The canonical bundle of a hermitian manifold

Gil Bor & Luis Hernández-Lamoneda *
Centro de Investigación en Matemáticas, A.C. (CIMAT)
Guanajuato, México

7 october, 1998

Abstract

This note contains a simple formula (Proposition 1 in Section 3) for the curvature of the canonical line bundle on a hermitian manifold, using the Levi-Civita connection (instead of the more usual hermitian connection, compatible with the holomorphic structure). As an immediate application of this formula we derive the following: *the six-sphere does not admit a complex structure, orthogonal with respect to any metric in a neighborhood of the round one*. Moreover, we obtain such a neighborhood in terms of explicit bounds on the eigen-values of the curvature operator. This extends a theorem of LeBrun.

Keywords: Hermitian manifold, almost-complex structure, canonical bundle, curvature.
AMS Subject classification number: 53C55.

1 Introduction

First, some standard definitions. An *almost-complex structure* on an even-dimensional manifold M^{2n} is a smooth endomorphism $J : TM \to TM$, such that $J^2 = -Id$. The standard example is $M = \mathbb{C}^n$ with J given by the usual scalar multiplication by i. A *holomorphic map* between two almost-complex manifolds (M_1, J_1) and (M_2, J_2) is a smooth map $f : M_1 \to M_2$ satisfying $df \circ J_1 = J_2 \circ df$. An almost-complex structure is said to be *integrable,* or is called simply a *complex structure,* if it is locally holomorphicaly diffeomorphic to the standard example; in other words, for each $x \in M$ there exist neighborhoods $U \subset M$, $x \in U$, and $V \subset \mathbb{C}^n$, and a holomorphic diffeomorphism $f : U \to V$.

Given an even-dimensional manifold, how is one to decide if it admits a complex structure? There are some, more or less obvious, necessary conditions

*Both authors received support from grants 0329P-E of CONACyT and E130.728 of CONACyT-CSIC

(e.g. the existence of an almost-complex structure, which can be tested by characteristic classes), but in general there is no known answer to this question. A well-known example, so far undecided, is the 6-sphere (this is the only interesting dimension, because in all other dimensions $n \neq 2, 6$, the n-sphere does not admit even an almost-complex structure). This space admits a non-integrable almost-complex structure, but it is unknown as yet if it admits a complex structure.

A related question is that of the existence of an *orthogonal* complex structure. Here the set-up is the following: given an even-dimensional *riemannian* manifold (M, g), one is looking for an integrable almost-complex structure J which is orthogonal with respect to g; that is, $g(X, Y) = g(JX, JY)$, for all $X, Y \in T_m M$ and $m \in M$. One calls such a pair (g, J) a *hermitian* structure. The problem here is then that of extending a given riemannian structure to a hermitian one.

One way of analyzing the problem of the existence of orthogonal complex structures is to consider the space of all orthogonal almost-complex structures. These are sections of a bundle over M, whose fiber at a point of the manifold consists of all (linear) orthogonal complex structures on the tangent space at that point. The total space Z of this bundle is called the *twistor space* associated to (M, g) and it admits a tautological almost-complex structure. Then the idea is to translate differential geometric problems on M to complex-geometric problems on Z. For example, an orthogonal almost-complex structure on M is given, by definition, by a section of $Z \to M$; it will be *integrable* if the section is *holomorphic*, thus embedding M as a complex sub-manifold of Z. In other words, the problem of orthogonal complex structures on M is translated into that of certain complex submanifolds of Z. This approach leads to the proof of C. LeBrun of non-existence of an orthogonal complex structure on S^6 relative to the round metric [2]. The twistor space Z in this case turns out to be Kähler, so that an orthogonal complex structure on S^6 would give an embedding of S^6 as a complex submanifold of a Kähler one, thus inheriting a Kähler structure, which is clearly impossible for S^6 (since $H^2(S^6) = 0$). For more information on this approach to orthogonal complex structures we recommend the survey article of S. Salamon [3].

Here we suggest a different construction, considerably more elementary. This is based on the observation that the curvature of a connection on a complex line bundle is a closed two-form (representing the first Chern class of the line bundle, up to a constant), so one can try to use the given data (g, J) on M to construct a line bundle with connection whose curvature two-form is non-degenerate, i.e. a symplectic form. On certain manifolds this might be impossible (e.g. on a compact manifold with $H^2 = 0$), so if one uses a connection coming from the Levi-Civita connection on (M, g) then one obtains in this way a curvature obstruction for the existence of an orthogonal complex structure.

