = (6, 7, 1, -2)$, and $w = (5, -4, 5, 7)$. Then

$$u \cdot v = 1 \cdot 6 + (-2) \cdot 7 + 3 \cdot 1 + (-4) \cdot (-2) = 6 - 14 + 3 + 8 = 3$$

$$u \cdot w = 1 \cdot 5 + (-2) \cdot (-4) + 3 \cdot 5 + (-4) \cdot 7 = 5 + 8 + 15 - 28 = 0$$

Thus u and w are orthogonal.

Basic properties of the dot product in $\mathbf{R}^n$ (proved in Problem 2.17) follow.

Theorem 2.2: For any vectors $u, v, w \in \mathbf{R}^n$ and any scalar $k \in \mathbf{R}$,

(i) $(u + v) \cdot w = u \cdot w + v \cdot w$ (iii) $u \cdot v = v \cdot u$

(ii) $(ku) \cdot v = k(u \cdot v)$ (iv) $u \cdot u \geq 0$, and $u \cdot u = 0$ iff $u = 0$

Remark: The space $\mathbf{R}^n$ with the above operations of vector addition, scalar multiplication, and dot product is usually called *Euclidean n-space*.

2.6 NORM OF A VECTOR

Let $u = (u_1, u_2, \ldots, u_n)$ be a vector in $\mathbf{R}^n$. The *norm* (or *length*) of the vector u, written $\|u\|$, is defined to be the nonnegative square root of $u \cdot u$:

$$\|u\| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Since $u \cdot u \geq 0$, the square root exists. Also, if $u \neq 0$, then $\|u\| > 0$; and $\|0\| = 0$.

The above definition of the norm of a vector conforms to that of the length of a vector (arrow) in (Euclidean) geometry. Specifically, suppose $\mathbf{u}$ is a vector (arrow) in the plane $\mathbf{R}^2$ with endpoint $P(a, b)$ as shown in Fig. 2-3. Then $|a|$ and $|b|$ are the lengths of the sides of the right triangle formed by $\mathbf{u}$ and the horizontal and vertical directions. By the Pythagorean Theorem, the length $|\mathbf{u}|$ of $\mathbf{u}$ is

$$|\mathbf{u}| = \sqrt{a^2 + b^2}$$

This value is the same as the norm of u defined above.

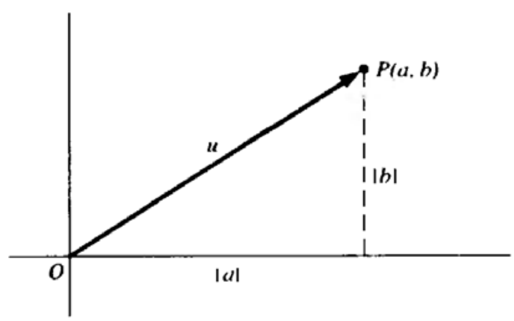


Fig. 2-3

Example 2.5. Suppose $u = (3, -12, -4)$. To find $\|u\|$, we first find $\|u\|^2 = u \cdot u$ by squaring the components of u and adding:

$$\|u\|^2 = 3^2 + (-12)^2 + (-4)^2 = 9 + 144 + 16 = 169$$

Then $\|u\| = \sqrt{169} = 13$.

A vector u is a unit vector if $\|u\| = 1$ or, equivalently, if $u \cdot u = 1$. Now if v is any nonzero vector, then

$$\hat{v} \equiv \frac{1}{\|v\|} v = \frac{v}{\|v\|}$$

is a unit vector in the same direction as v. (The process of finding $\hat{v}$ is called *normalizing* v.) For example,

$$\hat{v} = \frac{v}{\|v\|} = \left(\frac{2}{\sqrt{102}}, \frac{-3}{\sqrt{102}}, \frac{8}{\sqrt{102}}, \frac{-5}{\sqrt{102}}\right)$$

is the unit vector in the direction of the vector $v = (2, -3, 8, -5)$.

We now state a fundamental relationship (proved in Problem 2.22) known as the Cauchy–Schwarz inequality.

Theorem 2.3 (Cauchy–Schwarz): For any vectors u, v in $\mathbf{R}^n$, $|u \cdot v| \le \|u\| \, \|v\|$.

Using the above inequality, we prove (Problem 2.33) the following result known as the triangle inequality or as Minkowski's inequality.

Theorem 2.4 (Minkowski): For any vectors u, v in $\mathbf{R}^n$, $\|u + v\| \le \|u\| + \|v\|$.

Distance, Angles, Projections

Let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ be vectors in $\mathbf{R}^n$. The *distance* between u and v, denoted by $d(u, v)$, is defined as

$$d(u, v) \equiv \|u - v\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

We show that this definition corresponds to the usual notion of Euclidean distance in the plane $\mathbf{R}^2$. Consider $u = (a, b)$ and $v = (c, d)$ in $\mathbf{R}^2$. As shown in Fig. 2-4, the distance between the points $P(a, b)$ and $Q(c, d)$ is

$$d = \sqrt{(a - c)^2 + (b - d)^2}$$

On the other hand, by the above definition,

$$d(u, v) = \|u - v\| = \|(a - c, b - d)\| = \sqrt{(a - c)^2 + (b - d)^2}$$

Both give the same value.

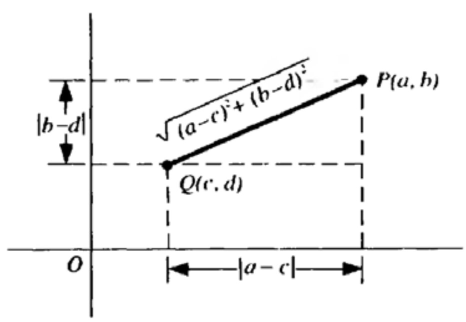


Fig. 2-4

Using the Cauchy–Schwarz inequality, we can define the angle θ between any two nonzero vectors u, v in $\mathbf{R}^n$ by

$$\cos \theta = \frac{u \cdot v}{\|u\| \, \|v\|}$$

Note that if $u \cdot v = 0$, then $\theta = 90^\circ$ (or $\theta = \pi/2$). This then agrees with our previous definition of orthogonality.

Example 2.6. Suppose $u = (1, -2, 3)$ and $v = (3, -5, -7)$. Then

$$d(u, v) = \sqrt{(1 - 3)^2 + (-2 + 5)^2 + (3 + 7)^2} = \sqrt{4 + 9 + 100} = \sqrt{113}$$

To find $\cos \theta$, where θ is the angle between u and v, we first find

$$u \cdot v = 3 + 10 - 21 = -8 \qquad \|u\|^2 = 1 + 4 + 9 = 14 \qquad \|v\|^2 = 9 + 25 + 49 = 83$$

Then

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = -\frac{-8}{\sqrt{14}\sqrt{83}}$$

Let u and $v \neq 0$ be vectors in $\mathbf{R}^n$. The *(vector) projection of u onto v* is the vector

$$\text{proj}\,(u, v) = \frac{u \cdot v}{\|v\|^2}\, v$$

We show how this definition conforms to the notion of vector projection in physics. Consider the vectors $\mathbf{u}$ and $\mathbf{v}$ in Fig. 2-5. The (perpendicular) projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector $\mathbf{u}^*$, of magnitude

$$|\mathbf{u}^*| = |\mathbf{u}|\cos\theta = |\mathbf{u}|\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$$

To obtain $\mathbf{u}^*$, we multiply its magnitude by the unit vector in the direction of $\mathbf{v}$:

$$\mathbf{u}^* = |\mathbf{u}^*|\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v}$$

This agrees with the above definition of proj (u, v).

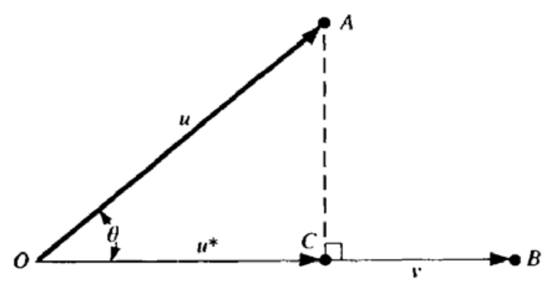


Fig. 2-5

Example 2.7. Suppose $u = (1, -2, 3)$ and $v = (2, 5, 4)$. To find proj (u, v), we first find

$$u \cdot v = 2 - 10 + 12 = 4 \qquad \text{and} \qquad \|v\|^2 = 4 + 25 + 16 = 45$$

Then

$$\text{proj}\,(u, v) = \frac{u \cdot v}{\|v\|^2}\, v = \frac{4}{45}(2, 5, 4) = \left(\frac{8}{45}, \frac{20}{45}, \frac{16}{45}\right) = \left(\frac{8}{45}, \frac{4}{9}, \frac{16}{45}\right)$$

2.7 LOCATED VECTORS, HYPERPLANES, AND LINES IN $\mathbf{R}^n$

This section distinguishes between an n-tuple $P(a_1, a_2, \ldots, a_n) \equiv P(a_i)$ viewed as a point in $\mathbf{R}^n$ and an n-tuple $v = [c_1, c_2, \ldots, c_n]$ viewed as a vector (arrow) from the origin O to the point $C(c_1, c_2, \ldots, c_n)$. Any pair of points $P = (a_i)$ and $Q = (b_i)$ in $\mathbf{R}^n$ defines the *located vector* or *directed line segment* from P to Q, written $\overrightarrow{PQ}$. We identify $\overrightarrow{PQ}$ with the vector

$$v = Q - P = [b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n]$$

since $\overrightarrow{PQ}$ and v have the same magnitude and direction as shown in Fig. 2-6.

A hyperplane H in $\mathbf{R}^n$ is the set of points $(x_1, x_2, \ldots, x_n)$ that satisfy a nondegenerate linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

In particular, a hyperplane H in $\mathbf{R}^2$ is a line; and a hyperplane H in $\mathbf{R}^3$ is a plane. The vector $u = [a_1, a_2, \ldots, a_n] \neq 0$ is called a *normal* to H. This terminology is justified by the fact (Problem 2.33)

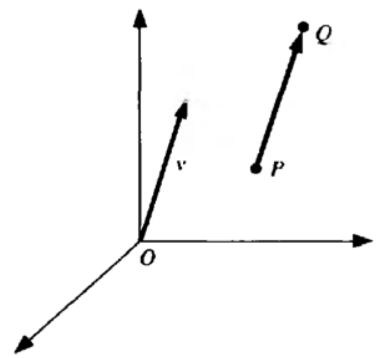


Fig. 2-6

that any directed line segment $\overrightarrow{PQ}$, where P, Q belong to H, is orthogonal to the normal vector u. This fact in $\mathbf{R}^3$ is shown in Fig. 2-7.

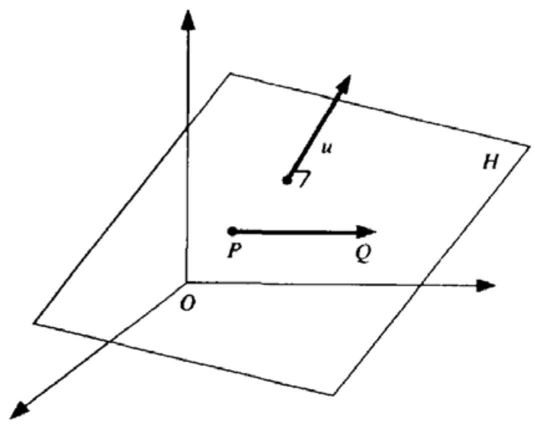


Fig. 2-7

The *line* L in $\mathbf{R}^n$ passing through the point $P(a_1, a_2, \ldots, a_n)$ and in the direction of the nonzero vector $u = [u_1, u_2, \ldots, u_n]$ consists of the points $X(x_1, x_2, \ldots, x_n)$ which satisfy

$$X = P + tu \qquad \text{or} \qquad \begin{cases} x_1 = a_1 + u_1 t \\ x_2 = a_2 + u_2 t \\ \ldots\ldots\ldots\ldots\ldots \\ x_n = a_n + u_n t \end{cases}$$

where the *parameter* t takes on all real numbers. (See Fig. 2-8.)

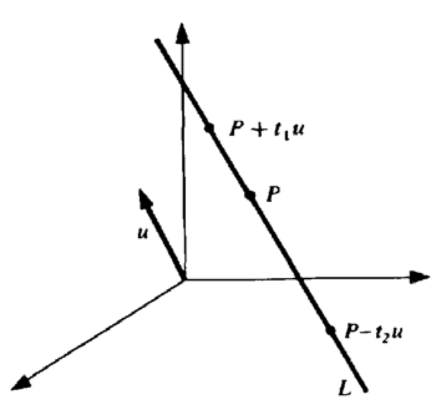


Fig. 2-8

Example 2.8

(*a*) Consider the hyperplane H in $\mathbf{R}^4$ which passes through the point $P(1, 3, -4, 2)$ and is normal to the vector $u = [4, -2, 5, 6]$. Its equation must be of the form

$$4x - 2y + 5z + 6t = k$$

Substituting P into this equation, we obtain

$$4(1) - 2(3) + 5(-4) + 6(2) = k \qquad \text{or} \qquad 4 - 6 - 20 + 12 = k \qquad \text{or} \qquad k = -10$$

Thus $4x - 2y + 5z + 6t = -10$ is the equation of H.

