

Chains of path geometries on surfaces: theory and examples

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Abstract

We derive the equations of chains for path geometries on surfaces by solving the equivalence problem of a related structure: sub-Riemannian geometry of signature $(1, 1)$ on a contact 3-manifold. This approach is significantly simpler than the standard method of solving the full equivalence problem for path geometry. We then use these equations to give a characterization of projective path geometries in terms of their chains (the chains projected to the surface coincide with the paths) and study the chains of four examples of homogeneous path geometries. In one of these examples (horocycles in the hyperbolic planes) the projected chains are bicircular quartics.

Contents

1	Introduction	2
1.1	A quick reminder about path geometries on surfaces	2
1.2	An abstract reformulation of path geometry	4
1.3	Chains of path geometries via the Fefferman metric	5
1.4	Contents of the article	6
2	The Fefferman metric for CR 3-manifolds (revisited)	7
2.1	Example: left-invariant CR structures on SU_2	11
3	The Fefferman metric for path geometries	11
3.1	Chains of $y'' = f(x, y, y')$	15
3.2	Chains of projective path geometries	17

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4	Examples of path geometries and their chains	18
4.1	The flat path geometry	18
4.1.1	First proof of Proposition 4.1: via the Heisenberg group .	19
4.1.2	Second proof of Proposition 4.1: via the Euclidean group	21
4.2	Circles of fixed radius	23
4.3	Hooke ellipses of fixed area	27
4.4	Horocycles in the hyperbolic plane	31
	References	32

1 Introduction

1.1 A quick reminder about path geometries on surfaces

A *path geometry* on a surface consists of a surface Σ (a 2-dimensional differentiable manifold) together with a non-degenerate 2-parameter family of *unparametrized* curves in Σ .¹An *equivalence* of path geometries on two surfaces is a diffeomorphism of the surfaces which maps the paths of one surface onto those of the other. A *symmetry* of a path geometry on a surface is a self-equivalence.

The basic example is $\Sigma = \mathbb{R}P^2$ (the 2-dimensional real projective plane) equipped with the family of straight lines in it. A path geometry² which is locally equivalent to this example is called *flat*. A less obvious flat example is given by all parabolas whose focus is at the origin (‘Kepler parabolas’; here $\Sigma := \mathbb{R}^2 \setminus \{0\}$). It is doubly covered by straight lines via the (complex) quadratic map $z \mapsto z^2$.

Every path geometry is given locally by the graphs of solutions of a second-order ODE $y'' = f(x, y, y')$. Conversely, a path geometry determines the ODE up to so-called point transformations, that is, changes of coordinate $(x, y) \mapsto (\tilde{x}, \tilde{y})$. The flat example of straight lines in $\mathbb{R}P^2$ corresponds to $y'' = 0$. A path geometry is *projective* if its paths are the (unparametrized) geodesics of a torsion-free affine connection on Σ . Such path geometries correspond to ODEs $y'' = f(x, y, y')$ where f is polynomial in y' of degree at most 3. Note that, somewhat surprisingly, this condition is independent of the coordinates x, y used on Σ . Thus a ‘generic’ path geometry is not projective, and in particular, non-flat. A non-projective example is the path geometry in \mathbb{R}^2 whose paths are all circles of a fixed radius.

A path geometry on a surface Σ defines a *dual* path geometry on the path space Σ^* , whose paths are parametrized by Σ : for each point $x \in \Sigma$ the corresponding path in Σ^* consists of all paths in Σ passing through x . Clearly, the dual of a flat path geometry is flat as well, an example of a *self-dual* path geometry. The path geometry of circles of fixed radius in \mathbb{R}^2 is an example of

¹This definition will be reformulated below more abstractly and precisely; in particular, the non-degeneracy condition will be spelled out.

²We shall henceforth usually drop the qualifier “on a surface” since that is the only situation this article considers.

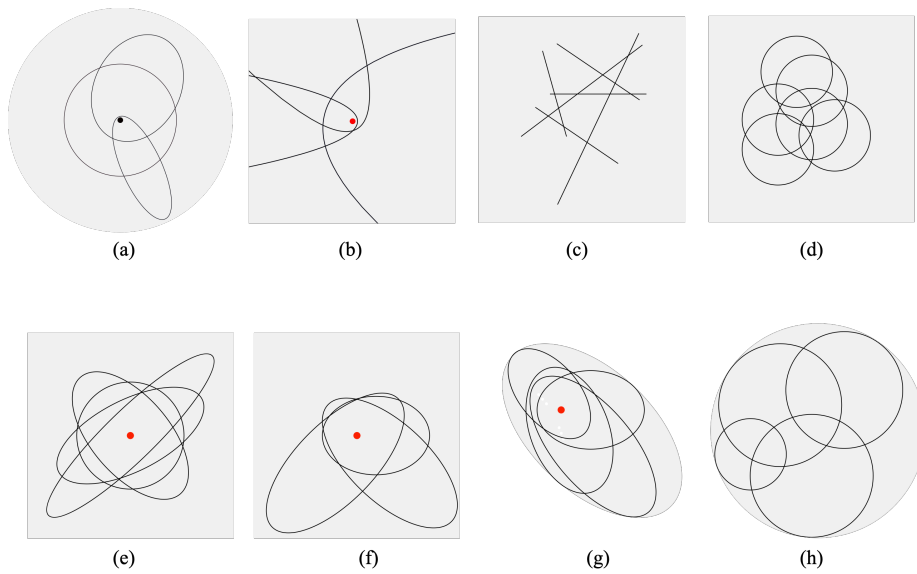


Figure 1. A gallery of 2D path geometries: (a) Kepler ellipses of fixed major axis. (b) Kepler parabolas. (c) Straight lines. (d) Circles of fixed radius. (e) Hooke ellipses of fixed area. (f) Kepler ellipses of fixed minor axis. (g) Kepler ellipses tangent to a fixed Kepler ellipse. (h) Circles tangent to a fixed circle (horocycles). Can you find the equivalent and dual geometries? (Answer: $a = e = f, b = c = g$ (flat); $a^* = h, b^* = b, d^* = d$.)

a self-dual non-projective path geometry. A projective path geometry is flat if and only if its dual is projective as well.

