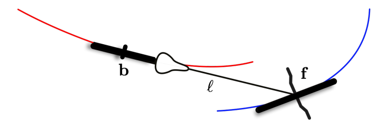
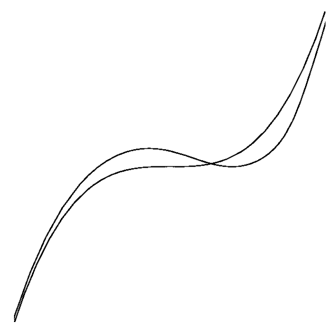
# Notes no. 3

April 2, 2024

## 1 Bicycle tracks




A simple mathematical model for bicycle motion consists of a directed line segment in $\mathbb{R}^2$ of length $\ell$ (the 'bicycle frame'), with end pts $\mathbf{b}$ and $\mathbf{f}$ (the 'back' and 'front' wheels), moving so that (1) its length is unchanged, $|\mathbf{f}(t) - \mathbf{b}(t)| = \ell$ for all $t$, and (2) the back end moves in the direction of the frame, ie $\mathbf{b}'(t)$ is a multiple of $\mathbf{f}(t) - \mathbf{b}(t)$ for all $t$. Condition (2) is called *the no-skid condition.* The pair of curves traced by the front and back ends of the line segment are the bicycle *front* and *back track* (resp.).



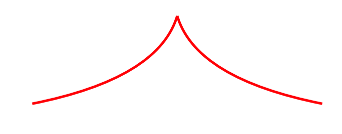
**Exercise 3.1.** Here is an example of such a pair of curves. Can you tell which is the front and which is the back track?

If the back track $\mathbf{b}(t)$ is given then clearly the front track is $\mathbf{f}(t) = \mathbf{b}(t) + \ell\mathbf{b}'(t)/|\mathbf{b}'(t)|$, provided $\mathbf{b}'(t)$ does not vanish. Suppose the front track $\mathbf{f}(t) = (x(t), y(t))$ is given. Can we determine the back track $\mathbf{b}(t)$? From condition (1) we can write $\mathbf{b}(t) = \mathbf{f}(t) + \ell\mathbf{r}(t)$, for some $\mathbf{r}(t) \in S^1$, ie $|\mathbf{r}(t)| = 1$ for all $t$. This takes care of condition (1). How about the no-skid condition (condition (2))?

**Proposition.** *Let* $\mathbf{b}(t) = \mathbf{f}(t) + \ell\mathbf{r}(t)$*, where* $\mathbf{f}(t) = (x(t), y(t))$ *and* $\mathbf{r}(t) = (\cos\theta(t), \sin\theta(t))$*. Then the no skid condition is equivalent to*

$$\ell\theta' = x'\sin\theta - y'\cos\theta. \tag{1}$$

The last equation is the *bicycle equation.*



**Exercise 3.2.** Solve the bicycle eqn for $\mathbf{f}(t) = (t, 0)$. Draw the back track $\mathbf{b}(t)$ for $\theta(0) \neq 0$. (*Ans.* A tractrix.)

Let us fix two points along the front track, say $\mathbf{f}(t_0), \mathbf{f}(t_1)$. For each $\theta_0$, solving the bicycle eqn (1) with the initial condition $\theta(0) = \theta_0$, defines a terminal condition $\theta_1 = \theta(t_1)$. This defines a map $M : S^1 \to S^1$, $e^{i\theta_0} \mapsto e^{i\theta_1}$.

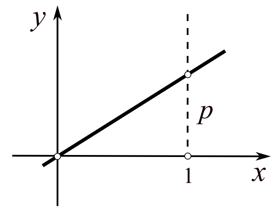
**Exercise 3.3.** Show that $M$ is a diffeomorphism (a smooth map with a smooth inverse). Suggestion: use the basic existence and uniqueness theorem for solutions of ODE.

**Proposition.** *$M$ is a Möbius transformation.*

**A reminder on Möbius transformations.** This is a class of diffeomorphisms of $S^1$, forming a group (ie the composition of two such maps is also a Mobius transformation, same for the inverse). They are also defined for $S^n$, $n > 1$, but here we are concerned only with $S^1$.

Consider a 2 by 2 invertible real matrix $g \in \mathrm{GL}_2(\mathbb{R})$,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc \neq 0.$$



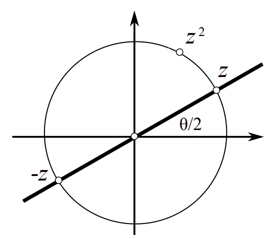

It maps each 1-dimensional subspace of $\mathbb{R}^2$ (a line through the origin) to the same kind of object, so $g$ acts on $\mathbb{RP}^1$, the space of 1-dimensional subspaces of $\mathbb{R}^2$, the *projectivization* of the linear action on $\mathbb{R}^2$. We assign to a line through the origin in the $xy$ plane, distinct from the $y$-axis, tits slope $p$, ie its intersection with the line $x = 1$ is $(1, p)$.

