# $SO(3)$ invariant Yang-Mills fields which are not self-dual.

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UC Berkeley, 1989

## Introduction

This note considers $SO(3)$ invariant solutions to the Yang-Mills equations on the standard four-sphere $S^4$ in five-dimensional Euclidean space $\mathbb{R}^5$, with $SU(2)$ as a gauge group. The $SO(3)$ action on $S^4$ is induced by the irreducible representation of $SO(3)$ on $\mathbb{R}^5$. Since a generic orbit of this action is three- dimensional, the Yang-Mills equations reduce to a system of ordinary differential equations with certain boundary conditions.

Our work is motivated by the following question:

> Are all solutions to the SU(2)-Yang-Mills equations over $S^4$ (anti-)self dual?

Sibner, Sibner, and Uhlenbeck[1] recently answered "no" to this question, by proving the existence of a non self-dual solution to these equations. Their proof is far from constructive. We believe that by using our symmetry we can find more explicit non self-dual connections of any Chern class.

The Yang-Mills (YM) equations are the Euler-Lagrange equations for the Yang-Mills action functional

$$YM(\omega) = \int_{S^4} \|F\|^2.$$

Here $F$ is the curvature of $\omega$, a connection defined on some principal $SU(2)$ bundle $P$ over $S^4$. Isomorphism classes of such bundles are classified by their second Chern number $c_2$, an integer. A connection $\omega$ is (anti-)self dual if and only if it minimizes $YM$ among all smooth connections defined on $P$, in which case $YM(\omega) = 8\pi^2|c_2|$.

In order to define invariant connections it is neccessary to lift the $SO(3)$ action to an action by $SU(2)$, the universal cover of SO(3). The decomposition of $SU(2)$ bundles with $SU(2)$ symmetry into (equivariant) isomorphism classes is finer than that of all $SU(2)$ principle bundles. In fact, we will show that the isomorphism classes of these bundles with symmetry are indexed by pairs of integers $(n_-, n_+)$ , each equivalent to 1 mod 4, and with

$$c_2 = \frac{n_-^2 - n_+^2}{8}.$$

Thus, there can be a number of distinct bundles with symmetry for a given Chern number. In case of $c_2 = 0$, there are an infinite number, but only one of these contains a self dual connection. So if we can show that the Yang-Mills functional attains its infimum on these equivariant components, then we obtain the existence of an infinite family of non (anti-)self dual connections. So far, we have not succeeded.

In this note we set the groundwork for this project by showing how to classify the invariant connections, how to write their Yang-Mills equations as ODE's on an interval, and how to get the correct boundary conditions for this system of ODE's. The trickiest part is the boundary conditions, which encode the smoothness of the connection. In the final section we present several possible approaches for proving the existence of the minimum.

Some of this groundwork can be found in Urakawa [6], and in Harnad [2]. These authors work much more generally than we do, but do not work out any details of our specific symmetry.

Our investigation of this specific symmetry was inspired by the paper of Avron et al [3]. For each half-integer $J$ (representing the total spin of a fermionic system) they construct a family of quaternionic line bundles with connections over $S^4$ with this symmetry. They show that all but one of these connections is not self-dual.

## §1 Equivariant bundles

The equivariant YM set-up consists of

a) a principle $G$ bundle $P \to X$ with a group $S$ (for symmetry) acting on $P$ by bundle automorphisms,

b) an $S$ invariant connection $\omega$ on $P$ such that

c) $\omega$ satisfies the YM equations $D * D\omega = 0$.

In this note $X = S^4$, $G = SU(2)$, $S = SU(2)$ and the $S$ action is required to project to the action on $S^4$ given by the irreducible representation of $S$ on $\mathbb{R}^5$.

We consider first the lifting problem: classify all possible lifts of the given $S$ action on $S^4$ to bundle automorphisms of a principal $G$ bundle $P \to S^4$. (The $S$ action on $S^4$ has an ineffective kernel $\pm 1$, so it is actually an $S/\pm 1 \cong SO(3)$ action; we look for $S$ actions on $P$ instead of $SO(3)$ actions because there are more of them: every $SO(3)$ action defines an $S$ action via the projection $S \to SO(3)$ , but not conversely).