A flat path geometry admits an 8-dimensional (local) group of symmetries (the projective group $\mathrm{PSL}_3(\mathbb{R})$). Conversely, a path geometry admitting an 8-dimensional local group of symmetries is necessarily flat (a theorem of Sophus Lie). The *sub-maximal* symmetry dimension, i.e. the maximum dimension of the local symmetry group of a non-flat path geometry, is 3. The path geometry of circles with a fixed radius is sub-maximal. Its symmetry group is the Euclidean group. Another sub-maximal example is given by central ellipses (‘Hooke ellipses’) of fixed area, where the symmetry group $\mathrm{SL}_2(\mathbb{R})$ acts by its standard linear action on \mathbb{R}^2 (here $\Sigma = \mathbb{R}^2 \setminus \{0\}$). In contrast to the previous example of circles with fixed radius, this example is projective and non-self-dual: its dual is the hyperbolic plane and the paths are the horocycles (in the Poincaré disk or upper half-plane model horocycles are the circles tangent to the boundary; see Section 4.4 below).

The subject was studied extensively in the second half of the 19th century by Roger Liouville (a relative of the more famous Joseph Liouville), Sophus Lie and his student Arthur Tresse, who produced a local classification, over the complex numbers, of sub-maximal path geometries (i.e. those admitting a 3-dimensional group of symmetries) [24]. This classification has been since refined over the real numbers [13]. The only non-flat projective items on the list is the above mentioned case of Hooke ellipses of fixed area and Hooke hyperbolas of fixed discriminant (see Table 2 in the Appendix of [3] for several equivalent models of these path geometries).

1.2 An abstract reformulation of path geometry

We describe here briefly a more abstract and rigorous reformulation of path geometries on surfaces, useful also for introducing chains. For further details we recommend V. I. Arnol’d’s book [1, Chapter 1, Section 6].

Given a surface Σ , let $\mathbb{P}T\Sigma$ be the (3-dimensional) total space of its projectivized tangent bundle. That is, a point in $\mathbb{P}T\Sigma$ corresponds to a point in Σ together with a tangent line at the point (a 1-dimensional linear subspace of the tangent space at the point). There is a standard contact distribution D on $\mathbb{P}T\Sigma$, given by the ‘skating’ condition: “the point moves along the line”, or “the line rotates about the point.” The fibers of the base point projection $\mathbb{P}T\Sigma \rightarrow \Sigma$ are integral curves of D . Their tangents form the vertical line field $L_1 \subset D$. A path $\gamma \subset \Sigma$ is lifted to $\mathbb{P}T\Sigma$ by mapping a point on γ to the tangent line to γ at this point. The lifted curve is clearly an integral curve of D , as it satisfies the skating condition. The non-degeneracy assumption on a path geometry on Σ is that the *lifted paths form a smooth 1-dimensional foliation of $\mathbb{P}T\Sigma$* , transverse to L_1 (in D); equivalently, the tangent lines to the lifted curves form a smooth line field $L_2 \subset D$, complementary to L_1 , so that $D = L_1 \oplus L_2$.

We thus arrive at an abstract reformulation of a path geometry:

Definition 1.1. *A (2-dimensional) path geometry is a smooth 3-manifold M*

together with an (ordered) pair of smooth line fields $L_1, L_2 \subset TM$, spanning a contact distribution $D = L_1 \oplus L_2$. The path geometry dual to (M, L_1, L_2) is (M, L_2, L_1) .

Remark 1.2. Another common name for (M, L_1, L_2) is a *para-CR structure*, due to the formal similarity with a (Levi–non-degenerate) CR structure (M, D, J) . The latter is a contact distribution D on a 3-manifold M together with a complex structure $J \in \text{End}(D)$, i.e. $J^2 = -\text{id}_D$; equivalently, it is a splitting $D \otimes \mathbb{C} = D^{1,0} \oplus D^{0,1}$, the direct sum of a conjugate pair of complex line bundles (the $\pm i$ -eigenbundles of $J \otimes \mathbb{C}$).

In the real-analytic setting, CR and para-CR structures have a common complexification: a complex 3-manifold together with a pair of (complex) line fields spanning a (complex) contact distribution.

Remark 1.3. Some authors define a path geometry as a 2-parameter family of curves on a surface Σ , a unique curve through any given point of Σ in any given direction (see, e.g. the first paragraph of [14], or the “fancy formulation” of Section 8.6 of [18]). Definition 1.1 is more precise and general: first, the surface Σ is recovered from (M, L_1, L_2) as the space of integral curves of L_1 , which may exist as a smooth surface only locally. Second, even if Σ exists, the set of directions at a given $x \in \Sigma$ for which a curve exists may be only an open subset in $\mathbb{P}T_x\Sigma$. For example, for the path geometry of Hooke (or central) ellipses in $\mathbb{R}^2 \setminus \{0\}$ a curve exists only in non-radial directions. Third, there may be more than one curve in a given direction. For example, for circles of fixed radius in \mathbb{R}^2 , there are *two* circles passing through each point in a given direction. This can be remedied by considering instead *oriented* circles of fixed radius and the *spherized* tangent bundle STR^2 (TR^2 , with the zero section removed, mod \mathbb{R}^+) instead of $\mathbb{P}TR^2$. An analogous remedy applies to the aforementioned path geometry of horocycles in the hyperbolic plane.

We shall not dwell here further on these details and refer the interested reader to Sections 4.2.3 and 4.4.3 of [6], where our notion of a path geometry on a surface is called both a *generalized path geometry* and a *Lagrangian contact structure* on a 3-manifold; the two notions differ in higher dimension.

1.3 Chains of path geometries via the Fefferman metric

In the CR case there is a well-known, naturally associated 4-parameter family of curves on M , called *chains*, one chain for each given point in M in a given direction transverse to the contact distribution. They are considered the CR analog of geodesics in Riemannian geometry (see the recent article [12] for a variational formulation). Chains were introduced by É. Cartan while solving the equivalence problem of CR geometry [10, 19] and were studied extensively by many authors, such as Chern–Moser [11] and C. Fefferman [16], who showed that they arise from a natural construction, considerably simpler than Cartan’s, nowadays called the *Fefferman metric*: a conformal Lorentzian metric, i.e. of signature $(3, 1)$, defined on the total space of a certain circle bundle over M . The

chains of the CR structure are then the projections onto M of the non-vertical null geodesics of the Fefferman metric.

