Chapter 5

Vector Spaces

5.1 INTRODUCTION

This chapter introduces the underlying algebraic structure of linear algebra—that of a finite-dimensional vector space. The definition of a vector space involves an arbitrary field whose elements are called *scalars*. The following notation will be used (unless otherwise stated or implied):

K	the field of scalars
a, b, c or k	the elements of K
V	the given vector space
u, v, w	the elements of V

Nothing essential is lost if the reader assumes that K is the real field **R** or the complex field **C**.

Length and orthogonality are not covered in this chapter since they are not considered as part of the fundamental structure of a vector space. They will be included as an additional structure in Chapter 6.

5.2 VECTOR SPACES

The following defines the notion of a vector space or linear space.

Definition: Let K be a given field and let V be a nonempty set with rules of addition and scalar multiplication which assigns to any $u, v \in V$ a *sum* $u + v \in V$ and to any $u \in V$, $k \in K$ a *product* $ku \in V$. Then V is called a *vector space over* K (and the elements of V are called *vectors*) if the following axioms hold (see Problem 5.3).

[A_1] For any vectors $u, v, w \in V$, $(u + v) + w = u + (v + w)$.

[A_2] There is a vector in V, denoted by 0 and called the *zero vector*, for which $u + 0 = u$ for any vector $u \in V$.

[A_3] For each vector $u \in V$ there is a vector in V, denoted by $-u$, for which $u + (-u) = 0$.

[A_4] For any vectors $u, v \in V$, $u + v = v + u$.

[M_1] For any scalar $k \in K$ and any vectors $u, v \in V$, $k(u + v) = ku + kv$.

[M_2] For any scalars $a, b \in K$ and any vector $u \in V$, $(a + b)u = au + bu$.

[M_3] For any scalars $a, b \in K$ and any vector $u \in V$, $(ab)u = a(bu)$.

[M_4] For the unit scalar $1 \in K$, $1u = u$ for any vector $u \in V$.

The above axioms naturally split into two sets. The first four are only concerned with the additive structure of V and can be summarized by saying that V is a *commutative group* under addition. It follows that any sum of vectors of the form

$$v_1 + v_2 + \cdots + v_m$$

requires no parentheses and does not depend upon the order of the summands, the zero vector 0 is unique, the *negative* $-u$ of u is unique, and the *cancellation law* holds; that is, for any vectors u, v, $w \in V$.

$$u + w = v + w \quad \text{implies} \quad u = v$$

Also, *subtraction* is defined by

$$u - v = u + (-v)$$

On the other hand, the remaining four axioms are concerned with the "action" of the field K on V. Observe that the labelling of the axioms reflects this splitting. Using these additional axioms we prove (Problem 5.1) the following simple properties of a vector space.

Theorem 5.1: Let V be a vector space over a field K.

(i) For any scalar $k \in K$ and $0 \in V$, $k0 = 0$.

(ii) For $0 \in K$ and any vector $u \in V$, $0u = 0$.

(iii) If $ku = 0$, where $k \in K$ and $u \in V$, then $k = 0$ or $u = 0$.

(iv) For any $k \in K$ and any $u \in V$, $(-k)u = k(-u) = -ku$.

5.3 EXAMPLES OF VECTOR SPACES

This section lists a number of important examples of vector spaces which will be used throughout the text.

Space K^n

Let K be an arbitrary field. The notation K^n is frequently used to denote the set of all n-tuples of elements in K. Here K^n is viewed as a vector space over K where vector addition and scalar multiplication are defined by

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

and

$$k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$$

The zero vector in K^n is the n-tuple of zeros,

$$0 = (0, 0, \ldots, 0)$$

and the negative of a vector is defined by

$$-(a_1, a_2, \ldots, a_n) = (-a_1, -a_2, \ldots, -a_n)$$

The proof that K^n is a vector space is identical to the proof of Theorem 2.1, which we now regard as stating that $\mathbf{R}^n$ with the operations defined there is a vector space over $\mathbf{R}$.

Matrix Space $\mathbf{M}_{m,n}$

The notation $\mathbf{M}_{m,n}$, or simply $\mathbf{M}$, will be used to denote the set of all $m \times n$ matrices over an arbitrary field K. Then $\mathbf{M}_{m,n}$ is a vector space over K with respect to the usual operations of matrix addition and scalar multiplication. (See Theorem 3.1.)

Polynomial Space P(t)

Let $\mathbf{P}(t)$ denote the set of all polynomials

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_s t^s \quad (s = 0, 1, 2, \ldots)$$

with coefficients a_i in some field K. Then $\mathbf{P}(t)$ is vector space over K with respect to the usual operations of addition of polynomials and multiplication of a polynomial by a constant.

Function Space F(X)

Let X be any nonempty set and let K be an arbitrary field. Consider the set $F(X)$ of all functions from X into K. [Note that $F(X)$ is nonempty since X is nonempty.] The sum of two functions f, $g \in F(X)$ is the function $f + g \in F(X)$ defined by

$$(f + g)(x) = f(x) + g(x) \qquad \forall x \in X$$

and the product of a scalar $k \in K$ and a function $f \in F(X)$ is the function $kf \in F(X)$ defined by

$$(kf)(x) = kf(x) \qquad \forall x \in X$$

(The symbol $\forall$ means "for every.") Then $F(X)$ with the above operations is a vector space over K (Problem 5.5).

The zero vector in $F(X)$ is the zero function $\mathbf{0}$ which maps each $x \in X$ into $0 \in K$, that is,

$$\mathbf{0}(x) = 0 \qquad \forall x \in X$$

Also, for any function $f \in F(X)$, the function $-f$ defined by

$$(-f)(x) = -f(x) \qquad \forall x \in X$$

is the negative of the function f.

Fields and Subfields

Suppose E is a field which contains a subfield K. Then E may be viewed as a vector space over K as follows. Let the usual addition in E be the vector addition, and let the scalar product kv of $k \in K$ and $v \in E$ be the product of k and v as elements of the field E. Then E is a vector space over K, that is, the above eight axioms of a vector space are satisfied by E and K.

5.4 SUBSPACES

Let W be a subset of a vector space V over a field K. W is called a *subspace* of V if W is itself a vector space over K with respect to the operations of vector addition and scalar multiplication on V. Simple criteria for identifying subspaces follow (see Problem 5.4 for proof).

Theorem 5.2: Suppose W is a subset of a vector space V. Then W is a subspace of V if and only if the following hold:

(i) $0 \in W$

(ii) W is *closed under vector addition*, that is:
For every $u, v \in W$, the sum $u + v \in W$.

(iii) W is *closed under scalar multiplication*, that is:
For every $u \in W$, $k \in K$, the multiple $ku \in W$.

Conditions (ii) and (iii) may be combined into one condition as in (ii) below (see Problem 5.5 for proof).

Corollary 5.3: W is a subspace of V if and only if:

(i) $0 \in W$

(ii) $au + bv \in W$ for every $u, v \in W$ and $a, b \in K$.

Example 5.1

(*a*) Let V be any vector space. Then the set $\{0\}$ consisting of the zero vector alone, and also the entire space V are subspaces of V.

(*b*) Let W be the xy plane in $\mathbf{R}^3$ consisting of those vectors whose third component is 0; or, in other words $W = \{(a, b, 0): a, b \in \mathbf{R}\}$. Note $0 = (0, 0, 0) \in W$ since the third component of 0 is 0. Further, for any vectors $u = (a, b, 0)$ and $v = (c, d, 0)$ in W and any scalar $k \in \mathbf{R}$, we have

$$u + v = (a + c, b + d, 0) \qquad \text{and} \qquad ku = (ka, kb, 0)$$

belong to W. Thus W is a subspace of V.

(*c*) Let $V = \mathbf{M}_{n,n}$ the space of $n \times n$ matrices. Then the subset W_1 of (upper) triangular matrices and the subset W_2 of symmetric matrices are subspaces of V since they are nonempty and closed under matrix addition and scalar multiplication.

(*d*) Recall that $\mathbf{P}(t)$ denotes the vector space of polynomials. Let $\mathbf{P}_n(t)$ denote the subset of $\mathbf{P}(t)$ that consists of all polynomials of degree $\leq n$, for a fixed n. Then $\mathbf{P}_n(t)$ is a subspace of $\mathbf{P}(t)$. This vector space $\mathbf{P}_n(t)$ will occur very often in our examples.

Example 5.2. Let U and W be subspaces of a vector space V. We show that the intersection $U \cap W$ is also a subspace of V. Clearly $0 \in U$ and $0 \in W$ since U and W are subspaces; whence $0 \in U \cap W$. Now suppose $u, v \in U \cap W$. Then $u, v \in U$ and $u, v \in W$ and, since U and W are subspaces,

$$u + v,\ ku \in U \qquad \text{and} \qquad u + v,\ ku \in W$$

for any scalar k. Thus $u + v, ku \in U \cap W$ and hence $U \cap W$ is a subspace of V.

The result in the preceding example generalizes as follows.

Theorem 5.4: The intersection of any number of subspaces of a vector space V is a subspace of V.

Recall that any solution u of a system $AX = B$ of linear equations in n unknowns may be viewed as a point in K^n; and thus the solution set of such a system is a subset of K^n. Suppose the system is homogeneous, i.e., suppose the system has the form $AX = 0$. Let W denote its solution set. Since $A0 = 0$, the zero vector $0 \in W$. Moreover, if u and v belong to W, i.e., if u and v are solutions of $AX = 0$, then $Au = 0$ and $Av = 0$. Therefore, for any scalars a and b in K, we have

$$A(au + bv) = aAu + bAv = a0 + b0 = 0 + 0 = 0$$

Thus $au + bv$ is also a solution of $AX = 0$ or, in other words, $au + bv \in W$. Accordingly, by the above Corollary 5.3, we have proved:

Theorem 5.5: The solution set W of a homogenous system $AX = 0$ in n unknowns is a subspace of K^n.

We emphasize that the solution set of a nonhomogenous system $AX = B$ is not a subspace of K^n. In fact, the zero vector 0 does not belong to its solution set.

5.5 LINEAR COMBINATIONS, LINEAR SPANS

Let V be a vector space over a field K and let $v_1, v_2, \ldots, v_m \in V$. Any vector in V of the form

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m$$

where the $a_i \in K$, is called a *linear combination* of $v_1, v_2, \ldots, v_m$. The set of all such linear combinations, denoted by

$$\text{span}\,(v_1, v_2, \ldots, v_m)$$

is called the *linear span* of $v_1, v_2, \ldots, v_m$.

Generally, for any subset S of V, span $S = \{0\}$ when S is empty and span S consists of all the linear combinations of vectors in S.

The following theorem is proved in Problem 5.16.

Theorem 5.6: Let S be a subset of a vector space V.

(i) Then span S is a subspace of V which contains S.

(ii) If W is a subspace of V containing S, then span $S \subseteq W$.

On the other hand, given a vector space V, the vectors $u_1, u_2, \ldots, u_r$ are said to *span* or *generate* or to form a *spanning set* of V if

$$V = \text{span}\,(u_1, u_2, \ldots, u_r)$$

In other words, $u_1, u_2, \ldots, u_r$ span V if, for every $v \in V$, there exist scalars $a_1, a_2, \ldots, a_r$ such that

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_r u_r$$

that is, if v is a linear combination of $u_1, u_2, \ldots, u_r$.

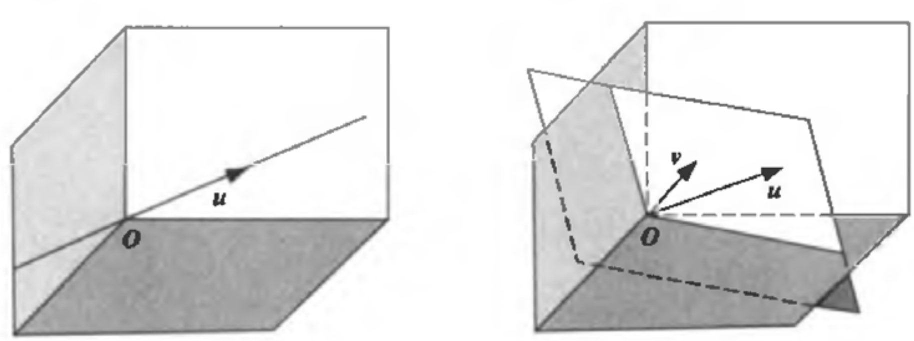
Example 5.3

(*a*) Consider the vector space $\mathbf{R}^3$. The linear span of any nonzero vector $u \in \mathbf{R}^3$ consists of all scalar multiples of u; geometrically, span u is the line through the origin and the endpoint of u as shown in Fig. 5-1(*a*). Also, for any two vectors $u, v \in \mathbf{R}^3$ which are not multiples of each other, span (u, v) is the plane through the origin and the endpoints of u and v as shown in Fig. 5-1(*b*).

(*b*) The vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ span the vector space $\mathbf{R}^3$. Specifically, for any vector $u = (a, b, c)$ in $\mathbf{R}^3$, we have

$$u = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = ae_1 + be_2 + ce_3$$

That is, u is a linear combination of e_1, e_2, e_3.



(*a*) (*b*)

Fig. 5-1

(c) The polynomials $1, t, t^2, t^3, \ldots$ span the vector space $\mathbf{P}(t)$ of all polynomials, that is,

$$\mathbf{P}(t) = \text{span}\,(1, t, t^2, t^3, \ldots)$$

In other words, any polynomial is a linear combination of 1 and powers of t. Similarly, the polynomials $1, t, t^2, \ldots, t^n$ span the vector space $\mathbf{P}_n(t)$ of all polynomials of degree $\leq n$.

Row Space of a Matrix

Let A be an arbitrary $m \times n$ matrix over a field K:

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

The rows of A,

$$R_1 = (a_{11}, a_{12}, \ldots, a_{1n}), \ldots, R_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$$

may be viewed as vectors in K^n and hence they span a subspace of K^n called the *row space* of A and denoted by rowsp A. That is,

$$\text{rowsp } A = \text{span}\,(R_1, R_2, \ldots, R_m)$$

Analogously, the columns of A may be viewed as vectors in K^m and hence span a subspace of K^m called the *column space* of A and denoted by colsp A. Alternatively, colsp A = rowsp A^T.

Now suppose we apply an elementary row operation on A,

$$\text{(i)}\quad R_i \leftrightarrow R_j, \qquad \text{(ii)}\quad kR_i \rightarrow R_i,\ k \neq 0, \qquad \text{or} \qquad \text{(iii)}\quad kR_j + R_i \rightarrow R_i$$

and obtain a matrix B. Then each row of B is clearly a row of A or a linear combination of rows of A. Hence the row space of B is contained in the row space of A. On the other hand, we can apply the inverse elementary row operation on B and obtain A; hence the row space of A is contained in the row space of B. Accordingly, A and B have the same row space. This leads us to the following theorem.

Theorem 5.7: Row equivalent matrices have the same row space.

In particular, we prove the following fundamental results about row equivalent matrices (proved in Problems 5.51 and 5.52, respectively).

Theorem 5.8: Row canonical matrices have the same row space if and only if they have the same nonzero rows.

Theorem 5.9: Every matrix is row equivalent to a unique matrix in row canonical form.

We apply the above results in the next example.

Example 5.4. Show that the subspace U of $\mathbf{R}^4$ spanned by the vectors

$$u_1 = (1, 2, -1, 3) \qquad u_2 = (2, 4, 1, -2) \qquad \text{and} \qquad u_3 = (3, 6, 3, -7)$$

and the subspace W of $\mathbf{R}^4$ spanned by the vectors

$$v_1 = (1, 2, -4, 11) \qquad \text{and} \qquad v_2 = (2, 4, -5, 14)$$

are equal; that is, $U = W$.

Method 1. Show that each u_i is a linear combination of v_1 and v_2, and show that each v_i is a linear combination of u_1, u_2, and u_3. Observe that we have to show that six systems of linear equations are consistent.

Method 2. Form the matrix A whose rows are the u_i, and row reduce A to row canonical form:

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{8}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now form the matrix B whose rows are v_1 and v_2, and row reduce B to row canonical form:

$$B = \begin{pmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{8}{3} \end{pmatrix}$$

Since the nonzero rows of the reduced matrices are identical, the row spaces of A and B are equal and so $U = W$.

5.6 LINEAR DEPENDENCE AND INDEPENDENCE

The following defines the notion of linear dependence and independence. This concept plays an essential role in the theory of linear algebra and in mathematics in general.

Definition: Let V be a vector space over a field K. The vectors $v_1, \ldots, v_m \in V$ are said to be *linearly dependent over K*, or simply *dependent*, if there exist scalars $a_1, \ldots, a_m \in K$, not all of them 0, such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0 \qquad (*)$$

Otherwise, the vectors are said to be *linearly independent over K*, or simply *independent*.

Observe that the relation $(*)$ will always hold if the a's are all 0. If this relation holds only in this case, that is,

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0 \quad \text{implies} \quad a_1 = 0, \ldots, a_m = 0$$

then the vectors are linearly independent. On the other hand, if the relation $(*)$ also holds when one of the a's is not 0, then the vectors are linearly dependent.

A set $\{v_1, v_2, \ldots, v_m\}$ of vectors is said to be linearly dependent or independent according as the vectors $v_1, v_2, \ldots, v_m$ are linearly dependent or independent. An infinite set S of vectors is linearly dependent if there exist vectors $u_1, \ldots, u_k$ in S which are linearly dependent; otherwise S is linearly independent.

The following remarks follow from the above definitions.

Remark 1: If 0 is one of the vectors $v_1, \ldots, v_m$, say $v_1 = 0$, then the vectors must be linearly dependent; for

$$1v_1 + 0v_2 + \cdots + 0v_m = 1 \cdot 0 + 0 + \cdots + 0 = 0$$

and the coefficient of v_1 is not 0.

Remark 2: Any nonzero vector v is, by itself, linearly independent; for

$$kv = 0, \quad v \neq 0 \quad \text{implies} \quad k = 0$$

Remark 3: If two of the vectors $v_1, v_2, \ldots, v_m$ are equal or one is a scalar multiple of the other, say $v_1 = kv_2$, then the vectors are linearly dependent. For

$$v_1 - kv_2 + 0v_3 + \cdots + 0v_m = 0$$

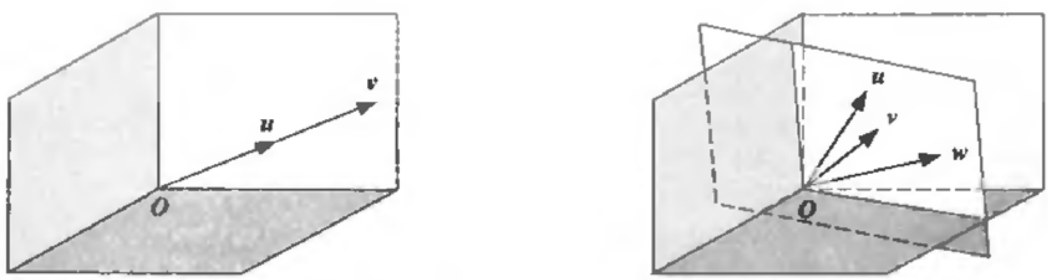
and the coefficient of v_1 is not 0.

Remark 4: Two vectors v_1 and v_2 are linearly dependent if and only if one of them is a multiple of the other.

Remark 5: If the set $\{v_1, \ldots, v_m\}$ is linearly independent, then any rearrangement of the vectors $\{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\}$ is also linearly independent.

Remark 6: If a set S of vectors is linearly independent, then any subset of S is linearly independent. Alternatively, if S contains a linearly dependent subset, then S is linearly dependent.

Remark 7: In the real space $\mathbf{R}^3$, linear dependence of vectors can be described geometrically as follows: (*a*) Any two vectors u and v are linearly dependent if and only if they lie on the same line through the origin as shown in Fig. 5-2(*a*). (*b*) Any three vectors u, v, and w are linearly dependent if and only if they lie on the same plane through the origin as shown in Fig. 5-2(*b*).



(*a*) u and v are linearly dependent

(*b*) u, v, and w are linearly dependent

Fig. 5-2

Other examples of linearly dependent and independent vectors follow.

Example 5.5

(*a*) The vectors $u = (1, -1, 0)$, $v = (1, 3, -1)$, and $w = (5, 3, -2)$ are linearly dependent since

$$3(1, -1, 0) + 2(1, 3, -1) - (5, 3, -2) = (0, 0, 0)$$

That is, $3u + 2v - w = 0$.

(*b*) We show that the vectors $u = (6, 2, 3, 4)$, $v = (0, 5, -3, 1)$, and $w = (0, 0, 7, -2)$ are linearly independent. For suppose $xu + yv + zw = 0$ where x, y and z are unknown scalars. Then

$$\begin{aligned}(0, 0, 0, 0) &= x(6, 2, 3, 4) + y(0, 5, -3, 1) + z(0, 0, 7, -2) \\ &= (6x, 2x + 5y, 3x - 3y + 7z, 4x + y - 2z)\end{aligned}$$

and so, by the equality of the corresponding components,

$$\begin{aligned}6x \qquad\qquad &= 0 \\ 2x + 5y \qquad &= 0 \\ 3x - 3y + 7z &= 0 \\ 4x + y - 2z &= 0\end{aligned}$$

The first equation yields $x = 0$; the second equation with $x = 0$ yields $y = 0$; and the third equation with $x = 0, y = 0$ yields $z = 0$. Thus

$$xu + yv + zw = 0 \quad \text{implies} \quad x = 0,\ y = 0,\ z = 0$$

Accordingly u, v, and w are linearly independent.

Linear Combinations and Linear Dependence

The notions of linear combinations and linear dependence are closely related. Specifically, for more than one vector, we show that the vectors $v_1, v_2, \ldots, v_m$ are linearly dependent if and only if one of them is a linear combination of the others.

For suppose, say, v_i is a linear combination of the others:

$$v_i = a_1 v_1 + \cdots + a_{i-1} v_{i-1} + a_{i+1} v_{i+1} + \cdots + a_m v_m$$

Then by adding $-v_i$ to both sides, we obtain

$$a_1 v_1 + \cdots + a_{i-1} v_{i-1} - v_i + a_{i+1} v_{i+1} + \cdots + a_m v_m = 0$$

where the coefficient of v_i is not 0; hence the vectors are linearly dependent. Conversely, suppose the vectors are linearly dependent, say,

$$b_1 v_1 + \cdots + b_j v_j + \cdots + b_m v_m = 0 \quad \text{where} \quad b_j \neq 0$$

Then

$$v_j = -b_j^{-1} b_1 v_1 - \cdots - b_j^{-1} b_{j-1} v_{j-1} - b_j^{-1} b_{j+1} v_{j+1} - \cdots - b_j^{-1} b_m v_m$$

and so v_j is a linear combination of the other vectors.

We now formally state a slightly stronger statement than that above (see Problem 5.36 for the proof); this result has many important consequences.

Lemma 5.10: Suppose two or more nonzero vectors $v_1, v_2, \ldots, v_m$ are linearly dependent. Then one of the vectors is a linear combination of the preceding vectors, that is, there exists a $k > 1$ such that

$$v_k = c_1 v_1 + c_2 v_2 + \cdots + c_{k-1} v_{k-1}$$

Example 5.6. Consider the following matrix in echelon form:

$$A = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 4 & -4 & 4 & -4 & 4 \\ 0 & 0 & 0 & 0 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe that rows R_2, R_3, and R_4 have 0s in the second column (below the pivot element in R_1) and hence any linear combination of R_2, R_3, and R_4 must have a 0 as its second component. Thus R_1 cannot be a linear combination of the nonzero rows below it. Similarly, rows R_3 and R_4 have 0s in the third column below the pivot element in R_2; hence R_2 cannot be a linear combination of the nonzero rows below it. Finally, R_3 cannot be a multiple of R_4 since R_4 has a 0 in the fifth column below the pivot in R_3. Viewing the nonzero rows from the bottom up, R_4, R_3, R_2, R_1, no row is a linear combination of the previous rows. Thus the rows are linearly independent by Lemma 5.10.