We begin by giving a more detailed description of the $S$ action on $S^4$. The space of $3\times 3$ traceless symmetric matrices $Q$ forms a real 5-dimensional vector space, $\mathbb{R}^5$, with norm $\|Q\|^2 = tr(Q^2)$. $SO(3)$ acts orthogonally on this vector space by conjugation of matrices, and hence $S$ acts via its projection to $SO(3)$. The action on $S^4$ is obtained by restricting this action to the four-sphere $S^4 = \{tr(Q^2) = 1\} \subset \mathbb{R}^5$.

This action on $S^4$ has two orbit types. The principal (generic) type consists of matrices $Q \in S^4$ with distinct eigenvalues. The singular orbits, of which there are two, consist of matrices with double eigenvalues. As a principal isotropy group we can take the the subgroup $\Gamma$ of $S$ which leaves the three coordinate axes (the eigenspaces of a generic matrix) invariant. This is a finite subgroup of $S$, and so the generic orbit is three dimensional. Each singular orbits is isomorphic to a real two-dimensional projective space , since, given two eigenvalues, one of which is degenerate, the symmetric matrix is uniquely determined by the choice of the nondegenerate eigenspace. On one singular orbit the nondegenerate eigenvalue is positive, and on the other it is negative.

For the rest of the paper we will identify $S$ with the group of unit quaternions. Then

$$\Gamma = \{\pm 1, \pm i, \pm j, \pm k\}.$$

Let

$$\Delta = \{diag(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1, \lambda_1 + \lambda_2 + \lambda_3 = 0\}$$

be the fixed-point set of $\Gamma$ in $S^4$.

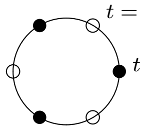
$\Delta$ is a great circle in $S^4$ parameterized by

$$c_t = \frac{\sqrt{3}}{2} diag(\cos(t), \cos(t + 2\pi/3), \cos(t - 2\pi/3)), 0 \le t \le 2\pi.$$

The arc $c_t, 0 \le t \le \pi/3$ is a global slice for the $S$ action: it intersects each orbit precisely once. The two endpoints $c_0$ and $c_{\pi/3}$ correspond to the two singular orbits, the interior of the arc to the principal orbits. Let $S_t$ be the isotropy subgroup of $c_t$. We have

$$S_0 = \{e^{i\theta}, je^{i\theta}\}, S_{\pi/3} = \{e^{j\theta}, ie^{j\theta}\}, S_t = \Gamma = S_0 \cap S_{\pi/3}, 0 < t < \pi/3.$$

Note that the two singular orbits intersect $\Delta$ at the 6 points $t = 0, \pi/3, 2\pi/3, \ldots$ etc., 3 per orbit (see figure).




A lift of the $S$ action to a principle $G$ bundle is described by the following data. Let $u_t, 0 \le t \le \pi/3$, be a lift of $c_t$ to $P$. This defines the holonomy representations $\lambda_t : S_t \to G$ by

$$s \cdot u_t = u_t \lambda_t(s), s \in S_t, 0 \le t \le \pi/3.$$

A different choice of $u_t$ differs by a curve $g_t \in G$. This results in the conjugation of $\lambda_t$ by $g_t$. This construction is reversible:

**proposition 1** *There is a 1-1 correspondence between*

*a) Isomorphism classes of $S$ equivariant principle $G$ bundles $P$ over $S^4$ (with the given $S$ action on $S^4$), and*

*b) conjugacy classes of homomorphisms $\lambda_t : S_t \to G$.*

See [5] for details.