Similarly, to each path geometry (M, L_1, L_2) one can associate a natural 4-parameter family of curves on M , a unique curve through any given point in M in any given direction transverse to the contact distribution $D := L_1 \oplus L_2$. The study of this natural class of curves is quite recent. The earliest reference we know of is a 2005 article of A. Čap and V. Žádník [7] (path geometries on surfaces appear there in Section 2 as 3-dimensional *Lagrangian contact structures*). See also Sections 5.3.7–8 and 5.3.13–14 of [6]. Both references define chains using the associated Cartan geometry. However, as in the CR case, there is a significant shortcut via the Fefferman metric. This is a conformal metric of signature $(2, 2)$ on the total space of an \mathbb{R}^* -bundle over M , and the chains are the projections onto M of non-vertical null geodesics of the Fefferman metric. In this article we explain this construction and use it to give several concrete examples.

Remark 1.4. As mentioned in Remark 1.3, path geometries on surfaces generalize in higher dimension to either (generalized) path geometries or Lagrangian contact geometries. The Fefferman-type construction of a conformal structure described here generalizes in higher dimensions to Lagrangian contact structures but not to path geometries.

1.4 Contents of the article

In the next section we re-derive, as a warm-up and a reminder, the Fefferman metric for a CR structure (M, D, J) . The construction appeared first in Fefferman’s article [16] for a CR manifold embedded as a real hypersurface in a complex manifold, followed by intrinsic constructions, first direct ones in [15, 21], then more advanced constructions that use the full solution of the equivalence problem for CR structures (Cartan bundle and connection), such as [5, 7, 22]. We view instead a CR structure as a conformal class of sub-Riemannian geometries of contact type, solve the equivalence problem of sub-Riemannian geometries of contact type following [17]—which is much simpler than that for CR geometry, use a sub-Riemannian metric on D to define a Lorentzian metric on $\mathbb{S}D$ (the spherization of D), then show that conformally equivalent sub-Riemannian metrics on D induce conformally equivalent Lorentzian metrics on $\mathbb{S}D$. In retrospect, our construction can be regarded as a shorter version of [15, 21], using [17]. It is still too complicated conceptually for our taste, and the below formula (9) for the metric appears a bit like magic, but this method is the best we have so far and is quite easy to work with.

Once the construction of the Fefferman metric for CR geometry is understood, we construct in Section 3 in a similar fashion the Fefferman metric for a path geometry. As far as we know, our derivation is new, and before this article the only available construction of the Fefferman metric for path geometry has been via the solution of the full equivalence problem for such a structure (see, e.g., [7, 22]), which is considerably more involved than our derivation.

In Section 3.2 we prove the following theorem, apparently new:

Theorem 1. *A path geometry on a surface Σ is projective if and only if the chains on $\mathbb{P}T\Sigma$ project to the paths in Σ .*

In the last section we study in some detail the chains of four homogeneous path geometries mentioned above: straight lines, circles of fixed radius, central ellipses of fixed area and horocycles in the hyperbolic plane.

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2 The Fefferman metric for CR 3-manifolds (revisited)

Let (M, D, J) be a CR 3-manifold, i.e. $D \subset TM$ is a contact 2-distribution (that is, $[D, D] = TM$) and $J \in \text{End}(D)$ satisfies $J^2 = -\text{id}_D$. Canonically associated to the CR structure is the circle bundle $\mathbb{S}D \rightarrow M$ (D with the zero section removed, mod \mathbb{R}^+) and a conformal class of metrics of signature $(3, 1)$ on $\mathbb{S}D$, the *Fefferman metric*. It depends on the *second-order jet* of the CR structure, so is not so easy to see. The fibers of $\mathbb{S}D \rightarrow M$ are null geodesics, and the projections of the non-vertical null geodesics to M are the *chains* of the CR structure, forming a 4 parameter family of curves on M .

The construction. Fix a positive contact form η^3 on M , i.e. a 1-form satisfying

$$D = \text{Ker}(\eta^3), \tag{1}$$

$$d\eta^3(X, JX) > 0 \text{ for every } X \in D, X \neq 0. \tag{2}$$

Remark 2.1. A general contact manifold does not admit necessarily a global contact form (a 1-form whose kernel is D) but the contact structure of a CR manifold does, using the orientation of D induced by J . If M is connected then any global contact form is either positive or negative.

Recall that the coframe bundle $\pi : F^* \rightarrow M$ is the principal $\text{GL}_3(\mathbb{R})$ -bundle whose fiber at a point $x \in M$ consists of all linear isomorphisms $u : T_x M \rightarrow \mathbb{R}^3$. The *tautological 1-form* on F^* is the \mathbb{R}^3 -valued 1-form ω whose value at $u \in F^*$ is $u \circ (d\pi)_u$.

Now a positive contact form η^3 on M defines a positive-definite inner product on D , $\langle X, Y \rangle := d\eta^3(X, JY)$. An *adapted coframe* is an extension of η^3 to a coframe $\eta = (\eta^1, \eta^2, \eta^3)^t$ (we view elements of \mathbb{R}^3 as *column vectors*), satisfying

$$d\eta^3 = \eta^1 \wedge \eta^2 \tag{3}$$

$$\langle \cdot, \cdot \rangle = [(\eta^1)^2 + (\eta^2)^2] \Big|_D. \tag{4}$$

It is easy to show that for a fixed η^3 these 2 equations define a circle's worth of coframes at each $x \in M$. Thus, let $S^1 \subset \text{GL}_3(\mathbb{R})$ be the set of matrices of the form

$$\begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $B \subset F^*$ the set of coframes adapted to η^3 . Then $B \rightarrow M$ is a principal S^1 -subbundle, an S^1 -reduction of F^* , whose local sections consist of adapted coframes.