The argument in the above example can be used for the nonzero rows of any echelon matrix. Thus we have the following very useful result (proved in Problem 5.37).

Theorem 5.11: The nonzero rows of a matrix in echelon form are linearly independent.

5.7 BASIS AND DIMENSION

First we state two equivalent ways (Problem 5.30) to define a basis of a vector space V.

Definition A: A set $S = \{u_1, u_2, \ldots, u_n\}$ of vectors is a *basis* of V if the following two conditions hold:

(1) $u_1, u_2, \ldots, u_n$ are linearly independent,

(2) $u_1, u_2, \ldots, u_n$ span V.

Definition B: A set $S = \{u_1, u_2, \ldots, u_n\}$ of vectors is a *basis* of V if every vector $v \in V$ can be written uniquely as a linear combination of the basis vectors.

A vector space V is said to be of *finite dimension n* or to be *n-dimensional*, written

$$\dim V = n$$

if V has such a basis with n elements. This definition of dimension is well defined in view of the following theorem (proved in Problem 5.40).

Theorem 5.12: Let V be a finite-dimensional vector space. Then every basis of V has the same number of elements.

The vector space $\{0\}$ is defined to have dimension 0. When a vector space is not of finite dimension, it is said to be of *infinite dimension*.

Example 5.7

(*a*) Consider the vector space $\mathbf{M}_{2,3}$ of all 2×3 matrices over a field K. Then the following six matrices form a basis of $\mathbf{M}_{2,3}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

More generally, in the vector space $\mathbf{M}_{r,s}$ of $r \times s$ matrices let E_{ij} be the matrix with ij-entry 1 and 0 elsewhere. Then all such matrices E_{ij} form a basis of $\mathbf{M}_{r,s}$, called the *usual basis* of $\mathbf{M}_{r,s}$. Then $\dim \mathbf{M}_{r,s} = rs$. In particular, $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, 0, \ldots, 0, 1)$ form the usual basis for K^n.

(*b*) Consider the vector space $\mathbf{P}_n(t)$ of polynomials of degree $\le n$. The polynomials $1, t, t^2, \ldots, t^n$ form a basis of $\mathbf{P}_n(t)$, and so $\dim \mathbf{P}_n(t) = n + 1$.

Theorem 5.12, the fundamental theorem on dimension, is a consequence of the following "replacement lemma" (proved in Problem 5.39):

Lemma 5.13: Suppose $\{v_1, v_2, \ldots, v_n\}$ spans V, and suppose $\{w_1, w_2, \ldots, w_m\}$ is linearly independent. Then $m \le n$, and V is spanned by a set of the form

$$\{w_1, \ldots, w_m, v_{i_1}, \ldots, v_{i_{n-m}}\}$$

Thus, in particular, any $n + 1$ or more vectors in V are linearly dependent.

Observe in the above lemma that we have replaced m of the vectors in the spanning set by the m independent vectors and still retained a spanning set.

The following theorems (proved in Problems 5.41, 5.42, and 5.43, respectively) will be frequently used.

Theorem 5.14: Let V be a vector space of finite dimension n.

(i) Any $n + 1$ or more vectors in V are linearly dependent.

(ii) Any linearly independent set $S = \{u_1, u_2, \ldots, u_n\}$ with n elements is a basis of V.

(iii) Any spanning set $T = \{v_1, v_2, \ldots, v_n\}$ of V with n elements is a basis of V.

Theorem 5.15: Suppose S spans a vector space V.

(i) Any maximum number of linearly independent vectors in S form a basis of V.

(ii) Suppose one deletes from S each vector which is a linear combination of preceding vectors in S. Then the remaining vectors form a basis of V.

Theorem 5.16: Let V be a vector space of finite dimension and let $S = \{u_1, u_2, \ldots, u_r\}$ be a set of linearly independent vectors in V. Then S is part of a basis of V, that is, S may be extended to a basis of V.

Example 5.8

(a) Consider the following four vectors in $\mathbf{R}^4$:

$$(1, 1, 1, 1) \qquad (0, 1, 1, 1) \qquad (0, 0, 1, 1) \qquad (0, 0, 0, 1)$$

Note that the vector will form a matrix in echelon form; hence the vectors are linearly independent. Furthermore, since $\dim \mathbf{R}^4 = 4$, the vectors form a basis of $\mathbf{R}^4$.

(b) Consider the following $n + 1$ polynomials in $\mathbf{P}_n(t)$:

$$1, t - 1, (t - 1)^2, \ldots, (t - 1)^n$$

The degree of $(t - 1)^k$ is k; hence no polynomial can be a linear combination of preceding polynomials. Thus the polynomials are linearly independent. Furthermore, they form a basis of $\mathbf{P}_n(t)$ since $\dim \mathbf{P}_n(t) = n + 1$.

Dimension and Subspaces

The following theorem (proved in Problem 5.44) gives the basic relationship between the dimension of a vector space and the dimension of a subspace.

Theorem 5.17: Let W be a subspace of an n-dimensional vector space V. Then $\dim W \le n$. In particular if $\dim W = n$, then $W = V$.

Example 5.9. Let W be a subspace of the real space $\mathbf{R}^3$. Now $\dim \mathbf{R}^3 = 3$; hence by Theorem 5.17 the dimension of W can only be 0, 1, 2, or 3. The following cases apply:

(i) $\dim W = 0$, then $W = \{0\}$, a point;

(ii) $\dim W = 1$, then W is a line through the origin;

(iii) $\dim W = 2$, then W is a plane through the origin;

(iv) $\dim W = 3$, then W is the entire space $\mathbf{R}^3$.

Rank of a Matrix

Let A be an arbitrary $m \times n$ matrix over a field K. Recall that the row space of A is the subspace of K^n spanned by its rows, and the column space of A is the subspace of K^m spanned by its columns.

The *row rank* of the matrix A is equal to the maximum number of linearly independent rows or, equivalently, to the dimension of the row space of A. Analogously, the *column rank* of A is equal to the maximum number of linearly independent columns or, equivalently, to the dimension of the column space of A.

Although rowsp A is a subspace of K^n, and colsp A is a subspace of K^m, where n may not equal m, we have the following important result (proved in Problem 5.53).

Theorem 5.18: The row rank and the column rank of any matrix A are equal.

Definition: The *rank* of the matrix A, written rank A, is the common value of its row rank and column rank.

The rank of a matrix may be easily found by using row reduction as illustrated in the next example.

Example 5.10. Suppose we want to find a basis and the dimension of the row space of

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{pmatrix}$$

We reduce A to echelon form using the elementary row operations:

$$A \sim \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 4 & -6 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Recall that row equivalent matrices have the same row space. Thus the nonzero rows of the echelon matrix, which are independent by Theorem 5.11, form a basis of the row space of A. Thus dim rowsp $A = 2$ and so rank $A = 2$.

5.8 LINEAR EQUATIONS AND VECTOR SPACES

Consider a system of m linear equations in n unknowns $x_1, \ldots, x_n$ over a field K:

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \tag{5.1}$$

or the equivalent matrix equation

$$AX = B$$

where $A = (a_{ij})$ is the coefficient matrix, and $X = (x_i)$ and $B = (b_i)$ are the column vectors consisting of the unknowns and of the constants, respectively. Recall that the *augmented matrix* of the system is defined to be the matrix

$$(A, B) = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{pmatrix}$$

Remark 1: The linear equations (*5.1*) are said to be dependent or independent according as the corresponding vectors, i.e., the rows of the augmented matrix, are dependent or independent.

Remark 2: Two systems of linear equations are equivalent if and only if the corresponding augmented matrices are row equivalent, i.e., have the same row space.

Remark 3: We can always replace a system of equations by a system of independent equations, such as a system in echelon form. The number of independent equations will always be equal to the rank of the augmented matrix.

Observe that the system (*5.1*) is also equivalent to the vector equation

$$x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \dots \\ a_{m2} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \dots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

The above comment gives us the following basic existence theorem.

Theorem 5.19: The following three statements are equivalent.

(*a*) The system of linear equations $AX = B$ has a solution.

(*b*) B is a linear combination of the columns of A.

(*c*) The coefficient matrix A and the augmented matrix (A, B) have the same rank.

Recall that an $m \times n$ matrix A may be viewed as a function $A : K^n \to K^m$. Thus a vector B belongs to the image of A if and only if the equation $AX = B$ has a solution. This means that the image (range) of the function A, written Im A, is precisely the column space of A. Accordingly,

$$\dim(\operatorname{Im} A) = \dim(\operatorname{colsp} A) = \operatorname{rank} A$$

We use this fact to prove (Problem 5.59) the following basic result on homogeneous systems of linear equations.

Theorem 5.20: The dimension of the solution space W of the homogeneous system of linear equations $AX = 0$ is $n - r$ where n is the number of unknowns and r is the rank of the coefficient matrix A.

In case the system $AX = 0$ is in echelon form, then it has precisely $n - r$ free variables, say, $x_{i_1}, x_{i_2}, \ldots, x_{i_{n-r}}$. Let v_j be the solution obtained by setting $x_{i_j} = 1$ (or any nonzero constant) and the remaining free variables equal to 0. Then the solutions $v_1, \ldots, v_{n-r}$ are linearly independent (Problem 5.58) and hence they form a basis for the solution space.

Example 5.11. Suppose we want to find the dimension and a basis of the solution space W of the following system:

$$\begin{aligned} x + 2y + 2z - s + 3t &= 0 \\ x + 2y + 3z + s + t &= 0 \\ 3x + 6y + 8z + s + 5t &= 0 \end{aligned}$$

First reduce the system to echelon form:

$$\begin{aligned} x + 2y + 2z - s + 3t &= 0 \\ z + 2s - 2t &= 0 \\ 2z + 4s - 4t &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y + 2z - s + 3t &= 0 \\ z + 2s - 2t &= 0 \end{aligned}$$

The system in echelon form has 2 (nonzero) equations in 5 unknowns; and hence the system has $5 - 2 = 3$ free variables which are y, s and t. Thus $\dim W = 3$. To obtain a basis for W, set:

(i) $y = 1, s = 0, t = 0$ to obtain the solution $v_1 = (-2, 1, 0, 0, 0)$,

(ii) $y = 0, s = 1, t = 0$ to obtain the solution $v_2 = (5, 0, -2, 1, 0)$,

(iii) $y = 0, s = 0, t = 1$ to obtain the solution $v_3 = (-7, 0, 2, 0, 1)$.

The set $\{v_1, v_2, v_3\}$ is a basis of the solution space W.

Two Basis-Finding Algorithms

Suppose we are given vectors $u_1, u_2, \ldots, u_r$ in K^n. Let

$$W = \text{span}(u_1, u_2, \ldots, u_r)$$

the subspace of K^n spanned by the given vectors. Each algorithm below finds a basis (and hence the dimension) of W.

Algorithm 5.8A (Row space algorithm)

Step 1. Form the matrix A whose *rows* are the given vectors.

Step 2. Row reduce A to an echelon form.

Step 3. Output the nonzero rows of the echelon matrix.

The above algorithm essentially already appeared in Example 5.10. The next algorithm is illustrated in Example 5.12 and uses the above result on nonhomogeneous systems of linear equations.

Algorithm 5.8B (Casting-Out algorithm)

Step 1. Form the matrix M whose *columns* are the given vectors.

Step 2. Row reduce M to echelon form.

Step 3. For each column C_k in the echelon matrix without a pivot, delete (cast-out) the vector v_k from the given vectors.

Step 4. Output the remaining vectors (which correspond to columns with pivots).

Example 5.12. Let W be the subspace of $\mathbf{R}^5$ spanned by the following vectors:

$$v_1 = (1, 2, 1, -2, 3) \qquad v_2 = (2, 5, -1, 3, -2) \qquad v_3 = (1, 3, -2, 5, -5)$$

$$v_4 = (3, 1, 2, -4, 1) \qquad v_5 = (5, 6, 1, -1, -1)$$

We use Algorithm *5.8B* to find the dimension and a basis of W.

First form the matrix M whose columns are the given vectors, and reduce the matrix to echelon form:

$$M = \begin{pmatrix} 1 & 2 & 1 & 3 & 5 \\ 2 & 5 & 3 & 1 & 6 \\ 1 & -1 & -2 & 2 & 1 \\ -2 & 3 & 5 & -4 & -1 \\ 3 & -2 & -5 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 3 & 5 \\ 0 & 1 & 1 & -5 & -4 \\ 0 & -3 & -3 & -1 & -4 \\ 0 & 7 & 7 & 2 & 9 \\ 0 & -8 & -8 & -8 & -16 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 3 & 5 \\ 0 & 1 & 1 & 7 & -4 \\ 0 & 0 & 0 & -16 & -16 \\ 0 & 0 & 0 & 37 & 37 \\ 0 & 0 & 0 & -48 & -48 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 3 & 5 \\ 0 & 1 & 1 & 7 & -4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe that the pivots in the echelon matrix appear in columns C_1, C_2, and C_4. The fact that column C_3 does not have a pivot means that the system $xv_1 + yv_2 = v_3$ has a solution and hence v_3 is a linear combination of v_1 and v_2. Similarly, the fact that column C_5 does not have a pivot means that v_5 is a linear combination of preceding vectors. Accordingly, the vectors v_1, v_2 and v_4, which correspond to the columns in the echelon matrix with pivots, form a basis of W and $\dim W = 3$.

5.9 SUMS AND DIRECT SUMS

Let U and W be subsets of a vector space V. The sum of U and W, written $U + W$, consists of all sums $u + w$ where $u \in U$ and $w \in W$. That is,

$$U + W = \{u + w : u \in U, w \in W\}$$

Now suppose U and W are subspaces of a vector space V. Note that $0 = 0 + 0 \in U + W$, since $0 \in U$, $0 \in W$. Furthermore, suppose $u + w$ and $u' + w'$ belong to $U + W$, with $u, u' \in U$ and $w, w' \in W$. Then

$$(u + w) + (u' + w') = (u + u') + (w + w') \in U + W$$

and, for any scalar k,

$$k(u + w) = ku + kw \in U + W$$

Thus we have proven the following theorem.

Theorem 5.21: The sum $U + W$ of the subspaces U and W of V is also a subspace of V.

Recall that the intersection $U \cap W$ is also a subspace of V. The following theorem, proved in Problem 5.69, relates the dimensions of these subspaces.

Theorem 5.22: Let U and W be finite-dimensional subspaces of a vector space V. Then $U + W$ has finite dimension and

$$\dim (U + W) = \dim U + \dim W - \dim (U \cap W)$$

Example 5.13. Suppose U and W are the xy and yz planes, respectively, in $\mathbf{R}^3$. That is,

$$U = \{(a, b, 0)\} \qquad \text{and} \qquad W = \{(0, b, c)\}$$

Note $\mathbf{R}^3 = U + W$; hence $\dim (U + W) = 3$. Also $\dim U = 2$ and $\dim W = 2$. By Theorem 5.22,

$$3 = 2 + 2 - \dim (U \cap W) \qquad \text{or} \qquad \dim (U \cap W) = 1$$

This agrees with the facts that $U \cap W$ is the y axis (Fig. 5-3) and the y axis has dimension 1.

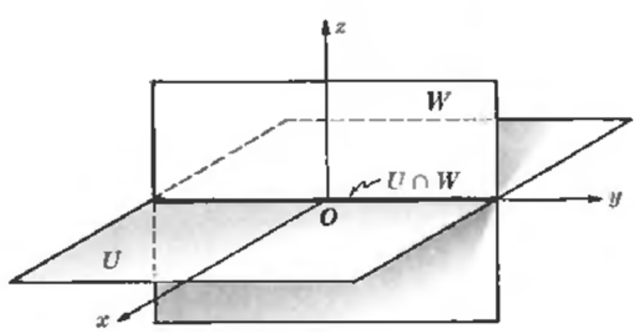


Fig. 5-3

Direct Sums

The vector space V is said to be the *direct sum* of its subspaces U and W, denoted by

$$V = U \oplus W$$

if every vector $v \in V$ can be written in one and only one way as $v = u + w$ where $u \in U$ and $w \in W$.

The following theorem, proved in Problem 5.70, characterizes such a decomposition.

Theorem 5.23: The vector space V is the direct sum of its subspaces U and W if and only if (i) $V = U + W$ and (ii) $U \cap W = \{0\}$.

Example 5.14

(*a*) In the vector space $\mathbf{R}^3$, let U be the xy plane and let W be the yz plane:

$$U = \{(a, b, 0): a, b \in \mathbf{R}\} \qquad \text{and} \qquad W = \{(0, b, c): b, c \in \mathbf{R}\}$$

Then $\mathbf{R}^3 = U + W$ since every vector in $\mathbf{R}^3$ is the sum of a vector in U and a vector in W. However, $\mathbf{R}^3$ is not the direct sum of U and W since such sums are not unique; for example,

$$(3, 5, 7) = (3, 1, 0) + (0, 4, 7) \qquad \text{and also} \qquad (3, 5, 7) = (3, -4, 0) + (0, 9, 7)$$

(*b*) In $\mathbf{R}^3$, let U be the xy plane and let W be the z axis:

$$U = \{(a, b, 0): a, b \in \mathbf{R}\} \qquad \text{and} \qquad W = \{(0, 0, c): c \in \mathbf{R}\}$$

Now any vector $(a, b, c) \in \mathbf{R}^3$ can be written as the sum of a vector in U and a vector in V in one and only one way:

$$(a, b, c) = (a, b, 0) + (0, 0, c)$$

Accordingly, $\mathbf{R}^3$ is the direct sum of U and W, that is, $\mathbf{R}^3 = U \oplus W$. Alternatively, $\mathbf{R}^3 = U \oplus W$ since $\mathbf{R}^3 = U + W$ and $U \cap W = \{0\}$.

General Direct Sums

The notion of a direct sum is extended to more than one factor in the obvious way. That is, V is the *direct sum* of subspaces $W_1, W_2, \ldots, W_r$, written

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$$

if every vector $v \in V$ can be written in one and only one way as

$$v = w_1 + w_2 + \cdots + w_r$$

where $w_1 \in W_1, w_2 \in W_2, \ldots, w_r \in W_r$.

The following theorems apply.

Theorem 5.24: Suppose $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$. Also, for each i, suppose S_i is a linearly independent subset of W_i. Then

(*a*) The union $S = \bigcup_i S_i$ is linearly independent in V.

(*b*) If S_i is a basis of W_i, then $S = \bigcup_i S_i$ is a basis of V.

(*c*) $\dim V = \dim W_1 + \dim W_2 + \cdots + \dim W_r$

Theorem 5.25: Suppose $V = W_1 + W_2 + \cdots + W_r$ (where V has finite dimension) and suppose

$$\dim V = \dim W_1 + \dim W_2 + \cdots + \dim W_r$$

Then $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$.

5.10 COORDINATES

Let V be an n-dimensional vector space over a field K, and suppose

$$S = \{u_1, u_2, \ldots, u_n\}$$

is a basis of V. Then any vector $v \in V$ can be expressed uniquely as a linear combination of the basis vectors in S, say

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$$

These n scalars $a_1, a_2, \ldots, a_n$ are called the *coordinates* of v relative to the basis S; and they form the n-tuple $[a_1, a_2, \ldots, a_n]$ in K^n, called the *coordinate vector* of v relative to S. We denote this vector by $[v]_S$, or simply $[v]$, when S is understood. Thus

$$[v]_S = [a_1, a_2, \ldots, a_n]$$

Observe that brackets [...], not parentheses (...), are used to denote the coordinate vector.

Example 5.15

(*a*) Consider the vector space $\mathbf{P}_2(t)$ of polynomials of degree ≤ 2. The polynomials

$$p_1 = 1 \qquad p_2 = t - 1 \qquad p_3 = (t-1)^2 = t^2 - 2t + 1$$

form a basis S of $\mathbf{P}_2(t)$. Let $v = 2t^2 - 5t + 6$. The coordinate vector of v relative to the basis S is obtained as follows.

Set $v = xp_1 + yp_2 + zp_3$ using unknown scalars x, y, z and simplify:

$$\begin{aligned} 2t^2 - 5t + 6 &= x(1) + y(t-1) + z(t^2 - 2t + 1) \\ &= x + yt - y + zt^2 - 2zt + z \\ &= zt^2 + (y - 2z)t + (x - y + z) \end{aligned}$$

Then set the coefficients of the same powers of t equal to each other:

$$\begin{aligned} x - y + z &= 6 \\ y - 2z &= -5 \\ z &= 2 \end{aligned}$$

The solution of the above system is $x = 3$, $y = -1$, $z = 2$. Thus

$$v = 3p_1 - p_2 + 2p_3 \qquad \text{and so} \qquad [v] = [3, -1, 2]$$

(*b*) Consider real space $\mathbf{R}^3$. The vectors

$$u_1 = (1, -1, 0) \qquad u_2 = (1, 1, 0) \qquad u_3 = (0, 1, 1)$$

form a basis S of $\mathbf{R}^3$. Let $v = (5, 3, 4)$. The coordinates of v relative to the basis S are obtained as follows.

Set $v = xu_1 + yu_2 + zu_3$, that is, set v as a linear combination of the basis vectors using unknown scalars x, y, z:

$$\begin{aligned} (5, 3, 4) &= x(1, -1, 0) + y(1, 1, 0) + z(0, 1, 1) \\ &= (x, -x, 0) + (y, y, 0) + (0, z, z) \\ &= (x + y, -x + y + z, z) \end{aligned}$$

Then set the corresponding components equal to each other to obtain the equivalent system of linear equations

$$x + y = 5 \qquad -x + y + z = 3 \qquad z = 4$$

The solution of the system is $x = 3, y = 2, z = 4$. Thus

$$v = 3u_1 + 2u_2 + 4u_3 \qquad \text{and so} \qquad [v]_S = [3, 2, 4]$$

Remark: There is a geometrical interpretation of the coordinates of a vector v relative to a basis S for real space $\mathbf{R}^n$. We illustrate this by using the following basis of $\mathbf{R}^3$ appearing in Example 5.15.

$$S = \{u_1 = (1, -1, 0), u_2 = (1, 1, 0), u_3 = (0, 1, 1)\}$$

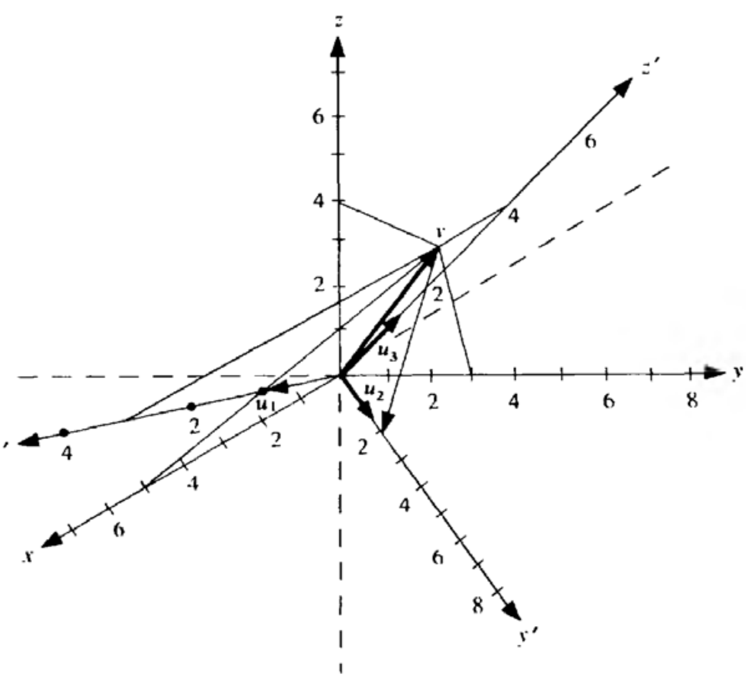
First consider the space $\mathbf{R}^3$ with the usual x, y, and z axes. Then the basis vectors determine a new coordinate system of $\mathbf{R}^3$, say with x', y', and z' axes, as shown in Fig. 5-4. That is:

(1) The x' axis is in the direction of u_1.

(2) The y' axis is in the direction of u_2.

(3) The z' axis is in the direction of u_3.