A short calculation yields the following:

**proposition 2** *The conjugacy classes of homomorphisms $\lambda_t : S_t \to G$ are given by the following list:*

*a)* $\lambda_t \equiv 1$;

*b)* $\lambda_t(\gamma) = \gamma, \gamma \in \Gamma$,
$\lambda_0(e^{i\theta}) = e^{in_+\theta}$,
$\lambda_{\pi/3}(e^{j\theta}) = e^{jn_-\theta}$,
*where* $n_\pm \equiv 1 \pmod 4$, *i.e.* $n_\pm = 1, -3, 5, -7 \ldots$ *etc.*

**Remark:** In case (b), the lift $u_t$ can be chosen as follows: consider the action $p \to \gamma \cdot p\gamma^{-1}$ of $\Gamma$ on $P|_\Delta$. It is easily seen from the definitions that the fixed point set of this action intersects each fiber of $P|_\Delta$ in two antipodal points, and so forms a 2:1 covering space of the circle $\Delta$. Define $u_t$ to be a local section of this 2:1 covering defined over $c_t, 0 \leq t \leq \pi/3$. (A global section may not exist.) Clearly there are two such choices.

This completely solves the lifting problem.

## §2 Invariant connections

In this section we determine an explicit form for the $S$-invariant connections. This is based on propositions 1 and 2 describing the possible $S$ actions. We will soon see that the only connection having the symmetry of type (a) of proposition 2 is the flat connection on the trivial bundle. Thus, we will be concentrating on connections having symmetry type (b).

If $\omega$ is an $S$ invariant connection on $P$ then it is determined by its values along a lift $u_t$ of the section $c_t$ of the $S$ action on $S^4$. When the symmetry of $P$ is of type (b) we fix this lift according to the remark at the end of the previous section . In case the symmetry is of type (a) we take $u_t$ to be a horizontal lift of $c_t$.

For a symmetry generator $X \in$ Lie algebra of $S$ we define

$$\phi_t(X) = \omega(X_{u_t}) \in \text{Lie algebra of } G.$$

This is the " vertical" component, or Higgs field, corresponding to the $X$ action at $u_t$. Note that in case (b) the definition of $\phi_t$ does not depend on which of the two lifts of $c_t$ we take, for they differ by right multiplication by $-1$, and this acts trivially on $Lie(G)$.

More generally, define $\phi\ : P \to\ Hom(Lie(S), Lie(G))$ by

$$\phi_p(X) = \omega(X_p).$$

If $\omega$ is $S$-invariant, this has the symmetries $\phi_{pg} = g \cdot \phi_p$, and $\phi_{sp} = \phi_p \cdot s$, where $g\cdot$ and $\cdot s$ mean the composition with the adjoint actions of these elements on the appropriate Lie algebra. These imply the following symmetries in our situation:

i) $\phi_t(\gamma \cdot X) = \lambda_t(\gamma) \cdot \phi_t(X)$,

ii) $\omega(\dot{u_t}) = \lambda_t(\gamma) \cdot \omega(\dot{u_t})$.

where $\lambda_t$ is given by proposition 2. In case (a) of that proposition these equivariance conditions easily imply that $\phi_t \equiv 0$. Since we chose $\omega(\dot{u_t}) \equiv 0$, $\omega$ is the flat connection on the trivial bundle. We therefore concentrate on case (b). Now $Lie(G) = Lie(S) = \mathbb{R}^3$, so that $\phi_t \in\ Hom(\mathbb{R}^3, \mathbb{R}^3)$, and $\Gamma$ acts on each of these $\mathbb{R}^3$'s by reflections about the $i, j, k$ axes. Thus the equivariance condition (i) implies that $\phi_t$ fixes these axes. Then

$$\phi_t = a_1(t) \cdot i \otimes \sigma^1 + a_2(t) \cdot j \otimes \sigma^2 + a_3(t) \cdot k \otimes \sigma^3$$

where $\{\sigma^1, \sigma^2, \sigma^3\}$ is the dual basis to $\{i, j, k\}$ and $a_i$ are real valued functions on $[0, \pi/3]$. Similarly, the $\Gamma$ equivariance property (ii) implies that $\omega(\dot{u}_t) = 0$ (i.e. $u_t$ is horizontal after all) since the only $\Gamma$ invariant vector of $\mathbb{R}^3$ is 0. So $a_1, a_2, a_3$ completely describe the $S$ invariant connection $\omega$.

The boundary conditions satisfied by the $a_i$ at $t = 0$ and $t = \pi/3$ are determined by using some additional symmetries of $\phi : \Delta \to \mathbb{R}^3 \cong 3 \times 3$ diagonal matrices; namely

1) $W$ equivariance, where $W = N/\Gamma$ and $N = N(\Gamma)$ is the normalizer of $\Gamma$ in $S$,

2) $S_t$ equivariance for $t = 0$ and $t = \pi/3$ .

