$G = SU_3$ y sus representaciones

idea: métodos "infinitesimales", pasamos al alg. de Lie de G, $su_3 = T_e G \cong \mathbb{R}^8$.

$\rho : G \to GL(V) \subset End(V)$

$\Rightarrow \rho' = d\rho(e) : \mathfrak{g} \to End(V)$ homo. de alg.

Ref:
- Fulton + Harris §12. ($SL_3(\mathbb{C})$).

Vez pasada: ρ' determina ρ si G es conexo.

- $G < GL_n(\mathbb{R}) \Rightarrow \mathfrak{g} = T_e G$ es un subalg. de Lie (cerrado bajo corchete) de $End(\mathbb{R}^n) = Mat_{n \times n}(\mathbb{R})$

$\cap$

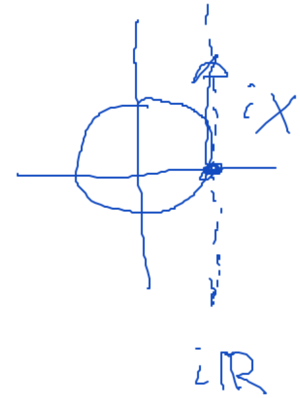
$End(\mathbb{R}^n)$

alg. de Lie $[A, B] = AB - BA$

$G = S^1$, $\rho_n : S^1 \to GL_1(\mathbb{C}) = \mathbb{C}^*$, $\rho_n(u) = u^n$

$u \in \mathbb{C}$

$|u| = 1$



$\rho_n' : i\mathbb{R} \to \mathbb{C} = \mathrm{End}(\mathbb{C})$

$iX \in i\mathbb{R}$, $X \in \mathbb{R}$, $x(t) = e^{itX}$, $x(0) = 1$, $x'(0) = iX$

$\rho_n'(iX) := \frac{d}{dt}\Big|_{t=0} \rho_n(x(t)) = \frac{d}{dt}\Big|_{t=0} e^{intX} =$ $\rho_n' : \mathbb{C} \to \mathbb{C}$

$= inX \implies \rho_n' = in \in (i\mathbb{R})^*$

$(i\mathbb{R})^*$

$\rho' : \cancel{i\mathbb{R}} \to$ real.

complejo

$\begin{cases} \rho' : \mathfrak{g} \to \mathrm{End}(V) \\ \rho : G \to GL(V) \end{cases}$

extendemos

$\rho' : \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(V)$

homo. comp. de alg. de Lie complejas.

$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$

$G = SU_2$, $H =$ los elem. diag. $= \left\{ \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \,\middle|\, \begin{matrix} u \in \mathbb{C} \\ |u| = 1 \end{matrix} \right\} \approx S^1$ "toro max."

$\mathfrak{h} = \left\{ \begin{pmatrix} ia & \\ & -ia \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\}$

$G = SU_3$, $H =$ " $= \left\{ \begin{pmatrix} u_1 & & \\ & u_2 & \\ & & u_3 \end{pmatrix} \,\middle|\, \begin{matrix} |u_i| = 1 \\ u_1 u_2 u_3 \end{matrix} \right\} = S^1 \times S^1$

$\underset{\substack{\dim = 1 \\ \mathbb{R}}}{\mathfrak{h}} \subset \underset{\substack{\dim = 3 \\ \mathbb{R}}}{\mathfrak{g}}$

$\mathfrak{h}_{\mathbb{C}} = \left\{ \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix} \,\middle|\, h \in \mathbb{C} \right\}$

$\mathfrak{g}_{\mathbb{C}} = \mathfrak{su}_2 \oplus i\,\mathfrak{su}_2 = \mathfrak{sl}_2(\mathbb{C}) =$ mat 2×2 tr $= 0$

$\rho : SU_2 \to GL(V)$

$\rho' : \underset{\substack{\cup \\ \mathfrak{su}_2}}{\mathfrak{sl}_2(\mathbb{C})} \to \mathrm{End}(V)$, | $V = \bigoplus V_\alpha$