A natural complex line bundle to consider, for a given complex structure, is the so-called canonical line bundle $K := \Lambda^{n,0}(M)$ – the bundle of $(n,0)$-forms, or the top exterior power of the holomorphic cotangent bundle. Now there are two natural ways to use the hermitian structure on M to equip K with a connection. First, the complex structure on M induces a holomorphic structure on K and the riemannian metric on M induces a hermitian metric on K; these two in turn determine uniquely a canonical hermitian connection (a metric-preserving connection whose $(0,1)$-part coincides with the $\bar{\partial}$-operator of the complex structure on M; see for example Griffiths and Harris [1], p. 73). The other choice of a connection on K comes from the Levi-Civita connection on TM, extended (by the Leibniz rule) to the bundle of exterior n-forms $\Lambda^n(M)$, complexified, then projected orthogonally to the sub-bundle $K \subset \Lambda^n_{\mathbb{C}}(M)$.

Unless the orthogonal complex structure happens to be Kähler (i.e. the Kähler 2-form $\omega = g(J\cdot,\cdot)$ is closed), these two choices of a connection are different. We make here the second choice, the one coming from the Levi-Civita connection, as it seems to us more natural from a Riemannian geometric point of view, e.g. for relating the resulting curvature 2-form of the canonical bundle with the Riemann curvature tensor of (M,g).

The outcome then is a rather simple formula for the curvature of the canonical line bundle on a hermitian manifold (Proposition 1 of Section 3). From this formula it becomes obvious that a complex structure compatible with the round metric on the sphere will render the curvature 2-form of the corresponding canonical line bundle a symplectic form (in fact Kähler), and that this property will be maintained for nearby metrics (Corollaries 2 and 3 of Section 4). [1]

We shall now outline the details of the calculation indicated above. We need to recall first some standard terminology.

Let $E \to M$ be a complex hermitian vector bundle over a differentiable manifold, with a hermitian connection $D : \Gamma(E) \to \Gamma(T^*(M) \otimes E)$, i.e.

$$d\langle s_1, s_2\rangle = \langle Ds_1, s_2\rangle + \langle s_1, Ds_2\rangle$$

for any two sections $s_1, s_2 \in \Gamma(E)$.

The curvature R of (E,D) is defined by first extending D to $\Gamma(\Lambda^k(M)\otimes E) \to \Gamma(\Lambda^{k+1}(M) \otimes E)$,

$$D(\alpha \otimes s) := d\alpha \otimes s + (-1)^k \alpha \otimes Ds,$$

then

$$R := D^2 \in \Gamma(\Lambda^2(M) \otimes \mathrm{End}(E)).$$

[1] Claude LeBrun has informed us recently that his proof also extends to metrics near the round one, but this requires embedding the usual twistor space inside a larger one. Also, after completing the work described here we found two articles ([4] and [5]) containing ideas close to ours.

If $E_0 \subset E$ is a sub-bundle then there is an induced hermitian connection on E_0 as follows: let s_0 be a section of E_0, and let $(E_0)^\perp$ be the orthogonal complement of E_0 in E, then decompose orthogonally

$$Ds_0 = D_0 s_0 + \Phi s_0,$$

with

$$D_0 s_0 \in \Gamma(T^*(M) \otimes E_0), \quad \Phi s_0 \in \Gamma(T^*(M) \otimes (E_0)^\perp).$$

One then verifies easily that D_0 defines a hermitian connection on E_0 and that Φ is "tensorial", i.e. a section of $T^*(M) \otimes \mathrm{Hom}(E_0, (E_0)^\perp)$, called *the second fundamental form* of E_0 in E.

Now there is a well-known formula for the curvature of (E_0, D_0) in terms of the curvature R of (E, D) and the second fundamental form Φ of E_0 in E. It is given by

$$\Omega = \pi_0 \circ R \circ \pi_0^* + \Phi^* \wedge \Phi, \tag{1}$$

where $\pi_0 : E \to E_0$ is orthogonal projection. The (easy) calculation can be found for example in [1], p.78.

In our case, starting with the Levi-Civita connection on $\Lambda^n_{\mathbb{C}}(M)$ and projecting onto the canonical line-bundle $K = \Lambda^{n,0}(M) \subset \Lambda^n_{\mathbb{C}}(M)$, we find out the following:

1. $\pi_0 \circ R \circ \pi_0^* = i\mathcal{R}(\omega)$, where ω is the Kähler form and $\mathcal{R}$ is the interpretation of the Riemann curvature tensor of M as an operator in $\mathrm{End}(\Lambda^2(M))$ (see the corollary to Lemma 1 in Section 3).