Furthermore, the unit length in each of the axes will be equal, respectively, to the length of the corresponding basis vector. Then each vector $v = (a, b, c)$ or, equivalently, the point $P(a, b, c)$ in $\mathbf{R}^3$ will have new coordinates with respect to the new x', y', z' axes. These new coordinates are precisely the coordinates of v with respect to the basis S.



$v = (5, 3, 4) = [3, 2, 4]$

Fig. 5-4

Isomorphism of V with K^n

Consider a basis $S = \{u_1, u_2, \ldots, u_n\}$ of a vector space V over a field K. We have shown above that to each vector $v \in V$ there corresponds a unique n-tuple $[v]_S$ in K^n. On the other hand, for any n-tuple

$[c_1, c_2, \ldots, c_n] \in K^n$, there corresponds the vector $c_1u_1 + c_2u_2 + \cdots + c_nu_n$ in V. Thus the basis S induces a one-to-one correspondence between the vectors in V and the n-tuples in K^n. Furthermore, suppose

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n \qquad \text{and} \qquad w = b_1u_1 + b_2u_2 + \cdots + b_nu_n$$

Then

$$v + w = (a_1 + b_1)u_1 + (a_2 + b_2)u_2 + \cdots + (a_n + b_n)u_n$$
$$kv = (ka_1)u_1 + (ka_2)u_2 + \cdots + (ka_n)u_n$$

where k is a scalar. Accordingly,

$$[v + w]_S = [a_1 + b_1, \ldots, a_n + b_n] = [a_1, \ldots, a_n] + [b_1, \ldots, b_n] = [v]_S + [w]_S$$

and

$$[kv]_S = [ka_1, ka_2, \ldots, ka_n] = k[a_1, a_2, \ldots, ka_n] = k[v]_S$$

Thus the above one-to-one correspondence between V and K^n preserves the vector space operations of vector addition and scalar multiplication; we then say that V and K^n are *isomorphic*, written $V \cong K^n$. We state this result formally.

Theorem 5.26: Let V be an n-dimensional vector space over a field K. Then V and K^n are isomorphic.

The next example gives a practical application of the above result.

Example 5.16. Suppose we want to determine whether or not the following matrices are linearly independent:

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 4 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & -4 \\ 6 & 5 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & 8 & -11 \\ 16 & 10 & 9 \end{pmatrix}$$

The coordinate vectors of the above matrices relative to the usual basis [Example 5.7(*a*)] of $\mathbf{M}_{2,3}$ are as follows:

$$[A] = (1, 2, -3, 4, 0, 1) \qquad [B] = (1, 3, -4, 6, 5, 4) \qquad [C] = (3, 8, -11, 16, 10, 9)$$

Form the matrix M whose rows are the above coordinate vectors:

$$M = \begin{pmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 1 & 3 & -4 & 6 & 5 & 4 \\ 3 & 8 & -11 & 16 & 10 & 9 \end{pmatrix}$$

Row reduce M to echelon form:

$$M \sim \begin{pmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 0 & 1 & -1 & 2 & 5 & 3 \\ 0 & 2 & -2 & 4 & 10 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 0 & 1 & -1 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the echelon matrix has only two nonzero rows, the coordinate vectors $[A]$, $[B]$, and $[C]$ span a subspace of dimension 2 and so are linearly dependent. Accordingly, the original matrices A, B, and C are linearly dependent.

5.11 CHANGE OF BASIS

Section 5.10 showed that we can represent each vector in a vector space V by means of an n-tuple once we have selected a basis S of V. We ask the following natural question: How does our representation change if we select another basis? For the answer, we must first redefine some terms. Specifically, suppose $a_1, a_2, \ldots, a_n$ are the coordinates of a vector v relative to a basis S of V. Then we will represent

v by its *coordinate column vector*, denoted and defined by

$$[v]_S = \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{pmatrix} = (a_1, a_2, \ldots, a_n)^T$$

We emphasize that in this section $[v]_S$ is an $n \times 1$ matrix, not simply an element of K^n. (The meaning of $[v]_S$ will always be clear by its context.)

Suppose $S = \{u_1, u_2, \ldots, u_n\}$ is a basis of a vector space V and suppose $S' = \{v_1, v_2, \ldots, v_n\}$ is another basis. Since S is a basis, each vector in S' can be written uniquely as a linear combination of the elements in S. Say

$$\begin{aligned} v_1 &= c_{11}u_1 + c_{12}u_2 + \cdots + c_{1n}u_n \\ v_2 &= c_{21}u_1 + c_{22}u_2 + \cdots + c_{2n}u_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ v_n &= c_{n1}u_1 + c_{n2}u_2 + \cdots + c_{nn}u_n \end{aligned}$$

Let P denote the transpose of the above matrix of coefficients:

$$P = \begin{pmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{pmatrix}$$

That is, $P = (p_{ij})$ where $p_{ij} = c_{ji}$. Then P is called the *change-of-basis matrix* (or *transition matrix*) from the "old basis" S to the "new basis" S'.

> **Remark:** Since the vectors $v_1, v_2, \ldots, v_n$ in S' are linearly independent, the matrix P is invertible (Problem 5.84). In fact (Problem 5.80), its inverse P^{-1} is the change-of-basis matrix from the basis S' back to the basis S.

Example 5.17. Consider the following two bases of $\mathbf{R}^2$:

$$S = \{u_1 = (1, 2), u_2 = (3, 5)\} \qquad E = \{e_1 = (1, 0), e_2 = (0, 1)\}$$

From $u_1 = e_1 + 2e_2$, $u_2 = 3e_1 + 5e_2$ it follows that

$$\begin{aligned} e_1 &= -5u_1 + 2u_2 \\ e_2 &= 3u_1 - u_2 \end{aligned}$$

Writing the coefficients of u_1 and u_2 as columns gives us the change-of-basis matrix P from the basis S to the usual basis E:

$$P = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$$

Furthermore, since E is the usual basis,

$$\begin{aligned} u_1 &= (1, 2) = e_1 + 2e_2 \\ u_2 &= (3, 5) = 3e_1 + 5e_2 \end{aligned}$$

Writing the coefficients of e_1 and e_2 as columns gives us the change-of-basis matrix Q from the basis E back to the basis S:

$$Q = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$$

Observe that P and Q are inverses:

$$PQ = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

The next theorem (proved in Problem 5.19) tells us how the coordinate (column) vectors are affected by a change of basis.

Theorem 5.27: Let P be the change-of-basis matrix from a basis S to a basis S' in a vector space V. Then, for any vector $v \in V$, we have

$$P[v]_{S'} = [v]_S \qquad \text{and hence} \qquad P^{-1}[v]_S = [v]_{S'}$$

Remark: Although P is called the change-of-basis matrix from the old basis S to the new basis S', it is P^{-1} which transforms the coordinates of v relative to the original basis S into the coordinates of v relative to the new basis S'.

We illustrate the above theorem in the case dim $V = 3$. Suppose P is the change-of-basis matrix from the basis $S = \{u_1, u_2, u_3\}$ to the basis $S' = \{v_1, v_2, v_3\}$; say

$$\begin{aligned} v_1 &= a_1u_1 + a_2u_2 + a_3u_3 \\ v_2 &= b_1u_1 + b_2u_2 + b_3u_3 \\ v_3 &= c_1u_1 + c_2u_2 + c_3u_3 \end{aligned}$$

Hence

$$P = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

Now suppose $v \in V$ and, say $v = k_1v_1 + k_2v_2 + k_3v_3$. Then, substituting for v_1, v_2, v_3 from above, we obtain

$$\begin{aligned} v &= k_1(a_1u_1 + a_2u_2 + a_3u_3) + k_2(b_1u_1 + b_2u_2 + b_3u_3) + k_3(c_1u_1 + c_2u_2 + c_3u_3) \\ &= (a_1k_1 + b_1k_2 + c_1k_3)u_1 + (a_2k_1 + b_2k_2 + c_2k_3)u_2 + (a_3k_1 + b_3k_2 + c_3k_3)u_3 \end{aligned}$$

Thus

$$[v]_{S'} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \qquad \text{and} \qquad [v]_S = \begin{pmatrix} a_1k_1 + b_1k_2 + c_1k_3 \\ a_2k_1 + b_2k_2 + c_2k_3 \\ a_3k_1 + b_3k_2 + c_3k_3 \end{pmatrix}$$

Accordingly,

$$P[v]_{S'} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} a_1k_1 + b_1k_2 + c_1k_3 \\ a_2k_1 + b_2k_2 + c_2k_3 \\ a_3k_1 + b_3k_2 + c_3k_3 \end{pmatrix} = [v]_S$$

Also, multiplying the above equation by P^{-1}, we have

$$P^{-1}[v]_S = P^{-1}P[v]_{S'} = I[v]_{S'} = [v]_{S'}$$

Remark: Suppose $S = \{u_1, u_2, \ldots, u_n\}$ is a basis of a vector space V over a field K, and suppose $P = (p_{ij})$ is any nonsingular matrix over K. Then the n vectors

$$v_i = p_{1i}u_1 + p_{2i}u_2 + \cdots + p_{ni}u_n \qquad i = 1, 2, \ldots, n$$

are linearly independent (Problem 5.84) and hence form another basis S' of V. Moreover, P will be the change-of-basis matrix from S to the new basis S'.

Solved Problems

VECTOR SPACES

5.1. Prove Theorem 5.1.

(i) By axiom $[A_2]$, with $u = 0$, we have $0 + 0 = 0$. Hence by axiom $[M_1]$,

$$k0 = k(0 + 0) = k0 + k0$$

Adding $-k0$ to both sides gives the desired result.

(ii) By a property of K, $0 + 0 = 0$. Hence by axiom $[M_2]$, $0u = (0 + 0)u = 0u + 0u$. Adding $-0u$ to both sides yields the required result.

(iii) Suppose $ku = 0$ and $k \neq 0$. Then there exists a scalar k^{-1} such that $k^{-1}k = 1$; hence

$$u = 1u = (k^{-1}k)u = k^{-1}(ku) = k^{-1}0 = 0$$

(iv) Using $u + (-u) = 0$, we obtain $0 = k0 = k(u + (-u)) = ku + k(-u)$. Adding $-ku$ to both sides gives $-ku = k(-u)$.

Using $k + (-k) = 0$, we obtain $0 = 0u = (k + (-k))u = ku + (-k)u$. Adding $-ku$ to both sides yields $-ku = (-k)u$. Thus $(-k)u = k(-u) = -ku$.

5.2. Show that for any scalar k and any vectors u and v, $k(u - v) = ku - kv$.

Use the definition of subtraction, $u - v \equiv u + (-v)$, and the result $k(-v) = -kv$ to obtain:

$$k(u - v) = k(u + (-v)) = ku + k(-v) = ku + (-kv) = ku - kv$$

5.3. Let V be the set of all functions from a nonempty set X into a field K. For any functions $f, g \in V$ and any scalar $k \in K$, let $f + g$ and kf be the functions in V defined as follows:

$$(f + g)(x) = f(x) + g(x) \qquad \text{and} \qquad (kf)(x) = kf(x) \qquad \forall x \in X$$

Prove that V is a vector space over K.

Since X is nonempty, V is also nonempty. We now need to show that all the axioms of a vector space hold.

$[A_1]$ Let $f, g, h \in V$. To show that $(f + g) + h = f + (g + h)$, it is necessary to show that the function $(f + g) + h$ and the function $f + (g + h)$ both assign the same value to each $x \in X$. Now,

$$((f + g) + h)(x) = (f + g)(x) + h(x) = (f(x) + g(x)) + h(x) \qquad \forall x \in X$$
$$(f + (g + h))(x) = f(x) + (g + h)(x) = f(x) + (g(x) + h(x)) \qquad \forall x \in X$$

But $f(x)$, $g(x)$, and $h(x)$ are scalars in the field K where addition of scalars is associative; hence

$$(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$$

Accordingly, $(f + g) + h = f + (g + h)$.

$[A_2]$ Let $\mathbf{0}$ denote the zero function: $\mathbf{0}(x) = 0$, $\forall x \in X$. Then for any function $f \in V$,

$$(f + \mathbf{0})(x) = f(x) + \mathbf{0}(x) = f(x) + 0 = f(x) \qquad \forall x \in X$$

Thus $f + \mathbf{0} = f$, and $\mathbf{0}$ is the zero vector in V.

$[A_3]$ For any function $f \in V$, let $-f$ be the function defined by $(-f)(x) = -f(x)$. Then,

$$(f + (-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = \mathbf{0}(x) \qquad \forall x \in X$$

Hence $f + (-f) = \mathbf{0}$.

$[A_4]$ Let $f, g \in V$. Then

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x) \qquad \forall x \in X$$

Hence $f + g = g + f$. [Note that $f(x) + g(x) = g(x) + f(x)$ follows from the fact that $f(x)$ and $g(x)$ are scalars in the field K where addition is commutative.]

$[M_1]$ Let $f, g \in K$. Then

$$(k(f + g))(x) = k((f + g)(x)) = k(f(x) + g(x)) = kf(x) + kg(x)$$
$$= (kf)(x) + kg)(x) = (kf + kg)(x) \qquad \forall x \in X$$

Hence $k(f + g) = kf + kg$. (Note that $k(f(x) + g(x)) = kf(x) + kg(x)$ follows from the fact that $k, f(x)$ and $g(x)$ are scalars in the field K where multiplication is distributive over addition.)

$[M_2]$ Let $f \in V$ and $a, b \in K$. Then

$$((a + b)f)(x) = (a + b)f(x) = af(x) + bf(x) = (af)(x) + bf(x)$$
$$= (af + bf)(x), \qquad \forall x \in X$$

Hence $(a + b)f = af + bf$.

$[M_3]$ Let $f \in V$ and $a, b \in K$. Then,

$$((ab)f)(x) = (ab)f(x) = a(bf(x)) = a(bf)(x) = (a(bf))(x) \qquad \forall x \in X$$

Hence $(ab)f = a(bf)$.

$[M_4]$ Let $f \in V$. Then, for the unit $1 \in K$, $(1f)(x) = 1f(x)) = f(x)$, $\forall x \in X$. Hence $1f = f$.

Since all the axioms are satisfied, V is a vector space over K.

SUBSPACES

5.4. Prove Theorem 5.2.

For W to be a subspace, conditions (i), (ii), and (iii) are clearly necessary; we now show them to be sufficient. By (i), W is nonempty; and by (ii) and (iii), the operations of vector addition and scalar multiplication are well defined for W. Moreover, the axioms $[A_1]$, $[A_4]$, $[M_1]$, $[M_2]$, $[M_3]$ and $[M_4]$ hold in W since the vectors in W belong to V. Hence we need only show that $[A_2]$ and $[A_3]$ also hold in W. Now, $[A_2]$ obviously holds, because the zero vector of V is also the zero vector of W. Finally, if $v \in W$, then $(-1)v = -v \in W$ and $v + (-v) = 0$; i.e., $[A_3]$ holds.

5.5. Prove Corollary 5.3.

Suppose W satisfies (i) and (ii). Then, by (i), W is nonempty. Furthermore, if $v, w \in W$ then, by (ii), $v + w = 1v + 1w \in W$; and if $v \in W$ and $k \in K$ then, by (ii), $kv = kv + 0v \in W$. Thus by Theorem 5.2, W is a subspace of V.

Conversely, if W is a subspace of V then clearly (i) and (ii) hold in W.

5.6. Show that W is a subspace of $\mathbf{R}^3$ where $W = \{(a, b, c): a + b + c = 0\}$, i.e., W consists of those vectors each with the property that the sum of its components is zero.

$0 = (0, 0, 0) \in W$ since $0 + 0 + 0 = 0$. Suppose $v = (a, b, c)$, $w = (a', b', c')$ belong to W, i.e., $a + b + c = 0$ and $a' + b' + c' = 0$. Then for any scalars k and k',

$$kv + k'w = k(a, b, c) + k'(a', b', c') = (ka, kb, kc) + (k'a', k'b', k'c') = (ka + k'a', kb + k'b', kc + k'c')$$

and furthermore

$$(ka + k'a') + (kb + k'b') + (kc + k'c') = k(a + b + c) + k'(a' + b' + c') = k0 + k'0 = 0$$

Thus $kv + k'w \in W$, and so W is a subspace of $\mathbf{R}^3$.

5.7. Let V be the vector space of all square $n \times n$ matrices over a field K. Show that W is a subspace of V where:

(*a*) W consists of the symmetric matrices, i.e., all matrices $A = (a_{ij})$ for which $a_{ji} = a_{ij}$;

(*b*) W consists of all matrices which commute with a given matrix T; that is

$$W = \{A \in V : AT = TA\}.$$

(*a*) $0 \in W$ since all entries of 0 are 0 and hence equal. Now suppose $A = (a_{ij})$ and $B = (b_{ij})$ belong to W, i.e., $a_{ji} = a_{ij}$ and $b_{ji} = b_{ij}$. For any scalars $a, b \in K$, $aA + bB$ is the matrix whose ij-entry is $aa_{ij} + bb_{ij}$. But $aa_{ji} + bb_{ji} = aa_{ij} + bb_{ij}$. Thus $aA + bB$ is also symmetric, and so W is a subspace of V.

(*b*) $0 \in W$ since $0T = 0 = T0$. Now suppose $A, B \in W$; that is, $AT = TA$ and $BT = TB$. For any scalars $a, b \in K$,

$$\begin{aligned}(aA + bB)T &= (aA)T + (bB)T = a(AT) + b(BT) = a(TA) + b(TB) \\ &= T(aA) + T(bB) = T(aA + bB)\end{aligned}$$

Thus $aA + bB$ commutes with T, i.e., belongs to W; hence W is a subspace of V.

5.8. Let V be the vector space of all 2×2 matrices over the real field $\mathbf{R}$. Show that W is not a subspace of V where:

(*a*) W consists of all matrices with zero determinant;

(*b*) W consists of all matrices A for which $A^2 = A$.

(*a*) $\left[\text{Recall that } \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.\right]$ The matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ belong to W since $\det(A) = 0$ and $\det(B) = 0$. But $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ does not belong to W since $\det(A + B) = 1$. Hence W is not a subspace of V.

(*b*) The unit matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ belongs to W since

$$I^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

But $2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ does not belong to W since

$$(2I)^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \neq 2I$$

Hence W is not a subspace of V.

5.9. Let V be the vector space of all functions from the real field $\mathbf{R}$ into $\mathbf{R}$. Show that W is a subspace of V where W consists of the odd functions, i.e., those functions f for which $f(-x) = -f(x)$.

Let $\mathbf{0}$ denote the zero function: $\mathbf{0}(x) = 0$, for every $x \in \mathbf{R}$. $\mathbf{0} \in W$ since $\mathbf{0}(-x) = 0 = -0 = -\mathbf{0}(x)$. Suppose $f, g \in W$, i.e., $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Then for any real numbers a and b,

$$(af + bg)(-x) = af(-x) + bg(-x) = -af(x) - bg(x) = -(af(x) + bg(x)) = -(af + bg)(x)$$

Hence $af + bg \in W$, and so W is a subspace of V.

5.10. Let V be the vector space of polynomials $a_0 + a_1 t + a_2 t^2 + \cdots + a_s t^s$ $(s = 0, 1, 2, \ldots)$, with coefficients $a_i \in \mathbf{R}$. Determine whether or not W is a subspace of V where:

(a) W consists of all polynomials with integral coefficients;

(b) W consists of all polynomials with degree ≤ 3;

(c) W consists of all polynomials with only even powers of t.

(d) W consists of all polynomials having $\sqrt{-1}$ as a root.

(a) No, since scalar multiples of vectors in W do not always belong to W. For example,

$$v = 3 + 5t + 7t^2 \in W \qquad \text{but} \qquad \tfrac{1}{2}v = \tfrac{3}{2} + \tfrac{5}{2}t + \tfrac{7}{2}t^2 \notin W$$

(Observe that W is "closed" under vector addition, i.e., sums of elements in W belong to W.)

(b), (c), and (d). Yes. For, in each case, W is nonempty, the sum of elements in W belong to W, and the scalar multiples of any element in W belong to W.

5.11 Prove Theorem 5.4.

Let $\{W_i : i \in I\}$ be a collection of subspaces of V and let $W = \bigcap (W_i : i \in I)$. Since each W_i is a subspace, $0 \in W_i$ for every $i \in I$. Hence $0 \in W$. Suppose $u, v \in W$. Then $u, v \in W_i$ for every $i \in I$. Since each W_i is a subspace, $au + bv \in W_i$, for each $i \in I$. Hence $au + bv \in W$. Thus W is a subspace of V.

LINEAR COMBINATIONS, LINEAR SPANS

5.12. Express $v = (1, -2, 5)$ in $\mathbf{R}^3$ as a linear combination of the vectors u_1, u_2, u_3 where $u_1 = (1, -3, 2)$, $u_2 = (2, -4, -1)$, $u_3 = (1, -5, 7)$.

First set

$$(1, -2, 5) = x(1, -3, 2) + y(2, -4, -1) + z(1, -5, 7) = (x + 2y + z, -3x - 4y - 5z, 2x - y + 7z)$$

Form the equivalent system of equations and reduce to echelon form:

$$\begin{aligned} x + 2y + z &= 1 \\ -3x - 4y - 5z &= -2 \\ 2x - y + 7z &= 5 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y + z &= 1 \\ 2y - 2z &= 1 \\ -5y + 5z &= 3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y + z &= 1 \\ 2y - 2z &= 1 \\ 0 &= 11 \end{aligned}$$

The system does not have a solution. Thus v is not a linear combination of u_1, u_2, u_3.

5.13. Express the polynomial $v = t^2 + 4t - 3$ over $\mathbf{R}$ as a linear combination of the polynomials $p_1 = t^2 - 2t + 5$, $p_2 = 2t^2 - 3t$, $p_3 = t + 3$.

Set v as a linear combination of p_1, p_2, p_3 using unknowns x, y, z:

$$\begin{aligned} t^2 + 4t - 3 &= x(t^2 - 2t + 5) + y(2t^2 - 3t) + z(t + 3) \\ &= xt^2 - 2xt + 5x + 2yt^2 - 3yt + zt + 3z \\ &= (x + 2y)t^2 + (-2x - 3y + z)t + (5x + 3z) \end{aligned}$$

Set coefficients of the same powers of t equal to each other, and reduce the system to echelon form:

$$\begin{aligned} x + 2y \qquad &= 1 \\ -2x - 3y + z &= 4 \\ 5x \qquad + 3z &= -3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y \qquad &= 1 \\ y + z &= 6 \\ -10y + 3z &= -8 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y \qquad &= 1 \\ y + z &= 6 \\ 13z &= 52 \end{aligned}$$

The system is in triangular form and has a solution. Solving by back-substitution yields $x = -3$, $y = 2$, $z = 4$. Thus $v = -3p_1 + 2p_2 + 4p_3$.

5.14. Write the matrix $E = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$ as a linear combination of the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$$

Set E as a linear combination of A, B, C using the unknowns x, y, z: $E = xA + yB + zC$.

$$\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + z\begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} x & x \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ y & y \end{pmatrix} + \begin{pmatrix} 0 & 2z \\ 0 & -z \end{pmatrix} = \begin{pmatrix} x & x + 2z \\ x + y & y - z \end{pmatrix}$$

Form the equivalent system of equations by setting corresponding entries equal to each other:

$$x = 3 \qquad x + y = 1 \qquad x + 2z = 1 \qquad y - z = -1$$

Substitute $x = 3$ in the second and third equations to obtain $y = -2$ and $z = -1$. Since these values also satisfy the last equation, they form a solution of the system. Hence $E = 3A - 2B - C$.

5.15. Find a condition on a, b, c so that $w = (a, b, c)$ is a linear combination of $u = (1, -3, 2)$ and $v = (2, -1, 1)$, that is, so that w belongs to span (u, v).