(*b*) Consider the line L in $\mathbf{R}^4$ passing through the point $P(1, 2, 3, -4)$ in the direction of $u = [5, 6, -7, 8]$. A parametric representation of L is as follows:

$$\begin{aligned} x_1 &= 1 + 5t \\ x_2 &= 2 + 6t \\ x_3 &= 3 - 7t \\ x_4 &= -4 + 8t \end{aligned} \qquad \text{or} \qquad (1 + 5t, 2 + 6t, 3 - 7t, -4 + 8t)$$

Note $t = 0$ yields the point P on L.

Curves in $\mathbf{R}^n$

Let D be an interval (finite or infinite) in the real line $\mathbf{R}$. A continuous function $F: D \to \mathbf{R}^n$ is a curve in $\mathbf{R}^n$. Thus to each $t \in D$ there is assigned the following point (vector) in $\mathbf{R}^n$:

$$F(t) = [F_1(t), F_2(t), \ldots, F_n(t)]$$

Moreover, the derivative (if it exists) of $F(t)$ yields the vector

$$V(t) = dF(t)/dt = [dF_1(t)/dt, dF_2(t)/dt, \ldots, dF_n(t)/dt]$$

which is tangent to the curve. Normalizing $V(t)$ yields

$$\mathbf{T}(t) = \frac{V(t)}{\| V(t) \|}$$

which is the unit tangent vector to the curve. [Unit vectors with geometrical significance are often notated in bold type; compare Section 2.8.]

Example 2.9. Consider the following curve C in $\mathbf{R}^3$:

$$F(t) = [\sin t, \cos t, t]$$

Taking the derivative of $F(t)$ [or each component of $F(t)$] yields

$$V(t) = [\cos t, -\sin t, 1]$$

which is a vector tangent to the curve. We normalize $V(t)$. First we obtain

$$\| V(t) \|^2 = \cos^2 t + \sin^2 t + 1 = 1 + 1 = 2$$

Then

$$\mathbf{T}(t) = \frac{V(t)}{\| V(t) \|} = \left[\frac{\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

which is the unit tangent vector to the curve.

2.8 SPATIAL VECTORS, ijk NOTATION IN $\mathbf{R}^3$

Vectors in $\mathbf{R}^3$, called spatial vectors, appear in many applications, especially in physics. In fact, a special notation is frequently used for such vectors as follows:

$$\mathbf{i} = (1, 0, 0) \text{ denotes the unit vector in the } x \text{ direction,}$$
$$\mathbf{j} = (0, 1, 0) \text{ denotes the unit vector in the } y \text{ direction,}$$
$$\mathbf{k} = (0, 0, 1) \text{ denotes the unit vector in the } z \text{ direction.}$$

Then any vector $u = (a, b, c)$ in $\mathbf{R}^3$ can be expressed uniquely in the form

$$u = (a, b, c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Since $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors and are mutually orthogonal, we have

$$\mathbf{i}\cdot\mathbf{i} = 1, \qquad \mathbf{j}\cdot\mathbf{j} = 1, \qquad \mathbf{k}\cdot\mathbf{k} = 1 \qquad \text{and} \qquad \mathbf{i}\cdot\mathbf{j} = 0, \qquad \mathbf{i}\cdot\mathbf{k} = 0, \qquad \mathbf{j}\cdot\mathbf{k} = 0$$

The various vector operations discussed previously may be expressed in the above notation as follows. Suppose $u = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $v = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Then

$$u + v = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k} \qquad u \cdot v = a_1 b_1 + a_2 b_2 + a_3 b_3$$
$$cu = ca_1\mathbf{i} + ca_2\mathbf{j} + ca_3\mathbf{k} \qquad \|u\| = \sqrt{u \cdot u} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

where c is a scalar.

Example 2.10. Suppose $u = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ and $v = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$.

(*a*) To find $u + v$, add corresponding components yielding

$$u + v = 7\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$$

(*b*) To find $3u - 2v$, first multiply the vectors by the scalars, and then add:

$$3u - 2v = (9\mathbf{i} + 15\mathbf{j} - 6\mathbf{k}) + (-8\mathbf{i} + 6\mathbf{j} - 14\mathbf{k}) = 4\mathbf{i} + 21\mathbf{j} - 20\mathbf{k}$$

(*c*) To find $u \cdot v$, multiply corresponding components and then add:

$$u \cdot v = 12 - 15 - 14 = -17$$

(*d*) To find $\|u\|$, square each component and then add to get $\|u\|^2$. That is,

$$\|u\|^2 = 9 + 25 + 4 = 38 \qquad \text{and hence} \qquad \|u\| = \sqrt{38}$$

Cross Product

There is a special operation for vectors u, v in $\mathbf{R}^3$, called the *cross product* and denoted by $u \times v$. Specifically, suppose

$$u = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \text{and} \qquad v = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

Then

$$u \times v = (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

Note $u \times v$ is a vector; hence $u \times v$ is also called *vector product* or *outer product* of u and v.

Using determinant notation (Chapter 7), where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, the cross product may also be expressed as follows:

$$u \times v = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

or, equivalently,

$$u \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Two important properties of the cross product follow (see Problem 2.56).

Theorem 2.5: Let u, v, w be vectors in $\mathbf{R}^3$.

(i) The vector $u \times v$ is orthogonal to both u and v.

(ii) The absolute value of the "triple product" $u \cdot v \times w$ represents the volume of the parallelepiped formed by the vectors u, v, and w (as shown in Fig. 2-9).

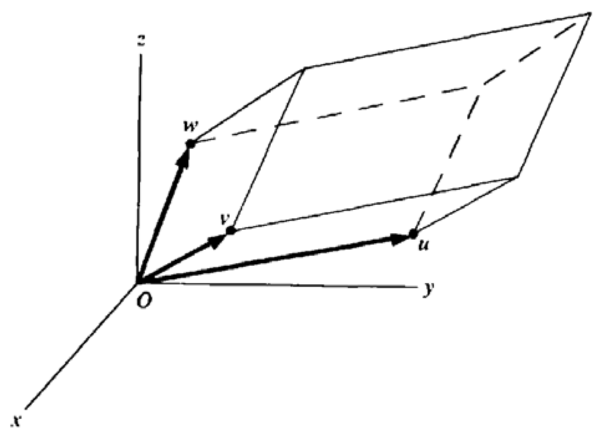


Fig. 2-9

Example 2.11

(*a*) Suppose $u = 4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ and $v = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$. Then

$$u \times v = \begin{vmatrix} 3 & 6 \\ 5 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & 6 \\ 2 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}\mathbf{k} = -39\mathbf{i} + 24\mathbf{j} + 14\mathbf{k}$$

(*b*)
$$(2, -1, 5) \times (3, 7, 6) = \left(\begin{vmatrix} -1 & 5 \\ 7 & 6 \end{vmatrix}, -\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 3 & 7 \end{vmatrix} \right) = (-41, 3, 17)$$

(Here we find the cross product without using the **ijk** notation.)

(*c*) The cross products of the vectors **i**, **j**, **k** are as follows:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}, \quad \text{and} \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

In other words, if we view the triple (**i**, **j**, **k**) as a cyclic permutation, i.e., as arranged around a circle in the counterclockwise direction as in Fig. 2-10, then the product of two of them in the given direction is the third one but the product of two of them in the opposite direction is the negative of the third one.

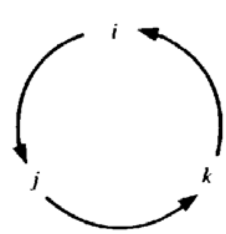


Fig. 2-10

2.9 COMPLEX NUMBERS

The set of complex numbers is denoted by **C**. Formally, a complex number is an ordered pair (a, b) of real numbers; equality, addition, and multiplication of complex numbers are defined as follows:

$$(a, b) = (c, d) \quad \text{iff} \quad a = c \text{ and } b = d$$
$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b)(c, d) = (ac - bd, ad + bc)$$

We identify the real number a with the complex number $(a, 0)$:

$$a \leftrightarrow (a, 0)$$

This is possible since the operations of addition and multiplication of real numbers are preserved under the correspondence

$$(a, 0) + (b, 0) = (a + b, 0) \quad \text{and} \quad (a, 0)(b, 0) = (ab, 0)$$

Thus we view **R** as a subset of **C** and replace $(a, 0)$ by a whenever convenient and possible.

The complex number $(0, 1)$, denoted by i, has the important property that

$$i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1 \quad \text{or} \quad i = \sqrt{-1}$$

Furthermore, using the fact

$$(a, b) = (a, 0) + (0, b) \quad \text{and} \quad (0, b) = (b, 0)(0, 1)$$

we have

$$(a, b) = (a, 0) + (b, 0)(0, 1) = a + bi$$

The notation $z = a + bi$, where $a \equiv \text{Re } z$ and $b \equiv \text{Im } z$ are called, respectively, the *real* and *imaginary parts* of the complex number z, is more convenient than (a, b). For example, the sum and product of two complex numbers $z = a + bi$ and $w = c + di$ can be obtained by simply using the commutative and distributive laws and $i^2 = -1$:

$$z + w = (a + bi) + (c + di) = a + c + bi + di = (a + c) + (b + d)i$$
$$zw = (a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (bc + ad)i$$

Warning: The above use of the letter i for $\sqrt{-1}$ has no relationship whatsoever to the vector notation $\mathbf{i} = (1, 0, 0)$ introduced in Section 2.8.

The conjugate of the complex number $z = (a, b) = a + bi$ is denoted and defined by

$$\bar{z} = \overline{a + bi} = a - bi$$

Then $z\bar{z} = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2$. If, in addition, $z \neq 0$, then the inverse z^{-1} of z and division of w by z are given, respectively, by

$$z^{-1} = \frac{\bar{z}}{z\bar{z}} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2} i \quad \text{and} \quad \frac{w}{z} = wz^{-1}$$

where $w \in \mathbf{C}$. We also define

$$-z = -1z \quad \text{and} \quad w - z = w + (-z)$$

Just as the real numbers can be represented by the points on a line, the complex numbers can be represented by the points in the plane. Specifically, we let the point (a, b) in the plane represent the complex number $z = a + bi$, i.e., whose real part is a and whose imaginary part is b. (See Fig. 2-11.) The *absolute value* of z, written $|z|$, is defined as the distance from z to the origin:

$$|z| = \sqrt{a^2 + b^2}$$

Note that $|z|$ is equal to the norm of the vector (a, b). Also, $|z| = \sqrt{z\bar{z}}$.

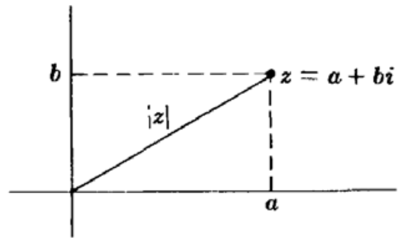


Fig. 2-11

Example 2.12. Suppose $z = 2 + 3i$ and $w = 5 - 2i$. Then

$$z + w = (2 + 3i) + (5 - 2i) = 2 + 5 + 3i - 2i = 7 + i$$

$$zw = (2 + 3i)(5 - 2i) = 10 + 15i - 4i - 6i^2 = 16 + 11i$$

$$\bar{z} = \overline{2 + 3i} = 2 - 3i \qquad \text{and} \qquad \bar{w} = \overline{5 - 2i} = 5 + 2i$$

$$\frac{w}{z} = \frac{5 - 2i}{2 + 3i} = \frac{(5 - 2i)(2 - 3i)}{(2 + 3i)(2 - 3i)} = \frac{4 - 19i}{13} = \frac{4}{13} - \frac{19}{13}i$$

$$|z| = \sqrt{4 + 9} = \sqrt{13} \qquad \text{and} \qquad |w| = \sqrt{25 + 4} = \sqrt{29}$$

Remark: We emphasize that the set $\mathbf{C}$ of complex numbers with the above operations of addition and multiplication is a *field*, like $\mathbf{R}$.

2.10 VECTORS IN $\mathbf{C}^n$

The set of all n-tuples of complex numbers, denoted by $\mathbf{C}^n$, is called *complex n-space*. Just as in the real case, the elements of $\mathbf{C}^n$ are called *points* or *vectors*, the elements of $\mathbf{C}$ are called *scalars*, and *vector addition* in $\mathbf{C}^n$ and *scalar multiplication* on $\mathbf{C}^n$ are given by

$$(z_1, z_2, \ldots, z_n) + (w_1, w_2, \ldots, w_n) = (z_1 + w_1, z_2 + w_2, \ldots, z_n + w_n)$$

$$z(z_1, z_2, \ldots, z_n) = (zz_1, zz_2, \ldots, zz_n)$$

where $z_i, w_i, z \in \mathbf{C}$.