In (1) $W \cong$ the group of permutations of $\{1, 2, 3\}$. It acts on $\mathbb{R}^3$ by permutation of the coordinates and on the circle $\Delta$ by reflections, i.e. (23) acts by $t \to -t$, (13) acts by $t \to \pi/3 - t$, etc. To verify this $W$ equivariance recall the general fact that for any $S$ action, $W$ acts on the fixed point set of $\Gamma$. In our case, $S$ acts on $P$ and $Hom(Lie(S), Lie(G))$ by conjugation. The $\Gamma$ fixed point set in $P$ is a double cover of $\Delta$ (see the remark at the end of section 1), and the fixed point set in $Hom(Lie(S), Lie(G))$ is "diagonal matrices"$\cong \mathbb{R}^3$. Since $\phi$ is $\pm 1$ invariant it descends to a $W$ equivariant map on $\Delta$. Finally, calculate that $N(\Gamma)$ projects to the subgroup of $SO(3)$ generated by $90^0$ rotations about the coordinate axes.

The above neccesary conditions, (1) and (2), are also sufficient for $\phi_t$ to represent a smooth $S$ invariant connection (see [5] for details):

**proposition 3** *There is a 1-1 correspondence between non-flat $S$ invariant connections on $S^4$ and maps $\phi = (a_1, a_2, a_3) : [0, \pi/3] \to \mathbb{R}^3$ which can be extended smoothly to a map of the circle $\Delta \supset [0, \pi/3]$ subject to the following conditions: At 0:*

- $a_1(0) = n_+ \equiv 1 \pmod 4$,
- $a_1(t) = a_1(-t)$,
- $a_2(t) = a_3(-t)$, *and*
- *if* $n_+ \neq 1$ *then* $a_2(0) = a_3(0) = 0$.

*At* $\pi/3$*:*

- $a_2(\pi/3) = n_- \equiv 1 \pmod 4$,
- $a_2(\pi/3 + t) = a_2(\pi/3 - t)$,
- $a_1(\pi/3 + t) = a_3(\pi/3 - t)$, *and*
- *if* $n_- \neq 1$ *then* $a_1(\pi/3) = a_3(\pi/3) = 0$.

## §3 The reduced YM equations

The YM equations in terms of $a_1, a_2, a_3$ are :

$$\frac{1}{4}\frac{d}{dt}(K_1 \dot{a}_1) = a_1\left(\frac{1}{K_1} + \frac{a_3^2}{K_2} + \frac{a_2^2}{K_3}\right) - a_2 a_3\left(\frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}\right), \ldots etc. \quad (1)$$

where "...etc." stands for another 2 similar equations that we get by cyclic permutations of 1,2,3. The $K_i$ are functions of t defined by

$$K_1(t) = \tfrac{3}{\sin t} - 4\sin t,$$
$$K_2(t) = K_1(t + 2\pi/3),$$
$$K_3(t) = K_1(t - 2\pi/3).$$

The (anti)self-dual equations are

$$\frac{1}{2}K_1\dot{a_1} = \pm(a_1 - a_2a_3), \ldots etc. \tag{2}$$

where "+" stands for the self-dual, and "−" for the anti self-dual equations.

To derive these equations we write the curvature along the cross-section $c_t$ in terms of the $a_i{}'s$

$$F = i[2(a_2a_3 - a_1)\sigma^2 \wedge \sigma^3 - \dot{a_1}\sigma^1 \wedge dt] + \ldots etc.$$

which gives the (anti-)self-dual equations (the coefficients $K_i$ arise from the fact that the $\sigma^i$ are not orthonormal). The YM equations can be derived from variation of the YM action

$$\int_{S^4} \|F\|^2 \propto \int_0^{\pi/3} dt[\frac{4(a_2a_3 - a_1)^2}{K_1} + K_1\dot{a_1}^2 + \ldots etc.]$$

with respect to $a_i$, or directly by evaluating $D * F = 0$ in terms of $a_i$.