2. The second fundamental form $\Phi \in \Lambda^1(M) \otimes \mathrm{Hom}(\Lambda^{n,0}, (\Lambda^{n,0})^\perp)$ is of type $(1, 0)$, hence $\Phi^* \wedge \Phi$ is non-positive (see Lemmas 2 and 3 in Section 3; see next section, Definition 2, for the sign convention).

The first fact does not require even the integrability of the orthogonal almost-complex structure, i.e. it holds also for almost-hermitian manifolds. The second one does depend on the integrability (in fact, it can be shown to be equivalent to the integrability of the almost-complex structure).

We use these two basic results to deduce rather easily the non-degeneracy of the 2-form Ω in the proof of the above mentioned theorem of LeBrun, as well as its extension to metrics which are nearby the round one (see Section 4).

2 Some definitions and notation

First, to make sense of Formula (1) in the Introduction, we need to review some terminology.

Let V be a real $2n$-dimensional vector space with a euclidean inner product $(\cdot, \cdot)$ and a linear orthogonal almost-complex structure J. We extend the inner product $(\cdot,\cdot)$ on V in the usual way to the real exterior algebra $\Lambda^*(V^*)$, by declaring the k-forms $\{\eta_{i_1}\wedge\ldots\wedge\eta_{i_k} | 1\leq i_1<\ldots<i_k\leq 2n\}$ an orthonormal basis of $\Lambda^k(V^*)$, where $\{\eta_1,\ldots,\eta_{2n}\}$ is the dual basis of an orthonormal basis of V.

We denote also by $(\cdot,\cdot)$ the *complex-linear* extension of the euclidean inner product $(\cdot,\cdot)$ to the complexified vector spaces $\Lambda^k_{\mathbb{C}}(V^*)=\Lambda^k(V^*)\otimes\mathbb{C}$. The *hermitian* inner-product on these spaces is thus given by $\langle\phi,\psi\rangle=(\phi,\overline{\psi})$.

Next, let W be a complex vector space with an hermitian inner product $\langle\cdot,\cdot\rangle$ and denote by $\mathrm{End}_{\mathbb{C}}(W)$ the complex-linear endomorphisms of W. Denote by $\mathrm{End}(V)$ the real endomorphisms of V.

All tensor products, unless denoted otherwise, are over the reals.

Definition 1 *Let V and W be as above, and $\alpha,\beta\in V^*\otimes\mathrm{End}_{\mathbb{C}}(W)$ two endomorphism-valued 1-forms.*

1. *The wedge product $\alpha\wedge\beta\in\Lambda^2(V^*)\otimes\mathrm{End}_{\mathbb{C}}(W)$ is defined by*

$$\alpha\wedge\beta(X,Y):=\alpha(X)\circ\beta(Y)-\alpha(Y)\circ\beta(X).$$

Equivalently, if $\alpha=a\otimes A$, $\beta=b\otimes B$, where $a,b\in V^*$ and $A,B\in\mathrm{End}_{\mathbb{C}}(W)$, then $\alpha\wedge\beta:=(a\wedge b)\otimes(A\circ B)$.

2. *The adjoint $\alpha^*\in V^*\otimes\mathrm{End}_{\mathbb{C}}(W)$ is defined by*

$$\alpha^*(X)=[\alpha(X)]^*.$$

Equivalently, for $\alpha=a\otimes A$, $\alpha^*=a\otimes A^*$.

Note that when extending the notation to complex forms in $V^*_{\mathbb{C}}\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}(W)$, one has that $\alpha^*(Z)=[\alpha(\bar{Z})]^*$, $Z\in V_{\mathbb{C}}$, so that if $\alpha=\phi\otimes A$, where $\phi\in V^*_{\mathbb{C}}$, then $\alpha^*=\bar{\phi}\otimes A^*$. (Proof: if $Z=X+iY$, then $\alpha^*(Z)=\alpha^*(X)+i\alpha^*(Y)=[\alpha(X)]^*+i[\alpha(Y)]^*=[\alpha(X)-i\alpha(Y)]^*=[\alpha(\bar{Z})]^*$.) Hence if α is of type $(1,0)$ then α^* is of type $(0,1)$ etc.

Next, we need to make some convention concerning **positivity** (watch for a confusing error in [1], pp. 29 & 79, around this definition).