Example 2.13

(*a*) $(2 + 3i, 4 - i, 3) + (3 - 2i, 5i, 4 - 6i) = (5 + i, 4 + 4i, 7 - 6i)$

(*b*) $2i(2 + 3i, 4 - i, 3) = (-6 + 4i, 2 + 8i, 6i)$

Now let u and v be arbitrary vectors in $\mathbf{C}^n$:

$$u = (z_1, z_2, \ldots, z_n) \qquad v = (w_1, w_2, \ldots, w_n) \qquad z_i, w_i \in \mathbf{C}$$

The *dot*, or *inner*, product of u and v is defined as follows:

$$u \cdot v = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n$$

Note that this definition reduces to the previous one in the real case, since $w_i = \bar{w}_i$ when w_i is real. The norm of u is defined by

$$\|u\| = \sqrt{u \cdot u} = \sqrt{z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n} = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$$

Observe that $u \cdot u$ and so $\|u\|$ are real and positive when $u \neq 0$ and 0 when $u = 0$.

Example 2.14. Let $u = (2 + 3i, 4 - i, 2i)$ and $v = (3 - 2i, 5, 4 - 6i)$. Then

$$\begin{aligned} u \cdot v &= (2 + 3i)\overline{(3 - 2i)} + (4 - i)\overline{(5)} + (2i)\overline{(4 - 6i)} \\ &= (2 + 3i)(3 + 2i) + (4 - i)(5) + (2i)(4 + 6i) \\ &= 13i + 20 - 5i - 12 + 8i = 8 + 16i \end{aligned}$$

$$\begin{aligned} u \cdot u &= (2 + 3i)\overline{(2 + 3i)} + (4 - i)\overline{(4 - i)} + (2i)\overline{(2i)} \\ &= (2 + 3i)(2 - 3i) + (4 - i)(4 + i) + (2i)(-2i) \\ &= 13 + 17 + 4 = 34 \end{aligned}$$

$$\|u\| = \sqrt{u \cdot u} = \sqrt{34}$$

The space $\mathbf{C}^n$ with the above operations of vector addition, scalar multiplication, and dot product, is called *complex Euclidean n-space.*

Remark: If $u \cdot v$ were defined by $u \cdot v = z_1 w_1 + \cdots + z_n w_n$, then it is possible for $u \cdot u = 0$ even though $u \neq 0$, e.g., if $u = (1, i, 0)$. In fact, $u \cdot u$ may not even be real.

Solved Problems

VECTORS IN $\mathbf{R}^n$

2.1. Let $u = (2, -7, 1)$, $v = (-3, 0, 4)$, $w = (0, 5, -8)$. Find (*a*) $3u - 4v$, (*b*) $2u + 3v - 5w$.

First perform the scalar multiplication and then the vector addition.

(*a*) $3u - 4v = 3(2, -7, 1) - 4(-3, 0, 4) = (6, -21, 3) + (12, 0, -16) = (18, -21, -13)$

(*b*) $$\begin{aligned} 2u + 3v - 5w &= 2(2, -7, 1) + 3(-3, 0, 4) - 5(0, 5, -8) \\ &= (4, -14, 2) + (-9, 0, 12) + (0, -25, 40) \\ &= (4 - 9 + 0, -14 + 0 - 25, 2 + 12 + 40) = (-5, -39, 54) \end{aligned}$$

2.2. Compute:

$$(a)\quad 2\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}, \qquad (b)\quad -2\begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix} + 4\begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

First perform the scalar multiplication and then the vector addition.

(*a*) $$2\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix} + \begin{pmatrix} -6 \\ -9 \\ 12 \end{pmatrix} = \begin{pmatrix} -4 \\ -11 \\ 18 \end{pmatrix}$$

(*b*) $$-2\begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix} + 4\begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -10 \\ -6 \\ 8 \end{pmatrix} + \begin{pmatrix} -4 \\ 20 \\ 8 \end{pmatrix} + \begin{pmatrix} -9 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -23 \\ 17 \\ 19 \end{pmatrix}$$

2.3. Find x and y if (*a*) $(x, 3) = (2, x + y)$; (*b*) $(4, y) = x(2, 3)$.

(*a*) Since the two vectors are equal, the corresponding components are equal to each other:

$$x = 2 \qquad 3 = x + y$$

Substitute $x = 2$ into the second equation to obtain $y = 1$. Thus $x = 2$ and $y = 1$.

(*b*) Multiply by the scalar x to obtain $(4, y) = x(2, 3) = (2x, 3x)$. Set the corresponding components equal to each other:

$$4 = 2x \qquad y = 3x$$

Solve the linear equations for x and y: $x = 2$ and $y = 6$.

2.4. Prove Theorem 2.1.

Let u_i, v_i, and w_i be the ith components of u, v, and w, respectively.

(i) By definition, $u_i + v_i$ is the ith component of $u + v$ and so $(u_i + v_i) + w_i$ is the ith component of $(u + v) + w$. On the other hand, $v_i + w_i$ is the ith component of $v + w$ and so $u_i + (v_i + w_i)$ is the ith component of $u + (v + w)$. But u_i, v_i, and w_i are real numbers for which the associative law holds, that is,

$$(u_i + v_i) + w_i = u_i + (v_i + w_i) \qquad \text{for} \qquad i = 1, \ldots, n$$

Accordingly, $(u + v) + w = u + (v + w)$ since their corresponding components are equal.

(ii) Here, $0 = (0, 0, \ldots, 0)$; hence

$$\begin{aligned} u + 0 &= (u_1, u_2, \ldots, u_n) + (0, 0, \ldots, 0) \\ &= (u_1 + 0, u_2 + 0, \ldots, u_n + 0) = (u_1, u_2, \ldots, u_n) = u \end{aligned}$$

(iii) Since $-u = -1(u_1, u_2, \ldots, u_n) = (-u_1, -u_2, \ldots, -u_n)$,

$$\begin{aligned} u + (-u) &= (u_1, u_2, \ldots, u_n) + (-u_1, -u_2, \ldots, -u_n) \\ &= (u_1 - u_1, u_2 - u_2, \ldots, u_n - u_n) = (0, 0, \ldots, 0) = 0 \end{aligned}$$

(iv) By definition, $u_i + v_i$ is the ith component of $u + v$, and $v_i + u_i$ is the ith component of $v + u$. But u_i and v_i are real numbers for which the commutative law holds, that is,

$$u_i + v_i = v_i + u_i \qquad i = 1, \ldots, n$$

Hence $u + v = v + u$ since their corresponding components are equal.

(v) Since $u_i + v_i$ is the ith component of $u + v$, $k(u_i + v_i)$ is the ith component of $k(u + v)$. Since ku_i and kv_i are the ith components of ku and kv, respectively, $ku_i + kv_i$ is the ith component of $ku + kv$. But k, u_i, and v_i are real numbers; hence

$$k(u_i + v_i) = ku_i + kv_i \qquad i = 1, \ldots, n$$

Thus $k(u + v) = ku + kv$, as corresponding components are equal.

(vi) Observe that the first plus sign refers to the addition of the two scalars k and k' whereas the second plus sign refers to the vector addition of the two vectors ku and $k'u$.

By definition, $(k + k')u_i$ is the ith component of the vector $(k + k')u$. Since ku_i and $k'u_i$ are the ith components of ku and $k'u$, respectively, $ku_i + k'u_i$ is the ith component of $ku + k'u$. But k, k', and u_i are real numbers; hence

$$(k + k')u_i = ku_i + k'u_i \qquad i = 1, \ldots, n$$

Thus $(k + k')u = ku + k'u$, as corresponding components are equal.

(vii) Since $k'u_i$ is the ith component of $k'u$, $k(k'u_i)$ is the ith component of $k(k'u)$. But $(kk')u_i$ is the ith component of $(kk')u$ and, since k, k' and u_i are real numbers,

$$(kk')u_i = k(k'u_i) \qquad i = 1, \ldots, n$$

Hence $(kk')u = k(k'u)$, as corresponding components are equal.

(viii) $1 \cdot u = 1(u_1, u_2, \ldots, u_n) = (1u_1, 1u_2, \ldots, 1u_n) = (u_1, u_2, \ldots, u_n) = u$.

VECTORS AND LINEAR EQUATIONS

2.5. Convert the following vector equation to an equivalent system of linear equations and solve:

$$\begin{pmatrix} 1 \\ -6 \\ 5 \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + z\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

Multiply the vectors on the right by the unknown scalars and then add:

$$\begin{pmatrix} 1 \\ -6 \\ 5 \end{pmatrix} = \begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix} + \begin{pmatrix} 2y \\ 5y \\ 8y \end{pmatrix} + \begin{pmatrix} 3z \\ 2z \\ 3z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 2x + 5y + 2z \\ 3x + 8y + 3z \end{pmatrix}$$

Set corresponding components of the vectors equal to each other, and reduce the system to echelon form:

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 5y + 2z &= -6 \\ 3x + 8y + 3z &= 5 \end{aligned} \qquad \begin{aligned} x + 2y + 3z &= 1 \\ y - 4z &= -8 \\ 2y - 6z &= 2 \end{aligned} \qquad \begin{aligned} x + 2y + 3z &= 1 \\ y - 4z &= -8 \\ 2z &= 18 \end{aligned}$$

The system is triangular, and back-substitution yields the unique solution $x = -82$, $y = 28$, $z = 9$.

2.6. Write the vector $v = (1, -2, 5)$ as a linear combination of the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$, and $u_3 = (2, -1, 1)$.

We want to express v in the form $v = xu_1 + yu_2 + zu_3$ with x, y, and z as yet unknown. Thus we have

$$\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + z\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} x + y + 2z \\ x + 2y - z \\ x + 3y + z \end{pmatrix}$$

(It is more convenient to write the vectors as columns than as rows when forming linear combinations.) Setting corresponding components equal to each other we obtain:

$$\begin{aligned} x + y + 2z &= 1 \\ x + 2y - z &= -2 \\ x + 3y + z &= 5 \end{aligned} \quad \text{or} \quad \begin{aligned} x + y + 2z &= 1 \\ y - 3z &= -3 \\ 2y - z &= 4 \end{aligned} \quad \text{or} \quad \begin{aligned} x + y + 2z &= 1 \\ y - 3z &= -3 \\ 5z &= 10 \end{aligned}$$

The unique solution of the triangular form is $x = -6$, $y = 3$, $z = 2$; thus $v = -6u_1 + 3u_2 + 2u_3$.

2.7. Write the vector $v = (2, 3, -5)$ as a linear combination of $u_1 = (1, 2, -3)$, $u_2 = (2, -1, -4)$, and $u_3 = (1, 7, -5)$.

Find the equivalent system of equations and then solve. First:

$$\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + y\begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} + z\begin{pmatrix} 1 \\ 7 \\ -5 \end{pmatrix} = \begin{pmatrix} x + 2y + z \\ 2x - y + 7z \\ -3x - 4y - 5z \end{pmatrix}$$

Setting corresponding components equal to each other we obtain

$$\begin{aligned} x + 2y + z &= 2 \\ 2x - y + 7z &= 3 \\ -3x - 4y - 5z &= -5 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y + z &= 2 \\ -5y + 5z &= -1 \\ 2y - 2z &= 1 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y + z &= 2 \\ -5y + 5z &= -1 \\ 0 &= 3 \end{aligned}$$

The third equation, $0 = 3$, indicates that the system has no solution. Thus v cannot be written as a linear combination of the vectors u_1, u_2 and u_3.

2.8. Determine whether the vectors $u_1 = (1, 1, 1)$, $u_2 = (2, -1, 3)$, and $u_3 = (1, -5, 3)$ are linearly dependent or linearly independent.

Recall that u_1, u_2, u_3 are linearly dependent or linearly independent according as the vector equation $xu_1 + yu_2 + zu_3 = 0$ has a nonzero solution or only the zero solution. Thus first set a linear combination of the vectors equal to the zero vector:

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = x\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}2\\-1\\3\end{pmatrix} + z\begin{pmatrix}1\\-5\\3\end{pmatrix} = \begin{pmatrix}x+2y+z\\x-y-5z\\x+3y+3z\end{pmatrix}$$

Set corresponding components equal to each other, and reduce the system to echelon form:

$$\begin{aligned}x+2y+z&=0\\x-y-5z&=0\\x+3y+3z&=0\end{aligned} \quad\text{or}\quad \begin{aligned}x+2y+z&=0\\-3y-6z&=0\\y+2z&=0\end{aligned} \quad\text{or}\quad \begin{aligned}x+2y+z&=0\\y+2z&=0\end{aligned}$$

The system in echelon form has a free variable; hence the system has a nonzero solution. Thus the original vectors are linearly dependent. (We do not need to solve the system to determine linear dependence or independence; we only need to know if a nonzero solution exists.)

