Notes no. 2

March 21, 2024

1 Envelopes of curves

1.1 Fireworks

Consider the family of trajectories of objects (fireworks) which are being ejected simultaneously from a point, with fixed initial velocity, in all possible directions.

Q: What is the "envelope" of all these trajectories?

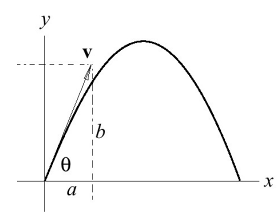
A: an inverted parabola, with a focus at the launching point.

Solution. Place the launching point of the objects at the origin of a vertical plane (the xy plane). If the initial velocity vector of a trajectory is $\mathbf{v} = (a, b)$, with $v = \|\mathbf{v}\| = \sqrt{a^2 + b^2} = const.$, then the trajectory, parametrized by time t, is given by

$$x = at, \quad y = -\frac{t^2}{2} + bt,$$

(supposing the acceleration of gravity is 1; which can be arranged by adjusting the unit of time).

Eliminating t from these two equations, and using $\tan\theta = b/a$, we get



$$y = -\frac{x^2}{2v^2}(1 + \tan^2\theta) + x(\tan\theta).$$

This is a parabola P_θ in the xy plane (its coefficients depend on the launching angle θ).

Denote by $\mathcal{E}$ the envelope of the family of parabolas $\{P_\theta\}$. Here is how one gets an equation for $\mathcal{E}$, somewhat informal (for more details one can consult for example the Vector Calculus book of Courant y John, vol. 2, Chap. 3.5).

We write 1st the equation of P_θ in the form

$$F(x, y, \theta) = y + \frac{x^2}{2v^2}(1 + \tan^2\theta) - x(\tan\theta) = 0. \tag{1}$$

Next, we parametrize the envelope $\mathcal{E}$ with the parameter θ of the curve P_θ that is tangent to $\mathcal{E}$; that is $F(x(\theta), y(\theta), \theta) = 0$. Taking derivative of the last equation of this equation wrt θ, one gets $F_x x' + F_y y' + F_\theta = 0$. Next, the velocity vector $(x'(\theta), y'(\theta))$ is *tangent* to P_θ at the point $(x(\theta), y(\theta))$, hence it is *perpendicular* to the gradient of the function defining P_θ in the point $(x(\theta), y(\theta))$; that is esto es, $F_x x' + F_y y' = 0$. Substructing this equation from $F_x x' + F_y y' + F_\theta = 0$, we obtain $F_\theta = 0$.

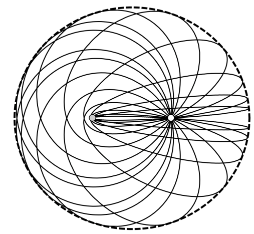
In summary, the equation for $\mathcal{E}$ is obtained by eliminating θ from the pair of equations in 3 variables

$$F(x, y, \theta) = 0, \quad F_\theta(x, y, \theta) = 0.$$

In our case, the equation $F_\theta = 0$ is $\tan(\theta) = v^2/x$. Sustituting this in $F = 0$ (equation (1)), the equation for $\mathcal{E}$ is

$$y = -\frac{x^2}{2v^2} + \frac{v^2}{2}. \tag{2}$$

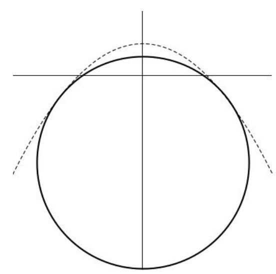
This is an equation of a parabola, with focus at the origen, whose "width" (at $y = 0$, the *latus rectum*) is $2v^2$ and whose "height" (the focal distance) is $v^2/2$ (this 1:4 ratio is for all parabolas, as they are all similar).



Note. In fact, the parabolic trajectories are an approximation to the true motion, where the acceleration due to gravity is not constant, but varies according to the distance to the center of the earth (the inverse square law). The trajectories are then *ellipses* instead of parabolas (for v not too big, $v < v_{esc} \approx$ 40,000 kmh), with one of their foci at the center of the earth (the other depend on the direction of launching). The envelope of the family of trajectories in this case turns out to be another elliptical trajectory, with one focus at the center of the earth and the other the launching point.

Excercise 2.1.* Prove the statement of the last sentence. (*Hint.* This family of elliptical trajectories all have the same size of major axis.)

1.2 An alternative derivation for the equation of the fireworks envelope $\mathcal{E}$ (without calculus!)



Let us imagine that all fireworks are launched at the same time $t = 0$ from the origin $(0, 0)$ in all directions, with the same initial velocity v. At some later time $t > 0$, they will all be on a curve given by

$$x = at, \quad y = -\frac{t^2}{2} + bt, \quad a^2 + b^2 = v^2.$$

Solving for a and b in the 1st two equations, then substituting in the 3rd, we obtain

$$x^2 + \left(y + \frac{t^2}{2}\right)^2 = (vt)^2. \tag{3}$$

This is the equation of a *circle*, centred at $(0, -t^2/2)$ with radius vt. It follows that the center of the circle is "falling" at the same time that the radius increases. In the beginning, when $0 \le t \le 2v$, we have $vt - t^2/2 > 0$, so that part of the circle is above the x-axis. But later, when $t > 2v$, the circle is totally below the x-axis.