Definition 2 1. *A 2-form $\omega\in\Lambda^2(V^*)$ is called positive, $\omega>0$, if $B(X,Y)=\omega(X,JY)$ is a symmetric positive bilinear form. Equivalently: ω is positive if it is a real 2-form of type $(1,1)$ (that's the "symmetric" requirement) and $\omega(X',\bar{X}')/i>0$ for all non-zero $X'\in V^{1,0}$, where $V_{\mathbb{C}}=V^{1,0}\oplus V^{0,1}$ is the decomposition of the complexification of V into $\pm i$ eigen-spaces of J. Obviously, a positive (or negative) 2-form is non-degenerate.*

2. *Now let $\Omega \in \Lambda^2(V^*) \otimes \mathrm{End}_{\mathbb{C}}(W)$ be a 2-form on V with values in* anti-hermitian *endomorphisms on W, so $i\Omega$ is an hermitian-valued 2-form (we have in mind the curvature of a hermitian connection). Then Ω is called positive, $\Omega > 0$, if $\langle i\Omega w, w\rangle$ is a positive 2-form for all non-zero $w \in W$. Equivalently, $\Omega > 0$ if it is an $\mathrm{End}(W)$-valued $(1,1)$-form such that $\Omega(X', \overline{X'})$ is a positive hermitian operator for all non-zero $X' \in V^{1,0}$. We define similarly $\Omega \geq 0$, $\Omega < 0$, etc.*

A word of caution: According to the last definition, the Kähler form $\omega = (J\cdot, \cdot)$ on V is a real positive 2-form, whereas $i\omega$ is an imaginary negative form.

Definition 3 *Let $A \in \mathrm{End}(V)$ be an antisymmetric endomorphism on V, i.e. $(Av, w) = -(v, Aw)$ for all $v, w \in V$. Define*

1. *$\hat{A} \in \Lambda^2(V^*)$ by $\hat{A}(v, w) = (Av, w)$.*

2. *$A^* \in \mathrm{End}(V^*)$ by $(A^*\eta)(v) = \eta(Av)$, as well as its extension to $\Lambda^*(V^*)$ as a derivation:*

$$A^*(\alpha \wedge \beta) = (A^*\alpha) \wedge \beta + \alpha \wedge (A^*\beta).$$

We use throughout the article the shorthand notation $\Lambda^k(M)$ for the bundle of alternating k-forms $\Lambda^k(T^*M)$.

Definition 4 1. *Let $R \in \Lambda^2(V^*) \otimes \mathrm{End}(V)$ (we have in mind the curvature tensor of the Levi-Civita connection on a riemannian manifold). Define $\mathcal{R} \in \mathrm{End}(\Lambda^2(V^*))$ as follows: if $R = \sum_j \alpha_j \otimes A_j$, where $\alpha_j \in \Lambda^2(V^*)$ and $A_j \in \mathrm{End}(V)$, then*

$$\mathcal{R}(\beta) = -\sum_j \alpha_j(\hat{A}_j, \beta), \qquad \beta \in \Lambda^2(V^*).$$

2. *Applying this definition to the curvature tensor of a riemannian manifold $R \in \Gamma(\Lambda^2(M) \otimes \mathrm{End}(TM))$, we obtain the so-called* curvature operator *$\mathcal{R} \in \Gamma(\mathrm{End}(\Lambda^2(T^*M)))$.*

Another word of caution concerning sign conventions: we have made the choice of signs in the above definitions so as to make $\mathcal{R}$ coincide with the curvature operator as defined in riemannian geometry. Thus, for example, the round sphere has a positive curvature operator (in fact, it is the identity operator). This is also tied up with our definition $R = D^2$, where there seems to be a conflict in the literature. In complex geometry it is usual to define the curvature of a connection by D^2, as we did in the Introduction. Thus, the curvature of the canonical bundle of $\mathbb{C}P^1$ is i times the area form. In riemannian geometry on the other hand, probably for historical reasons, the curvature tensor of the Levi-Civita connection is defined by the formula $\nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, which amounts to defining $R = -D^2$. Our sign choice in Definition 4 is made so as to reconcile this conflict.

3 Three lemmas on hermitian structures

The first lemma is quite simple, and has probably appeared elsewhere. The second is essentially in Griffiths and Harris ([1], p.79, after overcoming the positivity confusion). The third is a curious fact about the Levi-Civita connection on a hermitian manifold, probably known, though we could not find it in the literature.

Let V be, as in the last section, a euclidean $2n$-dimensional real vector space with an orthogonal almost-complex structure J, and let $\omega = (J\cdot, \cdot)$ denote the associated Kähler 2-form.

Lemma 1 *Let $A \in \mathrm{End}(V)$ be an antisymmetric endomorphism of V and let $\pi_0 : \Lambda^n_{\mathbb{C}}(V^*) \to \Lambda^{n,0}(V^*)$ denote orthogonal projection. Then*

$$\pi_0 \circ A^* \circ \pi_0^* = i(\hat{A}, \omega),$$

where $A^* \in \mathrm{End}(\Lambda^n_{\mathbb{C}}(V^*))$ and $\hat{A} \in \Lambda^2(V^*)$ are given above in Definition 3.