Set $w = xu + yv$ using unknowns x and y:

$$(a, b, c) = x(1, -3, 2) + y(2, -1, 1) = (x + 2y, -3x - y, 2x + y)$$

Form the equivalent system and reduce to echelon form:

$$\begin{aligned} x + 2y &= a \\ -3x - y &= b \\ 2x + y &= c \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y &= a \\ 5y &= 3a + b \\ -3y &= -2a + c \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y &= a \\ 5y &= 3a + b \\ 0 &= -a + 3b + 5c \end{aligned}$$

The system is consistent if and only if $a - 3b - 5c = 0$ and hence w is a linear combination of u and v when $a - 3b - 5c = 0$.

5.16. Prove Theorem 5.6.

Suppose S is empty. By definition span $S = \{0\}$. Hence span $S = \{0\}$ is a subspace and $S \subseteq$ span S. Suppose S is not empty, and $v \in S$. Then $1v = v \in$ span S; hence S is a subset of span S. Thus span S is not empty since S is not empty. Now suppose $v, w \in$ span S, say

$$v = a_1v_1 + \cdots + a_mv_m \qquad \text{and} \qquad w = b_1w_1 + \cdots + b_nw_n$$

where $v_i, w_j \in S$ and a_i, b_j are scalars. Then

$$v + w = a_1v_1 + \cdots + a_mv_m + b_1w_1 + \cdots + b_nw_n$$

and, for any scalar k,

$$kv = k(a_1v_1 + \cdots + a_mv_m) = ka_1v_1 + \cdots + ka_mv_m$$

belong to span S since each is a linear combination of vectors in S. Thus span S is a subspace of V.

Now suppose W is a subspace of V containing S and suppose $v_1, \ldots, v_m \in S \subseteq W$. Then all multiples $a_1v_1, \ldots, a_mv_m \in W$, where $a_i \in K$, and hence the sum $a_1v_1 + \cdots + a_mv_m \in W$. That is, W contains all linear combinations of elements of S. Consequently, span S is a subspace of W, as claimed.

LINEAR DEPENDENCE

5.17. Determine whether $u = 1 - 3t + 2t^2 - 3t^3$ and $v = -3 + 9t - 6t^2 + 9t^3$ are linearly dependent.

Two vectors are linearly dependent if and only if one is a multiple of the other. In this case, $v = -3u$.

5.18. Determine whether or not the following vectors in $\mathbf{R}^3$ are linearly dependent:

$$u = (1, -2, 1),\ v = (2, 1, -1),\ w = (7, -4, 1).$$

Method 1. Set a linear combination of the vectors equal to the zero vector using unknown scalars x, y, and z:

$$x(1, -2, 1) + y(2, 1, -1) + z(7, -4, 1) = (0, 0, 0)$$

Then

$$(x, -2x, x) + (2y, y, -y) + (7z, -4z, z) = (0, 0, 0)$$

or

$$(x + 2y + 7z, -2x + y - 4z, x - y + z) = (0, 0, 0)$$

Set corresponding components equal to each other to obtain the equivalent homogeneous system, and reduce to echelon form:

$$\begin{aligned} x + 2y + 7z &= 0 \\ -2x + y - 4z &= 0 \\ x - y + z &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y + 7z &= 0 \\ 5y + 10z &= 0 \\ -3y - 6z &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y + 7z &= 0 \\ y + 2z &= 0 \end{aligned}$$

The system, in echelon form, has only two nonzero equations in the three unknowns; hence the system has a nonzero solution. Thus the original vectors are linearly dependent.

Method 2. Form the matrix whose rows are the given vectors, and reduce to echelon form using the elementary row operations:

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 7 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 10 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the echelon matrix has a zero row, the vectors are linearly dependent. (The three given vectors span a space of dimension 2.)

5.19. Consider the vector space $\mathbf{P}(t)$ of polynomials over $\mathbf{R}$. Determine whether the polynomials u, v, and w are linearly dependent where $u = t^3 + 4t^2 - 2t + 3$, $v = t^3 + 6t^2 - t + 4$, $w = 3t^3 + 8t^2 - 8t + 7$.

Set a linear combination of the polynomials u, v and w equal to the zero polynomial using unknown scalars x, y, and z; that is, set $xu + yv + zw = 0$. Thus

$$x(t^3 + 4t^2 - 2t + 3) + y(t^3 + 6t^2 - t + 4) + z(3t^3 + 8t^2 - 8t + 7) = 0$$

or
$$xt^3 + 4xt^2 - 2xt + 3x + yt^3 + 6yt^2 - yt + 4y + 3zt^3 + 8zt^2 - 8zt + 7z = 0$$

or
$$(x + y + 3z)t^3 + (4x + 6y + 8z)t^2 + (-2x - y - 8z)t + (3x + 4y + 7z) = 0$$

Set the coefficients of the powers of t each equal to 0 and reduce the system to echelon form:

$$\begin{aligned} x + y + 3z &= 0 \\ 4x + 6y + 8z &= 0 \\ -2x - y - 8z &= 0 \\ 3x + 4y + 7z &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x + y + 3z &= 0 \\ 2y - 4z &= 0 \\ y - 2z &= 0 \\ y - 2z &= 0 \end{aligned} \quad \text{or finally} \quad \begin{aligned} x + y + 3z &= 0 \\ y - 2z &= 0 \end{aligned}$$

The system in echelon form has a free variable and hence a nonzero solution. We have shown that $xu + yv + zw = 0$ does not imply that $x = 0$, $y = 0$, $z = 0$; hence the polynomials are linearly dependent.

5.20. Let V be the vector space of functions from $\mathbf{R}$ into $\mathbf{R}$. Show that $f, g, h \in V$ are linearly independent, where $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = t$.

Set a linear combination of the functions equal to the zero function 0 using unknown scalars x, y, and z: $xf + yg + zh = 0$; and then show that $x = 0$, $y = 0$, $z = 0$. We emphasize that $xf + yg + zh = 0$ means that, *for every value of* t, $xf(t) + yg(t) + zh(t) = 0$.

Thus, in the equation $x \sin t + y \cos t + zt = 0$, substitute

$$\begin{array}{llll} t = 0 & \text{to obtain} \quad x \cdot 0 + y \cdot 1 + z \cdot 0 = 0 & \text{or} & y = 0 \\ t = \pi/2 & \text{to obtain} \quad x \cdot 1 + y \cdot 0 + z\pi/2 = 0 & \text{or} & x + \pi z/2 = 0 \\ t = \pi & \text{to obtain} \quad x \cdot 0 + y(-1) + z \cdot \pi = 0 & \text{or} & -y + \pi z = 0 \end{array}$$

The last three equations have only the zero solution: $x = 0$, $y = 0$, $z = 0$. Hence f, g and h are linearly independent.

5.21. Let V be the vector space of 2×2 matrices over $\mathbf{R}$. Determine whether the matrices $A, B, C \in V$ are linearly dependent, where:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Set a linear combination of the matrices A, B, and C equal to the zero matrix using unknown scalars x, y, and z; that is, set $xA + yB + zC = 0$. Thus

$$x\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + y\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} x + y + z & x + z \\ x & x + y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Set corresponding entries equal to each other to obtain the following equivalent system of linear equations:

$$x + y + z = 0 \qquad x + z = 0 \qquad x = 0 \qquad x + y = 0$$

Solving the above system we obtain only the zero solution, $x = 0$, $y = 0$, $z = 0$. We have shown that $xA + yB + zC = 0$ implies $x = 0$, $y = 0$, $z = 0$; hence the matrices A, B, and C are linearly independent.

5.22. Suppose u, v, and w are linearly independent vectors. Show that $u + v$, $u - v$, and $u - 2v + w$ are also linearly independent.

Suppose $x(u + v) + y(u - v) + z(u - 2v + w) = 0$ where x, y, and z are scalars. Then

$$xu + xv + yu - yv + zu - 2zv + zw = 0$$

or

$$(x + y + z)u + (x - y - 2z)v + zw = 0$$

But u, v, and w are linearly independent; hence the coefficients in the above relation are each 0:

$$\begin{aligned} x + y + z &= 0 \\ x - y - 2z &= 0 \\ z &= 0 \end{aligned}$$

The only solution to the above system is $x = 0$, $y = 0$, $z = 0$. Hence $u + v$, $u - v$, and $u - 2v + w$ are linearly independent.

5.23. Show that the vectors $v = (1 + i, 2i)$ and $w = (1, 1 + i)$ in $\mathbf{C}^2$ are linearly dependent over the complex field $\mathbf{C}$ but are linearly independent over the real field $\mathbf{R}$.

Recall that two vectors are linearly dependent (over a field K) if and only if one of them is a multiple of the other (by an element in K). Since

$$(1+i)w = (1+i)(1, 1+i) = (1+i, 2i) = v$$

v and w are linearly dependent over $\mathbf{C}$. On the other hand, v and w are linearly independent over $\mathbf{R}$ since no real multiple of w can equal v. Specifically, when k is real, the first component of $kw = (k, k+ki)$ is real and it can never equal the first component $1+i$ of v which is complex.

BASIS AND DIMENSION

5.24. Determine whether $(1, 1, 1)$, $(1, 2, 3)$, and $(2, -1, 1)$ form a basis for the vector space $\mathbf{R}^3$.

The three vectors form a basis if and only if they are linearly independent. Thus form the matrix A whose rows are the given vectors, and row reduce to echelon form:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

The echelon matrix has no zero rows; hence the three vectors are linearly independent and so they form a basis for $\mathbf{R}^3$.

5.25. Determine whether $(1, 1, 1, 1)$, $(1, 2, 3, 2)$, $(2, 5, 6, 4)$, $(2, 6, 8, 5)$ form a basis of $\mathbf{R}^4$.

Form the matrix whose rows are the given vectors, and row reduce to echelon form:

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 4 & 6 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The echelon matrix has a zero row; hence the four vectors are linearly dependent and do not form a basis of $\mathbf{R}^4$.

5.26. Consider the vector space $\mathbf{P}_n(t)$ of polynomials in t of degree $\le n$. Determine whether or not $1+t, t+t^2, t^2+t^3, \ldots, t^{n-1}+t^n$ form a basis of $\mathbf{P}_n(t)$.

The polynomials are linearly independent since each one is of degree higher than the preceding ones. However, there are only n polynomials and $\dim \mathbf{P}_n(t) = n+1$. Thus the polynomials do not form a basis of $\mathbf{P}_n(t)$.

5.27. Let V be the vector space of real 2×2 matrices. Determine whether

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis for V.

The coordinate vectors (see Section 5.10) of the matrices relative to the usual basis are, respectively,

$$[A] = (1, 1, 0, 0) \qquad [B] = (0, 1, 1, 0) \qquad [C] = (0, 0, 1, 1) \qquad [D] = (0, 0, 0, 1)$$

The coordinate vectors form a matrix in echelon form and hence they are linearly independent. Thus the four corresponding matrices are linearly independent. Moreover, since $\dim V = 4$, they form a basis for V.

5.28. Let V be the vector space of 2×2 symmetric matrices over K. Show that $\dim V = 3$. [Recall that $A = (a_{ij})$ is symmetric iff $A = A^T$ or, equivalently, $a_{ij} = a_{ji}$.]

An arbitrary 2×2 symmetric matrix is of the form $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ where $a, b, c \in K$. (Note that there are three "variables.") Setting

(i) $a = 1, b = 0, c = 0,$ (ii) $a = 0, b = 1, c = 0,$ (iii) $a = 0, b = 0, c = 1$

we obtain the respective matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We show that $\{E_1, E_2, E_3\}$ is a basis of V, that is, that (*a*) it spans V, and (*b*) it is linearly independent.

(*a*) For the above arbitrary matrix A in V, we have

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = aE_1 + bE_2 + cE_3$$

Thus $\{E_1, E_2, E_3\}$ spans V.

(*b*) Suppose $xE_1 + yE_2 + zE_3 = 0$, where x, y, z are unknown scalars. That is, suppose

$$x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x & y \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Setting corresponding entries equal to each other, we obtain $x = 0, y = 0, z = 0$. In other words,

$$xE_1 + yE_2 + zE_3 = 0 \quad \text{implies} \quad x = 0, y = 0, z = 0$$

Accordingly, $\{E_1, E_2, E_3\}$ is linearly independent.

Thus $\{E_1, E_2, E_3\}$ is a basis of V and so the dimension of V is 3.

5.29. Consider the complex field $\mathbf{C}$ which contains the real field $\mathbf{R}$ which contains the rational field $\mathbf{Q}$. (Thus $\mathbf{C}$ is a vector space over $\mathbf{R}$, and $\mathbf{R}$ is a vector space over $\mathbf{Q}$.)

(*a*) Show that $\mathbf{C}$ is a vector space of dimension 2 over $\mathbf{R}$.

(*b*) Show that $\mathbf{R}$ is a vector space of infinite dimension over $\mathbf{Q}$.

(*a*) We claim that $\{1, i\}$ is a basis of $\mathbf{C}$ over $\mathbf{R}$. For if $v \in \mathbf{C}$, then $v = a + bi = a \cdot 1 + b \cdot i$ where $a, b \in \mathbf{R}$; that is, $\{1, i\}$ spans $\mathbf{C}$ over $\mathbf{R}$. Furthermore, if $x \cdot 1 + y \cdot i = 0$ or $x + yi = 0$, where $x, y \in \mathbf{R}$, then $x = 0$ and $y = 0$; that is, $\{1, i\}$ is linearly independent over $\mathbf{R}$. Thus $\{1, i\}$ is a basis of $\mathbf{C}$ over $\mathbf{R}$, and so $\mathbf{C}$ is of dimension 2 over $\mathbf{R}$.

(*b*) We claim that, for any n, the set $\{1, \pi, \pi^2, \ldots, \pi^n\}$ is linearly independent over $\mathbf{Q}$. For suppose $a_0 1 + a_1\pi + a_2\pi^2 + \cdots + a_n\pi^n = 0$, where the $a_i \in \mathbf{Q}$, and not all the a_i are 0. Then π is a root of a nonzero polynomial over $\mathbf{Q}$: $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. But it can be shown that π is a transcendental number, i.e., that π is not a root of any nonzero polynomial over $\mathbf{Q}$. Accordingly, the $n + 1$ real numbers $1, \pi, \pi^2, \ldots, \pi^n$ are linearly independent over $\mathbf{Q}$. Thus, for any finite n, $\mathbf{R}$ cannot be of dimension n over $\mathbf{Q}$; $\mathbf{R}$ is of infinite dimension over $\mathbf{Q}$.

5.30. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a subset of a vector space V. Show that the following two conditions are equivalent: (*a*) S is linearly independent and spans V, and (*b*) every vector $v \in V$ can be written uniquely as a linear combination of the vectors in S.

Suppose (*a*) holds. Since S spans V, the vector v is a linear combination of the u_i; say,

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

Suppose we also have

$$v = b_1u_1 + b_2u_2 + \cdots + b_nu_n$$

Subtracting, we get

$$0 = v - v = (a_1 - b_1)u_1 + (a_2 - b_2)u_2 + \cdots + (a_n - b_n)u_n$$

But the u_i are linearly independent; hence the coefficients in the above relation are each 0:

$$a_1 - b_1 = 0,\ a_2 - b_2 = 0, \ldots, a_n - b_n = 0$$

Therefore, $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$; hence the representation of v as a linear combination of the u_i is unique. Thus (*a*) implies (*b*).

Suppose (*b*) holds. Then S spans V. Suppose

$$0 = c_1u_1 + c_2u_2 + \cdots + c_nu_n$$

However, we do have

$$0 = 0u_1 + 0u_2 + \cdots + 0u_n$$

By hypothesis, the representation of 0 as a linear combination of the u_i is unique. Hence each $c_i = 0$ and the u_i are linearly independent. Thus (*b*) implies (*a*).

DIMENSION AND SUBSPACES

5.31. Find a basis and the dimension of the subspace W of $\mathbf{R}^3$ where:

(*a*) $W = \{(a, b, c): a + b + c = 0\}$, (*b*) $W = \{(a, b, c): a = b = c\}$,

(*c*) $W = (a, b, c): c = 3a\}$

(*a*) Note $W \neq \mathbf{R}^3$ since, e.g., $(1, 2, 3) \notin W$. Thus $\dim W < 3$. Note $u_1 = (1, 0, -1)$ and $u_2 = (0, 1, -1)$ are two independent vectors in W. Thus $\dim W = 2$ and so u_1 and u_2 form a basis of W.

(*b*) The vector $u = (1, 1, 1) \in W$. Any vector $w \in W$ has the form $w = (k, k, k)$. Hence $w = ku$. Thus u spans W and $\dim W = 1$.

(*c*) $W \neq \mathbf{R}^3$ since, e.g., $(1, 1, 1) \notin W$. Thus $\dim W < 3$. The vectors $u_1 = (1, 0, 3)$ and $u_2 = (0, 1, 0)$ belong to W and are linearly independent. Thus $\dim W = 2$ and u_1, u_2 form a basis of W.

5.32. Find a basis and the dimension of the subspace W of $\mathbf{R}^4$ spanned by

$$u_1 = (1, -4, -2, 1), \quad u_2 = (1, -3, -1, 2), \quad u_3 = (3, -8, -2, 7).$$

Apply the Row Space Algorithm *5.8A*. Form a matrix where the rows are the given vectors, and row reduce it to echelon form:

$$\begin{pmatrix} 1 & -4 & -2 & 1 \\ 1 & -3 & -1 & 2 \\ 3 & -8 & -2 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 4 & 4 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The nonzero rows in the echelon matrix form a basis of W and so $\dim W = 2$. In particular, this means that the original three vectors are linearly dependent.

5.33. Let W be the subspace of $\mathbf{R}^4$ spanned by the vectors

$$u_1 = (1, -2, 5, -3), \quad u_2 = (2, 3, 1, -4) \quad u_3 = (3, 8, -3, -5).$$

(*a*) Find a basis and the dimension of W. (*b*) Extend the basis of W to a basis of the whole space $\mathbf{R}^4$.

(*a*) Form the matrix A whose rows are the given vectors, and row reduce it to echelon form:

$$A = \begin{pmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The nonzero rows $(1, -2, 5, -3)$ and $(0, 7, -9, 2)$ of the echelon matrix form a basis of the row space of A and hence of W. Thus, in particular, $\dim W = 2$.

(*b*) We seek four linearly independent vectors which include the above two vectors. The four vectors $(1, -2, 5, -3)$, $(0, 7, -9, 2)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$ are linearly independent (since they form an echelon matrix), and so they form a basis of $\mathbf{R}^4$ which is an extension of the basis of W.

5.34. Let W be the subspace of $\mathbf{R}^5$ spanned by the vectors $u_1 = (1, 2, -1, 3, 4)$, $u_2 = (2, 4, -2, 6, 8)$, $u_3 = (1, 3, 2, 2, 6)$, $u_4 = (1, 4, 5, 1, 8)$, and $u_5 = (2, 7, 3, 3, 9)$. Find a subset of the vectors which form a basis of W.

Method 1. Here we use the Casting-Out Algorithm *5.8B*. Form the matrix whose columns are the given vectors and reduce it to echelon form:

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The pivot positions are in columns C_1, C_3, C_5. Hence the corresponding vectors u_1, u_3, u_5 form a basis of W and $\dim W = 3$.

Method 2. Here we use a slight modification of the Row Reduction Algorithm *5.8A*. Form the matrix whose rows are the given vectors and reduce it to an "echelon" form but without interchanging any zero rows:

$$\begin{pmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{pmatrix}$$

The nonzero rows are the first, third, and fifth rows; hence u_1, u_3, u_5 form a basis of W. Thus, in particular, $\dim W = 3$.

5.35. Let V be the vector space of real 2×2 matrices. Find the dimension and a basis of the subspace W of V spanned by

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 5 & 12 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 4 \\ -2 & 5 \end{pmatrix}$$

The coordinate vectors (see Section 5.10) of the given matrices relative to the usual basis of V are as follows:

$$[A] = [1, 2, -1, 3] \qquad [B] = [2, 5, 1, -1] \qquad [C] = [5, 12, 1, 1] \qquad [D] = [3, 4, -2, 5]$$

Form a matrix whose rows are the coordinate vectors, and reduce it to echelon form:

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & -1 \\ 5 & 12 & 1 & 1 \\ 3 & 4 & -2 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -7 \\ 0 & 2 & 6 & -14 \\ 0 & -2 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & -18 \end{pmatrix}$$

The nonzero rows are linearly independent, hence the corresponding matrices $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 3 & -7 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 7 & -18 \end{pmatrix}$ form a basis of W and $\dim W = 3$. (Note also that the matrices A, B, and D form a basis of W.)

THEOREMS ON LINEAR DEPENDENCE, BASIS, AND DIMENSION

5.36. Prove Lemma 5.10.

Since the v_i are linearly dependent, there exist scalars $a_1, \ldots, a_m$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$. Let k be the largest integer such that $a_k \neq 0$. Then

$$a_1 v_1 + \cdots + a_k v_k + 0v_{k+1} + \cdots + 0v_m = 0 \qquad \text{or} \qquad a_1 v_1 + \cdots + a_k v_k = 0$$

Suppose $k = 1$; then $a_1 v_1 = 0$, $a_1 \neq 0$ and so $v_1 = 0$. But the v_i are nonzero vectors; hence $k > 1$ and

$$v_k = -a_k^{-1} a_1 v_1 - \cdots - a_k^{-1} a_{k-1} v_{k-1}$$

That is, v_k is a linear combination of the preceding vectors.

5.37. Prove Theorem 5.11.

Suppose $R_n, R_{n-1}, \ldots, R_1$ are linearly dependent. Then one of the rows, say R_m, is a linear combination of the preceding rows:

$$R_m = a_{m+1} R_{m+1} + a_{m+2} R_{m+2} + \cdots + a_n R_n \qquad (*)$$

Now suppose the kth component of R_m is its first nonzero entry. Then, since the matrix is in echelon form, the kth components of $R_{m+1}, \ldots, R_n$ are all 0, and so the kth component of $(*)$ is

$$a_{m+1} \cdot 0 + a_{m+2} \cdot 0 + \cdots + a_n \cdot 0 = 0$$

But this contradicts the assumption that the kth component of R_m is not 0. Thus $R_1, \ldots, R_n$ are linearly independent.

5.38. Suppose $\{v_1, \ldots, v_m\}$ spans a vector space V. Prove:

(a) If $w \in V$, then $\{w, v_1, \ldots, v_m\}$ is linearly dependent and spans V.

(b) If v_i is a linear combination of vectors $(v_1, v_2, \ldots, v_{i-1})$, then $\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m\}$ spans V.

(a) The vector w is a linear combination of the v_i since $\{v_i\}$ spans V. Accordingly, $\{w, v_1, \ldots, v_m\}$ is linearly dependent. Clearly, w with the v_i span V since the v_i by themselves span V. That is, $\{w, v_1, \ldots, v_m\}$ spans V.

(b) Suppose $v_i = k_1 v_1 + \cdots + k_{i-1} v_{i-1}$. Let $u \in V$. Since $\{v_i\}$ spans V, u is a linear combination of the v_i, say, $u = a_1 v_1 + \cdots + a_m v_m$. Substituting for v_i, we obtain

$$\begin{aligned} u &= a_1 v_1 + \cdots + a_{i-1} v_{i-1} + a_i(k_1 v_1 + \cdots + k_{i-1} v_{i-1}) + a_{i+1} v_{i+1} + \cdots + a_m v_m \\ &= (a_1 + a_i k_1) v_1 + \cdots + (a_{i-1} + a_i k_{i-1}) v_{i-1} + a_{i+1} v_{i+1} + \cdots + a_m v_m \end{aligned}$$

Thus $\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m\}$ spans V. In other words, we can delete v_i from the spanning set and still retain a spanning set.