We continue to denote by $\omega = (\omega^1, \omega^2, \omega^3)^t$ the restriction of the tautological 1-form on F^* to B . Then there are unique 1-form α and functions a_1, a_2 on B such that

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = - \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 & 0 \\ a_2 & -a_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^2 \wedge \omega^3 \\ \omega^3 \wedge \omega^1 \\ \omega^1 \wedge \omega^2 \end{pmatrix}. \quad (5)$$

(See equation (1) of [17]). Furthermore, there are unique functions b_1, b_2, K on B such that

$$d\alpha = b_1 \omega^2 \wedge \omega^3 + b_2 \omega^3 \wedge \omega^1 + K \omega^1 \wedge \omega^2. \quad (6)$$

(See equation (4) of [17]; in fact, K descends to M . Also, α is essentially the Webster connection form [25], a_1, a_2 its torsion, and K the Webster scalar curvature).

Define a Lorentzian metric on B by

$$g := \omega^1 \cdot \omega^1 + \omega^2 \cdot \omega^2 + \omega^3 \cdot \sigma, \quad (7)$$

where σ is a 1-form, to be determined later, and \cdot is the symmetric product of 1-forms.

Let $\mathbb{S}D$ be the ‘spherization’ (or ‘ray projectivization’) of D , the quotient of D , with the zero section removed, by the dilation action of \mathbb{R}^+ . There is an obvious S^1 -action on D , commuting with the \mathbb{R}^+ action, thus making $\mathbb{S}D$ a principal S^1 -bundle. Note that $\mathbb{S}D$, unlike B , is canonically associated to (M, D, J) : to define B we needed to choose the positive contact form η^3 . Define an isomorphism of principal S^1 -bundles

$$h : B \rightarrow \mathbb{S}D, \quad u \mapsto [u^{-1} \mathbf{e}_1], \quad (8)$$

where $\mathbf{e}_1 = (1, 0, 0)^t$. That is, $h(u) = [X] \in \mathbb{S}D$, where $X \in D$ is the unique vector in $T_x M$, $x = \pi(u)$, satisfying $u^1(X) = 1, u^2(X) = u^3(X) = 0$. We then use h to map the Lorentzian metric on B of equation (7) to a Lorentzian metric on $\mathbb{S}D$. In general, for arbitrary σ in formula (7), the resulting metric on $\mathbb{S}D$ depends on the choice of η^3 in a complicated way, but for a careful choice of σ the conformal class of the Lorentzian metric on $\mathbb{S}D$ is independent of the choice of η^3 .

Remark 2.2. There are other models for the underlying space of the Fefferman metric instead of $\mathbb{S}D$ (a matter of taste). For example, one can take the spherization of the dual bundle D^\vee , in which case the formula for the identification $B \rightarrow \mathbb{S}D^\vee$ is a little simpler: $u \mapsto [u^1|_D]$. Another model is the spherization of the canonical bundle $\Lambda^{2,0}D \subset \Lambda^2 T^*D \otimes \mathbb{C}$, as in [21]; the identification with B in this case is $u \mapsto [u^3 \wedge (u^1 + iu^2)|_D]$. Also, the metric on $\mathbb{S}D$ is invariant under the antipodal map in each fiber (a circle), and so it descends to the (full) projectivization $\mathbb{P}D$.

Theorem 2. *Let (M, D, J) be a CR 3-manifold, $\mathbb{S}D \rightarrow M$ the spherization of D and η^3 any positive contact 1-form, as in equations (1) and (2). Define a 1-form σ on the total space of the associated circle bundle $B \rightarrow M$,*

$$\sigma = \frac{4}{3}\alpha - \frac{1}{3}K\omega^3, \quad (9)$$

where α, K are defined via equations (5) and (6). Then the conformal class of the Lorentzian metric induced on $\mathbb{S}D$ by equation (7), via the isomorphism (8), is independent of the choice of η^3 . In fact, multiplying η^3 by a positive function rescales the induced metric on $\mathbb{S}D$ by the same factor.

Proof. If η^3 is a positive contact form on M , then any other positive contact form is of the form $\tilde{\eta}^3 = \lambda^2 \eta^3$, for some positive function $\lambda : M \rightarrow \mathbb{R}^+$. Changing η^3 to $\tilde{\eta}^3$ changes B to \tilde{B} , another S^1 -reduction of the coframe bundle of M , with corresponding metric \tilde{g} and isomorphism $\tilde{h} : \tilde{B} \rightarrow S$. We thus need to show that the composition $f := \tilde{h}^{-1} \circ h : B \rightarrow \tilde{B}$ satisfies $f^*(\tilde{g}) = \lambda^2 g$.

Let us pull-back λ to B by the projection $B \rightarrow M$, denoting the result by λ as well. Then

$$d\lambda = \lambda_i \omega^i, \quad d\lambda_i = \lambda_{i0} \alpha + \lambda_{ij} \omega^j, \quad (10)$$

for some functions $\lambda_i, \lambda_{ij}, \lambda_{i0}$ on B , $1 \leq i, j \leq 3$. (Note that by definition λ descends to M ; in general the λ_i do not, but λ_3 does.)

Lemma 2.3.

$$\lambda_{10} = -\lambda_2, \quad \lambda_{20} = \lambda_1, \quad \lambda_{12} - \lambda_{21} = \lambda_3.$$

Proof. These identities follow immediately from expanding $d(d\lambda) = 0$. \square

Now a section $\eta = (\eta^1, \eta^2, \eta^3) : M \rightarrow B$ of $B \rightarrow M$ is a coframe adapted to η^3 , so $f \circ \eta : M \rightarrow \tilde{B}$ is a section of $\tilde{B} \rightarrow M$, a coframe adapted to $\tilde{\eta}^3 = \lambda^2 \eta^3$.