Proof. Choose a unitary basis $\theta_1, \ldots, \theta_n$ for $(V^*)^{1,0}$, so that

$$\omega = i(\theta_1 \wedge \bar{\theta_1} + \ldots + \theta_n \wedge \bar{\theta_n}).$$

Now $\psi = \theta_1 \wedge \ldots \wedge \theta_n$ is a unitary element of $\Lambda^{n,0}(V^*)$, hence

$$\begin{aligned}
\pi_0 \circ A^* \circ \pi_0^* &= (A^*\psi, \bar{\psi}) = \\
&= ((A^*\theta_1) \wedge \theta_2 \wedge \ldots \wedge \theta_n \, , \, \bar{\theta}_1 \wedge \bar{\theta}_2 \wedge \ldots \wedge \bar{\theta}_n) \; + \\
&\quad + (\theta_1 \wedge (A^*\theta_2) \wedge \ldots \wedge \theta_n \, , \, \bar{\theta}_1 \wedge \bar{\theta}_2 \wedge \ldots \wedge \bar{\theta}_n) \; + \; \ldots \\
&= (A^*\theta_1, \bar{\theta_1}) + \cdots + (A^*\theta_n, \bar{\theta_n}).
\end{aligned}$$

Now, given any $\alpha, \beta \in V^*$, one can check easily from our definition of $\hat{A}$ that

$$(A^*\alpha, \beta) = -(\hat{A}, \alpha \wedge \beta),$$

hence

$$\pi_0 \circ A^* \circ \pi_0^* = -(\hat{A}, \theta_1 \wedge \bar{\theta_1} + \cdots + \theta_n \wedge \bar{\theta_n}) = i(\hat{A}, \omega),$$

as claimed. □

Corollary 1 *Let (M, g, J) be an almost-hermitian manifold and let $\pi_0 : \Lambda^2(M) \otimes \Lambda^n_{\mathbb{C}}(M) \to \Lambda^2(M) \otimes \Lambda^{n,0}(M)$ denote orthogonal projection in the second factor. Then*

$$\pi_0 \circ R \circ \pi_0^* = i\mathcal{R}(\omega),$$

where $\omega = g(J\cdot, \cdot)$ is the Kähler form, $R \in \Gamma(\Lambda^2(M) \otimes \mathrm{End}(\Lambda^n_{\mathbb{C}}(M))$ is the curvature of the connection induced on $\Lambda^n_{\mathbb{C}}(M)$ by the Levi-Civita connection on TM, and $\mathcal{R}$ is the curvature operator associated to the riemannian metric (as in Definition 4 above).

Proof. The main point to notice is that if the curvature tensor of a connection on TM is given (locally) by $\sum \alpha_j \otimes A_j$, where $\alpha_j \in \Gamma(\Lambda^2(M))$ and $A_j \in \Gamma(\mathrm{End}(TM))$, then the curvature tensor of the induced connection on $\Lambda^n_{\mathbb{C}}(M)$ is given by $-\sum \alpha_j \otimes A_j^*$, with A_j^* given by Definition 3. The result now follows immediately from the previous lemma and the definition of $\mathcal{R}$. $\square$

Lemma 2 *If $\Phi \in \Lambda^{1,0}(V^*) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(W)$, where W is a hermitian vector space, then $\Phi^* \wedge \Phi \le 0$.*

Proof. As noted above (Section 2, after Definition 1), Φ^* is of type $(0,1)$, hence $\Phi^* \wedge \Phi$ is of type $(1,1)$. Next, for any $X' \in V^{1,0}$,

$$\begin{aligned}(\Phi^* \wedge \Phi)(X', \bar{X}') &= \Phi^*(X')\Phi(\bar{X}') - \Phi^*(\bar{X}')\Phi(X') = \\ &= -\Phi^*(\bar{X}')\Phi(X') = -[\Phi(X')]^*\Phi(X'),\end{aligned}$$

and the claim follows since A^*A is a hermitian non-negative operator for any $A \in \mathrm{End}_{\mathbb{C}}(W)$. $\square$

Lemma 3 *Let M^{2n} be a riemannian manifold with an orthogonal complex structure (i.e. a hermitian manifold). Denote by ∇ the Levi-Civita connection on TM, as well as its extension to $\Lambda^*_{\mathbb{C}}(M)$ (using the Leibniz rule). Then the second fundamental form of the canonical bundle $K = \Lambda^{n,0}(M) \subset \Lambda^n_{\mathbb{C}}(M)$, with respect to the Levi-Civita connection, is of type $(1,0)$ (as in the previous Lemma).*