**Exercise 3.4.** Show that with this parametrization, $g$ acts on $\mathbb{RP}^1$ by

$$p \mapsto \frac{c + dp}{a + bp}.$$

In the coordinate $q = 1/p$, the formula is

$$q \mapsto \frac{aq + b}{cq + q}.$$




We can also identify $\mathbb{RP}^1$ with $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, by assigning to a line through the origin the point $z^2 \in S^1$, where $\pm z \in S^1$ are the intersection pts of the line with $S^1$.

**Exercise 3.5.** Let $z^2 = e^{i\theta}$, express the $p$ coordinate on $\mathbb{RP}^1$ in terms of $\theta$. (*Ans.* $p = \tan(\theta/2)$.)

Note that the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{RP}^1$ is not effective: the non-zero scalar multiples of the identity matrix act trivially on $\mathbb{RP}^1$. The quotient group is the *projective group*, $\mathrm{PGL}_2(\mathbb{R}) := \mathrm{GL}_2(\mathbb{R})/\mathbb{R}^*\mathrm{Id}$.

**Exercise 3.6.** (a) The action of $\mathrm{PGL}_2(\mathbb{R})$ on $\mathbb{RP}^1$ is triply transitive: for every two triples of *distinct* pts, $p_i, q_i \in \mathbb{RP}^1$, $i = 1, 2, 3$, there is a $g \in \mathrm{GL}_2(\mathbb{R})$ mapptng the 1st triple to the 2nd. (b) Furthermore, $g$ is unique up to multiplication of a non-zero scalar; ie, $[g] \in \mathrm{PGL}_2(\mathbb{R})$ is unique. (c) The $\mathrm{PGL}_2(\mathbb{R})$-action on $\mathbb{RP}^1$ preserves the *cross ratio*: $[x_1, x_2, x_2, x_4] := (x_1 - x_3)/(x_1 - x_4) \div (x_2 - x_3)/(x_2 - x_4)$. (d)$^*$ A diffeomorphism of $S^1$ which preserves the cross ratio is a Möbius transformation. (e) Every Mobius transformation is a composition of *dilation*, $x \mapsto ax$, $a \neq 0$, a *translation*, $x \mapsto x + b$, and *inversion*, $x \mapsto -1/x$.

**Proposition.** *A non-trivial Mobius transformation* $[g] \in \mathrm{PGL}_2(\mathbb{R})$ *may have 0,1, or 2 fixed points in* $S^1$. *If* $\det(g) = 1$ *then there are* $0, 1,$ *or* $2$ *fixed points according to* $|\mathrm{tr}(g)| < 2$, $= 2$ *or* $> 2$ *(resp.).*

*Proof.* The equation $g \cdot x = x$ is quadratic in $x$. Its discriminant is given by $|\mathrm{tr}(g)|$. □

**Definition.** A Mobius tranformation is called elliptic, parabolic or hyperbolic according to having 0, 1 or 2 fixed points (resp.).

**Exercise 3.7.** A Mobius transformation is elliptic iff it is conjugate to a rotation, $\theta \mapsto \theta + \theta_0$; it is parabolic iff it is conjugate to a translation, $x \mapsto x + x_0$; it is hyperbolic if it is conjugate to a dilation, $x \mapsto \lambda x$, $\lambda \neq 0, 1$. The corresponding matrices are

$$\begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix}, \quad \begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda & \\ 0 & 1/\lambda \end{pmatrix}.$$

Now given 4 functions $a(t), b(t), c(t), d(t)$, consider the linear system of ODE,

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \tag{2}$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \\ x_2(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a(t) & b(t) \\ \\ c(t) & d(t) \end{pmatrix}.$$

Let $g(t)$ be the fundamental system of solutions of (2), that is, the 1st and 2nd columns are the solutions satisfying $\mathbf{x}(0 = (1, 0)^t$ (1st column), $\mathbf{x}(0 = (0, 1)^t$ (2nd column). In other words, $g(0) = \mathrm{Id}$ and $\dot{g} = A(t)g$.

**Exercise 3.8.** If $\mathrm{tr}(A(t)) = 0$, ie $d(t) = -a(t)$, then $\det(g(t)) = 1$.

**Proposition.** *Let $\mathbf{x}(t) = (x_1(t), x_2(t))$ be a solution of (2). Then $p(t) := x_2(t)/x_1(t)$ is a solution of*

$$\dot{p} = c(t) - 2ap(t) - b(t)p^2. \tag{3}$$

The last type of eqn is called a *Ricatti equation.* It is the vector field induced on $\mathbb{RP}^1$ by the linear system (2).

**Exercise 3.9.** Write down the ODE on $\mathbb{RP}^1$ satisfied by $\theta(t)$, corresponding to the linear system (2). (Suggestion. Make the change of variable $p = \tan(\theta/2)$ in eqn (3).)

Now given a front track $\mathbf{f}(t) = (x(t), y(t))$, let

$$A(t) = -\frac{1}{2\ell} \begin{pmatrix} x(t) & y(t) \\ \\ y(t) & -x(t). \end{pmatrix}.$$

**Exercise 3.10.** Show that the Ricatti eqn ion $\mathbb{RP}^1$, in the coordinate $\theta$, is the bicycle eqn. This proves that the bicycle monodromy is a Mobius transformation.