Lemma 2.4. $f \circ \eta = \Lambda \eta$, where

$$\Lambda = \begin{pmatrix} \lambda & 0 & -2\lambda_2 \\ 0 & \lambda & 2\lambda_1 \\ 0 & 0 & \lambda^2 \end{pmatrix}.$$

Proof. It is enough to check that $\tilde{\eta} := \Lambda \eta$ satisfies equations (1)-(4) above, with $\tilde{\eta}^3 = \lambda^2 \eta^3$ instead of η^3 , as well as $\tilde{\eta}^1(\tilde{X}) = 1, \tilde{\eta}^2(\tilde{X}) = \tilde{\eta}^3(\tilde{X}) = 0$ for $\tilde{X} = X/\lambda$. \square

Lemma 2.5. $f^*\omega_{\tilde{B}} = \Lambda\omega_B$.

Proof. Let $\eta \in B$, $\tilde{\eta} = f(\eta)$. By Lemma 2.4, $\tilde{\eta} = \Lambda\eta$, hence $(f^*\omega_{\tilde{B}})_\eta = (\omega_{\tilde{B}})_{\tilde{\eta}} \circ (df)_\eta = \tilde{\eta} \circ (d\tilde{\pi})_{\tilde{\eta}} \circ (df)_\eta = \tilde{\eta} \circ d(\tilde{\pi} \circ f)_\eta = \tilde{\eta} \circ (d\pi)_\eta = \Lambda\eta \circ d\pi_\eta = \Lambda\omega_\eta$. \square

Notation. For sake of readability, we adopt henceforth the following abbreviated notation:

$$\omega := \omega_B, \quad g := g_B, \quad \dots, \quad \tilde{\omega} := f^*\omega_{\tilde{B}}, \quad \tilde{g} := f^*g_{\tilde{B}}, \quad \dots, \quad \text{etc.}$$

Thus, for example, Lemma 2.5 reads $\tilde{\omega} = \Lambda\omega$.

We proceed with the proof of Theorem 2. It is clearly enough to show an infinitesimal version of the claimed conformal invariance. Suppose $\lambda = \lambda(t)$ is differentiable and that it satisfies $\lambda(0) = 1$. Denote by a dot the derivative with respect to t at $t = 0$ of objects on \tilde{B} pulled back to B by f , e.g., $\dot{\lambda} = \lambda'(0)$, $\dot{\lambda}_i = \lambda'_i(0)$, $\dot{\lambda}_{ij} = \lambda'_{ij}(0)$, $\dot{g} = \left. \frac{d}{dt} \right|_{t=0} \tilde{g}$, etc. Then $\tilde{g} = \lambda^2 g$ if and only if $\dot{g} = 2\dot{\lambda}g$ (for all η^3 and $\lambda(t)$ satisfying $\lambda(0) = 1$). Now calculate using the previous lemmas:

$$\begin{aligned} \dot{\omega}^1 &= \dot{\lambda}\omega^1 - 2\dot{\lambda}_1\omega^3, & \dot{\omega}^2 &= \dot{\lambda}\omega^2 + 2\dot{\lambda}_2\omega^3, & \dot{\omega}^3 &= 2\dot{\lambda}\omega^3, \\ \dot{g} &= 2\dot{\lambda}g + (\dot{\sigma} - 4\dot{\lambda}_2\omega^1 + 4\dot{\lambda}_1\omega^2) \cdot \omega^3. \end{aligned}$$

Thus $\dot{g} = 2\dot{\lambda}g$ if and only if

$$\dot{\sigma} = 4(\dot{\lambda}_2\omega^1 - \dot{\lambda}_1\omega^2). \quad (11)$$

To calculate $\dot{\sigma}$, using formula (9), we need formulas for $\dot{\alpha}$ and \dot{K} . To find $\dot{\alpha}$ we find first a formula for $\tilde{\alpha}$. Write the structure equations (5) for $\tilde{\omega}$, substitute $\tilde{\omega} = \Lambda\omega$, and equate coefficients. The result is

$$\tilde{\alpha} = \alpha + 3\frac{\lambda_2}{\lambda}\omega^1 - 3\frac{\lambda_1}{\lambda}\omega^2 - \left[3\frac{(\lambda_1)^2 + (\lambda_2)^2}{\lambda^2} + \frac{\lambda_{11} + \lambda_{22}}{\lambda} \right] \omega^3.$$

Taking derivative with respect to t at $t = 0$ of the last formula, we get

$$\dot{\alpha} = 3\dot{\lambda}_2\omega^1 - 3\dot{\lambda}_1\omega^2 - (\dot{\lambda}_{11} + \dot{\lambda}_{22})\omega^3.$$

To find \dot{K} , there is a shortcut, avoiding an explicit formula for \tilde{K} , by noting first that K is defined by $d\alpha \equiv K\omega^1 \wedge \omega^2 \pmod{\alpha, \omega^3}$. Taking d of the above formula for $\tilde{\alpha}$, we get, using equations (10), $d\tilde{\alpha} \equiv -4(\dot{\lambda}_{11} + \dot{\lambda}_{22})\omega^1 \wedge \omega^2 \pmod{\alpha, \omega^3}$. Taking derivative with respect to t of $d\tilde{\alpha} \equiv \tilde{K}\tilde{\omega}^1 \wedge \tilde{\omega}^2 \pmod{\tilde{\alpha}, \tilde{\omega}^3}$, we get $d\dot{\alpha} \equiv (\dot{K} + 2\dot{\lambda}K)\omega^1 \wedge \omega^2 \pmod{\alpha, \omega^3}$, hence

$$\dot{K} = -2\dot{\lambda}K - 4(\dot{\lambda}_{11} + \dot{\lambda}_{22}).$$

Now let $\tilde{\sigma} = c_1\tilde{\alpha} + c_2\tilde{K}\tilde{\omega}^3$, with some constants c_1, c_2 . Then

$$\dot{\sigma} = c_1\dot{\alpha} + c_2(\dot{K}\omega^3 + K\dot{\omega}^3) = 3c_1(\dot{\lambda}_2\omega^1 - \dot{\lambda}_1\omega^2) - (c_1 + 4c_2)(\dot{\lambda}_{11} + \dot{\lambda}_{22})\omega^3.$$

Thus equation (11) is satisfied if $c_1 = 4/3, c_2 = -1/3$. \square

2.1 Example: left-invariant CR structures on SU_2

The left-invariant \mathfrak{su}_2 -valued Maurer–Cartan form on SU_2 is

$$\Theta = g^{-1}dg = \begin{pmatrix} i\theta^1 & \theta^2 + i\theta^3 \\ -\theta^2 + i\theta^3 & -i\theta^1 \end{pmatrix}. \quad (12)$$