Now, the points (x, y) inside the envelope $\mathcal{E}$ are reached *twice* by the fireworks, those outside it (the "safe zone") are never reached, hence the points of $\mathcal{E}$ are the points eached exactly once. That is, those are the points (x, y) for which equation (3) has a unique solution $t \ge 0$. This is an equation quadratic in t^2, whose discriminant is $v^4 + 2v^2y - x^2$. This vanishes when $y = (v^2 - x^2/v^2)/2$, which is equation (2). □

1.3 More fun with envelopes

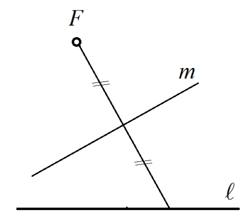
Excercise 2.2. Find the equation of the envelope of the family of line segments of length 1 connecting a point on the x-axis with a point of the y-axis. (Answer: the astroid $x^{2/3} + y^{2/3} = 1$. See the Gallery of Curves in Notes no. 1).

Excercise 2.3. Find the locus of midpoints of the segments of the previous exercise. But try 1st drawing this curve *before* making the calculation!

Excercise 2.4. Find the locus of the points that divide the segments of the last two exercises in some fixed proprotion.

Excercise 2.5. Find the envelope of family of ellipses with constant area, $(x/a)^2+(y/b)^2 = 1$, $ab = const.$

Excercise 2.6. Find the envelope of family of ellipses with constant sum of axes, $(x/a)^2 + (y/b)^2 = 1$, $a + b = const.$

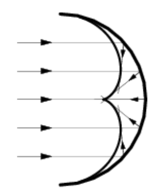


Excercise 2.7. Fix a point F and a line ℓ in the plane. Find the envelope of the perpendicular bisectors of the line segments connecting F with ℓ.

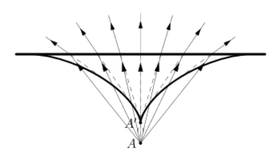
(*Answer.* Its a parabola with focus at F and directrz ℓ. This provides a practical method for constructing a parabola with origami.)

Excercise 2.8. Find a similar method to construct an ellipse.

Excercise 2.9. Find the evolute of a cycloid. (*Answer.* A congruent cycloid.)



Excercise 2.10. Find the envelope of parallel rays, after reflection by a semi-circular mirror.



Excercise 2.11.* Find the envelope of the refracted light rays ('cuastic') emanating from a point A below the x-axis, with indices of refraction $n_+ < n_-$ above and below the x-axis. In particular, find the location of the cusp A' of this envelope. This explains why pools seem shallower then they are. (Suggestion. Look up Snell's law).

2 The 4 vertex theorem

Theorem. *Every closed simple curve in the plane has at least 4 vertices (critical points of the curvature, where $\kappa' = 0$). It follows that the evolute of the curve has at least 4 cusps.*

Excercise 2.12. Find an example showing that the theorem is not true without the simplicity assumption.

Here we show the theorem only for a *convex* curve (each segment connecting two points of the curve is contained in the interior of the curve).

Proof. A closed curve C is compact, hence κ has at least two critical points, $c_{min}, c_{max} \in C$. We show that in at least one of the two arcs of C connecting c_{min} with c_{max} there is a local minimum, hence also a local maximum. If it is not true, then κ is a monotone function, along each of these two arcs, increasing or descending, depending on the sense in which the curve is parametrized. We now place the curve in the xyv plane so that c_{min} and c_{max} are on the x-axis and we parametrize the curve by arc length s, $0 \leq s \leq L$. We thus have $\kappa' \geq 0$ for $y \geq 0$ and $\kappa' \leq 0$ for $y \leq 0$, so that $\kappa' y \geq 0$ along the whole curve. Integrating $\kappa' y$ around the curve we obtain

$$0 \leq \int_0^L \kappa' y = \kappa y\Big|_0^L - \int_0^L \kappa y'.$$

Now $\kappa y\Big|_0^L = 0$ (for being a closed curve) and $\kappa y' = x''$, from the Frenet-Serret equations. Thus

$$0 \leq \int_0^L \kappa' y = -\int_0^L x'' = -x'\Big|_0^L = 0.$$

We conclude that $\kappa' y = 0$ identically, thus $\kappa' = 0$, since y vanishes only at c_{min} and c_{max}. $\square$

Note. This theorem has many other beautiful proofs. See for example page 81 of the book The geometry of billiards by S. Tabachnikov.