Diagramas de Young

$\to$ repns de S_d

$\to$ repns de $\boxed{GL_n(\mathbb{C})}$

$\cup$

$SL_n(\mathbb{C})$

$\cup$

$SU_n(\mathbb{C})$

Ref: Fulton & Harris

Tensores

$V = \mathbb{C}^n$ rep'n estandar de $GL_n(\mathbb{C})$

$$V \otimes V = S^2(V) \oplus \Lambda^2(V) \quad \text{son irred}$$

$(12) \in S_2$, $T \in V \otimes V$, $T = \sum v_i \otimes v_j$

$= \sum a_{ij} e_i \otimes e_j$

$$(12) \cdot T = \sum v_j \otimes v_i$$

$$= \sum a_{ji} e_i \otimes e_j = \sum a_{ij} e_j \otimes e_i$$

$$S^2(V) = \{ T \in V \otimes V \mid (12) \cdot T = T \}$$

$$\Lambda^2(V) = \{ \quad - \cdots \cdots = -T \}.$$

$GL_n(\mathbb{C}) \circlearrowright V \otimes V \circlearrowleft S_2$

$\boxed{\text{Las dos acciones conmutan}}$

$g \in GL(V)$, $\sigma \in S_2$

rep'n de [2]

$GL_n(\mathbb{C}) \times S_2$

$$\begin{array}{ccc} v_1 \otimes v_2 & \xmapsto{g} & (g v_1) \otimes (g v_2) \\ \downarrow \sigma^{-1} & & \downarrow \\ v_{\sigma(1)} \otimes v_{\sigma(2)} & \longmapsto & g v_{\sigma(1)} \otimes g v_{\sigma(2)} \end{array}$$

Repaso:

Para grupos finitos, rep'n de $G_1 \times G_2$ irred.

son $\rho_1 \otimes \rho_2$ en $V_1 \otimes V_2$

$$\underbrace{V \otimes V}_{\substack{\uparrow \\ b_{n,0} \\ GL(V) \times S_2}} = S^2(V) \otimes \underbrace{\mathbb{C}}_{\substack{\uparrow \\ \text{triv}}} \oplus \Lambda^2(V) \otimes \mathbb{C}_{\text{signo}}$$

Peter-Weyl. $A = \mathbb{C}[G]$

$$G \times G \curvearrowright A = \bigoplus_{\pi \in \hat{G}} \mathrm{End}(V_\pi) =$$

$$= \bigoplus_{\pi \in \hat{G}} \underset{\text{izq}}{V_\pi} \otimes \underset{\text{der.}}{V_\pi^*}$$

Meta: bajo $GL(V)\times S_d$

$$V^{\otimes d} = \underbrace{V\otimes \cdots \otimes V}_{d \text{ veces}} = \bigoplus_{\lambda\in S_d/\text{conj}} U_\lambda \otimes V_\lambda$$

rep. irred de $GL(V)$

rep. irred de S_d

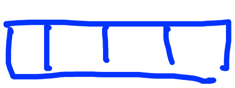
- Clases de conj. en $S_d \ni \sigma = c_1\, c_2\, c_3 \cdots$

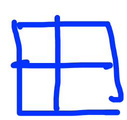
$S_4/_{\text{conj}} = \{\text{particiones de } \{1,2,3,4\}\}$

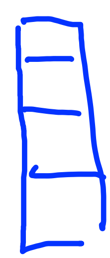
$= \left\{ 4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1 \right\}$

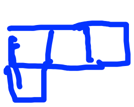
$c = (1, i_2 \dots i_k)$

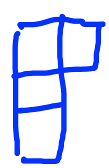
$\#\left(S_4/_{\text{conj.}}\right) = \#\hat{S}_4 = 5$

$4 \longleftrightarrow$ 

$2+2$ 



$3+1 \longleftrightarrow$ 

$2+1+1$ 

$1+1+1+1$

- Con cada diagrama, construimos una rep irred $V_d \subset A = \mathbb{C}[S_d]$

<u>Una receta:</u> un "tableau" = un diag. llenado.

1	2	3	4	5
6	7	8	9	10
11	12			
13				

← el "tableau"

$\lambda = 5+5+2+1$

$P_\lambda = \{\sigma \mid \sigma \text{ deja inv las filas}\} = \text{prod. de gps. sim}$

$Q_\lambda = \{ \text{ - - - - - - - columnas}\}$

$$a_\lambda = \sum_{\sigma \in P} \sigma$$

$$b_\lambda = \sum_{\sigma \in Q} \operatorname{sig}(\sigma)\sigma$$

$$c_\lambda = a_\lambda b_\lambda$$

$$V_\lambda = A c_\lambda \subset A$$ inv bajo la rep'n reg. izq.