Proof. In fact, the statement is true for all the sub-bundles $\Lambda^{k,0}(M) \subset \Lambda^k_{\mathbb{C}}(M)$, $k = 1, 2, \ldots, n$, and follows from the case $k = 1$. To see this, let $\theta_1, \ldots, \theta_n$ be a local framing of $\Lambda^{1,0}(M)$, and

$$\nabla\theta_i = \sum_j \left(\alpha_{ij} \otimes \theta_j + \beta_{ij} \otimes \bar{\theta}_j\right), \qquad \alpha_{ij}, \beta_{ij} \in \Lambda^1_{\mathbb{C}}(M).$$

Then for $k = 1$ the claim is that the 1-forms β_{ij} are of type $(1,0)$. If this is true, then for any $k \ge 1$,

$$\begin{aligned}\nabla\left(\theta_{i_1} \wedge \ldots \wedge \theta_{i_k}\right) &= (\nabla\theta_{i_1}) \wedge \theta_{i_2} \wedge \ldots \wedge \theta_{i_k} + \theta_{i_1} \wedge (\nabla\theta_{i_2}) \wedge \theta_{i_3} \wedge \ldots \wedge \theta_{i_k} + \ldots \\ &= \sum_j \beta_{i_1,j} \otimes (\bar{\theta}_j \wedge \theta_{i_2} \wedge \ldots \wedge \theta_{i_k}) + \beta_{i_2,j} \otimes (\theta_{i_1} \wedge \bar{\theta}_j \wedge \ldots \wedge \theta_{i_k}) + \ldots \\ &\quad \ldots + \left(\text{something in } \Lambda^1_{\mathbb{C}} \otimes_{\mathbb{C}} \Lambda^{k,0}\right)\end{aligned}$$

so that the second fundamental form of $\Lambda^{k,0}(M) \subset \Lambda^k_{\mathbb{C}}(M)$ is of type $(1,0)$.

Now for the case $k = 1$, i.e. to see that the 1-forms β_{ij} above are of type $(1,0)$, we argue as follows. First, we pick the frame $\theta_1, \ldots, \theta_n$ to be a *unitary* frame, i.e. $(\theta_i, \bar{\theta}_j) = \delta_{ij}$, It then follows that

$$0 = d(\theta_i, \theta_j) = (\nabla\theta_i, \theta_j) + (\theta_i, \nabla\theta_j) = \beta_{ij} + \beta_{ji},$$

i.e. $\beta_{ij} = -\beta_{ji}$.

Next, by the torsion-freeness of ∇, we have

$$d\theta_i = \text{anti-symmetrization of } \nabla\theta_i = \sum_j \left(\alpha_{ij} \wedge \theta_j + \beta_{ij} \wedge \bar{\theta}_j\right).$$

Now, for an integrable almost-complex structure, we have the vanishing of the $(0,2)$-component of $d : \Lambda^{1,0} \to \Lambda^2$, hence, taking $(0,2)$-components of the last equation we have

$$0 = \sum_j \beta''_{ij} \wedge \bar{\theta}_j,$$

where β''_{ij} denotes the $(0,1)$-component of β_{ij}. If we write $\beta''_{ij} = \sum_k \beta''_{ijk}\bar{\theta}_k$, then the last equation reads

$$0 = \beta''_{ijk} - \beta''_{ikj},$$

i.e. $\beta''_{ijk} = \beta''_{ikj}$, and this, combined with the previous $\beta''_{ijk} = -\beta''_{jik}$ yields $\beta''_{ij} = 0$ (we use here "the S_3-lemma": any tensor T_{ijk} which is symmetric in one pair of indices and anti-symmetric in another pair is identically zero). $\square$

In summary, we have obtained the following:

Proposition 1 *Let M^{2n} be a hermitian manifold (a riemannian manifold with an orthogonal complex structure). If we equip its canonical bundle $\Lambda^{n,0}(M)$ with the connection induced by the Levi-Civita connection of M, then its curvature Ω is given by the formula*

$$\Omega = i\mathcal{R}(\omega) + \Phi^* \wedge \Phi,$$

where ω is the Kähler 2-form associated with the hermitian structure, $\mathcal{R}$ is the curvature operator associated to the riemannian structure (see Definition 4 in Section 2), and

$$\Phi^* \wedge \Phi \leq 0.$$

This last inequality means, recalling our sign conventions of Definition 2 in Section 2, that $i(\Phi^* \wedge \Phi)$ is a non-positive real $(1,1)$-form.