5.39. Prove Lemma 5.13.

It suffices to prove the theorem in the case that the v_i are all not 0. (Prove!) Since $\{v_i\}$ spans V, we have by Problem 5.38 that

$$\{w_1, v_1, \ldots, v_n\} \qquad (1)$$

is linearly dependent and also spans V. By Lemma 5.10, one of the vectors in (1) is a linear combination of the preceding vectors. This vector cannot be w_1, so it must be one of the v's, say v_j. Thus by Problem 5.38 we can delete v_j from the spanning set (1) and obtain the spanning set

$$\{w_1; v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\} \tag{2}$$

Now we repeat the argument with the vector w_2. That is, since (2) spans V, the set

$$\{w_1, w_2, v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\} \tag{3}$$

is linearly dependent and also spans V. Again by Lemma 5.10, one of the vectors in (3) is a linear combination of the preceding vectors. We emphasize that this vector cannot be w_1 or w_2 since $\{w_1, \ldots, w_m\}$ is independent; hence it must be one of the v's, say v_k. Thus by the preceding problem we can delete v_k from the spanning set (3) and obtain the spanning set

$$\{w_1, w_2, v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$$

We repeat the argument with w_3 and so forth. At each step we are able to add one of the w's and delete one of the v's in the spanning set. If $m \le n$, then we finally obtain a spanning set of the required form:

$$\{w_1, \ldots, w_m, v_{i_1}, \ldots, v_{i_{n-m}}\}$$

Last, we show that $m > n$ is not possible. Otherwise, after n of the above steps, we obtain the spanning set $\{w_1, \ldots, w_n\}$. This implies that w_{n+1} is a linear combination of $w_1, \ldots, w_n$ which contradicts the hypothesis that $\{w_i\}$ is linearly independent.

5.40. Prove Theorem 5.12.

Suppose $\{u_1, u_2, \ldots, u_n\}$ is a basis of V, and suppose $\{v_1, v_2, \ldots\}$ is another basis of V. Since $\{u_i\}$ spans V, the basis $\{v_1, v_2, \ldots\}$ must contain n or less vectors, or else it is linearly dependent by Problem 5.39 (Lemma 5.13). On the other hand, if the basis $\{v_1, v_2, \ldots\}$ contains less than n elements, then $\{u_1, u_2, \ldots, u_n\}$ is linearly dependent by Problem 5.39. Thus the basis $\{v_1, v_2, \ldots\}$ contains exactly n vectors, and so the theorem is true.

5.41. Prove Theorem 5.14.

Suppose $B = \{w_1, w_2, \ldots, w_n\}$ is a basis of V.

(i) Since B spans V, any $n + 1$ or more vectors are linearly dependent by Lemma 5.13.

(ii) By Lemma 5.13, elements from B can be adjoined to S to form a spanning set of V with n elements. Since S already has n elements, S itself is a spanning set of V. Thus S is a basis of V.

(iii) Suppose T is linearly dependent. Then some v_i is a linear combination of the preceding vectors. By Problem 5.38, V is spanned by the vectors in T without v_i and there are $n - 1$ of them. By Lemma 5.13, the independent set B cannot have more than $n - 1$ elements. This contradicts the fact that B has n elements. Thus T is linearly independent and hence T is a basis of V.

5.42. Prove Theorem 5.15.

(i) Suppose $\{v_1, \ldots, v_m\}$ is a maximal linearly independent subset of S, and suppose $w \in S$. Accordingly $\{v_1, \ldots, v_m, w\}$ is linearly dependent. No v_k can be a linear combination of preceding vectors; hence w is a linear combination of the v_i. Thus $w \in \text{span } v_i$ and hence $S \subseteq \text{span } v_i$. This leads to

$$V = \text{span } S \subseteq \text{span } v_i \subseteq V$$

Thus $\{v_i\}$ spans V and, since it is linearly independent, it is a basis of V.

(ii) The remaining vectors form a maximal linearly independent subset of S and hence by part (i) it is a basis of V.

5.43. Prove Theorem 5.16.

Suppose $B = \{w_1, w_2, \ldots, w_n\}$ is a basis of V. Then B spans V and hence V is spanned by

$$S \cup B = \{u_1, u_2, \ldots, u_r, w_1, w_2, \ldots, w_n\}$$

By Theorem 5.15, we can delete from $S \cup B$ each vector which is a linear combination of preceding vectors to obtain a basis B' for V. Since S is linearly independent, no u_k is a linear combination of preceding vectors. Thus B' contains every vector in S. Thus S is part of the basis B' for V.

5.44. Prove Theorem 5.17.

Since V is of dimension n, any $n + 1$ or more vectors are linearly dependent. Furthermore, since a basis of W consists of linearly independent vectors, it cannot contain more than n elements. Accordingly, $\dim W \leq n$.

In particular, if $\{w_1, \ldots, w_n\}$ is a basis of W, then, since it is an independent set with n elements, it is also a basis of V. Thus $W = V$ when $\dim W = n$.

ROW SPACE AND RANK OF A MATRIX

5.45. Determine whether the following matrices have the same row space:

$$A = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{pmatrix}$$

Matrices have the same row space if and only if their row canonical forms have the same nonzero rows; hence row reduce each matrix to row canonical form:

$$A = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the nonzero rows of the reduced form of A and of the reduced form of C are the same, A and C have the same row space. On the other hand, the nonzero rows of the reduced form of B are not the same as the others, and so B has a different row space.

5.46. Show that $A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 4 & 3 \\ 1 & 1 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & -4 \\ 7 & 12 & 17 \end{pmatrix}$ have the same column space.

Observe that A and B have the same column space if and only if the transposes A^T and B^T have the same row space. Thus reduce A^T and B^T to row canonical form:

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 5 & 3 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 1 & -2 & 7 \\ 2 & -3 & 12 \\ 3 & -4 & 17 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Since A^T and B^T have the same row space, A and B have the same column space.

5.47. Consider the subspace $U = \text{span}(u_1, u_2, u_3)$ and $W = \text{span}(w_1, w_2, w_3)$ of $\mathbf{R}^3$ where:

$$u_1 = (1, 1, -1), \qquad u_2 = (2, 3, -1), \qquad u_3 = (3, 1, -5)$$
$$w_1 = (1, -1, -3), \qquad w_2 = (3, -2, -8), \qquad w_3 = (2, 1, -3)$$

Show that $U = W$.

Form the matrix A whose rows are the u_i, and row reduce A to row canonical form:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Next form the matrix B whose rows are the w_i and row reduce B to row canonical form:

$$B = \begin{pmatrix} 1 & -1 & -3 \\ 3 & -2 & -8 \\ 2 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since A and B have the same row canonical form, the row spaces of A and B are equal and so $U = W$.

5.48. Find the rank of the matrix A where:

$$(a) \quad A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{pmatrix}, \qquad (b) \quad A = \begin{pmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{pmatrix}$$

(a) Since row rank equals column rank, it is easier to form the transpose of A and then row reduce to echelon form:

$$A^t = \begin{pmatrix} 1 & 2 & -2 & -1 \\ 2 & 1 & -1 & 4 \\ -3 & 0 & 3 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & -3 & 3 & 6 \\ 0 & 6 & -3 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & 3 & 7 \end{pmatrix}$$

Thus rank $A = 3$.

(b) The two columns are linearly independent since one is not a multiple of the other. Thus rank $A = 2$.

5.49. Consider an arbitrary matrix $A = (a_{ij})$. Suppose $u = (b_1, \ldots, b_n)$ is a linear combination of the rows $R_1, \ldots, R_m$ of A; say $u = k_1 R_1 + \cdots + k_m R_m$. Show that

$$b_i = k_1 a_{1i} + k_2 a_{2i} + \cdots + k_m a_{mi} \qquad (i = 1, 2, \ldots, n)$$

where $a_{1i}, \ldots, a_{mi}$ are the entries of the ith column of A.

We are given $u = k_1 R_1 + \cdots + k_m R_m$; hence

$$\begin{aligned}(b_1, \ldots, b_n) &= k_1(a_{11}, \ldots, a_{1n}) + \cdots + k_m(a_{m1}, \ldots, a_{mn}) \\ &= (k_1 a_{11} + \cdots + k_m a_{m1}, \ldots, k_1 a_{m1} + \cdots + k_m a_{mn})\end{aligned}$$

Setting corresponding components equal to each other, we obtain the desired result.

5.50. Suppose $A = (a_{ij})$ and $B = (b_{ij})$ are echelon matrices with pivot entries:

$$a_{1j_1}, a_{2j_2}, \ldots, a_{rj_r} \qquad \text{and} \qquad b_{1k_1}, b_{2k_2}, \ldots, b_{sk_s}$$

$$A = \begin{pmatrix} a_{1j_1} & * & * & * & * & * & * \\ & & a_{2j_2} & * & * & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & a_{rj_r} & * & * \\ & & & & & & \end{pmatrix}, \qquad B = \begin{pmatrix} b_{1k_1} & * & * & * & * & * & * & * \\ & & b_{2k_2} & * & * & * & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & b_{sk_s} & * & * \\ & & & & & & & \end{pmatrix}$$

Suppose A and B have the same row space. Prove that the pivot entries of A and B are in the same positions, that is, prove that $j_1 = k_1, j_2 = k_2, \ldots, j_r = k_r$, and $r = s$.

Clearly $A = 0$ if and only if $B = 0$, and so we need only prove the theorem when $r \geq 1$ and $s \geq 1$. We first show that $j_1 = k_1$. Suppose $j_1 < k_1$. Then the j_1th column of B is zero. Since the first row of A is in the row space of B, we have by the preceding problem,

$$a_{1j_1} = c_1 0 + c_2 0 + \cdots + c_m 0 = 0$$

for scalars c_i. But this contradicts the fact that the pivot element $a_{1j_1} \neq 0$. Hence $j_1 \geq k_1$, and similarly $k_1 \geq j_1$. Thus $j_1 = k_1$.

Now let A' be the submatrix of A obtained by deleting the first row of A, and let B' be the submatrix of B obtained by deleting the first row of B. We prove that A' and B' have the same row space. The theorem will then follow by induction since A' and B' are also echelon matrices.

Let $R = (a_1, a_2, \ldots, a_n)$ be any row of A' and let $R_1, \ldots, R_m$ be the rows of B. Since R is in the row space of B, there exist scalars $d_1, \ldots, d_m$ such that $R = d_1R_1 + d_2R_2 + \cdots + d_mR_m$. Since A is in echelon form and R is not the first row of A, the j_1th entry of R is zero: $a_i = 0$ for $i = j_1 = k_1$. Furthermore, since B is in echelon form, all the entries in the k_1th column of B are 0 except the first: $b_{1k_1} \neq 0$, but $b_{2k_1} = 0, \ldots, b_{mk_1} = 0$. Thus

$$0 = a_{k_1} = d_1 b_{1k_1} + d_2 0 + \cdots + d_m 0 = d_1 b_{1k_1}$$

Now $b_{1k_1} \neq 0$ and so $d_1 = 0$. Thus R is a linear combination of $R_2, \ldots, R_m$ and so is in the row space of B'. Since R was any row of A', the row space of A' is contained in the row space of B'. Similarly, the row space of B' is contained in the row space of A'. Thus A' and B' have the same row space, and so the theorem is proved.

5.51. **Prove Theorem 5.8.**

Obviously, if A and B have the same nonzero rows then they have the same row space. Thus we only have to prove the converse.

Suppose A and B have the same row space, and suppose $R \neq 0$ is the ith row of A. Then there exist scalars $c_1, \ldots, c_s$ such that

$$R = c_1R_1 + c_2R_2 + \cdots + c_sR_s \tag{1}$$

where the R_i are the nonzero rows of B. The theorem is proved if we show that $R = R_i$, or $c_i = 1$ but $c_k = 0$ for $k \neq i$.

Let a_{ij_i} be the pivot entry in R, i.e., the first nonzero entry of R. By (*1*) and Problem 5.49,

$$a_{ij_i} = c_1 b_{1j_i} + c_2 b_{2j_i} + \cdots + c_s b_{sj_i} \tag{2}$$

But by the preceding problem b_{ij_i} is a pivot entry of B and, since B is row reduced, it is the only nonzero entry in the j_ith column of B. Thus from (*2*) we obtain $a_{ij_i} = c_i b_{ij_i}$. However, $a_{ij_i} = 1$ and $b_{ij_i} = 1$ since A and B are row reduced; hence $c_i = 1$.

Now suppose $k \neq i$, and b_{kj_k} is the pivot entry in R_k. By (*1*) and Problem 5.49,

$$a_{ij_k} = c_1 b_{1j_k} + c_2 b_{2j_k} + \cdots + c_s b_{sj_k} \tag{3}$$

Since B is row reduced, b_{kj_k} is the only nonzero entry in the j_kth column of B; hence by (*3*), $a_{ij_k} = c_k b_{kj_k}$. Furthermore, by the preceding problem a_{kj_k} is a pivot entry of A and, since A is row reduced, $a_{ij_k} = 0$. Thus $c_k b_{kj_k} = 0$ and, since $b_{kj_k} = 1$, $c_k = 0$. Accordingly $R = R_i$ and the theorem is proved.

5.52. Prove Theorem 5.9.

Suppose A is row equivalent to matrices A_1 and A_2 where A_1 and A_2 are in row canonical form. Then rowsp A = rowsp A_1 and rowsp A = rowsp A_2. Hence rowsp A_1 = rowsp A_2. Since A_1 and A_2 are in row canonical form, $A_1 = A_2$ by Theorem 5.8. Thus the theorem is proved.

5.53. Prove Theorem 5.18.

Let A be an arbitrary $m \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Let $R_1, R_2, \ldots, R_m$ denote its rows:

$$R_1 = (a_{11}, a_{12}, \ldots, a_{1n}), \ \ldots, \ R_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$$

Suppose the row rank is r and that the following r vectors form a basis for the row space:

$$S_1 = (b_{11}, b_{12}, \ldots, b_{1n}), \ S_2 = (b_{21}, b_{22}, \ldots, b_{2n}), \ \ldots, \ S_r = (b_{r1}, b_{r2}, \ldots, b_{rn})$$

Then each of the row vectors is a linear combination of the S_i:

$$\begin{aligned} R_1 &= k_{11}S_1 + k_{12}S_2 + \cdots + k_{1r}S_r \\ R_2 &= k_{21}S_1 + k_{22}S_2 + \cdots + k_{2r}S_r \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ R_m &= k_{m1}S_1 + k_{m2}S_2 + \cdots + k_{mr}S_r \end{aligned}$$

where the k_{ij} are scalars. Setting the ith components of each of the above vector equations equal to each other, we obtain the following system of equations, each valid for $i = 1, \ldots, n$:

$$\begin{aligned} a_{1i} &= k_{11}b_{1i} + k_{12}b_{2i} + \cdots + k_{1r}b_{ri} \\ a_{2i} &= k_{21}b_{1i} + k_{22}b_{2i} + \cdots + k_{2r}b_{ri} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{mi} &= k_{m1}b_{1i} + k_{m2}b_{2i} + \cdots + k_{mr}b_{ri} \end{aligned}$$

Thus, for $i = 1, \ldots, n$,

$$\begin{pmatrix} a_{1i} \\ a_{2i} \\ \cdots \\ a_{mi} \end{pmatrix} = b_{1i}\begin{pmatrix} k_{11} \\ k_{21} \\ \cdots \\ k_{m1} \end{pmatrix} + b_{2i}\begin{pmatrix} k_{12} \\ k_{22} \\ \cdots \\ k_{m2} \end{pmatrix} + \cdots + b_{ri}\begin{pmatrix} k_{1r} \\ k_{2r} \\ \cdots \\ k_{mr} \end{pmatrix}$$

In other words, each of the columns of A is a linear combination of the r vectors

$$\begin{pmatrix} k_{11} \\ k_{21} \\ \cdots \\ k_{m1} \end{pmatrix}, \begin{pmatrix} k_{12} \\ k_{22} \\ \cdots \\ k_{m1} \end{pmatrix}, \ldots, \begin{pmatrix} k_{1r} \\ k_{2r} \\ \cdots \\ k_{mr} \end{pmatrix}$$

Thus the column space of the matrix A has dimension at most r, i.e., column rank $\le r$. Hence, column rank $\le$ row rank.

Similarly (or considering the transpose matrix A^T) we obtain row rank $\le$ column rank. Thus the row rank and column rank are equal.

5.54. Suppose R is a row vector and A and B are matrices such that RB and AB are defined. Prove:

(a) RB is a linear combination of the rows of B.

(b) Row space of AB is contained in the row space of B.

(c) Column space of AB is contained in the column space of A.

(d) rank $AB \leq$ rank B and rank $AB \leq$ rank A.

(a) Suppose $R = (a_1, a_2, \ldots, a_m)$ and $B = (b_{ij})$. Let $B_1, \ldots, B_m$ denote the rows of B and $B^1, \ldots, B^n$ its columns. Then

$$\begin{aligned} RB &= (R \cdot B^1, R \cdot B^2, \ldots, R \cdot B^n) \\ &= (a_1 b_{11} + a_2 b_{21} + \cdots + a_m b_{m1}, a_1 b_{12} + a_2 b_{22} + \cdots + a_m b_{m2}, \ldots, a_1 b_{1n} + a_2 b_{2n} + \cdots + a_m b_{mn}) \\ &= a_1(b_{11}, b_{12}, \ldots, b_{1n}) + a_2(b_{21}, b_{22}, \ldots, b_{2n}) + \cdots + a_m(b_{m1}, b_{m2}, \ldots, b_{mn}) \\ &= a_1 B_1 + a_2 B_2 + \cdots + a_m B_m \end{aligned}$$

Thus RB is a linear combination of the rows of B, as claimed.

(b) The rows of AB are $R_i B$ where R_i is the ith row of A. Thus, by part (a), each row of AB is in the row space of B. Thus rowsp $AB \subseteq$ rowsp B, as claimed.

(c) Using part (b), we have:

$$\text{colsp } AB = \text{rowsp } (AB)^T = \text{rowsp } B^T A^T \subseteq \text{rowsp } A^T = \text{colsp A}$$

(d) The row space of AB is contained in the row space of B; hence rank $AB \leq$ rank B. Furthermore, the column space of AB is contained in the column space of A; hence rank $AB \leq$ rank A.

5.55. Let A be an n-square matrix. Show that A is invertible if and only if rank $A = n$.

Note that the rows of the n-square identity matrix I_n are linearly independent since I_n is in echelon form; hence rank $I_n = n$. Now if A is invertible then A is row equivalent to I_n; hence rank $A = n$. But if A is not invertible then A is row equivalent to a matrix with a zero row; hence rank $A < n$. That is, A is invertible if and only if rank $A = n$.

APPLICATIONS TO LINEAR EQUATIONS

5.56. Find the dimension and a basis of the solution space W of the system

$$\begin{aligned} x + 2y + z - 3t &= 0 \\ 2x + 4y + 4z - t &= 0. \\ 3x + 6y + 7z + t &= 0 \end{aligned}$$

Reduce the system to echelon form:

$$\begin{aligned} x + 2y + z - 3t &= 0 \\ 2z + 5t &= 0 \\ 4z + 10t &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y + z - 3t &= 0 \\ 2z + 5t &= 0 \end{aligned}$$

The free variables are y and t, and dim $W = 2$. Set:

(i) $y = 1, z = 0$ to obtain the solution $u_1 = (-2, 1, 0, 0)$

(ii) $y = 0, t = 2$ to obtain the solution $u_2 = (11, 0, -5, 2)$

Then $\{u_1, u_2\}$ is a basis of W. [The choice $y = 0, t = 1$ in (ii), would introduce fractions in the solution.]

5.57. Find a homogeneous system whose solution set W is spanned by

$$\{(1, -2, 0, 3), (1, -1, -1, 4), (1, 0, -2, 5)\}$$

Let $v = (x, y, z, t)$. Form the matrix M whose first rows are the given vectors and whose last row is v; and then row reduce to echelon form:

$$M = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 1 & -1 & -1 & 4 \\ 1 & 0 & -2 & 5 \\ x & y & z & t \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 2x + y & z & -3x + t \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2x + y + z & -5x - y + t \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The original first three rows show that W has dimension 2. Thus $v \in W$ if and only if the additional row does not increase the dimension of the row space. Hence we set the last two entries in the third row on the right equal to 0 to obtain the required homogeneous system

$$\begin{aligned} 2x + y + z \quad &= 0 \\ 5x + y \quad - t &= 0 \end{aligned}$$

5.58. Let $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ be the free variables of a homogeneous system of linear equations with n unknowns. Let v_j be the solution for which $x_{i_j} = 1$, and all other free variables $= 0$. Show that the solutions $v_1, v_2, \ldots, v_k$ are linearly independent.

Let A be the matrix whose rows are the v_i, respectively. We interchange column 1 and column i_1, then column 2 and column $i_2, \ldots$, and then column k and column i_k; and obtain the $k \times n$ matrix

$$B = (I, C) = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & c_{1,k+1} & \cdots & c_{1n} \\ 0 & 1 & 0 & \ldots & 0 & 0 & c_{2,k+1} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & c_{k,k+1} & \cdots & c_{kn} \end{pmatrix}$$

The above matrix B is in echelon form and so its rows are independent; hence rank $B = k$. Since A and B are column equivalent, they have the same rank, i.e., rank $A = k$. But A has k rows; hence these rows, i.e., the v_i, are linearly independent, as claimed.

5.59. Prove Theorem 5.20.

Suppose $u_1, u_2, \ldots, u_r$ form a basis for the column space of A (There are r such vectors since rank $A = r$.) By Theorem 5.19, each system $AX = u_i$ has a solution, say v_i. Hence

$$Av_1 = u_1, \; Av_2 = u_2, \ldots, Av_r = u_r \tag{1}$$

Suppose dim $W = s$ and $w_1, w_2, \ldots, w_s$ form a basis of W. Let

$$B = \{v_1, v_2, \ldots, v_r, w_1, w_2, \ldots, w_s\}$$

We claim that B is a basis of K^n. Thus we need to prove that B spans K^n and that B is linearly independent.

(*a*) *Proof that B spans K^n.* Suppose $v \in K^n$ and $Av = u$. Then $u = Av$ belongs to the column space of A and hence Av is a linear combination of the u_i. Say

$$Av = k_1 u_1 + k_2 u_2 + \cdots + k_r u_r \tag{2}$$

Let $v' = v - k_1 v_1 - k_2 v_2 - \cdots - k_r v_r$. Then, using (*1*) and (*2*),

$$\begin{aligned} A(v') &= A(v - k_1 v_1 - k_2 v_2 - \cdots - k_r v_r) \\ &= Av - k_1 Av_1 - k_2 Av_2 - \cdots - k_r Av_r \\ &= Av - k_1 u_1 - k_2 u_2 - \cdots - k_r u_r = Av - Av = 0 \end{aligned}$$

Thus v' belongs to the solution W and hence v' is a linear combination of the w_i. Say $v' = c_1 w_1 + c_2 w_2 + \cdots + c_s w_s$. Then

$$v = v' + \sum_{i=1}^{r} k_i v_i = \sum_{i=1}^{r} k_i v_i + \sum_{j=1}^{s} c_j w_j$$

Thus v is a linear combination of the elements in B, and hence B spans K^n.

(*b*) *Proof that B is linearly independent.* Suppose

$$a_1 v_1 + a_2 v_2 + \cdots + a_r v_r + b_1 w_1 + b_2 w_2 + \cdots + b_s w_s = 0 \tag{3}$$

Since $w_j \in W$, each $Aw_j = 0$. Using this fact and (*1*) and (*3*), we obtain

$$\begin{aligned} 0 = A(0) = A\left(\sum_{i=1}^{r} a_i v_i + \sum_{j=1}^{s} b_j w_j\right) &= \sum_{i=1}^{r} a_i Av_i + \sum_{j=1}^{s} b_j Aw_j \\ &= \sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j 0 = a_1 u_1 + a_2 u_2 + \cdots + a_r u_r \end{aligned}$$

Since $u_1, \ldots, u_r$ are linearly independent, each $a_i = 0$. Substituting this in (3) yields

$$b_1 w_1 + b_2 w_2 + \cdots + b_s w_s = 0$$

However, $w_1, \ldots, w_s$ are linearly independent. Thus each $b_j = 0$. Therefore, B is linearly independent.

Accordingly B is a basis of K^n. Since B has $r + s$ elements, we have $r + s = n$. Consequently, $\dim W = s = n - r$, as claimed.