2.9. Determine whether or not the vectors $(1, -2, -3)$, $(2, 3, -1)$, and $(3, 2, 1)$ are linearly dependent.

Set a linear combination (with coefficients x, y, z) of the vectors equal to the zero vector:

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = x\begin{pmatrix}1\\-2\\-3\end{pmatrix} + y\begin{pmatrix}2\\3\\-1\end{pmatrix} + z\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}x+2y+3z\\-2x+3y+2z\\-3x-y+z\end{pmatrix}$$

Set corresponding components equal to each other, and reduce the system to echelon form:

$$\begin{aligned}x+2y+3z&=0\\-2x+3y+2z&=0\\-3x-y+z&=0\end{aligned} \quad\text{or}\quad \begin{aligned}x+2y+3z&=0\\7y+8z&=0\\5y+10z&=0\end{aligned} \quad\text{or}\quad \begin{aligned}x+2y+3z&=0\\y+2z&=0\\7y+8z&=0\end{aligned} \quad\text{or}\quad \begin{aligned}x+2y+3z&=0\\y+2z&=0\\-6z&=0\end{aligned}$$

The homogeneous system is in triangular form, with no free variables; hence it has only the zero solution. Thus the original vectors are linearly independent.

2.10. Prove: Any $n + 1$ or more vectors in $\mathbf{R}^n$ are linearly dependent.

Suppose $u_1, u_2, \ldots, u_q$ are vectors in $\mathbf{R}^n$ and $q > n$. The vector equation

$$x_1u_1 + x_2u_2 + \cdots + x_qu_q = 0$$

is equivalent to a homogeneous system of n equations in $q > n$ unknowns. By Theorem 1.9, this system has a nonzero solution. Therefore $u_1, u_2, \ldots, u_q$ are linearly dependent.

2.11. Show that any set of q vectors that includes the zero vector is linearly dependent.

Denoting the vectors as $0, u_2, u_3, \ldots, u_q$, we have $1(0) + 0u_2 + 0u_3 + \cdots + 0u_q = 0$.

DOT (INNER) PRODUCT, ORTHOGONALITY

2.12. Compute $u \cdot v$ where $u = (1, -2, 3, -4)$ and $v = (6, 7, 1, -2)$.

Multiply corresponding components and add: $u \cdot v = (1)(6) + (-2)(7) + (3)(1) + (-4)(-2) = 3$.

2.13. Suppose $u = (3, 2, 1)$, $v = (5, -3, 4)$, $w = (1, 6, -7)$. Find: (*a*) $(u + v) \cdot w$, (*b*) $u \cdot w + v \cdot w$.

(a) First calculate $u + v$ by adding corresponding components:

$$u + v = (3 + 5,\ 2 - 3,\ 1 + 4) = 8,\ -1,\ 5)$$

Then compute $(u + v) \cdot w = (8)(1) + (-1)(6) + (5)(-7) = 8 - 6 - 35 = -33$.

(b) First find $u \cdot w = 3 + 12 - 7 = 8$ and $v \cdot w = 5 - 18 - 28 = -41$. Then

$$u \cdot w + v \cdot w = 8 - 41 = -33$$

[Note: As expected from Theorem 2.2(i), the two values are equal.]

2.14. Let $u = (1, 2, 3, -4)$, $v = (5, -6, 7, 8)$, and $k = 3$. Find: (a) $k(u \cdot v)$, (b) $(ku) \cdot v$, (c) $u \cdot (kv)$.

(a) First find $u \cdot v = 5 - 12 + 21 - 32 = -18$. Then $k(u \cdot v) = 3(-18) = -54$.

(b) First find $ku = (3(1), 3(2), 3(3), 3(-4)) = (3, 6, 9, -12)$. Then

$$(ku) \cdot v = (3)(5) + (6)(-6) + (9)(7) + (-12)(8) = 15 - 36 + 63 - 96 = -54$$

(c) First find $kv = (15, -18, 21, 24)$. Then

$$u \cdot (kv) = (1)(15) + (2)(-18) + (3)(21) + (-4)(24) = 15 - 36 + 63 - 96 = -54$$

2.15. Let $u = (5, 4, 1)$, $v = (3, -4, 1)$, and $w = (1, -2, 3)$. Which pair of these vectors, if any, are perpendicular?

Find the dot product of each pair of vectors:

$$u \cdot v = 15 - 16 + 1 = 0 \qquad v \cdot w = 3 + 8 + 3 = 14 \qquad u \cdot w = 5 - 8 + 3 = 0$$

Hence vectors u and v and vectors u and w are orthogonal, but vectors v and w are not.

2.16. Determine k so that the vectors u and v are orthogonal where $u = (1, k, -3)$ and $v = (2, -5, 4)$.

Compute $u \cdot v$, set it equal to 0, and solve for k. $u \cdot v = (1)(2) + (k)(-5) + (-3)(4) = 2 - 5k - 12 = 0$, or $-5k - 10 = 0$. Solving, $k = -2$.

2.17. Prove Theorem 2.2.

Let $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$, $w = (w_1, w_2, \ldots, w_n)$.

(i) Since $u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$,

$$\begin{aligned}(u + v) \cdot w &= (u_1 + v_1)w_1 + (u_2 + v_2)w_2 + \cdots + (u_n + v_n)w_n \\ &= u_1 w_1 + v_1 w_1 + u_2 w_2 + v_2 w_2 + \cdots + u_n w_n + v_n w_n \\ &= (u_1 w_1 + u_2 w_2 + \cdots + u_n w_n) + (v_1 w_1 + v_2 w_2 + \cdots + v_n w_n) \\ &= u \cdot w + v \cdot w\end{aligned}$$

(ii) Since $ku = (ku_1, ku_2, \ldots, ku_n)$,

$$(ku) \cdot v = ku_1 v_1 + ku_2 v_2 + \cdots + ku_n v_n = k(u_1 v_1 + u_2 v_2 + \cdots + u_n v_n) = k(u \cdot v)$$

(iii) $u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = v_1 u_1 + v_2 u_2 + \cdots + v_n u_n = v \cdot v$

(iv) Since u_i^2 is nonnegative for each i, and since the sum of nonnegative real numbers is nonnegative,

$$u \cdot u = u_1^2 + u_2^2 + \cdots + u_n^2 \geq 0$$

Furthermore, $u \cdot u = 0$ iff $u_i = 0$ for each i, that is, iff $u = 0$.

NORM (LENGTH) IN $\mathbf{R}^n$

2.18. Find $\|w\|$ if $w = (-3, 1, -2, 4, -5)$.

$\|w\|^2 = (-3)^2 + 1^2 + (-2)^2 + 4^2 + (-5)^2 = 9 + 1 + 4 + 16 + 25 = 55$; hence $\|w\| = \sqrt{55}$.

2.19. Determine k such that $\|u\| = \sqrt{39}$ where $u = (1, k, -2, 5)$.

$\|u\|^2 = 1^2 + k^2 + (-2)^2 + 5^2 = k^2 + 30$. Now solve $k^2 + 30 = 39$ and obtain $k = 3, -3$.

2.20. Normalize $w = (4, -2, -3, 8)$.

First find $\|w\|^2 = w \cdot w = 4^2 + (-2)^2 + (-3)^2 + 8^2 = 16 + 4 + 9 + 64 = 93$. Divide each component of w by $\|w\| = \sqrt{93}$ to obtain

$$\hat{w} = \frac{w}{\|w\|} = \left(\frac{4}{\sqrt{93}}, \frac{-2}{\sqrt{93}}, \frac{-3}{\sqrt{93}}, \frac{8}{\sqrt{93}}\right)$$

2.21. Normalize $v = (\frac{1}{2}, \frac{2}{3}, -\frac{1}{4})$.

Note that v and any positive multiple of v will have the same normalized form. Hence, first multiply v by 12 to "clear" fractions: $12v = (6, 8, -3)$. Then

$$\|12v\|^2 = 36 + 64 + 9 = 109 \qquad \text{and} \qquad \hat{v} = \widehat{12v} = \frac{12v}{\|12v\|} = \left(\frac{6}{\sqrt{109}}, \frac{8}{\sqrt{109}}, \frac{-3}{\sqrt{109}}\right)$$

2.22. Prove Theorem 2.3 (Cauchy–Schwarz). $|u \cdot v| \le \|u\|\,\|v\|$.

We shall prove the following stronger statement: $|u \cdot v| \le \sum_{i=1}^{n} |u_i v_i| \le \|u\|\|v\|$. First, if $u = 0$ or $v = 0$, then the inequality reduces to $0 \le 0 \le 0$ and is therefore true. Hence we need only consider the case in which $u \ne 0$ and $v \ne 0$, i.e., where $\|u\| \ne 0$ and $\|v\| \ne 0$. Furthermore, because

$$|u \cdot v| = \left|\sum u_i v_i\right| \le \sum |u_i v_i|$$

we need only prove the second inequality.

Now, for any real numbers $x, y \in \mathbf{R}$, $0 \le (x - y)^2 = x^2 - 2xy + y^2$ or, equivalently,

$$2xy \le x^2 + y^2 \tag{1}$$

Set $x = |u_i|/\|u\|$ and $y = |v_i|/\|v\|$ in (*1*) to obtain, for any i,

$$2\frac{|u_i|}{\|u\|}\frac{|v_i|}{\|v\|} \le \frac{|u_i|^2}{\|u\|^2} + \frac{|v_i|^2}{\|v\|^2} \tag{2}$$

But, by definition of the norm of a vector, $\|u\| = \sum u_i^2 = \sum |u_i|^2$ and $\|v\| = \sum v_i^2 = \sum |v_i|^2$. Thus summing (*2*) with respect to i and using $|u_i v_i| = |u_i||v_i|$, we have

$$2\frac{\sum |u_i v_i|}{\|u\|\|v\|} \le \frac{\sum |u_i|^2}{\|u\|^2} + \frac{\sum |v_i|^2}{\|v\|^2} = \frac{\|u\|^2}{\|u\|^2} + \frac{\|v\|^2}{\|v\|^2} = 2$$

that is,

$$\frac{\sum |u_i v_i|}{\|u\|\|v\|} \le 1$$

Multiplying both sides by $\|u\|\|v\|$, we obtain the required inequality.

2.23. Prove Theorem 2.4 (Minkowski). $\|u + v\| \le \|u\| + \|v\|$.

By the Cauchy–Schwarz inequality (Problem 2.22) and the other properties of the inner product,

$$\begin{aligned}\|u + v\|^2 &= (u + v) \cdot (u + v) = u \cdot u + 2(u \cdot v) + v \cdot v \\ &\le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2\end{aligned}$$

Taking the square roots of both sides yields the desired inequality.

2.24. Prove that the norm in $\mathbf{R}^n$ satisfies the following laws:

(a) $[N_1]$ For any vector u, $\|u\| \geq 0$; and $\|u\| = 0$ iff $u = 0$.

(b) $[N_2]$ For any vector u and any scalar k, $\|ku\| = |k|\,\|u\|$.

(c) $[N_3]$ For any vectors u and v, $\|u + v\| \leq \|u\| + \|v\|$.

(a) By Theorem 2.2, $u \cdot u \geq 0$, and $u \cdot u = 0$ iff $u = 0$. Since $\|u\| = \sqrt{u \cdot u}$, $[N_1]$ follows.

(b) Suppose $u = (u_1, u_2, \ldots, u_n)$ and so $ku = (ku_1, ku_2, \ldots, ku_n)$. Then

$$\|ku\|^2 = (ku_1)^2 + (ku_2)^2 + \cdots + (ku_n)^2 = k^2(u_1^2 + u_2^2 + \cdots + u_n^2) = k^2\|u\|^2$$

Taking square roots gives $[N_2]$.

(c) $[N_3]$ was proved in Problem 2.23.

2.25. Let $u = (1, 2, -2)$, $v = (3, -12, 4)$, and $k = -3$. (a) Find $\|u\|$, $\|v\|$, and $\|ku\|$. (b) Verify that $\|ku\| = |k|\,\|u\|$ and $\|u + v\| \leq \|u\| + \|v\|$.

(a) $\|u\| = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$, $\|v\| = \sqrt{9 + 144 + 16} = \sqrt{169} = 13$, $ku = (-3, -6, 6)$, and $\|ku\| = \sqrt{9 + 36 + 36} = \sqrt{81} = 9$.

(b) Since $|k| = |-3| = 3$, we have $|k|\,\|u\| = 3 \cdot 3 = 9 = \|ku\|$. Also $u + v = (4, -10, 2)$. Thus

$$\|u + v\| = \sqrt{16 + 100 + 4} = \sqrt{120} \leq 16 = 3 + 13 = \|u\| + \|v\|$$

DISTANCE, ANGLES, PROJECTIONS

2.26. Find the distance $d(u, v)$ between the vectors u and v where

(a) $u = (1, 7)$, $v = (6, -5)$,

(b) $u = (3, -5, 4)$, $v = (6, 2, -1)$,

(c) $u = (5, 3, -2, -4, -1)$, $v = (2, -1, 0, -7, 2)$.