The Maurer–Cartan equation $d\Theta = -\Theta \wedge \Theta$ gives

$$d\theta^1 = -2\theta^2 \wedge \theta^3, \quad d\theta^2 = -2\theta^3 \wedge \theta^1, \quad d\theta^3 = -2\theta^1 \wedge \theta^2. \quad (13)$$

For each $t \in [1, \infty)$ let

$$\eta^1 = \sqrt{t}\theta^1, \quad \eta^2 = \theta^2/\sqrt{t}, \quad \eta^3 = -\theta^3/2.$$

One can show that every left-invariant CR structure $D^{0,1} \subset TSU_2 \otimes \mathbb{C}$ is equivalent (via right translation), for a unique $t \in [1, \infty)$, to $\{\eta^1 + i\eta^2, \eta^3\}^\perp$. For $t = 1$ we obtain the standard ‘spherical’ CR structure on $SU_2 \simeq S^3$, for $t > 1$ these are non-spherical CR structures, distinct t determine inequivalent structures (see [3], Prop. 5.1). We use (13) to find

$$d\eta^1 = 4t\eta^2 \wedge \eta^3, \quad d\eta^2 = (4/t)\eta^3 \wedge \eta^1, \quad d\eta^3 = \eta^1 \wedge \eta^2.$$

Using this coframe we identify $B \simeq SU_2 \times S^1$ and $\omega = \bar{u} \cdot \eta$, where $u = e^{i\varphi}$. Explicitly,

$$\omega^1 = \sqrt{t}(\cos \theta)\theta^1 + \frac{1}{\sqrt{t}}(\sin \theta)\theta^2, \quad \omega^2 = -\sqrt{t}(\sin \theta)\theta^1 + \frac{1}{\sqrt{t}}(\cos \theta)\theta^2, \quad \omega^3 = -\frac{1}{2}\theta^3.$$

Inserting these into equations (5)–(6), we obtain

$$\alpha = \theta^4 - \left(t + \frac{1}{t}\right)\theta^3, \quad K = 2\left(t + \frac{1}{t}\right),$$

where $\theta^4 := d\varphi$. Inserting all this into equations (7)–(9), we get

$$g = t(\theta^1)^2 + \frac{1}{t}(\theta^2)^2 + \frac{1}{2}\left(t + \frac{1}{t}\right)(\theta^3)^2 - \frac{2}{3}\theta^3 \cdot \theta^4.$$

This is essentially formula (15) of [11]; the coefficient of our $\theta^3 \cdot \theta^4$ term can be made to agree with the cited formula by rescaling the φ coordinate by a constant. See also [11] for a study of the chains of this example via null geodesics of the Fefferman metric.

3 The Fefferman metric for path geometries

Let (M, L_1, L_2) be a path geometry, i.e., L_1, L_2 is a pair of line fields on a 3-manifold M , spanning a contact distribution $D := L_1 \oplus L_2$. Let us fix a contact form η^3 , that is, $D = \text{Ker}(\eta^3)$ (possibly defined only locally, see Remark 3.1

below). An *adapted coframe* (with respect to η^3) is an extension of η^3 to a (local) coframe (η^1, η^2, η^3) satisfying

$$d\eta^3 = \eta^1 \wedge \eta^2, \quad (14)$$

$$\eta^1|_{L_2} = \eta^2|_{L_1} = 0. \quad (15)$$

These equations define an \mathbb{R}^* -structure, i.e. an \mathbb{R}^* -principal subbundle $B \subset F^*$, whose local sections are the coframes adapted to η^3 , where $s \in \mathbb{R}^*$ acts by $(\eta^1, \eta^2, \eta^3) \mapsto (\eta^1/s, s\eta^2, \eta^3)$.

Let $D^* = D \setminus (L_1 \cup L_2) \subset D$, with spherization $\mathbb{S}D^* \subset \mathbb{S}D$. The Fefferman metric associated to the path geometry is a conformal pseudo-Riemannian metric of signature $(2, 2)$ on $\mathbb{S}D^*$. We shall define it in a manner similar to the CR case. The splitting $D = L_1 \oplus L_2$ defines an involution $J \in \text{End}(D)$, $J^2 = \text{id}$, by

$$J(X_1 + X_2) = X_1 - X_2, \quad X_i \in L_i, \quad i = 1, 2. \quad (16)$$

The contact form η^3 defines on D an area form, $d\eta^3|_D$, and an indefinite metric of signature $(1, 1)$, $\langle X, Y \rangle := d\eta^3(X, JY)$. Now $D^* = D^+ \cup D^-$, where D^\pm are the positive (resp. negative) vectors with respect to $\langle \cdot, \cdot \rangle$, and corresponding decomposition $\mathbb{S}D^* = \mathbb{S}D^+ \cup \mathbb{S}D^-$. Both $\mathbb{S}D^\pm$ are \mathbb{R}^* -principal bundles over M , where $s \in \mathbb{R}^*$ acts by $[X_1 + X_2] \mapsto [sX_1 + X_2/s]$, $X_i \in L_i$. Note that D^\pm are interchanged by J or by taking $-\eta^3$ instead of η^3 . There is an identification of \mathbb{R}^* -principal bundles,

$$h : B \rightarrow \mathbb{S}D^+, \quad u \mapsto [X], \quad \text{where } u^1(X) = u^2(X) = 1, u^3(X) = 0. \quad (17)$$

We shall define a pseudo-Riemannian metric of signature $(2, 2)$ on B , map it by h to $\mathbb{S}D^+$, then by J to $\mathbb{S}D^-$. As in the CR case, we show that the associated conformal class of metrics on $\mathbb{S}D^*$ is independent of the chosen contact form η^3 .