$V_\alpha =$ eigen esp. de $\mathfrak{h}$

$$\mathfrak{su}_2 \otimes \mathbb{C} = \mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{C} \right\} = \langle H, X, Y \rangle$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[H, X] = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix} - \begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix}\begin{pmatrix} 1 & \cdot \\ \cdot & -1 \end{pmatrix} = \begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix} - \begin{pmatrix} \cdot & -1 \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$= 2X$$

$$[H, Y] = -2Y$$

$$[X, Y] = \begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix} - \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix}\begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 1 & \\ & \end{pmatrix} - \begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \end{pmatrix} = H$$

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

V = rep de SU_2
= rep de $\mathfrak{su}_2$
= rep de $\mathfrak{sl}_2(\mathbb{C})$.
$= \bigoplus_{\lambda} V_\lambda$

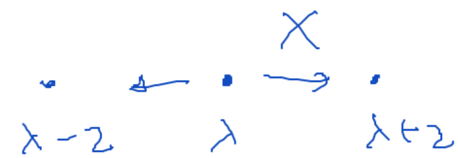
$$v \in V_\lambda \Rightarrow \rho'(H)v = H \cdot v = \lambda v.$$

$$HX \cdot v = (\underbrace{HX - XH}_{[H,X] = 2X} + XH)v = 2Xv + \lambda Xv = (\lambda + 2)Xv$$

$$\Rightarrow \boxed{Xv \in V_{\lambda+2}}$$

Resumen $V = \oplus V_\lambda$, $v \in V_\lambda$

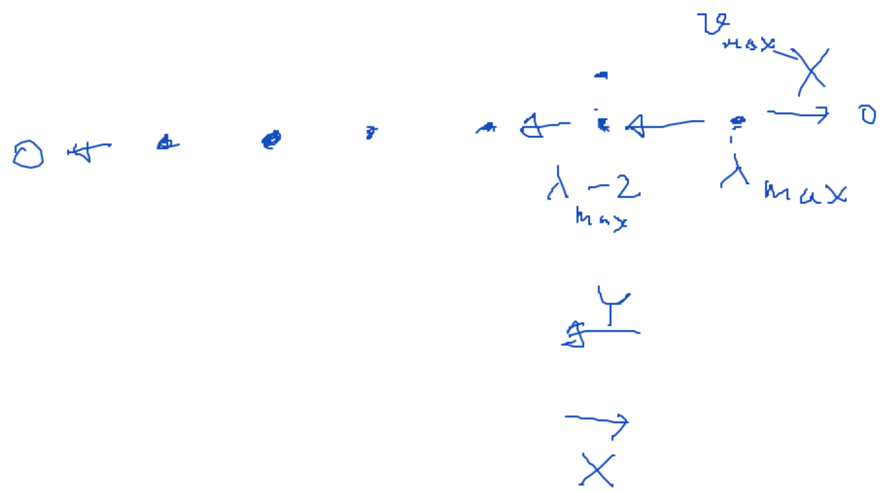
$\Rightarrow Hv = \lambda v$, $Xv \in V_{\lambda+2}$, $Yv \in V_{\lambda-2}$



$X \, Y^k \, v_{max}$

$X(Y v_{max}) =$

$=$



ρ rep'n de SU_2 en V

ρ' ------ $sl_2(\mathbb{C})$ en $V = \bigoplus V_\lambda$ .... v_λ

$\ldots, \lambda-4, \lambda-2, \lambda$

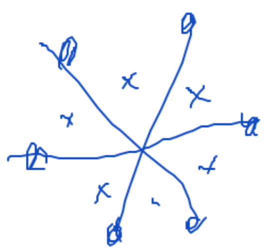
P.D., $-n, -n+2, \ldots, n$ (ej.)

$$H = \begin{pmatrix} n & & \\ & \ddots & \\ & & -n \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & n-2 & & \\ 0 & 0 & \ddots & \\ \vdots & & & \ddots \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{pmatrix}$$

Prox. Hacer lo mismo para SU_3.



"diag. de raíces"