In each case use the formula $d(u, v) = \|u - v\| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}$.

(a) $d(u, v) = \sqrt{(1 - 6)^2 + (7 + 5)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

(b) $d(u, v) = \sqrt{(3 - 6)^2 + (-5 - 2)^2 + (4 + 1)^2} = \sqrt{9 + 49 + 25} = \sqrt{83}$

(c) $d(u, v) = \sqrt{(5 - 2)^2 + (3 + 1)^2 + (-2 + 0)^2 + (-4 + 7)^2 + (-1 - 2)^2} = \sqrt{47}$

2.27. Find k such that $d(u, v) = 6$ where $u = (2, k, 1, -4)$ and $v = (3, -1, 6, -3)$.

First find

$$[d(u, v)]^2 = (2 - 3)^2 + (k + 1)^2 + (1 - 6)^2 + (-4 + 3)^2 = k^2 + 2k + 28$$

Now solve $k^2 + 2k + 28 = 6^2$ to obtain $k = 2, -4$.

2.28. From Problem 2.24, prove that the distance function $d(u, v)$ satisfies the following:

$[M_1]$ $d(u, v) \geq 0$; and $d(u, v) = 0$ iff $u = v$.

$[M_2]$ $d(u, v) = d(v, u)$.

$[M_3]$ $d(u, v) \leq d(u, w) + d(w, v)$ (triangle inequality).

$[M_1]$ follows directly from $[N_1]$. By $[N_2]$,

$$d(u, v) = \|u - v\| = \|(-1)(v - u)\| = |-1|\,\|v - u\| = \|v - u\| = d(v, u)$$

which is $[M_2]$. By $[N_3]$,

$$d(u, v) = \|u - v\| = \|(u - w) + (w - v)\| \le \|u - w\| + \|w - v\| = d(u, w) + d(w, v)$$

which is $[M_3]$.

2.29. Find $\cos\theta$, where θ is the angle between $u = (1, 2, -5)$ and $v = (2, 4, 3)$.

First find

$$u \cdot v = 2 + 8 - 15 = -5 \qquad \|u\|^2 = 1 + 4 + 25 = 30 \qquad \|v\|^2 = 4 + 16 + 9 = 29$$

Then

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = -\frac{5}{\sqrt{30}\sqrt{29}}$$

2.30. Find proj (u, v) where $u = (1, -3, 4)$ and $v = (3, 4, 7)$.

First find $u \cdot v = 3 - 12 + 28 = 19$ and $\|v\|^2 = 9 + 16 + 49 = 74$. Then

$$\text{proj}\,(u, v) = \frac{u \cdot v}{\|v\|^2} v = \frac{19}{74}(3, 4, 7) = \left(\frac{57}{74}, \frac{76}{74}, \frac{133}{74}\right) = \left(\frac{57}{74}, \frac{38}{37}, \frac{133}{74}\right)$$

POINTS, LINES, AND HYPERPLANES

This section distinguishes between an n-tuple $P(a_1, a_2, \ldots, a_n) \equiv P(a_i)$ viewed as a point in $\mathbf{R}^n$ and an n-tuple $v = [c_1, c_2, \ldots, c_n]$ viewed as a vector (arrow) from the origin O to the point $C(c_1, c_2, \ldots, c_n)$.

2.31. Find the vector v that is identified with the directed line segment $\overrightarrow{PQ}$ for the points (*a*) $P(2, 5)$ and $Q(-3, 4)$ in $\mathbf{R}^2$, (*b*) $P(1, -2, 4)$ and $Q(6, 0, -3)$ in $\mathbf{R}^3$.

(*a*) $v = Q - P = [-3 - 2, 4 - 5] = [-5, -1]$

(*b*) $v = Q - P = [6 - 1, 0 + 2, -3 - 4] = [5, 2, -7]$

2.32. Consider points $P(3, k, -2)$ and $Q(5, 3, 4)$ in $\mathbf{R}^3$. Find k so that $\overrightarrow{PQ}$ is orthogonal to the vector $u = [4, -3, 2]$.

First find $v = Q - P = [5 - 3, 3 - k, 4 + 2] = [2, 3 - k, 6]$. Next compute

$$u \cdot v = 4(2) - 3(3 - k) + 2(6) = 8 - 9 + 3k + 12 = 3k + 11$$

Lastly, set $u \cdot v = 0$ or $3k + 11 = 0$, from which $k = -11/3$.

2.33. Consider the hyperplane H in $\mathbf{R}^n$ which is the solution set of the linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \qquad (1)$$

where $u = [a_1, a_2, \ldots, a_n] \ne 0$. Show that the directed line segment $\overrightarrow{PQ}$ of any pair of points P, $Q \in H$ is orthogonal to the coefficient vector u; the vector u is said to be *normal* to the hyperplane H.

Let $w_1 = \overrightarrow{OP}$ and $w_2 = \overrightarrow{OQ}$; hence $v = w_2 - w_1 = \overrightarrow{PQ}$. By (*1*), $u \cdot w_1 = b$ and $u \cdot w_2 = b$. But then

$$u \cdot v = u \cdot (w_2 - w_1) = u \cdot w_2 - u \cdot w_1 = b - b = 0$$

Thus $v = \overrightarrow{PQ}$ is orthogonal to the normal vector u.

2.34. Find an equation of the hyperplane H in $\mathbf{R}^4$ which passes through $P(3, -2, 1, -4)$ and is normal to $u = [2, 5, -6, -2]$.

An equation of H is of the form $2x + 5y - 6z - 2w = k$ since it is normal to u. Substitute P into this equation to obtain $k = -2$. Thus an equation of H is $2x + 5y - 6z - 2w = -2$.

2.35. Find an equation of the plane H in $\mathbf{R}^3$ which contains $P(1, -5, 2)$ and is parallel to the plane H' determined by $3x - 7y + 4z = 5$.

H and H' are parallel if and only if their normals are parallel or antiparallel. Hence an equation of H is of the form $3x - 7y + 4z = k$. Substitute $P(1, -5, 2)$ into this equation to obtain $k = 46$. Thus the required equation is $3x - 7y + 4z = 46$.

2.36. Find a parametric representation of the line in $\mathbf{R}^4$ passing through $P(4, -2, 3, 1)$ in the direction $u = [2, 5, -7, 11]$.

The line L in $\mathbf{R}^n$ passing through the point $P(a_i)$ and in the direction of the nonzero vector $u = [u_i]$ consists of the points $X = (x_i)$ which satisfy the equation

$$X = P + tu \qquad \text{or} \qquad x_i = a_i + u_i t \qquad (\text{for } i = 1, 2, \ldots, n) \tag{1}$$

where the parameter t takes on all real values. Thus we obtain

$$\begin{cases} x = 4 + 2t \\ y = -2 + 5t \\ z = 3 - 7t \\ w = 1 + 11t \end{cases} \qquad \text{or} \qquad (4 + 2t, -2 + 5t, 3 - 7t, 1 + 11t)$$

2.37. Find a parametric equation of the line in $\mathbf{R}^3$ passing through the points $P(5, 4, -3)$ and $Q(1, -3, 2)$.

First compute $u = \overrightarrow{PQ} = [1 - 5, -3 - 4, 2 - (-3)] = [-4, -7, 5]$. Then use Problem 2.36 to obtain

$$x = 5 - 4t \qquad y = 4 - 7t \qquad z = -3 + 5t$$

2.38. Give a nonparametric representation for the line of Problem 2.37.

Solve each coordinate equation for t and equate the results

$$\frac{x - 5}{-4} = \frac{y - 4}{-7} = \frac{z + 3}{5}$$

or the pair of linear equations $7x - 4y = 19$ and $5x + 4z = 13$.

2.39. Find a parametric equation of the line in $\mathbf{R}^3$ perpendicular to the plane $2x - 3y + 7z = 4$ and intersecting the plane at the point $P(6, 5, 1)$.

Since the line is perpendicular to the plane, the line must be in the direction of the normal vector $u = [2, -3, 7]$ of the plane. Hence

$$x = 6 + 2t \qquad y = 5 - 3t \qquad z = 1 + 7t$$

2.40. Consider the following curve C in $\mathbf{R}^4$ where $0 \le t \le 4$:

$$F(t) = (t^2, 3t - 2, t^3, t^2 + 5)$$

Find the unit tangent vector $\mathbf{T}$ when $t = 2$.

Take the derivative of (each component of) $F(t)$ to obtain a vector V which is tangent to the curve:

$$V(t) = \frac{dF(t)}{dt} = (2t, 3, 3t^2, 2t)$$

Now find V when $t = 2$: $V = (4, 3, 12, 4)$. Normalize V to get the unit tangent vector $\mathbf{T}$ to the curve when $t = 2$. We have

$$\|V\|^2 = 16 + 9 + 144 + 16 = 185 \qquad \text{or} \qquad \|V\| = \sqrt{185}$$

Thus

$$\mathbf{T} = \left[\frac{4}{\sqrt{185}}, \frac{3}{\sqrt{185}}, \frac{12}{\sqrt{185}}, \frac{4}{\sqrt{185}}\right]$$

2.41. Let $\mathbf{T}(t)$ be the unit tangent vector to a curve C in $\mathbf{R}^n$. Show that $d\mathbf{T}(t)/dt$ is orthogonal to $\mathbf{T}(t)$.

We have $\mathbf{T}(t) \cdot \mathbf{T}(t) = 1$. Using the rule for differentiating a dot product, along with $d(1)/dt = 0$, we have

$$d[\mathbf{T}(t) \cdot \mathbf{T}(t)]/dt = \mathbf{T}(t) \cdot d\mathbf{T}(t)/dt + d\mathbf{T}(t)/dt \cdot \mathbf{T}(t) = 2\mathbf{T}(t) \cdot d\mathbf{T}(t)/dt = 0$$

Thus $d\mathbf{T}(t)/dt$ is orthogonal to $\mathbf{T}(t)$.

SPATIAL VECTORS (VECTORS IN $\mathbf{R}^3$), PLANES, LINES, CURVES, AND SURFACES IN $\mathbf{R}^3$

The following formulas will be used in Problems 2.42–2.53.

The equation of a plane through the point $P_0(x_0, y_0, z_0)$ with normal direction $N = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is

$$a(x - x_0) + b(y - y_0) + c(x - x_0) = 0 \tag{2.1}$$

The parametric equation of a line L through a point $P_0(x_0, y_0, z_0)$ in the direction of the vector $v = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is

$$x = at + x_0 \qquad y = bt + y_0 \qquad z = ct + z_0$$

or, equivalently,

$$f(t) = (at + x_0)\mathbf{i} + (bt + y_0)\mathbf{j} + (ct + z_0)\mathbf{k} \tag{2.2}$$

The equation of a normal vector N to a surface $F(x, y, z) = 0$ is

$$N = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \tag{2.3}$$

2.42. Find the equation of the plane with normal direction $N = 5\mathbf{i} - 6\mathbf{j} + 7\mathbf{k}$ and containing the point $P(3, 4, -2)$.

Substitute P and N in equation (2.1) to get

$$5(x - 3) - 6(y - 4) + 7(z + 2) = 0 \qquad \text{or} \qquad 5x - 6y + 7z = -23$$

2.43. Find a normal vector N to the plane $4x + 7y - 12z = 3$.

The coefficients of x, y, z give a normal direction; hence $N = 4\mathbf{i} + 7\mathbf{j} - 12\mathbf{k}$. (Any multiple of N also is normal to the plane.)

2.44. Find the plane H parallel to $4x + 7y - 12z = 3$ and containing the point $P(2, 3, -1)$.

H and the given plane have the same normal direction; that is, $N = 4\mathbf{i} + 7\mathbf{j} - 12\mathbf{k}$ is normal to H. Substitute P and N in equation (*a*) to get

$$4(x-2) + 7(y-3) - 12(z+1) = 0 \qquad \text{or} \qquad 4x + 7y - 12z = 41$$

2.45. Let H and K be, respectively, the planes $x + 2y - 4z = 5$ and $2x - y + 3z = 7$. Find $\cos\theta$ where θ is the angle between planes H and K.