Remark. There is an analogous statement for an *almost-Kähler* manifold, i.e. when the almost-complex structure is not necessarily integrable but the Kähler form is closed (so we have a symplectic manifold). The difference is that in this case the second fundamental form is of type $(0,1)$, hence the correction term $\Phi^* \wedge \Phi$ in the curvature formula is *non-negative*.

4 Some applications

As an immediate corollary to Proposition 1 we obtain the following result of LeBrun [2]:

Corollary 2 *There is no complex structure on S^6 which is orthogonal with respect to the round metric.*

Proof. For the round metric on a sphere, $\mathcal{R}$ is the identity operator. Therefore, for any orthogonal complex structure, the formula of Proposition 1 for the curvature of the canonical line bundle gives

$$\Omega = i\omega + \Phi^* \wedge \Phi \leq i\omega < 0.$$

It follows that the closed 2-form Ω is non-degenerate, i.e. symplectic, which is impossible since $H^2(S^6) = 0$. $\square$

The next corollary extends the above conclusion to a C^2-neighbourhood of the round metric on S^6.

Corollary 3 *Let g be a riemannian metric on S^6 satisfying the following conditions:*

- *The curvature operator $\mathcal{R}$ is* positive *(i.e. all its eigen values are positive).*
- *At each point $x \in S^6$, the ratio of the largest eigen-value λ_{max} of $\mathcal{R}$ to the lowest eigen-value λ_{min} satisfies $\lambda_{max}/\lambda_{min} < 7/5 = 1.4$.*

Then (S^6, g) does not admit an orthogonal complex structure.

The proof of the last corollary is based on the following

Lemma 4 *Let V be a $2n$-dimensional euclidean vector space with an orthogonal complex structure J, and let $\omega = (J\cdot, \cdot)$ be the associated Kähler form. Let Ω_0 be an imaginary $(1,1)$-form satisfying $\Omega_0 \leq i\omega$. (See Definition 2 in Section 2 for the sign convention. Note that in particular, since $i\omega < 0$, Ω_0 is also negative, hence non-degenerate).*

Then, if Ω is any imaginary 2-form satisfying

$$\|\Omega - \Omega_0\| < \frac{1}{2\sqrt{n}},$$

Ω is non-degenerate.

Proof. First, a brief reminder about norms. We use the euclidean norm on V to embed $\Lambda^2(V^*) \subset \mathrm{End}(V)$ as antisymmetric endomorphisms: $\alpha \mapsto A$, where A is given by $(Av, w) = \alpha(v, w)$. In fact, this is the inverse of our map of Definition 3 in Section 2, $A \mapsto \hat{A} = \alpha$.

Next, the euclidean structure on V induces a euclidean norm $\|\cdot\|_{\mathrm{E}}$ on $\mathrm{End}(V)$ by $\|A\|_{\mathrm{E}}^2 = \sum |A_{ij}|^2$, where A_{ij} are the components of an element $A \in \mathrm{End}(V)$ with respect to an orthonormal basis of V. This norm is *multiplicative*, i.e. $\|AB\|_{\mathrm{E}} \leq \|A\|_{\mathrm{E}}\|B\|_{\mathrm{E}}$. Using this multiplicativity property, one can show that if $A \in \mathrm{End}(V)$ satisfies $\|A\|_{\mathrm{E}} < 1$, then $I + A + A^2 + ...$ is convergent, thus giving an inverse to $I - A$.

Unfortunately, the norm $\|\cdot\|_{\mathrm{E}}$ induces on $\Lambda^2(V^*)$ a norm which differs by a constant from the standard norm on $\Lambda^2(V^*)$: for any 2-form β, $\|\beta\|_{\mathrm{E}} = \sqrt{2}\|\beta\|$. For example, the Kähler form ω has (standard) norm $\sqrt{n}$, whereas the corresponding endomorphism, namely J, has norm $\sqrt{2n}$. In what follows, we will work with the $\|\cdot\|_{\mathrm{E}}$ norm on 2-forms.