Teo: V_λ es irred., y esto da una biy. $S_d/\text{conj} \Leftrightarrow \hat{S}_d$

$$\lambda \longmapsto V_\lambda$$

$\underline{d=2}$ $\left\{ \boxed{}\boxed{} \,,\, \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \right\} = S_2/\text{conj}.$

$\lambda = \boxed{1}\boxed{2}$ $\quad P = S_2, \; a = e + (12) \quad S_2 = \{e, (12)\}$

$Q = \{e\}, \; b = e$

"simetrizador de Young" $\rightsquigarrow c = ab = e + (12)$ $\quad \mathbb{C}e$

$V_\lambda = Ac = \cancel{(\mathbb{C} \oplus \mathbb{C}(12))(e + (12))}$

$\cancel{= \mathbb{C} + \mathbb{C}(12)}$

$A = \{x + y(12) \mid x, y \in \mathbb{C}\}$ $\quad 1 + (12)$

$(x + y(12))(1 + (12)) = (x+y) + (x+y)(12) = \mathbb{C}c$

$= (x+y)(1 + (12))$

~~S_n~~ $\quad (12)c = (12)(1 + (12)) = (12) + 1$

$\therefore V_{\boxed{}\boxed{}} = \text{trivial}.$

$\lambda = \begin{ytableau} 1 \\ 2 \end{ytableau}$ $\qquad a = 1$

$\qquad\qquad b = 1 - (12) = c$

$V_{\ydiagram{1,1}} = \mathbb{C}\, c, \quad (12)\cdot c = (12)\big(1 - (12)\big)$

$\qquad\qquad = (12) - 1 = -c$

$\Rightarrow V_{\ydiagram{1,1}} = \text{signo}.$

$\underline{S_3}$: $\ydiagram{3}\,, \ \ydiagram{2,1}\,, \ \ydiagram{1,1,1}$

$V_{\ydiagram{3}} = \text{trivial} \qquad a = \sum_\sigma \sigma = c, \quad b = 1$

$\beta c = \sum_\sigma \beta\sigma = c, \ \forall \beta \in S_d$

$\underline{\Rightarrow A c = \mathbb{C}\, c, \text{ con acción triv}}$

$V_{\ydiagram{1,1,1}} = \text{signo} \qquad a = 1, \ b = \sum \pm \sigma = c$

$\beta \cdot c = \sum \ldots = \text{signo}(\beta)\, c$

$\Rightarrow A c = \mathbb{C}\, c, \text{ con acción signo}$

$6 = 1^2 + 1^2 + 2^2$

$(a\, b\, c \ldots)$

$a \mapsto b \mapsto c \mapsto \ldots$

$V_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} = ?$

$a = 1 + (12)$

$b = 1 - (13)$

$c = 1 + (12) - (13) - (321)$

$1 \cdot c = c$

$(12) \cdot c = (12) + 1 - (321) - (13) = c$

$(23) \cdot c = (23) + (132) - (123) - (12)$

$(31) \cdot c =$

$(123) \cdot c$

$(321) \cdot c$

$(23)(321) = (12)$

$(12)(13) = (132) = (321)$

$(12)(321) = (13)$

$(23)(12) = (132)$

$(23)(13) = (123)$

Prop: $\boxed{c_\lambda^2 = n_\lambda c_\lambda}$ $n_\lambda > 0$

$\Rightarrow$ es facil calcular $\underline{\underline{A c_\lambda}}$.

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$$SU_3 \subset SL_3(\mathbb{C}) \subset GL_3(\mathbb{C})$$

$\approx$ casi ($\cancel{\det}$) $\approx$

$$\mathbb{C}^3 \otimes \cdots \otimes \mathbb{C}^3$$

$$GL_3(\mathbb{C}) \otimes S_d$$

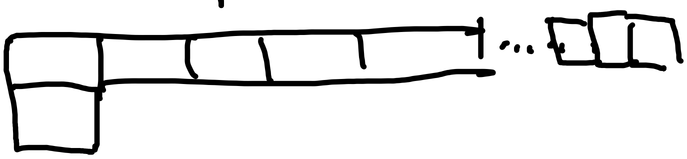
V: $\underbrace{\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3} = \underbrace{S^2(V)} \oplus \underbrace{V \otimes \Lambda^2 V} \oplus \underbrace{\Lambda^3(V)}$

bajo SU_3

- $\dim V_\lambda = \dfrac{d!}{\prod h}$ "gancho"

- $c_\lambda^2 = n_\lambda c_\lambda$, $n_\lambda = \dfrac{d!}{\dim V_\lambda} \in \mathbb{N}$ $\quad \tilde{c}_\lambda = c_\lambda / n_\lambda$

- la rep "estandar"



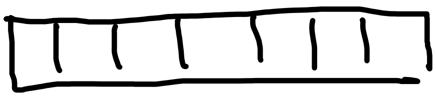
-  dual!