## §4 Chern numbers

We now would like to calculate the second Chern number $c_2$ corresponding to each of the equivariant bundles given in proposition 2b.

**proposition 4** *The second Chern number of a bundle $P$ with the $n_\pm$ symmetry given in proposition 2b is*

$$c_2 = \frac{n_-^2 - n_+^2}{8}. \tag{3}$$

*Proof:* This is a straightforward calculation similar to the one in [3]. We use one of the equivariant connections in proposition 3 and the formula for its curvature in the previous section. We than have

$$tr(F^2) \propto (\dot{a_1}(a_2a_3 - a_1) + \ldots)\sigma^1\sigma^2\sigma^3 dt \propto \frac{d}{dt}(a_1a_2a_3 - \frac{a_1^2 + a_2^2 + a_3^2}{2})\sigma^1\sigma^2\sigma^3 dt,$$

$$\Rightarrow c_2 \propto \int_{S^4} tr(F^2) \propto a_1a_2a_3 - \frac{a_1^2 + a_2^2 + a_3^2}{2}\Big|_0^{\pi/3} v \propto n_-^2 - n_+^2.$$

We can determine the missing constant by some more care in the above calculation or by doing a special case, say $n_+ = 1, n_- = -3$, corresponding to the lift to the quaternionic Hopf bundle $S^7 \to S^4$ ($c_2 = -1$) by the irreducible representation on $\mathbb{C}^4$. QED

**Remarks: 1.** It is possible to calculate $c_2$ somewhat more simply without using the explicit form of an equivariant connection. This is based on a general equivariant integration formula due to Atiyah and Bott[4], but the above calculation is more elementry.

**2.** The lifts in the general $n_\pm$ case can also be realized concretely. These arise by the "spectral splitting" of the trivial rank-n quaternionic vector bundle $S^4 \times \mathbb{H}^n$ with respect to a certain linear operator on its sections (the "quadrupole Hamiltonian" in [3]).

## §5 Existence of non-dual connections.

If a particular equivariant class of connections does not posses (anti-)self dual connections than any critical point of the YM functional restricted to this class (e.g. a minimum) is a YM connection which is not (anti-)self dual.

The only (anti-)self dual connections on a trivial bundle are flat. The only equivariant flat connection has $a_1 = a_2 = a_3 \equiv 1$ and $n_+ = n_- = 1$ (by the curvature formula in section 3). Therefore, any equivariant YM connection on a bundle with $n_+ = n_- \neq 1$ is not (anti-)self dual.

There are a number of possible approaches for proving the existence of such a non self-dual solution. We have not yet been able to push any of these through to fruition. We will review one fairly convincing approach in some detail, and discuss some others briefly.

The most convincing approach so far is the direct method of the calculus of variations. What one has to do is show that the set of invariant connections satisfy the Palais-Smale condition. Tom Parker is working on a general theorem to this effect.

In more detail , this approach runs as follows. Take $\omega^i$ to be a sequence of invariant connections all lying in one invariant component, and satisfying $YM(\omega^i) \to \inf YM$, where the infimum is taken over this component. Then show

1) a subsequence of this sequence converges to some $\omega^\infty$,

2) $\omega^\infty$ is invariant, lies in the same invariant component as the $\omega^i$, and enjoys some regularity,

3) $YM(\omega^\infty) \leq YM(\omega^i)$,

4) the validity of the principle of symmetric criticality [5], which states that if an invariant field is extremal among invariant fields, then it is extremal among all fields.

(2) and (3) imply that $\omega^\infty$ satisfies the hypothesis of the principle of symmetric criticality, and so, if we can complete these steps we are done. Steps (1), (3), and (4) can be completed.

Step (1) is essentially Uhlenbeck's famed compactness theorem, which says the following. Suppose we are handed a sequence of connections on which $YM$ is bounded. Then there exists a finite set $D$ of points of $S^4$, a subsequence of the sequence, and a sequence of gauge transformations defined on $S^4 \setminus D$, such that after applying the gauge transformations to the connections of the subsequence, the resulting subsequence converges to some $\omega^\infty$. This convergence is in the weak topology of the Sobolev space of $L^1_{2,loc}$ connections on $S^4 \setminus D$. Moreover, the points of $D$, so called "bubbling off points", are characterized by concentration of curvature density there. If a connection is invariant, then the curvature density is an invariant function. Consequently, for invariant sequences, blow up of curvature must happen along entire group orbits. But $D$ is a discrete point set, and for our symmetry, no orbits are discrete. Consequently $D$ is automatically empty. This shows (1).