SUMS, DIRECT SUMS, INTERSECTIONS

5.60. Let U and W be subspaces of a vector space V. Show that: (a) U and W are each contained in V; (b) $U + W$ is the smallest subspace of V containing U and W, that is, $U + W = \text{span}(U, W)$, the linear span of U and W; (c) $W + W = W$.

(a) Let $u \in U$. By hypothesis W is a subspace of V and so $0 \in W$. Hence $u = u + 0 \in U + W$. Accordingly, U is contained in $U + W$. Similarly, W is contained in $U + W$.

(b) Since $U + W$ is a subspace of V containing both U and W, it must also contain the linear span of U and W, that is, $\text{span}(U, W) \subseteq U + W$.

On the other hand, if $v \in U + W$ then $v = u + w = 1u + 1w$ where $u \in U$ and $w \in W$; hence v is a linear combination of elements in $U \cup W$ and so belongs to span (U, W). Consequently $U + W \subseteq \text{span}(U, W)$.

(c) Since W is a subspace of V, we have that W is closed under vector addition; hence $W + W \subseteq W$. By part (a), $W \subseteq W + W$. Hence $W + W = W$.

5.61. Give an example of a subset S of $\mathbf{R}^2$ such that: (a) $S + S \subset S$ (properly contained); (b) $S \subset S + S$ (properly contained); (c) $S + S = S$ but S is not a subspace of $\mathbf{R}^2$.

(a) Let $S = \{(0, 5), (0, 6), (0, 7), \ldots\}$. Then $S + S \subset S$.

(b) Let $S = \{(0, 0), (0, 1)\}$. Then $S \subset S + S$.

(c) Let $S = \{(0, 0), (0, 1), (0, 2), (0, 3), \ldots\}$. Then $S + S = S$.

5.62. Suppose U and W are distinct 4-dimensional subspaces of a vector space V where $\dim V = 6$. Find the possible dimensions of $U \cap W$.

Since U and W are distinct, $U + W$ properly contains U and W; consequently $\dim(U + W) > 4$. But $\dim(U + W)$ cannot be greater than 6, since $\dim V = 6$. Hence we have two possibilities: (i) $\dim(U + W) = 5$, or (ii) $\dim(U + W) = 6$. By Theorem 5.22,

$$\dim(U \cap W) = \dim U + \dim W - \dim(U + W) = 8 - \dim(U + W)$$

Thus (i) $\dim(U \cap W) = 3$, or (ii) $\dim(U \cap W) = 2$.

5.63. Consider the following subspaces of $\mathbf{R}^4$:

$$U = \text{span}\{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)\}$$
$$W = \text{span}\{(1, 2, 2, -2), (2, 3, 2, -3), (1, 3, 4, -3)\}$$

Find (a) $\dim(U + W)$ and (b) $\dim(U \cap W)$.

(a) $U + W$ is the space spanned by all six vectors. Hence form the matrix whose rows are the given six vectors, and then row reduce to echelon form:

$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the echelon matrix has three nonzero rows, $\dim(U + W) = 3$.

(*b*) First find $\dim U$ and $\dim W$. Form the two matrices whose rows are the generators of U and W, respectively, and then row reduce each to echelon form:

$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each of the echelon matrices has two nonzero rows, $\dim U = 2$ and $\dim W = 2$. Using Theorem 5.22, i.e., $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$, we have

$$3 = 2 + 2 - \dim(U \cap W) \qquad \text{or} \qquad \dim(U \cap W) = 1$$

5.64. Let U and W be the following subspaces of $\mathbf{R}^4$:

$$U = \{(a, b, c, d): b + c + d = 0\}, \qquad W = \{(a, b, c, d): a + b = 0, c = 2d\}$$

Find a basis and the dimension of: (*a*) U, (*b*) W, (*c*) $U \cap W$, (*d*) $U + W$.

(*a*) We seek a basis of the set of solutions (a, b, c, d) of the equation

$$b + c + d = 0 \qquad \text{or} \qquad 0 \cdot a + b + c + d = 0$$

The free variables are a, c, and d. Set:

(1) $a = 1, c = 0, d = 0$ to obtain the solution $v_1 = (1, 0, 0, 0)$

(2) $a = 0, c = 1, d = 0$ to obtain the solution $v_2 = (0, -1, 1, 0)$

(3) $a = 0, c = 0, d = 1$ to obtain the solution $v_3 = (0, -1, 0, 1)$

The set $\{v_1, v_2, v_3\}$ is a basis of U, and $\dim U = 3$.

(*b*) We seek a basis of the set of solutions (a, b, c, d) of the system

$$\begin{aligned} a + b &= 0 \\ c &= 2d \end{aligned} \qquad \text{or} \qquad \begin{aligned} a + b &= 0 \\ c - 2d &= 0 \end{aligned}$$

The free variables are b and d. Set

(1) $b = 1, d = 0$ to obtain the solution $v_1 = (-1, 1, 0, 0)$

(2) $b = 0, d = 1$ to obtain the solution $v_2 = (0, 0, 2, 1)$

The set $\{v_1, v_2\}$ is a basis of W, and $\dim W = 2$.

(*c*) $U \cap W$ consists of those vectors (a, b, c, d) which satisfy the conditions defining U and the conditions defining W, i.e., the three equations

$$\begin{aligned} b + c + d &= 0 \\ a + b \qquad &= 0 \\ c \qquad &= 2d \end{aligned} \qquad \text{or} \qquad \begin{aligned} a + b \qquad &= 0 \\ b + c + d &= 0 \\ c - 2d &= 0 \end{aligned}$$

The free variable is d. Set $d = 1$ to obtain the solution $v = (3, -3, 2, 1)$. Thus $\{v\}$ is a basis of $U \cap W$, and $\dim(U \cap W) = 1$.

(*d*) By Theorem 5.22,

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 3 + 2 - 1 = 4$$

Thus $U + W = \mathbf{R}^4$. Accordingly any basis of $\mathbf{R}^4$, say the usual basis, will be a basis of $U + W$.

5.65. Consider the following subspaces of $\mathbf{R}^5$:

$$U = \text{span}\{(1, 3, -2, 2, 3), (1, 4, -3, 4, 2), (2, 3, -1, -2, 9)\}$$
$$W = \text{span}\{(1, 3, 0, 2, 1), (1, 5, -6, 6, 3), (2, 5, 3, 2, 1)\}$$

Find a basis and dimension of (*a*) $U + W$, (*b*) $U \cap W$.

(*a*) $U + W$ is the space spanned by all six vectors. Hence form the matrix whose rows are the given six vectors, and then row reduce to echelon form:

$$\begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 9 \\ 1 & 3 & 0 & 2 & 1 \\ 1 & 5 & -6 & 6 & 3 \\ 2 & 5 & 3 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -3 & 3 & -6 & 3 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 2 & -4 & 4 & 0 \\ 0 & -1 & 7 & -2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 6 & 0 & -6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The set of nonzero rows of the echelon matrix,

$$\{(1, 3, -2, 2, 3), (0, 1, -1, 2, -1), (0, 0, 2, 0, -2)\}$$

is a basis of $U + W$; thus $\dim(U + W) = 3$.

(*b*) First find homogeneous systems whose solution sets are U and W, respectively. Form the matrix whose first three rows span U and whose last row is (x, y, z, s, t) and then row reduce to an echelon form:

$$\begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 9 \\ x & y & z & s & t \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -3 & 3 & -6 & 3 \\ 0 & -3x + y & 2x + z & -2x + s & -3x + t \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & -x + y + z & 4x - 2y + s & -6x + y + t \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Set the entries of the third row equal to 0 to obtain the homogeneous system whose solution space is U:

$$-x + y + z = 0 \qquad 4x - 2y + s = 0 \qquad -6x + y + t = 0$$

Now form the matrix whose first rows span W and whose last row is (x, y, z, s, t) and then row reduce to an echelon form:

$$\begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 1 & 5 & -6 & 6 & 3 \\ 2 & 5 & 3 & 2 & 1 \\ x & y & z & s & t \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 2 & -6 & 4 & 2 \\ 0 & -1 & 3 & -2 & -1 \\ 0 & -3x+y & z & -2x+s & -x+t \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & -9x+3y+z & 4x-2y+s & 2x-y+t \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Set the entries of the third row equal to 0 to obtain the homogeneous system whose solution space is W:

$$-9x + 3y + z = 0 \qquad 4x - 2y + s = 0 \qquad 2x - y + t = 0$$

Combine both of the above systems to obtain a homogeneous system whose solution space is $U \cap W$, and then solve:

$$\begin{cases} -x + y + z = 0 \\ 4x - 2y + s = 0 \\ -6x + y + t = 0 \\ -9x + 3y + z = 0 \\ 4x - 2y + s = 0 \\ 2x - y + t = 0 \end{cases} \quad \text{or} \quad \begin{cases} -x + y + z = 0 \\ 2y + 4z + s = 0 \\ -5y - 6z + t = 0 \\ -6y - 8z = 0 \\ 2y + 4z + s = 0 \\ y + 2z + t = 0 \end{cases}$$

$$\text{or} \quad \begin{cases} -x + y + z = 0 \\ 2y + 4z + s = 0 \\ 8z + 5s + 2t = 0 \\ 4z + 3s = 0 \\ s - 2t = 0 \end{cases} \quad \text{or} \quad \begin{cases} -x + y + z = 0 \\ 2y + 4z + s = 0 \\ 8z + 5s + 2t = 0 \\ s - 2t = 0 \end{cases}$$

There is one free variable, which is t; hence $\dim (U \cap W) = 1$. Setting $t = 2$, we obtain the solution $x = 1, y = 4, z = -3, s = 4, t = 2$. Thus $\{(1, 4, -3, 4, 2)\}$ is a basis of $U \cap W$.

5.66. Let U and W be the subspaces of $\mathbf{R}^3$ defined by

$$U = \{(a, b, c): a = b = c\} \qquad \text{and} \qquad W = \{(0, b, c)\}$$

(Note that W is the yz plane.) Show that $\mathbf{R}^3 = U \oplus W$.

Note first that $U \cap W = \{0\}$, for $v = (a, b, c) \in U \cap W$ implies that

$$a = b = c \qquad \text{and} \qquad a = 0 \qquad \text{which implies} \qquad a = 0, b = 0, c = 0$$

We also claim that $\mathbf{R}^3 = U + W$. For if $v = (a, b, c) \in \mathbf{R}^3$, then

$$v = (a, a, a) + (0, b - a, c - a) \qquad \text{where} \qquad (a, a, a) \in U \qquad \text{and} \qquad (0, b - a, c - a) \in W$$

Both conditions, $U \cap W = \{0\}$ and $\mathbf{R}^3 = U + W$, imply $\mathbf{R}^3 = U \oplus W$.

5.67. Let V be the vector space of n-square matrices over a field K.

(a) Show that $V = U \oplus W$ where U and W are the subspaces of symmetric and antisymmetric matrices, respectively. (Recall M is symmetric iff $M = M^T$, and M is antisymmetric iff $M^T = -M$.)

(b) Show that $V \neq U \oplus W$ where U and W are the subspaces of upper and lower triangular matrices, respectively. (Note $V = U + W$.)

(a) We first show that $V = U + W$. Let A be any arbitrary n-square matrix. Note that

$$A = \tfrac{1}{2}(A + A^T) + \tfrac{1}{2}(A - A^T)$$

We claim that $\tfrac{1}{2}(A + A^T) \in U$ and that $\tfrac{1}{2}(A - A^T) \in W$. For

$$(\tfrac{1}{2}(A + A^T))^T = \tfrac{1}{2}(A + A^T)^T = \tfrac{1}{2}(A^T + A^{TT}) = \tfrac{1}{2}(A + A^T)$$

that is, $\tfrac{1}{2}(A + A^T)$ is symmetric. Furthermore,

$$(\tfrac{1}{2}(A - A^T))^T = \tfrac{1}{2}(A - A^T)^T = \tfrac{1}{2}(A^T - A) = -\tfrac{1}{2}(A - A^T)$$

that is, $\tfrac{1}{2}(A - A^T)$ is antisymmetric.

We next show that $U \cap W = \{0\}$. Suppose $M \in U \cap W$. Then $M = M^T$ and $M^T = -M$, which implies $M = -M$ and hence $M = 0$. Thus $U \cap W = \{0\}$. Accordingly, $V = U \oplus W$.

(b) $U \cap W \neq \{0\}$ since $U \cap W$ consists of all the diagonal matrices. Thus the sum cannot be direct.

5.68. Suppose U and W are subspaces of a vector space V, and suppose that $S = \{u_i\}$ spans U and $S' = \{w_j\}$ spans W. Show that $S \cup S'$ spans $U + W$. (Accordingly, by induction, if S_i spans W_i for $i = 1, 2, \ldots, n$, then $S_1 \cup \cdots \cup S_n$ spans $W_1 + \cdots + W_n$.)

Let $v \in U + W$. Then $v = u + w$ where $u \in U$ and $w \in W$. Since S spans U, u is a linear combination of the u_i's, and since S' spans W, w is a linear combination of the w_j's:

$$u = a_1 u_{i_1} + a_2 u_{i_2} + \cdots + a_n u_{i_n} \qquad a_j \in K$$
$$w = b_1 w_{j_1} + b_2 w_{j_2} + \cdots + b_m w_{j_m} \qquad b_j \in K$$

Thus

$$v = u + w = a_1 u_{i_1} + a_2 u_{i_2} + \cdots + a_n u_{i_n} + b_1 w_{j_1} + b_2 w_{j_2} + \cdots + b_m w_{j_m}$$

Accordingly, $S \cup S' = \{u_i, v_j\}$ spans $U + W$.

5.69. Prove Theorem 5.22.

Observe that $U \cap W$ is a subspace of both U and W. Suppose $\dim U = m$, $\dim W = n$, and $\dim (U \cap W) = r$. Suppose $\{v_1, \ldots, v_r\}$ is a basis of $U \cap W$. By Theorem 5.16, we can extend $\{v_i\}$ to a basis of U and to a basis of W; say,

$$\{v_1, \ldots, v_r, u_1, \ldots, u_{m-r}\} \quad \text{and} \quad \{v_1, \ldots, v_r, w_1, \ldots, w_{n-r}\}$$

are bases of U and W, respectively. Let

$$B = \{v_1, \ldots, v_r, u_1, \ldots, u_{m-r}, w_1, \ldots, w_{n-r}\}$$

Note that B has exactly $m + n - r$ elements. Thus the theorem is proved if we can show that B is a basis of $U + W$. Since $\{v_i, u_j\}$ spans U and $\{v_i, w_k\}$ spans W, the union $B = \{v_i, u_j, w_k\}$ spans $U + W$. Thus it suffices to show that B is independent.

Suppose

$$a_1 v_1 + \cdots + a_r v_r + b_1 u_1 + \cdots + b_{m-r} u_{m-r} + c_1 w_1 + \cdots + c_{n-r} w_{n-r} = 0 \tag{1}$$

where a_i, b_j, c_k are scalars. Let

$$v = a_1 v_1 + \cdots + a_r v_r + b_1 u_1 + \cdots + b_{m-r} u_{m-r} \tag{2}$$

By (1), we also have that

$$v = -c_1 w_1 - \cdots - c_{n-r} w_{n-r} \tag{3}$$

Since $\{v_i, u_j\} \subseteq U$, $v \in U$ by (2); and since $\{w_k\} \subseteq W$, $v \in W$ by (3). Accordingly, $v \in U \cap W$. Now $\{v_i\}$ is a basis of $U \cap W$ and so there exist scalars $d_1, \ldots, d_r$ for which $v = d_1 v_1 + \cdots + d_r v_r$. Thus by (3) we have

$$d_1 v_1 + \cdots + d_r v_r + c_1 w_1 + \cdots + c_{n-r} w_{n-r} = 0$$

But $\{v_i, w_k\}$ is a basis of W and so is independent. Hence the above equation forces $c_1 = 0, \ldots, c_{n-r} = 0$. Substituting this into *(1)*, we obtain

$$a_1v_1 + \cdots + a_rv_r + b_1u_1 + \cdots + b_{m-r}u_{m-r} = 0$$

But $\{v_i, u_j\}$ is a basis of U and so is independent. Hence the above equation forces $a_1 = 0, \ldots, a_r = 0$, $b_1 = 0, \ldots, b_{m-r} = 0$.

Since the equation *(1)* implies that the a_i, b_j and c_k are all 0, $B = \{v_i, u_j, w_k\}$ is independent and the theorem is proved.

5.70. Prove Theorem 5.23.

Suppose $V = U \oplus W$. Then any $v \in V$ can be uniquely written in the form $v = u + w$, where $u \in U$ and $w \in W$. Thus, in particular, $V = U + W$. Now suppose $v \in U \cap W$. Then:

(1) $v = v + 0$ where $v \in U, 0 \in W$; and (2) $v = 0 + v$ where $0 \in U, v \in W$

Since such a sum for v must be unique, $v = 0$. Accordingly, $U \cap W = \{0\}$.

On the other hand, suppose $V = U + W$ and $U \cap W = \{0\}$. Let $v \in V$. Since $V = U + W$, there exist $u \in U$ and $w \in W$ such that $v = u + w$. We need to show that such a sum is unique. Suppose also that $v = u' + w'$ where $u' \in U$ and $w' \in W$. Then

$$u + w = u' + w' \quad \text{and so} \quad u - u' = w' - w$$

But $u - u' \in U$ and $w' - w \in W$; hence, by $U \cap W = \{0\}$,

$$u - u' = 0, w' - w = 0 \quad \text{and so} \quad u = u', w = w'$$

Thus such a sum for $v \in V$ is unique and $V = U \oplus W$.

5.71. Prove Theorem 5.24 (for two factors): Suppose $V = U \oplus W$. Suppose $S = \{u_1, \ldots, u_m\}$ and $S' = \{w_1, \ldots, w_n\}$ are linearly independent subsets of U and W, respectively. Then: (*a*) $S \cup S'$ is linearly independent in V; (*b*) if S is a basis of U and S' is a basis of W, then $S \cup S'$ is a basis of V; and (*c*) $\dim V = \dim U + \dim W$.

(*a*) Suppose $a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n = 0$, where a_i, b_j are scalars. Then

$$0 = (a_1u_1 + \cdots + a_mu_m) + (b_1w_1 + \cdots + b_nw_n) = 0 + 0$$

where $0, a_1u_1 + \cdots + a_mu_m \in U$ and $0, b_1w_1 + \cdots + b_nw_n \in W$. Since such a sum for 0 is unique, this leads to

$$a_1u_1 + \cdots + a_mu_m = 0 \qquad b_1w_1 + \cdots + b_nw_n = 0$$

Since S is linearly independent, each $a_i = 0$, and since S' is linearly independent, each $b_j = 0$. Thus $S \cup S'$ is linearly independent.

(*b*) By part (*a*), $S \cup S'$ is linearly independent, and, by Problem 5.68, $S \cup S'$ span V. Thus $S \cup S'$ is a basis of V.

(*c*) Follows directly from part (*b*).

COORDINATE VECTORS

5.72. Let S be the basis of $\mathbf{R}^2$ consisting of $u_1 = (2, 1)$ and $u_2 = (1, -1)$. Find the coordinate vector $[v]$ of v relative to S where $v = (a, b)$.

Set $v = xu_1 + yu_2$ to obtain $(a, b) = (2x + y, x - y)$. Solve $2x + y = a$ and $x - y = b$ to obtain $x = (a + b)/3$, $y = (a - 2b)/3$. Thus $[v] = [(a + b)/3, (a - 2b)/3]$.

5.73. Consider the vector space $\mathbf{P}_3(t)$ of real polynomials in t of degree ≤ 3.

(*a*) Show that $S = \{1, 1 - t, (1 - t)^2, (1 - t)^3\}$ is a basis of $\mathbf{P}_3(t)$.

(*b*) Find the coordinate vector $[u]$ of $u = 2 - 3t + t^2 + 2t^3$ relative to S.

(a) The degree of $(1-t)^k$ is k; hence no polynomial in S is a linear combination of preceding polynomials. Thus the polynomials are linearly independent and, since $\dim \mathbf{P}_3(t) = 4$, they form a basis of $\mathbf{P}_3(t)$.

(b) Set u as a linear combination of the basis vectors using unknowns x, y, z, s:

$$\begin{aligned} u = 2 - 3t + t^2 + 2t^3 &= x(1) + y(1-t) + z(1-t)^2 + s(1-t)^3 \\ &= x(1) + y(1-t) + z(1-2t+t^2) + s(1-3t+3t^2-t^3) \\ &= x + y - yt + z - 2zt + zt^2 + s - 3st + 3st^2 - st^3 \\ &= (x+y+z+s) + (-y-2z-3s)t + (z+3s)t^2 + (-s)t^3 \end{aligned}$$

Then set the coefficients of the same powers of t equal to each other:

$$x + y + z + s = 2 \qquad -y - 2z - 3s = -3 \qquad z + 3s = 1 \qquad -s = 2$$

Solving, $x = 2$, $y = -5$, $z = 7$, $s = -2$. Thus $[u] = [2, -5, 7, -2]$.

5.74. Consider the matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix}$ in the vector space V of 2×2 real matrices. Find the coordinate vector $[A]$ of the matrix A relative to $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, the usual basis of V.

We have

$$\begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

Thus $x = 2$, $y = 3$, $z = 4$, $t = -7$. Hence $[A] = [2, 3, 4, -7]$, whose components are the elements of A written row by row.

Remark: The above result is true in general, that is, if A is any $m \times n$ matrix in the vector space V of $m \times n$ matrices over a field K, then the coordinate vector $[A]$ of A relative to the usual basis of V is the mn coordinate vector in K^{mn} whose components are the elements of A written row by row.

CHANGE OF BASIS

This section will represent a vector of $v \in V$ relative to a basis S of V by its coordinate column vector,

$$[v]_S = \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{pmatrix} = [a_1, a_2, \ldots, a_n]^T$$

(which is an $n \times 1$ matrix).

5.75. Consider the following bases of $\mathbf{R}^2$:

$$S_1 = \{u_1 = (1, -2), u_2 = (3, -4)\} \qquad \text{and} \qquad S_2 = \{v_1 = (1, 3), v_2 = (3, 8)\}$$

(a) Find the coordinates of an arbitrary vector $v = (a, b)$ in $\mathbf{R}^2$ relative to the basis $S_1 = \{u_1, u_2\}$.

(b) Find the change-of-basis matrix P from S_1 to S_2.

(c) Find the coordinates of an arbitrary vector $v = (a, b)$ in $\mathbf{R}^2$ relative to the basis $S_2 = \{v_1, v_2\}$.

(d) Find the change-of-basis matrix Q from S_2 back to S_1.

(*e*) Verify that $Q = P^{-1}$.

(*f*) Show that $P[v]_{S_2} = [v]_{S_1}$ for any vector $v = (a, b)$. (See Theorem 5.27.)

(*g*) Show that $P^{-1}[v]_{S_1} = [v]_{S_2}$ for any vector $v = (a, b)$. (See Theorem 5.27.)