The angle θ between H and K is the same as the angle between a normal N to H and a normal N' to K. We have

$$N = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k} \qquad \text{and} \qquad N' = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

Then

$$N \cdot N' = 2 - 2 - 12 = -12 \qquad \|N\|^2 = 1 + 4 + 16 = 21 \qquad \|N'\|^2 = 4 + 1 + 9 = 14$$

Thus

$$\cos\theta = \frac{N \cdot N'}{\|N\|\,\|N'\|} = -\frac{12}{\sqrt{21}\sqrt{14}} = -\frac{12}{7\sqrt{6}}$$

2.46. Derive equation (*2.1*).

Let $P(x, y, z)$ be an arbitrary point in the plane. The vector v from P_0 to P is

$$v = P - P_0 = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

Since v is orthogonal to $N = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ (Fig. 2-12), we get the required formula

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

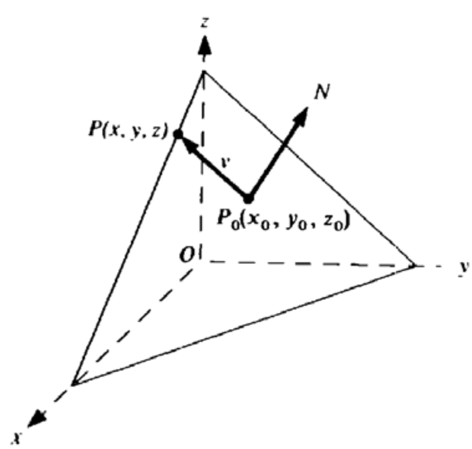


Fig. 2-12

2.47. Derive equation (*2.2*).

Let $P(x, y, z)$ be an arbitrary point on the line L. The vector w from P_0 to P is

$$w = P - P_0 = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} \tag{1}$$

Since w and v have the same direction (Fig. 2-13),

$$w = tv = t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k} \tag{2}$$

Equations (*1*) and (*2*) give us our result.

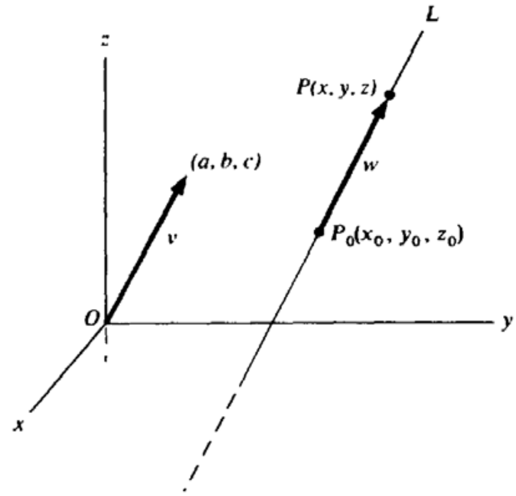


Fig. 2-13

2.48. Find the (parametric) equation of the line L through:

(a) Point $P(3, 4, -2)$ and in the direction of $v = 5\mathbf{i} - \mathbf{j} + 3\mathbf{k}$,

(b) Points $P(1, 3, 2)$ and $Q(2, 5, -6)$.

(a) Substitute in equation (2.2) to get

$$f(t) = (5t + 3)\mathbf{i} + (-t + 4)\mathbf{j} + (3t - 2)\mathbf{k}$$

(b) First find the vector v from P to Q: $v = Q - P = \mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$. Then use (2.2) with v and one of the given points, say P, to get

$$f(t) = (t + 1)\mathbf{i} + (2t + 3)\mathbf{j} + (-8t + 2)\mathbf{k}$$

2.49. Let H be the plane $3x + 5y + 7z = 15$. Find the equation of the line L perpendicular to H and containing the point $P(1, -2, 4)$.

Since L is perpendicular to the plane H, L must be in the same direction as the normal $N = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$ to H. Thus use (2.2) with N and P to get

$$f(t) = (3t + 1)\mathbf{i} + (5t - 2)\mathbf{j} + (7t + 4)\mathbf{k}$$

2.50. Consider a moving body B whose position at time t is given by $R(t) = t^3\mathbf{i} + 2t^2\mathbf{j} + 3t\mathbf{k}$. [Then $V(t) = dR(t)/dt$ denotes the velocity of B and $A(t) = dV(t)/dt$ denotes the acceleration of B.]

(a) Find the position of B when $t = 1$.
(c) Find the speed s of B when $t = 1$.

(b) Find the velocity v of B when $t = 1$.
(d) Find the acceleration a of B when $t = 1$.

(a) Substitute $t = 1$ into $R(t)$ to get $R(1) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(b) Take the derivative of $R(t)$ to get

$$V(t) = \frac{dR(t)}{dt} = 3t^2\mathbf{i} + 4t\mathbf{j} + 3\mathbf{k}$$

Substitute $t = 1$ in $V(t)$ to get $v = V(1) = 3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$.

(c) The speed s is the magnitude of the velocity v. Thus

$$s^2 = \|v\|^2 = 9 + 16 + 9 = 34 \qquad \text{and hence} \qquad s = \sqrt{34}$$

(d) Take the second derivative of $R(t)$ or, in other words, the derivative of $V(t)$ to get

$$A(t) = \frac{dV(t)}{dt} = 6t\mathbf{i} + 4\mathbf{j}$$

Substitute $t = 1$ in $A(t)$ to get $a = A(1) = 6\mathbf{i} + 4\mathbf{j}$.

2.51. Consider the surface $xy^2 + 2yz = 16$ in R^3. Find (a) a normal vector $N(x, y, z)$ to the surface and (b) the tangent plane H to the surface at point $P(1, 2, 3)$.

(a) Find the partial derivatives F_x, F_y, F_z where $F(x, y, z) = xy^2 + 2yz - 16$. We have

$$F_x = y^2 \qquad F_y = 2xy + 2z \qquad F_z = 2y$$

Thus, from equation (2.3), $N(x, y, z) = y^2\mathbf{i} + (2xy + 2z)\mathbf{j} + 2y\mathbf{k}$.

(b) A normal to the surface at point P is

$$N(P) = N(1, 2, 3) = 4\mathbf{i} + 10\mathbf{j} + 4\mathbf{k}$$

Thus $N = 2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ is also a normal vector at P. Substitute P and N into equation (2.1) to get

$$2(x - 1) + 5(y - 2) + 2(z - 3) = 0 \qquad \text{or} \qquad 2x + 5y + 2z = 18$$

2.52. Consider the ellipsoid $x^2 + 2y^2 + 3z^2 = 15$. Find the tangent plane H at point $P(2, 2, 1)$.

First find a normal vector [from equation (2.3)]

$$N(x, y, z) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} = 2x\mathbf{i} + 4y\mathbf{j} + 6z\mathbf{k}$$

Evaluate the normal vector $N(x, y, z)$ at P to get

$$N(P) = N(2, 2, 1) = 4\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$$

Thus $N = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ is normal to the ellipsoid at P. Substitute P and N into (2,1) to obtain H:

$$2(x - 2) + 4(y - 2) + 3(z - 1) = 0 \qquad \text{or} \qquad 2x + 4y + 3z = 15$$

2.53. Consider the equation $F(x, y, z) \equiv x^2 + y^2 - z = 0$, whose solution set $z = x^2 + y^2$ represents a surface S in $\mathbf{R}^3$. Find (a) a normal vector N to the surface S when $x = 2$, $y = 3$; and (b) the tangent plane H to the surface S when $x = 2$, $y = 3$.

(a) Use the fact that, when $F(x, y, z) = f(x, y) - z$, we have $F_x = f_x$, $F_y = f_y$, and $F_z = -1$. Then

$$N = (f_x, f_y, -1) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} = 4\mathbf{i} + 6\mathbf{j} - \mathbf{k}$$

(b) If $x = 2$, $y = 3$, then $z = 4 + 9 = 13$; hence $P(2, 3, 13)$ is the point on the surface S. Substitute P and $N = 4\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ into equation (2.1) to obtain H.

$$4(x - 2) + 6(y - 3) - (z - 13) = 0 \qquad \text{or} \qquad 4x + 6y - z = 13$$

CROSS PRODUCT

The cross product is defined only for vectors in $\mathbf{R}^3$.

2.54. Find $u \times v$ where (a) $u = (1, 2, 3)$ and $v = (4, 5, 6)$, (b) $u = (7, 3, 1)$ and $v = (1, 1, 1)$, (c) $u = (-4, 12, 2)$ and $v = (6, -18, 3)$.

The cross product of $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$ may be obtained as follows. Put the vector $v = (b_1, b_2, b_3)$ *under* the vector $u = (a_1, a_2, a_3)$ to form the array

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Then

$$u \times v = \left(\begin{vmatrix} \boxed{a_1} & a_2 & a_3 \\ \boxed{b_1} & b_2 & b_3 \end{vmatrix}, \; - \begin{vmatrix} a_1 & \boxed{a_2} & a_3 \\ b_1 & \boxed{b_2} & b_3 \end{vmatrix}, \; \begin{vmatrix} a_1 & a_2 & \boxed{a_3} \\ b_1 & b_2 & \boxed{b_3} \end{vmatrix} \right)$$

That is, cover the first column of the array and take the determinant to obtain the first component of $u \times v$; cover the second column and take the determinant backwards [compute $a_3 b_1 - a_1 b_3$ instead of $-(a_1 b_3 - a_3 b_1)$] to obtain the second component; and cover the third column and take the determinant to obtain the third component.

(a) $$u \times v = \left(\begin{vmatrix} \boxed{1} & 2 & 3 \\ \boxed{4} & 5 & 6 \end{vmatrix}, \; - \begin{vmatrix} 1 & \boxed{2} & 3 \\ 4 & \boxed{5} & 6 \end{vmatrix}, \; \begin{vmatrix} 1 & 2 & \boxed{3} \\ 4 & 5 & \boxed{6} \end{vmatrix} \right) = (12 - 15, 12 - 6, 5 - 8) = (-3, 6, -3)$$

(b) $$u \times v = \left(\begin{vmatrix} \boxed{7} & 3 & 1 \\ \boxed{1} & 1 & 1 \end{vmatrix}, \; - \begin{vmatrix} 7 & \boxed{3} & 1 \\ 1 & \boxed{1} & 1 \end{vmatrix}, \; \begin{vmatrix} 7 & 3 & \boxed{1} \\ 1 & 1 & \boxed{1} \end{vmatrix} \right) = (3 - 1, 1 - 7, 7 - 3) = (2, -6, 4)$$

(c) $$u \times v = \left(\begin{vmatrix} \boxed{-4} & 12 & 2 \\ \boxed{6} & -18 & 3 \end{vmatrix}, \; - \begin{vmatrix} -4 & \boxed{12} & 2 \\ 6 & \boxed{-18} & 3 \end{vmatrix}, \; \begin{vmatrix} -4 & 12 & \boxed{2} \\ 6 & -18 & \boxed{-3} \end{vmatrix} \right)$$
$$= (36 + 36, 12 + 12, 72 - 72) = (72, 24, 0)$$

2.55. Consider the vectors $u = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $v = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $w = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$. Find: (a) $u \times v$, (b) $u \times w$, (c) $v \times w$.

Use

$$v_1 \times v_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

where $v_1 = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $v_2 = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

(a) $$u \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 4 \\ 3 & 1 & -2 \end{vmatrix} = (6 - 4)\mathbf{i} + (12 + 4)\mathbf{j} + (2 + 9)\mathbf{k} = 2\mathbf{i} + 16\mathbf{j} + 11\mathbf{k}$$

(*Remark:* Observe that the $\mathbf{j}$ component is obtained by taking the determinant backwards.)

(b) $$u \times w = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 4 \\ 1 & 5 & 3 \end{vmatrix} = (-9 - 20)\mathbf{i} + (4 - 6)\mathbf{j} + (10 + 3)\mathbf{k} = -29\mathbf{i} - 2\mathbf{j} + 13\mathbf{k}$$

(c) $$v \times w = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & 5 & 3 \end{vmatrix} = (3 + 10)\mathbf{i} + (-2 - 9)\mathbf{j} + (15 - 1)\mathbf{k} = 13\mathbf{i} - 11\mathbf{j} + 14\mathbf{k}$$

2.56. Prove Theorem 2.5(i): The vector $u \times v$ is orthogonal to both u and v.

Suppose $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$. Then

$$\begin{aligned} u \cdot (u \times v) &= a_1(a_2 b_3 - a_3 b_2) + a_2(a_3 b_1 - a_1 b_3) + a_3(a_1 b_2 - a_2 b_1) \\ &= a_1 a_2 b_3 - a_1 a_3 b_2 + a_2 a_3 b_1 - a_1 a_2 b_3 + a_1 a_3 b_2 - a_2 a_3 b_1 = 0 \end{aligned}$$

Thus $u \times v$ is orthogonal to u. Similarly, $u \times v$ is orthogonal to v.

2.57. Find a unit vector u orthogonal to $v = (1, 3, 4)$ and $w = (2, -6, 5)$.

First find $v \times w$ which is orthogonal to v and w. The array $\begin{pmatrix} 1 & 3 & 4 \\ 2 & -6 & -5 \end{pmatrix}$ gives

$$v \times w = (-15 + 24, 8 + 5, -6 - 6) = (9, 13, -12)$$

Now normalize $u \times w$ to get $u = (9/\sqrt{394}, 13/\sqrt{394}, -12/\sqrt{394})$.