Proving the regularity of $\omega^\infty$ in step (2) is usually bound up with showing that it is a critical point. In regard to its invariance, the following observation is in order. Suppose that $\omega$ is an invariant connection and that $\omega'$ is obtained from $\omega$ by gauge transformation. Then $\omega'$ is itself invariant if and only if it equals $\omega$. This suggests that in applying the Uhlenbeck compactness result, we need not apply any gauge transformations.

In addition to this four-dimensional approach, there are one-dimensional approaches to existence. One approach is to apply the direct method of the calculus of variations to the one-dimensional action. The role played by the special endpoint conditions of proposition 3 is unclear from this 1-dimensional point of view. Another approach is to work directly with the one-dimensional ODE. The endpoints, 0 and $\pi/3$, are the only singular points for this ODE.

A fourth approach is to use the "ambi-twistor" method pioneered by Witten[7]. See also Isenberg, Yasskin and Green [8]. This is a generalization of the twistor method for self-dual solutions. Of the four approaches this is the most constructive.

## §6 Numerical evidence.

Notice that the reduced YM action and equations are invariant w.r.t. the reflection $t \to t+\pi$. (This interchanges dual and anti-self-dual solutions and flips the boundary conditions). It is therefore natural to consider reflection invariant solutions on the trivial bundle $n_+ = n_- \neq 1$. Combined with the $W$ symmetries of proposition 3 this means that a solution to the equations on $[0, \pi/6]$ with

$$a_1(\pi/6) - a_2(\pi/6) = a_1'(\pi/6) + a_2'(\pi/6) = a_3(\pi/6) = 0$$

extends to a smooth reflection-invariant solution on $[0, \pi/3]$ with $n_+ = n_-$.

A numerical integration of the equations for the case $n_+ = n_- = -3$ was done as follows: first a (truncated) power series expansion for the solutions was calculated at $t = 0$. This gives a family of solutions starting at $t = 0$ depending on the 3 parameters $a_1''(0), a_2'(0)$, and $a_2''(0)$. For example, the values $a_1''(0) = 2, a_2'(0) = \sqrt{3}, a_2''(0) = -1$ correspond to the anti-self dual solution $a_1(t) = -2\cos t - 1$. After "jumping-off" a small amount off the $t = 0$ singularity by the power series we begin a numerical integration (Runge-Kutta) up to $t = \pi/3$ where we check the above "reflectability" conditions. This method yields a solution for the values

$$a_1''(0) = 7, a_2'(0) = -4, a_2''(0) = -3$$

with approximately 1 precent error.

# References


[1] L. M. Sibner, R. J. Sibner, and K. Uhlenbeck, *Solutions to YM equations which are not self-dual,* Preprint (1989).

[2] J. Harnad, S. Shnider, and J.Tafel, *Group actions on principle bundles and dimesional reduction,* Lett. in math. Phys., **4** (1980), 107-113.

[3] J. E. Avron, L. Sadun, J. Segret, and B. Simon, *Chern numbers and Berry's Phase in Fermi systems,* Caltech preprint (1989).

[4] M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology,* Topology, **23** (No. 1) (1984),1-28.

[5] R. S. Palais, *The Principle of Symmetric Criticality,* Comm. Math. Phys., **69** (1979), 19-30.

[6] H. Urakawa, *Equivariant theory of YM connections over Riemannian manifolds of cohomogeneity one,* Indiana Univ. Math. J., **37** (No. 4188) (1988), 753-788.

[7] E.Witten ,*An interpetation of classical Yang-Mills theory,* Phys. Lett., **77B** (1978), 394.

[8] J. Isenberg, Ph. B. Yasskin, and P. G. Green, *Non-self-dual guage fields,* Phys. Lett., **78B** (1978), 462.