Remark 3.1. A general contact 3-manifold is naturally oriented. (Proof: choose a local contact form η^3 , then $\eta^3 \wedge d\eta^3$ is a volume form; multiplying η^3 by a non-vanishing λ multiplies this volume form by λ^2 , so does not change the associated orientation.) The Lie bracket of sections of D defines an isomorphism $\Lambda^2(D) \rightarrow TM/D$, but these isomorphic line bundles need not be trivial, i.e. there might not exist on M a global contact form (a non-vanishing section of $D^\perp \simeq (TM/D)^*$). In the CR case, J defines an orientation on $D \subset TM$, hence of TM/D (since TM is oriented), and a dual orientation of $D^\perp = (TM/D)^*$, so there is always a global contact form. This is not the case for a path geometry (e.g., $M = \mathbb{P}T\mathbb{R}^2$, equipped with the standard flat path geometry). But this topological difficulty is minor, we can still define B locally, then show that the conformal structures defined on $\mathbb{S}D^*$ restricted to open subsets of M agree on intersections. We shall not dwell on the details.

We shall now proceed with the plan outlined above, in the paragraph before Remark 3.1.

The structure equations for any \mathbb{R}^* -connection form α on $B \rightarrow M$ are

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = - \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} T_{23}^1 & T_{31}^1 & T_{12}^1 \\ T_{23}^2 & T_{31}^2 & T_{12}^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^2 \wedge \omega^3 \\ \omega^3 \wedge \omega^1 \\ \omega^1 \wedge \omega^2 \end{pmatrix},$$

where T_{jk}^i are some real functions on B (the coefficients of the torsion tensor of the connection). Starting from any such connection it is easy to show that it can be modified, in a unique way, by adding to α multiples of the ω^i , so as to render $T_{31}^1 = T_{12}^1 = T_{12}^2 = 0$ (in fact doing so also solves the equivalence problem for path geometry equipped with a fixed contact form). Taking the exterior derivative of $d\omega^3 = \omega^1 \wedge \omega^2$ shows that $T_{23}^2 = 0$ as well. The structure equations now become

$$\begin{aligned} d\omega^1 &= -\alpha \wedge \omega^1 + a_1 \omega^2 \wedge \omega^3 \\ d\omega^2 &= \alpha \wedge \omega^2 + a_2 \omega^3 \wedge \omega^1 \\ d\omega^3 &= \omega^1 \wedge \omega^2 \end{aligned} \tag{18}$$

for some functions a_1, a_2 on B . Taking exterior derivative of these equations we get

$$d\alpha = b_1 \omega^2 \wedge \omega^3 + b_2 \omega^3 \wedge \omega^1 + K \omega^1 \wedge \omega^2 \tag{19}$$

for some functions b_1, b_2, K on B (i.e. $d\alpha$ is semi-basic, containing no $\alpha \wedge \omega^i$ terms.)

Theorem 3. *Let (M, L_1, L_2) be a path geometry and $\mathbb{S}D^* \subset \mathbb{S}D$ the set of rays in $D = L_1 \oplus L_2$ not contained in $L_1 \cup L_2$. Then there is a canonically associated conformal class of metrics of signature $(2, 2)$ on $\mathbb{S}D^*$, called the Fefferman metric, defined as follows. Associated with each contact 1-form η^3 on M is an \mathbb{R}^* -reduction $B \rightarrow M$ of the coframe bundle of M , given by equations (14)-(15), a unique \mathbb{R}^* -connection form α on B satisfying equations (18) and the 1-form*

$$\sigma := -\frac{2}{3}\alpha + \frac{1}{6}K\omega^3, \tag{20}$$

where K is defined via equations (19), and where $\omega^1, \omega^2, \omega^3$ are the tautological 1-forms on the coframe bundle of M restricted to B . Then

$$g := \omega^1 \cdot \omega^2 + \omega^3 \cdot \sigma \tag{21}$$

is a pseudo-Riemannian metric on B of signature $(2, 2)$. There is also associated with η^3 a decomposition $\mathbb{S}D^* = \mathbb{S}D^+ \cup \mathbb{S}D^-$ and \mathbb{R}^* -isomorphisms $h : B \rightarrow \mathbb{S}D^+$, $J \circ h : B \rightarrow \mathbb{S}D^-$, where h is given by equation (17) and J by equation (16), such that the conformal class of the induced metric on $\mathbb{S}D^*$ is independent of the choice of η^3 ; in fact, multiplying η^3 by a smooth non-vanishing function rescales the induced metric on $\mathbb{S}D^*$ by the same factor.

Proof. The proof is very similar to the CR case. Here are the formulas that differ:

$$\begin{aligned}
\lambda_{10} &= \lambda_1, & \lambda_{20} &= -\lambda_2, & \lambda_{12} - \lambda_{21} &= \lambda_3, \\
\tilde{\alpha} &= \alpha + \frac{3\lambda_1}{\lambda}\omega^1 - \frac{3\lambda_2}{\lambda}\omega^2 - \left(\frac{\lambda_{12} + \lambda_{21}}{\lambda} + \frac{6\lambda_1\lambda_2}{\lambda^2}\right)\omega^3, \\
\dot{\alpha} &= 3\dot{\lambda}_1\omega^1 - 3\dot{\lambda}_2\omega^2 - \left(\dot{\lambda}_{12} + \dot{\lambda}_{21}\right)\omega^3, \\
d\dot{\alpha} &\equiv -4\left(\dot{\lambda}_{12} + \dot{\lambda}_{21}\right)\omega^1 \wedge \omega^2 \equiv \left(\dot{K} + 2\dot{\lambda}K\right)\omega^1 \wedge \omega^2 \pmod{\alpha, \omega^3}, \\
\dot{K} &= -2\dot{\lambda}K - 4\left(\dot{\lambda}_{12} + \dot{\lambda}_{21}\right), \\
\dot{g} &= 2\dot{\lambda}g + \left(\dot{\sigma} + 2\dot{\lambda}_1\omega^1 - 2\dot{\lambda}_2\omega^2\right).
\end{aligned}$$

□

Definition 3.2. A chain of a path geometry (M, L_1, L_2) is the projection to M of an unparametrized non-vertical null geodesic of the associated Fefferman conformal metric on $\mathbb{S}D^*$.

Proposition 3.3.