2.58. Prove *Lagrange's identity*, $\|u \times v\|^2 = (u \cdot u)(v \cdot v) - (u \cdot v)^2$.

If $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$, then

$$\|u \times v\|^2 = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \tag{1}$$

$$(u \cdot u)(v \cdot v) - (u \cdot v)^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \tag{2}$$

Expansion of the right-hand sides of (*1*) and (*2*) establishes the identity.

COMPLEX NUMBERS

2.59. Suppose $z = 5 + 3i$ and $w = 2 - 4i$. Find: (*a*) $z + w$, (*b*) $z - w$, (*c*) zw.

Use the ordinary rules of algebra together with $i^2 = -1$ to obtain a result in the standard form $a + bi$.

(*a*) $z + w = (5 + 3i) + (2 - 4i) = 7 - i$

(*b*) $z - w = (5 + 3i) - (2 - 4i) = 5 + 3i - 2 + 4i = 3 + 7i$

(*c*) $zw = (5 + 3i)(2 - 4i) = 10 - 14i - 12i^2 = 10 - 14i + 12 = 22 - 14i$

2.60. Simplify: (*a*) $(5 + 3i)(2 - 7i)$, (*b*) $(4 - 3i)^2$, (*c*) $(1 + 2i)^3$.

(*a*) $(5 + 3i)(2 - 7i) = 10 + 6i - 35i - 21i^2 = 31 - 29i$

(*b*) $(4 - 3i)^2 = 16 - 24i + 9i^2 = 7 - 24i$

(*c*) $(1 + 2i)^3 = 1 + 6i + 12i^2 + 8i^3 = 1 + 6i - 12 - 8i = -11 - 2i$

2.61. Simplify: (*a*) i^0, i^3, i^4, (*b*) i^5, i^6, i^7, i^8, (*c*) $i^{39}, i^{174}, i^{252}, i^{317}$.

(*a*) $i^0 = 1$, $i^3 = i^2(i) = (-1)(i) = -i$, $i^4 = (i^2)(i^2) = (-1)(-1) = 1$.

(*b*) $i^5 = (i^4)(i) = (1)(i) = i$, $i^6 = (i^4)(i^2) = (1)(i^2) = i^2 = -1$, $i^7 = i^3 = -i$, $i^8 = i^4 = 1$.

(*c*) Using $i^4 = 1$ and $i^n = i^{4q+r} = (i^4)^q i^r = 1^q i^r = i^r$, divide the exponent n by 4 to obtain the remainder r:

$$i^{39} = i^{4(9)+3} = (i^4)^9 i^3 = 1^9 i^3 = i^3 = -i \qquad i^{174} = i^2 = -1 \qquad i^{252} = i^0 = 1 \qquad i^{317} = i^1 = i$$

2.62. Find the complex conjugate of each of the following:

(*a*) $6 + 4i, 7 - 5i, 4 + i, -3 - i$; (*b*) $6, -3, 4i, -9i$.

(*a*) $\overline{6 + 4i} = 6 - 4i$, $\overline{7 - 5i} = 7 + 5i$, $\overline{4 + i} = 4 - i$, $\overline{-3 - i} = -3 + i$.

(*b*) $\bar{6} = 6$, $\overline{-3} = -3$, $\overline{4i} = -4i$, $\overline{-9i} = 9i$.

(Note that the conjugate of a real number is the original number, but the conjugate of a pure imaginary number is the negative of the original number.)

2.63. Find $z\bar{z}$ and $|z|$ when $z = 3 + 4i$.

For $z = a + bi$, use $z\bar{z} = a^2 + b^2$ and $z = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$.

$$z\bar{z} = 9 + 16 = 25 \qquad |z| = \sqrt{25} = 5$$

2.64. Simplify $\dfrac{2 - 7i}{5 + 3i}$.

To simplify a fraction z/w of complex numbers, multiply both numerator and denominator by $\bar{w}$, the conjugate of the denominator.

$$\frac{2 - 7i}{5 + 3i} = \frac{(2 - 7i)(5 - 3i)}{(5 + 3i)(5 - 3i)} = \frac{-11 - 41i}{34} = -\frac{11}{34} - \frac{41}{34}i$$

2.65. Prove: For any complex numbers $z, w \in \mathbf{C}$, (i) $\overline{z + w} = \bar{z} + \bar{w}$, (ii) $\overline{zw} = \bar{z}\bar{w}$, (iii) $\bar{\bar{z}} = z$.

Suppose $z = a + bi$ and $w = c + di$ where $a, b, c, d \in \mathbf{R}$.

(i)
$$\begin{aligned}\overline{z + w} &= \overline{(a + bi) + (c + di)} = \overline{(a + c) + (b + d)i}\\ &= (a + c) - (b + d)i = a + c - bi - di\\ &= (a - bi) + (c - di) = \bar{z} + \bar{w}\end{aligned}$$

(ii)
$$\begin{aligned}\overline{zw} &= \overline{(a + bi)(c + di)} = \overline{(ac - bd) + (ad + bc)i}\\ &= (ac - bd) - (ad + bc)i = (a - bi)(c - di) = \bar{z}\bar{w}\end{aligned}$$

(iii)
$$\bar{\bar{z}} = \overline{\overline{a + bi}} = \overline{a - bi} = a - (-b)i = a + bi = z$$

2.66. Prove: For any complex numbers $z, w \in \mathbf{C}$, $|zw| = |z||w|$.

By (ii) of Problem 2.65,

$$\begin{aligned}|zw|^2 &= (zw)(\overline{zw}) = (zw)(\bar{z}\bar{w})\\ &= (z\bar{z})(w\bar{w}) = |z|^2|w|^2\end{aligned}$$

The square root of both sides gives us the desired result.

2.67. Prove: For any complex numbers $z, w \in \mathbf{C}$, $|z + w| \le |z| + |w|$.

Suppose $z = a + bi$ and $w = c + di$ where $a, b, c, d \in \mathbf{R}$. Consider the vectors $u = (a, b)$ and $v = (c, d)$ in $\mathbf{R}^2$. Note that

$$|z| = \sqrt{a^2 + b^2} = \|u\| \qquad |w| = \sqrt{c^2 + d^2} = \|v\|$$

and

$$|z + w| = |(a + c) + (b + d)i| = \sqrt{(a + c)^2 + (b + d)^2} = \|(a + c, b + d)\| = \|u + v\|$$

By Minkowski's inequality (Problem 2.23), $\|u + v\| \le \|u\| + \|v\|$ and so

$$|z + w| = \|u + v\| \le \|u\| + \|v\| = |z| + |w|$$

2.68. Find the dot products $u \cdot v$ and $v \cdot u$ where (*a*) $u = (1 - 2i, 3 + i)$, $v = (4 + 2i, 5 - 6i)$, and (*b*) $u = (3 - 2i, 4i, 1 + 6i)$, $v = (5 + i, 2 - 3i, 7 + 2i)$.

Recall that the conjugates of the second vector appear in the dot product

$$(z_1, \ldots, z_n) \cdot (w_1, \ldots, w_n) = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$$

(*a*)
$$\begin{aligned}u \cdot v &= (1 - 2i)\overline{(4 + 2i)} + (3 + i)\overline{(5 - 6i)}\\ &= (1 - 2i)(4 - 2i) + (3 + i)(5 + 6i) = -10i + 9 + 23i = 9 + 13i\\ v \cdot u &= (4 + 2i)\overline{(1 - 2i)} + (5 - 6i)\overline{(3 + i)}\\ &= (4 + 2i)(1 + 2i) + (5 - 6i)(3 - i) = 10i + 9 - 23i = 9 - 13i\end{aligned}$$

(*b*)
$$\begin{aligned}u \cdot v &= (3 - 2i)\overline{(5 + i)} + (4i)\overline{(2 - 3i)} + (1 + 6i)\overline{(7 + 2i)}\\ &= (3 - 2i)(5 - i) + (4i)(2 + 3i) + (1 + 6i)(7 - 2i) = 20 + 35i\\ v \cdot u &= (5 + i)\overline{(3 - 2i)} + (2 - 3i)\overline{(4i)} + (7 + 2i)\overline{(1 + 6i)}\\ &= (5 + i)(3 + 2i) + (2 - 3i)(-4i) + (7 + 2i)(1 - 6i) = 20 - 35i\end{aligned}$$

In both cases, $v \cdot u = \overline{u \cdot v}$. This holds true in general, as seen in Problem 2.70.

2.69. Let $u = (7 - 2i, 2 + 5i)$ and $v = (1 + i, -3 - 6i)$. Find:

(*a*) $u + v$; (*b*) $2iu$; (*c*) $(3 - i)v$; (*d*) $u \cdot v$; (*e*) $\|u\|$ and $\|v\|$.

(*a*) $u + v = (7 - 2i + 1 + i, 2 + 5i - 3 - 6i) = (8 - i, -1 - i)$

(*b*) $2iu = (14i - 4i^2, 4i + 10i^2) = (4 + 14i, -10 + 4i)$

(*c*) $(3 - i)v = (3 + 3i - i - i^2, -9 - 18i + 3i + 6i^2) = (4 + 2i, -15 - 15i)$

(*d*)
$$\begin{aligned} u \cdot v &= (7 - 2i)\overline{(1 + i)} + (2 + 5i)\overline{(-3 - 6i)} \\ &= (7 - 2i)(1 - i) + (2 + 5i)(-3 + 6i) = 5 - 9i - 36 - 3i = -31 - 12i \end{aligned}$$

(*e*) $\|u\| = \sqrt{7^2 + (-2)^2 + 2^2 + 5^2} = \sqrt{82}$, $\|v\| = \sqrt{1^2 + 1^2 + (-3)^2 + (-6)^2} = \sqrt{47}$

2.70. Prove: For any vectors $u, v \in \mathbf{C}^n$ and any scalar $z \in \mathbf{C}$, (i) $u \cdot v = \overline{v \cdot u}$, (ii) $(zu) \cdot v = z(u \cdot v)$, (iii) $u \cdot (zv) = \bar{z}(u \cdot v)$.

Suppose $u = (z_1, z_2, \ldots, z_n)$ and $v = (w_1, w_2, \ldots, w_n)$.

(i) Using the properties of the conjugate,

$$\begin{aligned} \overline{v \cdot u} &= \overline{w_1\bar{z}_1 + w_2\bar{z}_2 + \cdots + w_n\bar{z}_n} = \overline{w_1\bar{z}_1} + \overline{w_2\bar{z}_2} + \cdots + \overline{w_n\bar{z}_n} \\ &= \bar{w}_1 z_1 + \bar{w}_2 z_2 + \cdots + \bar{w}_n z_n = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n = u \cdot v \end{aligned}$$

(ii) Since $zu = (zz_1, zz_2, \ldots, zz_n)$,

$$(zu) \cdot v = zz_1\bar{w}_1 + zz_2\bar{w}_2 + \cdots + zz_n\bar{w}_n = z(z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n) = z(u \cdot v)$$

(Compare with Theorem 2.2 on vectors in $\mathbf{R}^n$.)

(iii) **Method 1.** Since $zv = (zw_1, zw_2, \ldots, zw_n)$,

$$\begin{aligned} u \cdot (zv) &= z_1\overline{zw_1} + z_2\overline{zw_2} + \cdots + z_n\overline{zw_n} = z_1\bar{z}\bar{w}_1 + z_2\bar{z}\bar{w}_2 + \cdots + z_n\bar{z}\bar{w}_n \\ &= \bar{z}(z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n) = \bar{z}(u \cdot v) \end{aligned}$$

Method 2. Using (i) and (ii),

$$u \cdot (zv) = \overline{(zv) \cdot u} = \overline{z(v \cdot u)} = \bar{z}\overline{(v \cdot u)} = \bar{z}(u \cdot v)$$

Supplementary Problems

VECTORS IN $\mathbf{R}^n$

2.71. Let $u = (2, -1, 0, -3)$, $v = (1, -1, -1, 3)$, $w = (1, 3, -2, 2)$. Find: (*a*) $2u - 3v$; (*b*) $5u - 3v - 4w$; (*c*) $-u + 2v - 2w$; (*d*) $u \cdot v$, $u \cdot w$ and $v \cdot w$; (*e*) $\|u\|$, $\|v\|$, and $\|w\|$.

2.72. Determine x and y if: (*a*) $x(3, 2) = 2(y, -1)$; (*b*) $x(2, y) = y(1, -2)$.

2.73. Find $d(u, v)$ and $\text{proj}\,(u, v)$ when (*a*) $u = (1, -3)$, $v = (4, 1)$ and (*b*) $u = (2, -1, 0, 1)$, $v = (1, -1, 1, 2)$.

LINEAR COMBINATIONS, LINEAR INDEPENDENCE

2.74. Let

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Express v as a linear combination of u_1, u_2, u_3 where

(a) $v = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$, (b) $v = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$, (c) $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

2.75. Determine whether the following vectors u, v, w are linearly independent and, if not, express one of them as a linear combination of the others.

