***8.** (a) Let $f: S^1 \longrightarrow S^1$ be any smooth map. Prove that there exists a smooth map $g: \mathbf{R} \longrightarrow \mathbf{R}$ such that $f(\cos t, \sin t) = (\cos g(t), \sin g(t))$, and satisfying $g(2\pi) = g(0) + 2\pi q$ for some integer q. [HINT: First define g on $[0, 2\pi]$, and show that $g(2\pi) = g(0) + 2\pi q$. Now extend g by demanding $g(t + 2\pi) = g(t) + 2\pi q$.]
(b) Prove that $\deg_2(f) = q \bmod 2$.

$0, 2\pi$

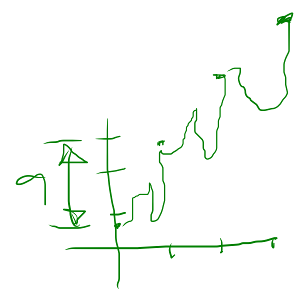
$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{g} & \mathbb{R} \\ \downarrow{\scriptstyle exp} & & \downarrow{\scriptstyle exp = e^{it}} \\ 1 \in S^1 & \xrightarrow{f} & S^1 \end{array} \qquad f(1) = e^{ig(0)}$$

a) $e^{ig(0)} = f(e^{i0}) = f(1)$
$= e^{i\theta_1}, \quad 0 \leq \theta_1 < 2\pi.$

$exp(t) = e^{it} = \cos t + i \sin t$
$= (\cos t, \sin t)$

$$\boxed{\begin{array}{l} \text{Lemma: } e^{i\theta_1} = e^{i\theta_2} \\ \iff \theta_1 - \theta_2 \in 2\pi\mathbb{Z}. \\ \left[\text{D: } e^{i(\theta_1 - \theta_2)} = 1.\right] \end{array}}$$

$f(1)$ θ_1 1



$\Rightarrow g(0) = \theta_1 + 2\pi q_1$

$e^{ig(2\pi)} = f(e^{i2\pi}) = f(1) = e^{i\theta_1} \Rightarrow g(2\pi) = \theta_1 + 2\pi q_2$

$\Rightarrow g(2\pi) = g(0) + 2\pi(q_2 - q_1)$

$\therefore$ si f tiene levantamiento $g \Rightarrow g(2\pi) = g(0) + 2\pi q$
$q \in \mathbb{Z}$

$\Rightarrow g(t + 2\pi) = g(t) + 2\pi q$

Falta P.D.: g existe.

$$\begin{array}{ccc} & & \exists!\, \tilde{f} \quad \mathbb{R} \ni \tilde{f}(x_0) \\ & \text{a.c.} & \downarrow exp \\ x_0 \in X & \xrightarrow[f]{cont} & S^1 \ni f(x_0) \end{array}$$

$M = \sup? \cancel{\max} \{ m \in [0, 2\pi] \mid \exists g: [0, m] \to \mathbb{R} \}$
$e^{ig(t)} = f(e^{it})$
$t \in [0, m]$

P.D. $M = 2\pi$.

$$\begin{array}{ccc} [0 \quad m \quad 2\pi] & \xrightarrow{g} & g(m) \\ \varphi \ \wr\wr \quad \downarrow exp & & \wr\wr \ \downarrow \quad \psi \\ S^1 \ni z & \xrightarrow{f} & w = f(z) \in S^1 \end{array}$$

$\psi \cdot g = f \circ \varphi$

$g := \psi^{-1} f \cdot \varphi$

<u>Lemma</u>: $\exp : \mathbb{R} \to S^1$, $t \mapsto e^{it}$, es

① difeo loc.

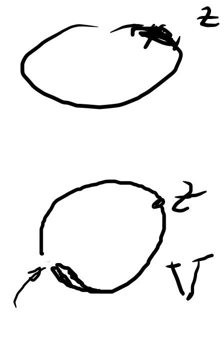
$f: X \to Y$, $\dim X = \dim Y$, y val. reg $\in \mathrm{Im}(f)$

② $\forall z \in S^1$, $\exists$ una vec $V \subset S^1$ de z, tq. $\exp^{-1}(V) = \coprod U_i$, $U_i \subset \mathbb{R}$ intervalo

y $\exp|_{U_i} : U_i \to V$ difeo.

$U_i = (\text{———})$ 2π

$V = S^1 \setminus \{-z\}$ $-z$



$df(x)$ iso. $x \in f^{-1}(y)$ $\Rightarrow$ f difeo loc. en una vecindad $\Rightarrow$ "stack of records"

<u>Si $M < 2\pi$</u>

$[\text{———}(\ )\,)$ M

$M = \sup\{m \geq 0 \mid g : (-m, m) \to \mathbb{R}$ levant. de $f\}$

$$\boxed{\pi_1(S^1) = \mathbb{Z}}$$

<u>Massey</u>.

$\Rightarrow$ $M = \infty$ (mismo argumento).