(*a*) Let $v = xu_1 + yu_2$ for unknowns x and y:

$$\begin{pmatrix} a \\ b \end{pmatrix} = x\begin{pmatrix} 1 \\ -2 \end{pmatrix} + y\begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad \text{or} \quad \begin{aligned} x + 3y &= a \\ -2x - 4y &= b \end{aligned} \quad \text{or} \quad \begin{aligned} x + 3y &= a \\ 2y &= 2a + b \end{aligned}$$

Solve for x and y in terms of a and b to get $x = -2a - \frac{3}{2}b$, $y = a + \frac{1}{2}b$. Thus

$$(a, b) = (-2a - \tfrac{3}{2}b)u_1 + (a + \tfrac{1}{2}b)u_2 \quad \text{or} \quad [(a, b)]_{S_1} = [-2a - \tfrac{3}{2}b, a + \tfrac{1}{2}b]^T$$

(*b*) Use (*a*) to write each of the basis vectors v_1 and v_2 of S_2 as a linear combination of the basis vectors u_1 and u_2 of S_1:

$$v_1 = (1, 3) = (-2 - \tfrac{9}{2})u_1 + (1 + \tfrac{3}{2})u_2 = (-\tfrac{13}{2})u_1 + (\tfrac{5}{2})u_2$$

$$v_2 = (3, 8) = (-6 - 12)u_1 + (3 + 4)u_2 = -18u_1 + 7u_2$$

Then P is the matrix whose columns are the coordinates of v_1 and v_2 relative to the basis S_1, that is,

$$P = \begin{pmatrix} -\frac{13}{2} & -18 \\ \frac{5}{2} & 7 \end{pmatrix}$$

(*c*) Let $v = xv_1 + yv_2$ for unknown scalars x and y:

$$\begin{pmatrix} a \\ b \end{pmatrix} = x\begin{pmatrix} 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} 3 \\ 8 \end{pmatrix} \quad \text{or} \quad \begin{aligned} x + 3y &= a \\ 3x + 8y &= b \end{aligned} \quad \text{or} \quad \begin{aligned} x + 3y &= a \\ -y &= b - 3a \end{aligned}$$

Solve for x and y to get $x = -8a + 3b$, $y = 3a - b$. Thus

$$(a, b) = (-8a + 3b)v_1 + (3a - b)v_2 \quad \text{or} \quad [(a, b)]S_2 = [-8a + 3b, 3a - b]^T$$

(*d*) Use (*c*) to express each of the basis vectors u_1 and u_2 of S_1 as a linear combination of the basis vectors v_1 and v_2 of S_2:

$$u_1 = (1, -2) = (-8 - 6)v_1 + (3 + 2)v_2 = -14v_1 + 5v_2$$

$$u_2 = (3, -4) = (-24 - 12)v_1 + (9 + 4)v_2 = -36v_1 + 13v_2$$

Write the coordinates of u_1 and u_2 relative to S_2 as columns to obtain $Q = \begin{pmatrix} -14 & -36 \\ 5 & 13 \end{pmatrix}$.

(*e*)
$$QP = \begin{pmatrix} -14 & -36 \\ 5 & 13 \end{pmatrix}\begin{pmatrix} -\frac{13}{2} & -18 \\ \frac{5}{2} & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

(*f*) Use (*a*), (*b*), and (*c*) to obtain

$$P[v]_{S_2} = \begin{pmatrix} -\frac{13}{2} & -18 \\ \frac{5}{2} & 7 \end{pmatrix}\begin{pmatrix} -8a + 3b \\ 3a - b \end{pmatrix} = \begin{pmatrix} -2a - \frac{3}{2}b \\ a + \frac{1}{2}b \end{pmatrix} = [v]_{S_1}$$

(*g*) Use (*a*), (*c*), and (*d*) to obtain

$$P^{-1}[v]_{S_1} = Q[v]_{S_1} = \begin{pmatrix} -14 & -36 \\ 5 & 13 \end{pmatrix}\begin{pmatrix} -2a - \frac{3}{2}b \\ a + \frac{1}{2}b \end{pmatrix} = \begin{pmatrix} -8a + 3b \\ 3a - b \end{pmatrix} = [v]_{S_2}$$

5.76. Suppose the following vectors form a basis S of K^n:

$$v_1 = (a_1, a_2, \ldots, a_n), \qquad v_2 = (b_1, b_2, \ldots, b_n), \qquad \ldots, \qquad v_n = (c_1, c_2, \ldots, c_n)$$

Show that the change-of-basis matrix from the usual basis $E = \{e_i\}$ of K^n to the basis S is the matrix P whose columns are the vectors $v_1, v_2, \ldots, v_n$, respectively.

Since $e_1, e_2, \ldots, e_n$ form the usual basis E of K^n, we have

$$\begin{aligned} v_1 &= (a_1, a_2, \ldots, a_n) = a_1e_1 + a_2e_2 + \cdots + a_ne_n \\ v_2 &= (b_1, b_2, \ldots, b_n) = b_1e_1 + b_2e_2 + \cdots + b_ne_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ v_n &= (c_1, c_2, \ldots, c_n) = c_1e_1 + c_2e_2 + \cdots + c_ne_n \end{aligned}$$

Writing the coordinates as columns we get

$$P = \begin{pmatrix} a_1 & b_1 & \ldots & c_1 \\ a_2 & b_2 & \ldots & c_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & \ldots & c_n \end{pmatrix}$$

as claimed.

5.77. Consider the basis $S = \{u_1 = (1, 2, 0), u_2 = (1, 3, 2), u_3 = (0, 1, 3)\}$ of $\mathbf{R}^3$. Find:

(*a*) The change-of-basis matrix P from the usual basis $E = \{e_1, e_2, e_3\}$ of $\mathbf{R}^3$ to the basis S,

(*b*) The change-of-basis matrix Q from the above basis S back to the usual basis E of $\mathbf{R}^3$.

(*a*) Since E is the usual basis, simply write the basis vectors of S as columns:

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{pmatrix}$$

(*b*) **Method 1.** Express each basis vector of E as a linear combination of the basis vectors of S by first finding the coordinates of an arbitrary vector $v = (a, b, c)$ relative to the basis S. We have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad \text{or} \quad \begin{aligned} x + y &= a \\ 2x + 3y + z &= b \\ 2y + 3z &= c \end{aligned}$$

Solve for x, y, z to get $x = 7a - 3b + c$, $y = -6a + 3b - c$, $z = 4a - 2b + c$. Thus

$$v = (a, b, c) = (7a - 3b + c)u_1 + (-6a + 3b - c)u_2 + (4a - 2b + c)u_3$$

or $$[v]_S = [(a, b, c)]_S = [7a - 3b + c,\ -6a + 3b - c,\ 4a - 2b + c]^T$$

Using the above formula for $[v]_S$ and then writing the coordinates of the e_i as columns yields

$$\begin{aligned} e_1 &= (1, 0, 0) = 7u_1 - 6u_2 + 4u_3 \\ e_2 &= (0, 1, 0) = -3u_1 + 3u_2 - 2u_3 \\ e_3 &= (0, 0, 1) = u_1 - u_2 + u_3 \end{aligned} \quad \text{and} \quad Q = \begin{pmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{pmatrix}$$

Method 2. Find P^{-1} by row reducing $M = (P \mid I)$ to the form $(I \mid P^{-1})$:

$$M = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -6 & 3 & -1 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & 1 \\ 0 & 1 & 0 & -6 & 3 & 1 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{array}\right)$$

Thus $Q = P^{-1} = \begin{pmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{pmatrix}$.

5.78. Suppose the x and y axes in the plane $\mathbf{R}^2$ are rotated counterclockwise 45° so that the new x' axis is along the line $y = x$, and the new y' axis is along the line $y = -x$. Find (*a*) the change-of-basis matrix P and (*b*) the new coordinates of the point $A(5, 6)$ under the given rotation.

(*a*) The unit vectors in the direction of the new x' and y' axes are, respectively,

$$u_1 = (\sqrt{2}/2, \sqrt{2}/2) \qquad \text{and} \qquad u_2 = (-\sqrt{2}/2, \sqrt{2}/2)$$

(The unit vectors in the direction of the original x and y axes are, respectively, the usual basis vectors for $\mathbf{R}^2$.) Thus write the coordinates of u_1 and u_2 as columns to obtain

$$P = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

(*b*) Multiply the coordinates of the point by P^{-1}:

$$\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 11\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$$

[Since P is orthogonal, P^{-1} is simply the transpose of P.]

5.79. Consider the bases $S = \{1, i\}$ and $S' = \{1 + i, 1 + 2i\}$ of the complex field $\mathbf{C}$ over the real field $\mathbf{R}$. Find (*a*) the change-of-basis matrix P from the S-basis to the S'-basis, and (*b*) find the change-of-basis matrix Q from the S'-basis back to the S-basis.

(*a*) We have

$$\begin{aligned} 1 + i &= 1(1) + 1(i) \\ 1 + 2i &= 1(1) + 2(i) \end{aligned} \qquad \text{and so} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

(*b*) Use the formula for the inverse of a 2×2 matrix to obtain $Q = P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$.

5.80. Suppose P is the change-of-basis matrix from a basis $\{u_i\}$ to a basis $\{w_i\}$, and suppose Q is the change-of-basis matrix from the basis $\{w_i\}$ back to the basis $\{u_i\}$. Prove that P is invertible and $Q = P^{-1}$.

Suppose, for $i = 1, 2, \ldots, n$,

$$w_i = a_{i1}u_1 + a_{i2}u_2 + \cdots + a_{in}u_n = \sum_{j=1}^{n} a_{ij}u_j \tag{1}$$

and, for $j = 1, 2, \ldots, n$,

$$u_j = b_{j1}w_1 + b_{j2}w_2 + \cdots + b_{jn}w_n = \sum_{k=1}^{n} b_{jk}w_k \tag{2}$$

Let $A = (a_{ij})$ and $B = (b_{jk})$. Then $P = A^T$ and $Q = B^T$. Substituting (*2*) into (*1*) yields

$$w_i = \sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{n} b_{jk}w_k\right) = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)w_k$$

Since the $\{w_i\}$ is a basis $\sum a_{ij}b_{jk} = \delta_{ik}$ where δ_{ik} is the Kronecker delta, that is, $\delta_{ik} = 1$ if $i = k$ but $\delta_{ik} = 0$ if $i \neq k$. Suppose $AB = (c_{ik})$. Then $c_{ik} = \delta_{ik}$. Accordingly, $AB = I$, and so

$$QP = B^TA^T = (AB)^T = I^T = I$$

Thus $Q = P^{-1}$.

5.81. Prove Theorem 5.27.

Suppose $S = \{u_1, \ldots, u_n\}$ and $S' = \{w_1, \ldots, w_n\}$, and suppose, for $i = 1, \ldots, n$,

$$w_i = a_{i1}u_1 + a_{i2}u_2 + \cdots + a_{in}u_n = \sum_{j=1}^{n} a_{ij}u_j$$

Then P is the n-square matrix whose jth row is

$$(a_{1j}, a_{2j}, \ldots, a_{nj}) \tag{1}$$

Also suppose $v = k_1w_1 + k_2w_2 + \cdots + k_nw_n = \sum_{i=1}^{n} k_iw_i$. Then

$$[v]_{S'} = [k_{1j}, k_{2j}, \ldots, k_{nj}]^T \tag{2}$$

Substituting for w_i in the equation for v, we obtain

$$v = \sum_{i=1}^{n} k_iw_i = \sum_{i=1}^{n} k_i\left(\sum_{j=1}^{n} a_{ij}u_j\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ij}k_i\right)u_j$$

$$= \sum_{j=1}^{n}(a_{1j}k_1 + a_{2j}k_2 + \cdots + a_{nj}k_n)u_j$$

Accordingly, $[v]_S$ is the column vector whose jth entry is

$$a_{1j}k_1 + a_{2j}k_2 + \cdots + a_{nj}k_n \tag{3}$$

On the other hand, the jth entry of $P[v]_{S'}$ is obtained by multiplying the jth row of P by $[v]_{S'}$, that is, (1) by (2). However, the product of (1) and (2) is (3); hence $P[v]_{S'}$ and $[v]_S$ have the same entries. Thus $P[v]_{S'} = [v]_{S'}$, as claimed.

Furthermore, multiplying the above by P^{-1} gives $P^{-1}[v]_S = P^{-1}P[v]_{S'} = [v]_{S'}$.

MISCELLANEOUS PROBLEMS

5.82. Consider a finite sequence of vectors $S = \{v_1, v_2, \ldots, v_n\}$. Let T be the sequence of vectors obtained from S by one of the following "elementary operations": (i) interchange two vectors, (ii) multiply a vector by a nonzero scalar, (iii) add a multiple of one vector to another. Show that S and T span the same space W. Also show that T is independent if and only if S is independent.

Observe that, for each operation, the vectors in T are linear combinations of vectors in S. On the other hand, each operation has an inverse of the same type (Prove!); hence the vectors in S are linear combinations of vectors in T. Thus S and T span the same space W. Also, T is independent if and only if $\dim W = n$, and this is true if and only if S is also independent.

5.83. Let $A = (a_{ij})$ and $B = (b_{ij})$ be row equivalent $m \times n$ matrices over a field K, and let $v_1, \ldots, v_n$ be any vectors in a vector space V over K. Let

$$\begin{array}{ll}
u_1 = a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n & w_1 = b_{11}v_1 + b_{12}v_2 + \cdots + b_{1n}v_n \\
u_2 = a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n & w_2 = b_{21}v_1 + b_{22}v_2 + \cdots + b_{2n}v_n \\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\
u_m = a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n & w_m = b_{m1}v_1 + b_{m2}v_2 + \cdots + b_{mn}v_n
\end{array}$$

Show that $\{u_i\}$ and $\{w_i\}$ span the same space.

Applying an "elementary operation" of Problem 5.82 to $\{u_i\}$ is equivalent to applying an elementary row operation to the matrix A. Since A and B are row equivalent, B can be obtained from A by a sequence of elementary row operations; hence $\{w_i\}$ can be obtained from $\{u_i\}$ by the corresponding sequence of operations. Accordingly, $\{u_i\}$ and $\{w_i\}$ span the same space.

5.84. Let $v_1, \ldots, v_n$ belong to a vector space V over a field K. Let

$$\begin{aligned} w_1 &= a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ w_2 &= a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ w_n &= a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nn}v_n \end{aligned}$$

where $a_{ij} \in K$. Let P be the n-square matrix of coefficients, i.e., let $P = (a_{ij})$.

(*a*) Suppose P is invertible. Show that $\{w_i\}$ and $\{v_i\}$ span the same space; hence $\{w_i\}$ is independent if and only if $\{v_i\}$ is independent.

(*b*) Suppose P is not invertible. Show that $\{w_i\}$ is dependent.

(*c*) Suppose $\{w_i\}$ is independent. Show that P is invertible.

(*a*) Since P is invertible, it is row equivalent to the identity matrix I. Hence by the preceding problem $\{w_i\}$ and $\{v_i\}$ span the same space. Thus one is independent if and only if the other is.

(*b*) Since P is not invertible, it is row equivalent to a matrix with a zero row. This means that $\{w_i\}$ spans a space which has a spanning set of less than n elements. Thus $\{w_i\}$ is dependent.

(*c*) This is the contrapositive of the statement of (*b*) and so it follows from (*b*).

5.85. Suppose that $A_1, A_2, \ldots$ are linearly independent sets of vectors, and that $A_1 \subseteq A_2 \subseteq \cdots$. Show that the union $A = A_1 \cup A_2 \cup \cdots$ is also linearly independent.

Suppose A is linearly dependent. Then there exist vectors $v_1, \ldots, v_n \in A$ and scalars $a_1, \ldots, a_n \in K$, not all of them 0, such that

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \qquad (1)$$

Since $A = \cup A_i$ and the $v_i \in A$, there exist sets $A_{i_1}, \ldots, A_{i_n}$ such that

$$v_1 \in A_{i_1}, v_2 \in A_{i_2}, \ldots, v_n \in A_{i_n}$$

Let k be the maximum index of the sets A_{i_j}: $k = \max(i_1, \ldots, i_n)$. It follows then, since $A_1 \subseteq A_2 \subseteq \cdots$, that each A_{i_j} is contained in A_k. Hence $v_1, v_2, \ldots, v_n \in A_k$ and so, by (*1*), A_k is linearly dependent, which contradicts our hypothesis. Thus A is linearly independent.

5.86. Let K be a subfield of a field L and L a subfield of a field E: that is, $K \subseteq L \subseteq E$. (Hence K is a subfield of E.) Suppose that E is of dimension n over L and L is of dimension m over K. Show that E is of dimension mn over K.

Suppose $\{v_1, \ldots, v_n\}$ is a basis of E over L and $\{a_1, \ldots, a_m\}$ is a basis of L over K. We claim that $\{a_iv_j : i = 1, \ldots, m, j = 1, \ldots, n\}$ is a basis of E over K. Note that $\{a_iv_j\}$ contains mn elements.

Let w be any arbitrary element in E. Since $\{v_1, \ldots, v_n\}$ spans E over L, w is a linear combination of the v_i with coefficients in L:

$$w = b_1v_1 + b_2v_2 + \cdots + b_nv_n \qquad b_i \in L \qquad (1)$$

Since $\{a_1, \ldots, a_m\}$ spans L over K, each $b_i \in L$ is a linear combination of the a_j with coefficients in K:

$$\begin{aligned} b_1 &= k_{11}a_1 + k_{12}a_2 + \cdots + k_{1m}a_m \\ b_2 &= k_{21}a_1 + k_{22}a_2 + \cdots + k_{2m}a_m \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_n &= k_{n1}a_1 + k_{n2}a_2 + \cdots + k_{nm}a_m \end{aligned}$$

where $k_{ij} \in K$. Substituting in (*1*), we obtain

$$\begin{aligned} w &= (k_{11}a_1 + \cdots + k_{1m}a_m)v_1 + (k_{21}a_1 + \cdots + k_{2m}a_m)v_2 + \cdots + (k_{n1}a_1 + \cdots + k_{nm}a_m)v_n \\ &= k_{11}a_1v_1 + \cdots + k_{1m}a_mv_1 + k_{21}a_1v_2 + \cdots + k_{2m}a_mv_2 + \cdots + k_{n1}a_1v_n + \cdots + k_{nm}a_mv_n \\ &= \sum_{i,j} k_{ji}(a_iv_j) \end{aligned}$$

where $k_{ji} \in K$. Thus w is a linear combination of the $a_i v_j$ with coefficients in K; hence $\{a_i v_j\}$ spans E over K.

The proof is complete if we show that $\{a_i v_j\}$ is linearly independent over K. Suppose, for scalars $x_{ji} \in K$, $\sum_{i,j} x_{ji}(a_i v_j) = 0$; that is,

$$(x_{11}a_1v_1 + x_{12}a_2v_1 + \cdots + x_{1m}a_mv_1) + \cdots + (x_{n1}a_1v_n + x_{n2}a_2v_n + \cdots + x_{nm}a_mv_n) = 0$$

or

$$(x_{11}a_1 + x_{12}a_2 + \cdots + x_{1m}a_m)v_1 + \cdots + (x_{n1}a_1 + x_{n2}a_2 + \cdots + x_{nm}a_m)v_n = 0$$

Since $\{v_1, \ldots, v_n\}$ is linearly independent over L and since the above coefficients of the v_i belong to L, each coefficient must be 0:

$$x_{11}a_1 + x_{12}a_2 + \cdots + x_{1m}a_m = 0, \ \ldots, \ x_{n1}a_1 + x_{n2}a_2 + \cdots + x_{nm}a_m = 0$$

But $\{a_1, \ldots, a_m\}$ is linearly independent over K; hence, since the $x_{ji} \in K$,

$$x_{11} = 0, \ x_{12} = 0, \ \ldots, \ x_{1m} = 0, \ \ldots, \ x_{n1} = 0, \ x_{n2} = 0, \ \ldots, \ x_{nm} = 0$$

Accordingly, $\{a_i v_j\}$ is linearly independent over K and the theorem is proved.

Supplementary Problems

VECTOR SPACES

5.87. Let V be the set of ordered pairs (a, b) of real numbers with addition in V and scalar multiplication on V defined by

$$(a, b) + (c, d) = (a + c, b + d) \qquad \text{and} \qquad k(a, b) = (ka, 0)$$

Show that V satisfies all of the axioms of a vector space except $[M_4]$: $1u = u$. Hence $[M_4]$ is not a consequence of the other axioms.

5.88. Show that the following axiom $[A_4]$ can be derived from the other axioms of a vector space.
$[A_4]$ For any vectors $u, v \in V$, $u + v = v + u$.

5.89. Let V be the set of infinite sequences $(a_1, a_2, \ldots)$ in a field K with addition in V and scalar multiplication on V defined by

$$(a_1, a_2, \ldots) + (b_1, b_2, \ldots) = (a_1 + b_1, a_2 + b_2, \ldots)$$
$$k(a_1, a_2, \ldots) = (ka_1, ka_2, \ldots)$$

where $a_i, b_j, k \in K$. Show that V is a vector space over K.

SUBSPACES

5.90. Determine whether or not W is a subspace of $\mathbf{R}^3$ where W consists of those vectors $(a, b, c) \in \mathbf{R}^3$ for which (*a*) $a = 2b$; (*b*) $a \le b \le c$; (*c*) $ab = 0$; (*d*) $a = b = c$; (*e*) $a = b^2$.

5.91. Let V be the vector space of n-square matrices over a field K. Show that W is a subspace of V if W consists of all matrices which are (*a*) antisymmetric ($A^T = -A$), (*b*) (upper) triangular, (*c*) diagonal, (*d*) scalar.

5.92. Let $AX = B$ be a nonhomogeneous system of linear equations in n unknowns over a field K. Show that the solution set of the system is not a subspace of K^n.

5.93. Discuss whether or not $\mathbf{R}^2$ is a subspace of $\mathbf{R}^3$.

5.94. Suppose U and W are subspaces of V for which $U \cup W$ is also a subspace. Show that either $U \subseteq W$ or $W \subseteq U$.

5.95. Let V be the vector space of all functions from the real field $\mathbf{R}$ into $\mathbf{R}$. Show that W is a subspace of V in each of the following cases.

(a) W consists of all bounded functions. [Here f: $\mathbf{R} \to \mathbf{R}$ is bounded if there exists $M \in \mathbf{R}$ such that $|f(x)| \le M$, $\forall x \in \mathbf{R}$.]

(b) W consists of all even functions. [Here f: $\mathbf{R} \to \mathbf{R}$ is even if $f(-x) = f(x)$, $\forall x \in \mathbf{R}$.]

(c) W consists of all continuous functions.

(d) W consists of all differentiable functions.

(e) W consists of all integrable functions in, say, the interval $0 \le x \le 1$.

(The last three cases require some knowledge of analysis.)

5.96. Let V be the vector space (Problem 5.106) of infinite sequences $(a_1, a_2, \ldots)$ in a field K. Show that W is a subspace of V where (a) W consists of all sequences with 0 as the first component, and (b) W consists of all sequences with only a finite number of nonzero components.

LINEAR COMBINATIONS, LINEAR SPANS

5.97. Show that the complex numbers $w = 2 + 3i$ and $z = 1 - 2i$ span the complex field $\mathbf{C}$ as a vector space over the real field $\mathbf{R}$.

5.98. Show that the polynomials $(1 - t)^3$, $(1 - t)^2$, $1 - t$, and 1 span the space $P_3(t)$ of polynomials of degree ≤ 3.

5.99. Find one vector in $\mathbf{R}^3$ which spans the intersection of U and W where U is the xy plane: $U = \{(a, b, 0)\}$, and W is the space spanned by the vectors $(1, 2, 3)$ and $(1, -1, 1)$.

5.100. Prove that span S is the intersection of all the subspaces of V containing S.

5.101. Show that span S = span $(S \cup \{0\})$. That is, by joining or deleting the zero vector from a set, we do not change the space spanned by the set.

5.102. Show that if $S \subseteq T$, then span $S \subseteq$ span T.

5.103. Show that span (span S) = span S.

5.104. Let $W_1, W_2, \ldots$ be subspaces of a vector space V for which $W_1 \subseteq W_2 \subseteq \cdots$. Let $W = W_1 \cup W_2 \cup \cdots$. (a) Show that W is a subspace of V. (b) Suppose S_i spans W_i for $i = 1, 2, \ldots$. Show that $S = S_1 \cup S_2 \cup \cdots$ spans W.