(a) $u = (1, 0, 1), v = (1, 2, 3), w = (3, 2, 5)$;
(b) $u = (1, 0, 1), v = (1, 1, 1), w = (0, 1, 1)$;
(c) $u = (1, 2), v = (1, -1), w = (2, 5)$;
(d) $u = (1, 0, 0, 1), v = (0, 1, 2, 1), w = (1, 2, 3, 4)$;
(e) $u = (1, 0, 0, 1), v = (0, 1, 2, 1), w = (1, 2, 4, 3)$.

2.76. Prove that Theorem 2.3 holds with equality iff u and v are linearly dependent.

LOCATED VECTORS, HYPERPLANES, LINES, CURVES

2.77. Find the (located) vector v from (a) $P(2, 3, -7)$ to $Q(1, -6, -5)$; (b) $P(1, -8, -4, 6)$ to $Q(3, -5, 2, -4)$.

2.78. Find an equation of the hyperplane in $\mathbf{R}^3$ which:

(a) Passes through $(2, -7, 1)$ and is normal to $(3, 1, -11)$;
(b) Contains $(1, -2, 2), (0, 1, 3)$, and $(0, 2, -1)$;
(c) Contains $(1, -5, 2)$ and is parallel to $3x - 7y + 4z = 5$.

2.79. Find a parametric representation of the line which:

(a) Passes through $(7, -1, 8)$ in the direction of $(1, 3, -5)$;
(b) Passes through $(1, 9, -4, 5)$ and $(2, -3, 0, 4)$;
(c) Passes through $(4, -1, 9)$ and is perpendicular to the plane $3x - 2y + z = 18$.

SPATIAL VECTORS (VECTORS IN $\mathbf{R}^3$); PLANES, LINES, CURVES, AND SURFACES IN $\mathbf{R}^3$

2.80. Find the equation of the plane H:

(a) With normal $N = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ and containing the point $P(1, 2, -3)$;
(b) Parallel to $4x + 3y - 2z = 11$ and containing the point $P(1, 2, -3)$.

2.81. Find a unit vector u which is normal to the plane:

(a) $3x - 4y - 12z = 11$; (b) $2x - y - 2z = 7$.

2.82. Find $\cos\theta$ where θ is the angle between the planes:

(a) $3x - 2y - 4z = 5$ and $x + 2y - 6z = 4$;
(b) $2x + 5y - 4z = 1$ and $4x + 3y + 2z = 1$.

2.83. Find the (parametric) equation of the line L:

(a) Through the point $P(2, 5, -3)$ and in the direction of $v = 4\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$;
(b) Through the points $P(1, 2, -4)$ and $Q(3, -7, 2)$;
(c) Perpendicular to the plane $2x - 3y + 7z = 4$ and containing $P(1, -5, 7)$.

2.84. Consider the following curve where $0 \le t \le 5$:

$$F(t) = t^3\mathbf{i} - t^2\mathbf{j} + (2t - 3)\mathbf{k}$$

(*a*) Find $F(t)$ when $t = 2$.

(*b*) Find the endpoints of the curve.

(*c*) Find the unit tangent vector $\mathbf{T}$ to the curve when $t = 2$.

2.85. Consider the curve $F(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$.

(*a*) Find the unit tangent vector $\mathbf{T}(t)$ to the curve.

(*b*) Find the unit normal vector $\mathbf{N}(t)$ to the curve by normalizing $U(t) = d\mathbf{T}(t)/dt$.

(*c*) Find the unit binormal vector $\mathbf{B}(t)$ to the curve using $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

2.86. Consider a moving body B whose position at time t is given by $\mathbf{R}(t) = t^2\mathbf{i} + t^3\mathbf{j} + 2t\mathbf{k}$. [Then $V(t) = dR(t)/dt$ denotes the velocity of B and $A(t) = dV(t)/dt$ denotes the acceleration of B.]

(*a*) Find the position of B when $t = 1$.

(*b*) Find the velocity v of B when $t = 1$.

(*c*) Find the speed s of B when $t = 1$.

(*d*) Find the acceleration a of B when $t = 1$.

2.87. Find a normal vector $\mathbf{N}$ and the tangent plane H to the surface at the given point:

(*a*) Surface: $x^2y + 3yz = 20$ and point $P(1, 3, 2)$;

(*b*) Surface: $x^2 + 3y^2 - 5z^2 = 16$ and point $P(3, -2, 1)$.

2.88. Given the surface $z = f(x, y) = x^2 + 2xy$. Find a normal vector N and the tangent plane H when $x = 3$, $y = 1$.

CROSS PRODUCT

The cross product is defined only for vectors in $\mathbf{R}^3$.

2.89. Given $u = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $v = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, $w = 4\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$. Find: (*a*) $u \times v$, (*b*) $u \times w$, (*c*) $v \times w$, (*d*) $v \times u$.

2.90. Find a unit vector w orthogonal to (*a*) $u = (1, 2, 3)$ and $v = (1, -1, 2)$; (*b*) $u = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $v = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

2.91. Prove the following properties of the cross product:

(*a*) $u \times v = -(v \times u)$

(*b*) $u \times u = 0$ for any vector u

(*c*) $(ku) \times v = k(u \times v) = u \times (kv)$

(*d*) $u \times (v + w) = (u \times v) + (u \times w)$

(*e*) $(v + w) \times u = (v \times u) + (w \times u)$

(*f*) $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$

COMPLEX NUMBERS

2.92. Simplify: (*a*) $(4 - 7i)(9 + 2i)$; (*b*) $(3 - 5i)^2$; (*c*) $\dfrac{1}{4 - 7i}$; (*d*) $\dfrac{9 + 2i}{3 - 5i}$; (*e*) $(1 - i)^3$.

2.93. Simplify: (*a*) $\dfrac{1}{2i}$; (*b*) $\dfrac{2 + 3i}{7 - 3i}$; (*c*) i^{15}, i^{25}, i^{34}; (*d*) $\left(\dfrac{1}{3 - i}\right)^2$.

2.94. Let $z = 2 - 5i$ and $w = 7 + 3i$. Find: (*a*) $z + w$; (*b*) zw; (*c*) z/w; (*d*) $\bar{z}, \bar{w}$; (*e*) $|z|, |w|$.

2.95. Let $z = 2 + i$ and $w = 6 - 5i$. Find: (*a*) z/w; (*b*) $\bar{z}, \bar{w}$; (*c*) $|z|, |w|$.

2.96. Show that (*a*) $\text{Re } z = \frac{1}{2}(z + \bar{z})$; (*b*) $\text{Im } z = (z - \bar{z})/2i$.

2.97. Show that $zw = 0$ implies $z = 0$ or $w = 0$.

VECTORS IN $\mathbf{C}^n$

2.98. Prove: For any vectors $u, v, w \in \mathbf{C}^n$:

(i) $(u + v) \cdot w = u \cdot w + v \cdot w$; (ii) $w \cdot (u + v) = w \cdot u + w \cdot v$.

2.99. Prove that the norm in $\mathbf{C}^n$ satisfies the following laws:

$[N_1]$ For any vector u, $\|u\| \geq 0$; and $\|u\| = 0$ iff $u = 0$.

$[N_2]$ For any vector u and any complex number z, $\|zu\| = |z| \|u\|$.

$[N_3]$ For any vectors u and v, $\|u + v\| \leq \|u\| + \|v\|$.

Answers to Supplementary Problems

2.71. (*a*) $2u - 3v = (1, 1, 3, -15)$ (*d*) $u \cdot v = -6, u \cdot w = -7, v \cdot w = 6$

(*b*) $5u - 3v - 4w = (3, -14, 11, -32)$ (*e*) $\|u\| = \sqrt{14}, \|v\| = 2\sqrt{3}, \|w\| = 3\sqrt{2}$

(*c*) $-u + 2v - 2w = (-2, -7, 2, 5)$

2.72. (*a*) $x = -1, y = -\frac{3}{2}$; (*b*) $x = 0, y = 0$ or $x = -2, y = -4$

2.73. (*a*) $d = 5$, $\text{proj}(u, v) = (\frac{4}{17}, \frac{1}{17})$; (*b*) $d = \sqrt{11}$, $\text{proj}(u, v) = (\frac{5}{7}, -\frac{5}{7}, \frac{5}{7}, \frac{10}{7})$

2.74. (*a*) $v = (\frac{9}{2})u_1 - 5u_2 + (\frac{3}{2})u_3$

(*b*) $v = u_2 + 2u_3$

(*c*) $v = [(a - 2b + c)/2]u_1 + (a + b - c)u_2 + [(c - a)/2]u_3$

2.75. (*a*) dependent; (*b*) independent; (*c*) dependent; (*d*) independent; (*e*) dependent.

2.77. (*a*) $v = (-1, -9, 2)$; (*b*) $v = (2, 3, 6, -10)$

2.78. (*a*) $3x + y - 11z = -12$; (*b*) $13x + 4y + z = 7$; (*c*) $3x - 7y + 4z = 46$

2.79. (*a*) $x = 7 + t, y = -1 + 3t, z = 8 - 5t$

(*b*) $x_1 = 1 + t, x_2 = 9 - 12t, x_3 = -4 + 4t, x_4 = 5 - t$

(*c*) $x = 4 + 3t, y = -1 - 2t, z = 9 + t$

2.80. (*a*) $3x - 4y + 5z = -20$; (*b*) $4x + 3y - 2z = 16$

2.81. (*a*) $u = (3\mathbf{i} - 4\mathbf{j} - 12\mathbf{k})/13$; (*b*) $u = (2\mathbf{i} - \mathbf{j} - 2\mathbf{k})/3$

2.82. (*a*) $23/(\sqrt{29}\sqrt{41})$; (*b*) $15/(\sqrt{45}\sqrt{29})$

2.83. (*a*) $x = 2 + 4t, y = 5 - 5t, z = -3 + 7t$
(*b*) $x = 1 + 2t, y = 2 - 9t, z = -4 + 6t$
(*c*) $x = 1 + 2t, y = -5 - 3t, z = 7 + 7t$

2.84. (*a*) $8\mathbf{i} - 4\mathbf{j} + \mathbf{k}$; (*b*) $-3\mathbf{k}$ and $125\mathbf{i} - 25\mathbf{j} + 7\mathbf{k}$; (*c*) $\mathbf{T} = (6\mathbf{i} - 2\mathbf{j} + \mathbf{k})/\sqrt{41}$

2.85. (*a*) $\mathbf{T}(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k})/\sqrt{2}$
(*b*) $\mathbf{N}(t) = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j}$
(*c*) $\mathbf{B}(t) = (\sin t)\mathbf{i} - (\cos t)\mathbf{j} + \mathbf{k})/\sqrt{2}$

2.86. (*a*) $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$; (*b*) $2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$; (*c*) $\sqrt{17}$; (*d*) $2\mathbf{i} + 6\mathbf{j}$

2.87. (*a*) $N = 6\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}, 6x + 7y + 9z = 45$
(*b*) $N = 6\mathbf{i} - 12\mathbf{j} - 10\mathbf{k}, 3x - 6y - 5z = 16$

2.88. $N = 8\mathbf{i} + 6\mathbf{j} - \mathbf{k}, 8x + 6y - z = 15$

2.89. (*a*) $2\mathbf{i} + 13\mathbf{j} + 23\mathbf{k}$ (*c*) $31\mathbf{i} - 16\mathbf{j} - 6\mathbf{k}$
(*b*) $-22\mathbf{i} + 2\mathbf{j} + 37\mathbf{k}$ (*d*) $-2\mathbf{i} - 13\mathbf{j} - 23\mathbf{k}$

2.90. (*a*) $(7, 1, -3)/\sqrt{59}$; (*b*) $(5\mathbf{i} + 11\mathbf{j} - 2\mathbf{k})/\sqrt{150}$

2.92. (*a*) $50 - 55i$; (*b*) $-16 - 30i$; (*c*) $(4 + 7i)/65$; (*d*) $(1 + 3i)/2$; (*e*) $-2 - 2i$

2.93. (*a*) $-\frac{1}{2}i$; (*b*) $(5 + 27i)/58$; (*c*) $-i, i, -1$; (*d*) $(4 + 3i)/50$

2.94. (*a*) $z + w = 9 - 2i$ (*d*) $\bar{z} = 2 + 5i, \bar{w} = 7 - 3i$
(*b*) $zw = 29 - 29i$ (*e*) $|z| = \sqrt{29}, |w| = \sqrt{58}$
(*c*) $z/w = (-1 - 41i)/58$

2.95. (*a*) $z/w = (7 + 16i)/61$; (*b*) $\bar{z} = 2 - i, \bar{w} = 6 + 5i$; (*c*) $|z| = \sqrt{5}, |w| = \sqrt{61}$

2.97. If $zw = 0$, then $|zw| = |z||w| = |0| = 0$. Hence $|z| = 0$ or $|w| = 0$; and so $z = 0$ or $w = 0$.