- $$\boxed{V^{\otimes d} = \bigoplus_\lambda \mathbb{S}_\lambda V \otimes V_\lambda}$$ $\boxed{\text{Weyl (?) reps of the classical gps.}}$

$$= [S^d(V) \otimes \text{triv}] \oplus \ldots \oplus [\Lambda^d(V) \otimes \text{signo}]$$

$$\mathbb{S}_\lambda V := c_\lambda V^{\otimes d} = \begin{cases} 0 & \dim V < \#\text{filas de } \lambda \\ \underline{\text{irred.}} & \end{cases}$$

funtor de Shur

$\lambda =$  $\Rightarrow c_\lambda = \sum \sigma$

$$\underbrace{V \otimes \cdots\cdots \otimes V}_{d} \supset S^d(V) = (c_\lambda)\, V^{\otimes d}$$

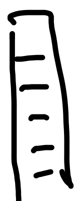
$\underline{d=3}$ $\quad c_\lambda = 1 + (12) + (23) + (13) + (123) + (321)$

$$c_\lambda \cdot v_1 \otimes v_2 \otimes v_3 = \, ?$$

$$(12) \cdot v_1 \otimes v_2 \otimes v_3 = v_2 \otimes v_1 \otimes v_3$$

$$(123) \cdot v_1 \otimes v_2 \otimes v_3 = v_3 \otimes v_1 \otimes v_2$$

$$c_\lambda (v_1 \otimes v_2 \otimes v_3) = 6 \left[\frac{1}{6} \sum_\sigma \sigma \cdot v_1 \otimes v_2 \otimes v_3 \right]$$

$\lambda =$  $\quad V_\lambda = \text{signo} \quad \dim = 1$

$$c_\lambda = \sum_\sigma \pm\sigma \quad \Big| \quad \underline{c_\lambda V^{\otimes d} = \Lambda^d(V)}$$

$$c_\lambda \cdot v_1 \otimes \cdots \otimes v_d = (\text{const})$$ (la antisimetrización del tensor)

claro, si $d > \dim V \Rightarrow \Lambda^d V = c_\lambda V^{\otimes d} = \mathbb{S}_{(1,1,1)} V = 0$

$\underline{d=3}$ $\quad V\otimes V\otimes V = \underbrace{S^3(V)\otimes \text{triv}}_{(3)}$

$$\oplus \underbrace{\mathbb{S}_{(2,1)}(V) \otimes V_{(2,1)}}_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} \quad \oplus \quad \underbrace{\Lambda^3(V)\otimes \text{signo}}_{(1,1,1)}$$

$= 0$ si $\dim V \leq 1$ $\qquad\qquad$ $= 0$ si $\dim V \leq 2$

$$c_\lambda = 1 + (12) - (13) - (321) = a_\lambda b_\lambda = (1+(12))(1-(13))$$

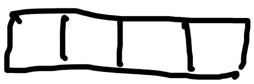
$$c_\lambda\, V\otimes V\otimes V = (1+(13))\left[(1-(12))\, V\otimes V\otimes V\right] =$$

$$= (1+(13))\left[\Lambda^2 V\otimes V\right] \subset V\otimes S^2 V$$

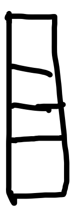
$$\boxed{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}}$$

$$(13)\,T = T, \ \ldots$$

$\boxed{d=4}$



$$\mathbb{S}_\lambda V = S^4 V$$



$$\mathbb{S}_\lambda V = \Lambda^4 V \qquad \dim V \geq 4$$

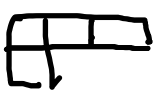
tipo curv

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$$

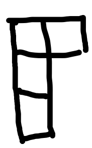
$$c_\lambda = (1+(12)+(34)+(12)(34))(1-(13)-(24)+(13)(24))$$

$$\dim V_\lambda = \frac{4!}{\prod h} = \frac{2\cdot\cancel{3}\cdot\cancel{4}}{\cancel{2}\cdot 2\cdot\cancel{3}} = 2$$

$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline\end{array}$$



$$c_\lambda = (1+(12)+(13)+(23)+(123)+(132))\cdot(1-(14))$$



$$\dim V_\lambda = \frac{\cancel{2}\cdot 3\cdot\cancel{4}}{\cancel{2}\cdot\cancel{4}} = 3$$

$$\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 & & \\ \cline{1-1}\end{array}$$

ej: $\mathbb{S}_\lambda V = \{T\in V^{\otimes 4} \mid$

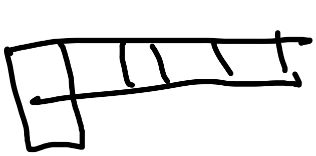
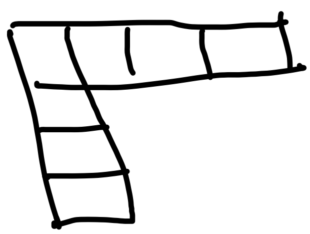
$(12)T=(34)T=-T$

$(13)(24)T = T$

$(1+(123)+(321))T=0$

("id de Bianchi")

Proy de final del curso:

- "Formula de Gancho"
- V_λ es irred. $\lambda \neq \mu$ $V_\lambda \not\cong V_\mu$
-  = la rep estandar
-  = ??
-  $\longrightarrow$ dual!

** • Frobenius formula par χ_λ

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