LINEAR DEPENDENCE AND INDEPENDENCE

5.105. Determine whether the following vectors in $\mathbf{R}^4$ are linearly dependent or independent:

(a) $(1, 3, -1, 4), (3, 8, -5, 7), (2, 9, 4, 23)$ (b) $(1, -2, 4, 1), (2, 1, 0, -3), (3, -6, 1, 4)$

5.106. Let V be the vector space of polynomials of degree ≤ 3 over $\mathbf{R}$. Determine whether $u, v, w \in V$ are linearly dependent or independent where:

(a) $u = t^3 - 4t^2 + 2t + 3$, $v = t^3 + 2t^2 + 4t - 1$, $w = 2t^3 - t^2 - 3t + 5$

(b) $u = t^3 - 5t^2 - 2t + 3$, $v = t^3 - 4t^2 - 3t + 4$, $w = 2t^3 - 7t^2 - 7t + 9$

5.107. Show that (*a*) the vectors $(1 - i, i)$ and $(2, -1 + i)$ in $\mathbf{C}^2$ are linearly dependent over the complex field $\mathbf{C}$ but are linearly independent over the real field $\mathbf{R}$; (*b*) the vectors $(3 + \sqrt{2}, 1 + \sqrt{2})$ and $(7, 1 + 2\sqrt{2})$ in $\mathbf{R}^2$ are linearly dependent over the real field $\mathbf{R}$ but are linearly independent over the rational field $\mathbf{Q}$.

5.108. Suppose $\{u_1, \ldots, u_r, w_1, \ldots, w_s\}$ is a linearly independent subset of V. Prove that span $u_i \cap$ span $w_j = \{0\}$. (Recall that span u_i is the subspace of V spanned by the u_i.)

5.109. Suppose $v_1, v_2, \ldots, v_n$ are linearly independent vectors. Prove the following:

(*a*) $\{a_1 v_1, a_2 v_2, \ldots, a_n v_n\}$ is linearly independent where each $a_i \neq 0$.

(*b*) $\{v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_n\}$ is linearly independent where $w = b_1 v_1 + \cdots + b_i v_i + \cdots + b_n v_n$ and $b_i \neq 0$.

5.110. Suppose $(a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})$ are linearly independent vectors in K^n, and suppose $v_1, \ldots, v_n$ are linearly independent vectors in a vector space V over K. Show that the vectors

$$w_1 = a_{11}v_1 + \cdots + a_{1n}v_n, \ldots, w_m = a_{m1}v_1 + \cdots + a_{mn}v_n$$

are also linearly independent.

5.111. Suppose A is any n-square matrix and suppose $u_1, u_2, \ldots, u_r$ are $n \times 1$ column vectors. Show that, if $Au_1, Au_2, \ldots, Au_r$ are linearly independent (column) vectors, then $u_1, u_2, \ldots, u_r$ are linearly independent.

BASIS AND DIMENSION

5.112. Find a subset of u_1, u_2, u_3, u_4 which gives a basis for $W = \text{span}(u_1, u_2, u_3, u_4)$ of $\mathbf{R}^5$ where:

(*a*) $u_1 = (1, 1, 1, 2, 3), u_2 = (1, 2, -1, -2, 1), u_3 = (3, 5, -1, -2, 5), u_4 = (1, 2, 1, -1, 4)$

(*b*) $u_1 = (1, -2, 1, 3, -1), u_2 = (-2, 4, -2, -6, 2), u_3 = (1, -3, 1, 2, 1), u_4 = (3, -7, 3, 8, -1)$

(*c*) $u_1 = (1, 0, 1, 0, 1), u_2 = (1, 1, 2, 1, 0), u_3 = (1, 2, 3, 1, 1), u_4 = (1, 2, 1, 1, 1)$

(*d*) $u_1 = (1, 0, 1, 1, 1), u_2 = (2, 1, 2, 0, 1), u_3 = (1, 1, 2, 3, 4), u_4 = (4, 2, 5, 4, 6)$

5.113. Let U and W be the following subspaces of $\mathbf{R}^4$:

$$U = \{(a, b, c, d)\colon b - 2c + d = 0\} \qquad W = \{(a, b, c, d)\colon a = d, b = 2c\}$$

Find a basis and the dimension of (*a*) U, (*b*) W, (*c*) $U \cap W$.

5.114. Find a basis and the dimension of the solution space W of each homogeneous system:

$$\begin{aligned} x + 2y - 2z + 2s - t &= 0 \\ x + 2y - z + 3s - 2t &= 0 \\ 2x + 4y - 7z + s + t &= 0 \end{aligned} \qquad \begin{aligned} x + 2y - z + 3s - 4t &= 0 \\ 2x + 4y - 2z - s + 5t &= 0 \\ 2x + 4y - 2z + 4s - 2t &= 0 \end{aligned}$$

(*a*) (*b*)

5.115. Find a homogeneous system whose solution space is spanned by the three vectors

$$(1, -2, 0, 3, -1) \qquad (2, -3, 2, 5, -3) \qquad (1, -2, 1, 2, -2)$$

5.116. Let V be the vector space of polynomials in t of degree $\leq n$. Determine whether or not each of the following is a basis of V:

(*a*) $\{1, 1 + t, 1 + t + t^2, 1 + t + t^2 + t^3, \ldots, 1 + t + t^2 + \cdots + t^{n-1} + t^n\}$

(*b*) $\{1 + t, t + t^2, t^2 + t^3, \ldots, t^{n-2} + t^{n-1}, t^{n-1} + t^n\}$

5.117. Find a basis and the dimension of the subspace W of $\mathbf{P}(t)$ spanned by the polynomials

(*a*) $u = t^3 + 2t^2 - 2t + 1$, $v = t^3 + 3t^2 - t + 4$, and $w = 2t^3 + t^2 - 7t - 7$

(*b*) $u = t^3 + t^2 - 3t + 2$, $v = 2t^3 + t^2 + t - 4$, and $w = 4t^3 + 3t^2 - 5t + 2$

5.118. Let V be the space of 2×2 matrices over $\mathbf{R}$. Find a basis and the dimension of the subspace W of V spanned by the matrices

$$\begin{pmatrix} 1 & -5 \\ -4 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ -1 & 5 \end{pmatrix} \qquad \begin{pmatrix} 2 & -4 \\ -5 & 7 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -7 \\ -5 & 1 \end{pmatrix}$$

ROW SPACE AND RANK OF A MATRIX

5.119. Consider the following subspaces of $\mathbf{R}^3$:

$$U_1 = \text{span}\,[(1, 1, -1), (2, 3, -1), (3, 1, -5)]$$
$$U_2 = \text{span}\,[(1, -1, -3), (3, -2, -8), (2, 1, -3)]$$
$$U_3 = \text{span}\,[(1, 1, 1), (1, -1, 3), (3, -1, 7)]$$

Determine which of the subspaces are identical.

5.120. Find the rank of each matrix:

$$\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -4 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 1 & -9 & -10 & -3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 5 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 3 & -7 \\ -6 & 1 \\ 5 & -8 \end{pmatrix}$$

(*a*) (*b*) (*c*) (*d*)

5.121. Show that if any row is deleted from a matrix in echelon (row canonical) form then the resulting matrix is still in echelon (row canonical) form.

5.122. Let A and B be arbitrary $m \times n$ matrices. Show that $\text{rank}\,(A + B) \le \text{rank}\,A + \text{rank}\,B$.

5.123. Give examples of 2×2 matrices A and B such that:

(*a*) $\text{rank}\,(A + B) < \text{rank}\,A, \text{rank}\,B$

(*b*) $\text{rank}\,(A + B) = \text{rank}\,A = \text{rank}\,B$

(*c*) $\text{rank}\,(A + B) > \text{rank}\,A, \text{rank}\,B$

SUMS, DIRECT SUMS, INTERSECTIONS

5.124. Suppose U and W are 2-dimensional subspaces of $\mathbf{R}^3$. Show that $U \cap W \neq \{0\}$.

5.125. Suppose U and W are subspaces of V and that $\dim U = 4$, $\dim W = 5$, and $\dim V = 7$. Find the possible dimensions of $U \cap W$.

5.126. Let U and W be subspaces of $\mathbf{R}^3$ for which $\dim U = 1$, $\dim W = 2$, and $U \not\subseteq W$. Show that $\mathbf{R}^3 = U \oplus W$.

5.127. Let U be the subspace of $\mathbf{R}^5$ spanned by

$$(1, 3, -3, -1, -4) \qquad (1, 4, -1, -2, -2) \qquad (2, 9, 0, -5, -2)$$

and let W be the subspace spanned by

$$(1, 6, 2, -2, 3) \qquad (2, 8, -1, -6, -5) \qquad (1, 3, -1, -5, -6)$$

Find (*a*) $\dim\,(U + W)$, (*b*) $\dim\,(U \cap W)$.

5.128. Let V be the vector space of polynomials over $\mathbf{R}$. Find (*a*) $\dim(U + W)$, (*b*) $\dim(U \cap W)$, where

$$U = \text{span}\,(t^3 + 4t^2 - t + 3,\ t^3 + 5t^2 + 5,\ 3t^3 + 10t^2 - 5t + 5)$$
$$W = \text{span}\,(t^3 + 4t^2 + 6,\ t^3 + 2t^2 - t + 5,\ 2t^3 + 2t^2 - 3t + 9)$$

5.129. Let U be the subspace of $\mathbf{R}^5$ spanned by

$$(1, -1, -1, -2, 0) \qquad (1, -2, -2, 0, -3) \qquad \text{and} \qquad (1, -1, -2, -2, 1)$$

and let W be the subspace spanned by

$$(1, -2, -3, 0, -2) \qquad (1, -1, -3, 2, -4) \qquad \text{and} \qquad (1, -1, -2, 2, -5)$$

(*a*) Find two homogeneous systems whose solution spaces are U and W, respectively.

(*b*) Find a basis and the dimension of $U \cap W$.

5.130. Let U_1, U_2, and U_3 be the following subspaces of $\mathbf{R}^3$:

$$U_1 = \{(a, b, c): a + b + c = 0\} \qquad U_2 = \{(a, b, c): a = c\} \qquad U_3 = \{(0, 0, c): c \in \mathbf{R}\}$$

Show that: (*a*) $\mathbf{R}^3 = U_1 + U_2$, (*b*) $\mathbf{R}^3 = U_2 + U_3$, (*c*) $\mathbf{R}^3 = U_1 + U_3$. When is the sum direct?

5.131. Suppose U, V, and W are subspaces of a vector space. Prove that

$$(U \cap V) + (U \cap W) \subseteq U \cap (V + W)$$

Find subspaces of $\mathbf{R}^2$ for which equality does not hold.

5.132. The sum of arbitrary nonempty subsets (not necessarily subspaces) S and T of a vector space V is defined by $S + T = \{s + t : s \in S, t \in T\}$. Show that this operation satisfies:

(*a*) Commutative law: $S + T = T + S$

(*b*) Associative law: $(S_1 + S_2) + S_3 = S_1 + (S_2 + S_3)$

(*c*) $S + \{0\} = \{0\} + S = S$

(*d*) $S + V = V + S = V$

5.133. Suppose $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V. Show that:

(*a*) $\text{span}\,(W_1, W_2, \ldots, W_r) = W_1 + W_2 + \cdots + W_r$

(*b*) If S_i spans W_i for $i = 1, \ldots, r$, then $S_1 \cup S_2 \cup \cdots \cup S_r$ spans $W_1 + W_2 + \cdots + W_r$.

5.134. Prove Theorem 5.24.

5.135. Prove Theorem 5.25.

5.136. Let U and W be vector spaces over a field K. Let V be the set of ordered pairs (u, w) where u belongs to U and w to W: $V = \{(u, w): u \in U, w \in W\}$. Show that V is a vector space over K with addition in V and scalar multiplication on V defined by

$$(u, w) + (u', w') = (u + u', w + w') \qquad \text{and} \qquad k(u, w) = (ku, kw)$$

where $u, u' \in U$, $w, w' \in W$, and $k \in K$. (This space V is called the *external direct sum* of U and W.)

5.137. Let V be the external direct sum of the vector spaces U and W over a field K. (See Problem 5.136.) Let $\hat{U} = \{(u, 0): u \in U\}$ and $\hat{W} = \{(0, w): w \in W\}$. Show that

(*a*) $\hat{U}$ and $\hat{W}$ are subspaces of V and that $V = \hat{U} \oplus \hat{W}$;

(*b*) U is isomorphic to $\hat{U}$ under the correspondence $u \leftrightarrow (u, 0)$, and that W is isomorphic to $\hat{W}$ under the correspondence $w \leftrightarrow (0, w)$;

(*c*) $\dim V = \dim U + \dim W$.

5.138. Suppose $V = U \oplus W$. Let $\hat{V}$ be the external direct product of U and W. Show that V is isomorphic to $\hat{V}$ under the correspondence $v = u + w \leftrightarrow (u, w)$.

COORDINATE VECTORS

5.139. Consider the basis $S = \{u_1 = (1, -2), u_2 = (4, -7)\}$ of $\mathbf{R}^2$. Find the coordinate vector $[v]$ of v relative to S where (*a*) $v = (3, 5)$, (*b*) $v = (1, 1)$, and (*c*) $v = (a, b)$.

5.140. Consider the vector space $\mathbf{P}_3(t)$ of polynomials of degree ≤ 3 and the basis $S = \{1, t + 1, t^2 + t, t^3 + t^2\}$ of $\mathbf{P}_3(t)$. Find the coordinate vector of v relative to S where (*a*) $v = 2 - 3t + t^2 + 2t^3$; (*b*) $v = 3 - 2t - t^2$; and (*c*) $v = a + bt + ct^2 + dt^3$.

5.141. Let S be the following basis of the vector space W of 2×2 real symmetric matrices:

$$\left\{\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}\right\}$$

Find the coordinate vector of the matrix $A \in W$ relative to the above basis S where (*a*) $A = \begin{pmatrix} 1 & -5 \\ -5 & 5 \end{pmatrix}$ and (*b*) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

CHANGE OF BASIS

5.142. Find the change-of-basis matrix P from the usual basis $E = \{(1, 0), (0, 1)\}$ of $\mathbf{R}^2$ to the basis S, the change-of-basis matrix Q from S back to E, and the coordinate vector of $v = (a, b)$ relative to S where

(*a*) $S = \{(1, 2), (3, 5)\}$ (*c*) $S = \{(2, 5), (3, 7)\}$

(*b*) $S = \{(1, -3), (3, -8)\}$ (*d*) $S = \{(2, 3), (4, 5)\}$

5.143. Consider the following bases of $\mathbf{R}^2$: $S = \{u_1 = (1, 2), u_2 = (2, 3)\}$ and $S' = \{v_1 = (1, 3), v_2 = (1, 4)\}$. Find: (*a*) the change-of-basis matrix P from S to S', and (*b*) the change-of-basis matrix Q from S' back to S.

5.144. Suppose that the x and y axes in the plane $\mathbf{R}^2$ are rotated counterclockwise 30° to yield new x' and y' axes for the plane. Find: (*a*) the unit vectors in the direction of the new x' and y' axes, (*b*) the change-of-basis matrix P for the new coordinate system, and (*c*) the new coordinates of each of the following points under the new coordinate system: $A(1, 3)$, $B(2, -5)$, $C(a, b)$.

5.145. Find the change-of-basis matrix P from the usual basis E of $\mathbf{R}^3$ to the basis S, the change-of-basis matrix Q from S back to E, and the coordinate vector of $v = (a, b, c)$ relative to S where S consists of the vectors:

(*a*) $u_1 = (1, 1, 0), u_2 = (0, 1, 2), u_3 = (0, 1, 1)$ (*c*) $u_1 = (1, 2, 1), u_2 = (1, 3, 4), u_3 = (2, 5, 6)$

(*b*) $u_1 = (1, 0, 1), u_2 = (1, 1, 2), u_3 = (1, 2, 4)$

5.146. Suppose S_1, S_2, and S_3 are bases of a vector space V, and suppose P is the change-of-basis matrix from S_1 to S_2 and Q is the change-of-basis matrix from S_2 to S_3. Prove that the product PQ is the change-of-basis matrix from S_1 to S_3.

MISCELLANEOUS PROBLEMS

5.147. Determine the dimension of the vector space W of n-square: (*a*) symmetric matrices over a field K, (*b*) antisymmetric matrices over K.

5.148. Let V be a vector space of dimension n over a field K, and let K be a vector space of dimension m over a subfield F. (Hence V may also be viewed as a vector space over the subfield F.) Prove that the dimension of V over F is mn.

5.149. Let $t_1, t_2, \ldots, t_n$ be symbols, and let K be any field. Let V be the set of expressions

$$a_1t_1 + a_2t_2 + \cdots + a_nt_n \qquad \text{where} \qquad a_i \in K$$

Define addition in V by

$$(a_1t_1 + a_2t_2 + \cdots + a_nt_n) + (b_1t_1 + b_2t_2 + \cdots + b_nt_n) = (a_1 + b_1)t_1 + (a_2 + b_2)t_2 + \cdots + (a_n + b_n)t_n$$

Define scalar multiplication on V by

$$k(a_1t_1 + a_2t_2 + \cdots + a_nt_n) = ka_1t_1 + ka_2t_2 + \cdots + ka_nt_n$$

Show that V is a vector space over K with the above operations. Also show that $\{t_1, \ldots, t_n\}$ is a basis of V where, for $i = 1, \ldots, n$,

$$t_i = 0t_1 + \cdots + 0t_{i-1} + 1t_i + 0t_{i+1} + \cdots + 0t_n$$

Answers to Supplementary Problems

5.90. (*a*) Yes.
(*b*) No; e.g., $(1, 2, 3) \in W$ but $-2(1, 2, 3) \notin W$.
(*c*) No; e.g., $(1, 0, 0), (0, 1, 0) \in W$, but not their sum.
(*d*) Yes.
(*e*) No; e.g., $(9, 3, 0) \in W$ but $2(9, 3, 0) \notin W$.

5.92. $X = 0$ is not a solution of $AX = B$.

5.93. No. Although one may "identify" the vector $(a, b) \in \mathbf{R}^2$ with, say, $(a, b, 0)$ in the xy plane in $\mathbf{R}^3$, they are distinct elements belonging to distinct, disjoint sets.

5.95. (*a*) Let $f, g \in W$ with M_f and M_g bounds for f and g, respectively. Then for any scalars $a, b \in \mathbf{R}$,

$$|(af + bg)(x)| = |af(x) + bg(x)| \le |af(x)| + |bg(x)| = |a||f(x)| + |b||g(x)| \le |a|M_f + |b|M_g$$

That is, $|a|M_f + |b|M_g$ is a bound for the function $af + bg$.

(*b*) $(af + bg)(-x) = af(-x) + bg(-x) = af(x) + bg(x) = (af + bg)(x)$.

5.99. $(2, -5, 0)$.

5.105. (*a*) Dependent, (*b*) Independent.

5.106. (*a*) Independent, (*b*) Dependent.

5.107. (*a*) $(2, -1 + i) = (1 + i)(1 - i, i)$; (*b*) $(7, 1 + 2\sqrt{2}) = (3 - \sqrt{2})(3 + \sqrt{2}, 1 + \sqrt{2})$.

5.112. (*a*) u_1, u_2, u_4; (*b*) u_1, u_3; (*c*) u_1, u_2, u_3, u_4; (*d*) u_1, u_2, u_3.

5.113. (*a*) Basis, $\{(1, 0, 0, 0), (0, 2, 1, 0), (0, -1, 0, 1)\}$; $\dim U = 3$.
(*b*) Basis, $\{(1, 0, 0, 1), (0, 2, 1, 0)\}$; $\dim W = 2$.
(*c*) Basis, $\{(0, 2, 1, 0)\}$; $\dim (U \cap W) = 1$. *Hint.* $U \cap W$ must satisfy all three conditions on a, b, c and d.

5.114. (*a*) Basis, $\{(2, -1, 0, 0, 0), (4, 0, 1, -1, 0), (3, 0, 1, 0, 1)\}$; $\dim W = 3$.
(*b*) Basis, $\{(2, -1, 0, 0, 0), (1, 0, 1, 0, 0)\}$; $\dim W = 2$.

5.115. $$\begin{cases} 5x + y - z - s \phantom{{}-t} = 0 \\ x + y - z \phantom{{}-s} - t = 0 \end{cases}$$

5.116. (*a*) Yes, (*b*) No. For $\dim V = n + 1$, but the set contains only n elements.

5.117. (*a*) $\dim W = 2$, (*b*) $\dim W = 3$

5.118. $\dim W = 2$

5.119. U_1 and U_2.

5.120. (*a*) 3, (*b*) 2, (*c*) 3, (*d*) 2

5.123. (*a*) $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$ (*c*) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(*b*) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

5.125. $\dim(U \cap W) = 2, 3$ or 4.

5.127. (*a*) $\dim(U + W) = 3$, (*b*) $\dim(U \cap W) = 2$.

5.128. (*a*) $\dim(U + W) = 3$, $\dim(U \cap W) = 1$.

5.129. (*a*) $\begin{cases} 3x + 4y - z \qquad - t = 0 \\ 4x + 2y \qquad + s \qquad = 0 \end{cases}$, $\begin{cases} 4x + 2y \qquad - s \qquad = 0 \\ 9x + 2y + z \qquad + t = 0 \end{cases}$

(*b*) $\{(1, -2, -5, 0, 0), (0, 0, 1, 0, -1)\}$. $\dim(U \cap W) = 2$.

5.130. The sum is direct in (*b*) and (*c*).

5.131. In $\mathbf{R}^2$, let U, V, and W be, respectively, the line $y = x$, the x axis, and the y axis.

5.139. (*a*) $[-41, 11]$, (*b*) $[-11, 3]$, (*c*) $[-7a - 4b, 2a + b]$

5.140. (*a*) $[4, -2, -1, 2]$, (*b*) $[4, -1, -1, 0]$, (*c*) $[a - b + c - d, b - c + d, c - d, d]$

5.141. (*a*) $[2, -1, 1]$, (*b*) $[3, 1, -2]$

5.142. (*a*) $P = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}, Q = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}, [v] = \begin{pmatrix} -5a + b \\ 2a - b \end{pmatrix}$

(*b*) $P = \begin{pmatrix} 1 & 3 \\ -3 & -8 \end{pmatrix}, Q = \begin{pmatrix} -8 & -3 \\ 3 & 1 \end{pmatrix}, [v] = \begin{pmatrix} -8a - 3b \\ 3a - b \end{pmatrix}$

(*c*) $P = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}, Q = \begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix}, [v] = \begin{pmatrix} -7a + 3b \\ 5a - 2b \end{pmatrix}$

(*d*) $P = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}, Q = \begin{pmatrix} -\frac{5}{2} & 2 \\ \frac{3}{2} & -1 \end{pmatrix}, [v] = \begin{pmatrix} (-\frac{5}{2})a + 2b \\ (\frac{3}{2})a - b \end{pmatrix}$

5.143. (*a*) $P = \begin{pmatrix} 3 & 5 \\ -1 & -1 \end{pmatrix}$, (*b*) $Q = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix}$

5.144. (*a*) $(\sqrt{3}/2, \frac{1}{2}), (-\frac{1}{2}, \sqrt{3}/2)$

(*b*) $P = \begin{pmatrix} \sqrt{3}/2 & -\frac{1}{2} \\ \frac{1}{2} & \sqrt{3}/2 \end{pmatrix}$

(c) $[A] = [(\sqrt{3} - 3)/2, (1 + 3\sqrt{3})/2]$,
$[B] = [(2\sqrt{3} + 5)/2, (2 - 5\sqrt{3})/2]$,
$[C] = [(\sqrt{3}a - b)/2, (a + \sqrt{3}b)/2]$

5.145. Since E is the usual basis, simply let P be the matrix whose columns are u_1, u_2, u_3. Then $Q = P^{-1}$ and $[v] = P^{-1}v = Qv$.

(a) $P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}, [v] = \begin{pmatrix} a \\ a - b + c \\ -2a + 2b - c \end{pmatrix}$

(b) $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}, Q = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}, [v] = \begin{pmatrix} -2b + c \\ 2a + 3b - 2c \\ -a - b + c \end{pmatrix}$

(c) $P = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 1 & 4 & 6 \end{pmatrix}, Q = \begin{pmatrix} -2 & 2 & -1 \\ -7 & 4 & -1 \\ 5 & -3 & 1 \end{pmatrix}, [v] = \begin{pmatrix} -2a + 2b - c \\ -7a + 4b - c \\ 5a - 3b + c \end{pmatrix}$

5.147. (a) $n(n + 1)/2$, (b) $n(n - 1)/2$

5.148. *Hint:* The proof is almost identical to that given in Problem 5.86 for the special case when V is an extension field of K.