Now, we can diagonalize ω and Ω_0 simultaneously (over $\mathbb{C}$), obtaining

$$\omega = i\sum \theta_j \wedge \bar{\theta}_j, \quad \Omega_0 = \sum \lambda_j \theta_j \wedge \bar{\theta}_j,$$

with $\{\theta_j\}$ a unitary frame, and the condition $\Omega_0 \leq i\omega$ implies $\lambda_j \leq -1$. Then

$$\Omega_0^{-1} = \sum \frac{1}{\lambda_j} \theta_j \wedge \bar{\theta}_j,$$

thus

$$\|\Omega_0^{-1}\|_{\mathrm{E}}^2 = 2\sum |\frac{1}{\lambda_j}|^2 \leq 2n.$$

Now,

$$\Omega = \Omega_0 + (\Omega - \Omega_0) = \Omega_0 \left(I + \Omega_0^{-1}(\Omega - \Omega_0)\right),$$

and our condition of $\|\Omega - \Omega_0\| < 1/(2\sqrt{n})$ translates to

$$\|(\Omega - \Omega_0)\|_{\mathrm{E}} < \frac{1}{\sqrt{2n}},$$

hence

$$\|\Omega_0^{-1}(\Omega - \Omega_0)\|_{\mathrm{E}} \leq \|\Omega_0^{-1}\|_{\mathrm{E}} \, \|(\Omega - \Omega_0)\|_{\mathrm{E}} < \sqrt{2n} \cdot \frac{1}{\sqrt{2n}} = 1,$$

and so Ω is non-degenerate. □

Proof of Corollary 3. Let us suppose there is a complex structure on S^6 which is orthogonal with respect to a metric g whose curvature operator satisfies the said conditions. From Proposition 1, the curvature of the associated canonical line bundle is given by

$$\Omega = i\mathcal{R}(\omega) + \Phi^* \wedge \Phi = i(\mathcal{R}\omega - \omega) + \Omega_0,$$

where $\Omega_0 = i\omega + \Phi^* \wedge \Phi \leq i\omega$. Now we apply the previous lemma. We conclude that Ω is non-degenerate provided $\|\mathcal{R}\omega - \omega\| < 1/(2\sqrt{3})$ (pointwise).

Now, by rescaling the metric if necessary (this does not affect of course the orthogonality of the complex structure), we can bring the eigen-values of $\mathcal{R}$ to the range $(5/6, 7/6)$, so that the eigen-values of $\mathcal{R} - I$ are in the range $(-1/6, 1/6)$. This implies that $\|(\mathcal{R} - I)\alpha\| < (1/6)\|\alpha\|$ (pointwise) for any $\alpha \in \Lambda^2(M)$, so in particular

$$\|\mathcal{R}\omega - \omega\| < \frac{1}{6}\|\omega\| = \frac{1}{6}\sqrt{3} = \frac{1}{2\sqrt{3}}.$$

And so, according to the lemma above, the closed 2-form Ω is non-degenerate, i.e. symplectic, which is impossible since $H^2(S^6) = 0$. $\square$

Remarks.
1. It is tempting to generalize Corollary 3 to the case of a hermitian structure on a $2n$-dimensional manifold with a positive curvature operator satisfying $\lambda_{max}/\lambda_{min} < (2n+1)/(2n-1)$. Unfortunately, such a generalization is useless, because of the well-known "sphere-theorem", which implies that the universal cover of a complete riemannian manifold satisfying our curvature bound is a sphere, on which a complex structure is in question only in dimension 6 (because in all dimensions except 2 and 6 the n-sphere does not admit even an almost-complex structure), so we are back to our case.
2. However, we believe that one should be able to use Proposition 1 beyond what we have done here, because of the following argument. The condition of orthogonality of a complex structure with respect to a riemannian metric is obviously conformally invariant. On the other hand, the curvature restriction in Corollary 3 is *not* conformally invariant. Thus, Corollary 3 can be improved by including any metric on S^6 which is conformal to a metric satisfying the given curvature condition, but one hopes for a more explicit condition, say in terms of the Weyl tensor. So far, we were not able to derive such a condition.
3. Another direction in which one could possibly use Proposition 1 is by applying it to some specific classes of hermitian structures. In such cases one may be able to give a more delicate estimate of the terms in the formula of Proposition 1, especially the $\mathcal{R}\omega$ term.

References

[1] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, 1994.

[2] C. LeBrun, *Orthogonal complex structures on S^6*, Proc. Amer. Math. Soc. **101** (1987), 136-138.

[3] S. Salamon, *Orthogonal complex structures*, Diff. Geom. & App., Proc. Conf., Aug. 28 – Sept. 1 , 1995, Brno, Czech Republic, Massaryk University, Brno 1996, 103-117.

[4] K. Sekigawa and L. Vanhecke, *Almost-hermitian manifolds with vanishing first Chern classes or Chern numbers*, Rend. Sem. Mat. Univ. Pol. Torino **50, 2** (1992), 195-208.

[5] C. M. Wood, *Harmonic almost-complex structures*, Compositio Mathematica **99** (1995), 183-212.