1. The \mathbb{R}^* -action on $\mathbb{S}D^*$ is by conformal isometries.
2. For every point in M , in every given direction transverse to D , there is a unique chain passing through this point in the given direction.
3. The fibers of $\mathbb{S}D^* \rightarrow M$ are null geodesics and project to constant curves on M .

Proof. (1) For every contact form η^3 , the map $h : B \rightarrow \mathbb{S}D^+$ (by definition, a conformal isometry) is \mathbb{R}^* -equivariant, hence it is enough to verify that the pseudo-Riemannian metric on B given by equations (20)–(21) is \mathbb{R}^* -invariant. This follows from the \mathbb{R}^* -invariance of α, ω^3, K and the \mathbb{R}^* -equivariance $R_s^*\omega^1 = \omega^1/s, R_s^*\omega^2 = s\omega^2$.

(2) Let $x \in M$ and $v \in T_xM, v \notin D_x$. Pick a contact form η^3 and work on the associated bundle B . The fiber B_x consists of coframes $u = (u^1, u^2, u^3)$ on T_xM adapted to η^3 , as in equations (14)–(15). We show that for every $u \in B_x$ there is a unique lift $\tilde{v} \in T_uB$ of v which is null. Now \tilde{v} is a lift of v if and only if $\omega^i(\tilde{v}) = u^i(v), i = 1, 2, 3$. It remains to determine $\sigma(\tilde{v})$. Now $\omega^3(\tilde{v}) = u^3(v) \neq 0$ and, by formula (21), \tilde{v} is null if and only if $\omega^1(\tilde{v})\omega^2(\tilde{v}) + \omega^3(\tilde{v})\sigma(\tilde{v}) = 0$, i.e. $\sigma(\tilde{v}) = -u^1(v)u^2(v)/u^3(v)$. This shows that $v \in T_xM$ has a unique null lift at $u \in B_x$. The null geodesic through u in the direction of \tilde{v} projects to a chain through x in the direction of v . This proves existence of the required chain. As for uniqueness, we need to show that if we repeat the above at another point of B_x , say $s \cdot u \in B_x$, we obtain the same chain. We use the fact that s acts on B by isometries R_s , mapping \tilde{v} to the unique null-lift of v at $s \cdot u$, and the

null geodesic through u tangent to \tilde{v} to the null geodesic through $s \cdot u$ in the direction of $(R_s)_* \tilde{v}$. Since R_s commutes with the projection $B \rightarrow M$, the two null geodesics project to the same chain in M .

(3) The vertical distribution of $B \rightarrow M$ is given by $\omega^1 = \omega^2 = \omega^3 = 0$, thus $g = \omega^1 \cdot \omega^2 + \omega^3 \cdot \sigma$ restricted to the fibers vanishes, so these fibers are null curves. We proceed to show that they are null geodesics.

As shown in part (1) above, the principal \mathbb{R}^* -action on B is isometric. Let ζ denote an infinitesimal generator of this action (i.e. a nonzero vertical null Killing vector field on B). The fibers of $B \rightarrow M$ are the integral curves of ζ , hence to show that these fibers are null geodesics it is enough to show that $\nabla_\zeta \zeta = 0$, or in index notation,

$$\zeta^b \nabla_b \zeta^a = 0.$$

Lowering an index of $\nabla_b \zeta^a$ (using g), splitting $\nabla_b \zeta_a$ into its symmetric and antisymmetric parts, and contracting with ζ^b gives

$$\zeta^b \nabla_b \zeta_a = \zeta^b \cdot \frac{1}{2} (\nabla_a \zeta_b + \nabla_b \zeta_a) + \zeta^b \cdot \frac{1}{2} (\nabla_a \zeta_b - \nabla_b \zeta_a). \quad (22)$$

The quantity $\frac{1}{2} (\nabla_a \zeta_b + \nabla_b \zeta_a)$ in the first term is $(\mathcal{L}_\zeta g)_{ab}$, but, per part (1), g is ζ -invariant—that is, $\mathcal{L}_\zeta g = 0$ —and so the first term vanishes.

The quantity $\frac{1}{2} (\nabla_a \zeta_b - \nabla_b \zeta_a)$ in the second term is $(d\zeta^b)_{ab}$, so the second term is $-\iota_\zeta (d\zeta^b)$, where ι_ζ denotes interior multiplication by ζ . Since ζ generates the \mathbb{R}^* -action on $B \rightarrow M$ and α is a connection form thereon, $\alpha(\zeta)$ is a nonzero constant, and by rescaling ζ by a nonzero constant we may as well assume $\alpha(\zeta) = 3$. Lowering an index with g (equations (20)–(21)) then gives $\zeta^b = -\omega^3$, so the third equation of (18) yields $-\iota_\zeta (d\zeta^b) = \iota_\zeta d\omega^3 = \iota_\zeta (\omega^1 \wedge \omega^2) = 0$. \square

Remark 3.4. In fact, chains come equipped with a preferred projective structure (see, e.g., [6, Theorem 5.3.7], which applies to all so-called parabolic contact structures), but we do not need that structure here.

3.1 Chains of $y'' = f(x, y, y')$

Here $\Sigma = J^0(\mathbb{R}, \mathbb{R}) = \mathbb{R}^2$, with coordinates (x, y) , and $M = J^1(\mathbb{R}, \mathbb{R}) = \mathbb{R}^3$, with coordinates (x, y, p) and contact distribution $D = \text{Ker}(dy - p dx)$. The paths in Σ are graphs of solutions $y(x)$ to $y'' = f(x, y, y')$, and their lifts to M are graphs of their first jets, $(x, y(x)) \mapsto (x, y(x), y'(x))$. So here

$$L_1 = \text{Span}(\partial_p), \quad L_2 = \text{Span}[\partial_x + p\partial_y + f(x, y, p)\partial_p].$$

We fix the contact form $\eta^3 := dy - p dx$. An adapted coframe on M , satisfying equations (14)–(15), is

$$\eta^1 := dp - f dx, \quad \eta^2 := -dx, \quad \eta^3 := dy - p dx. \quad (23)$$

Any other adapted coframe is of the form $s \cdot \eta = (\eta^1/s, s\eta^2, \eta^3)^t$, $s : M \rightarrow \mathbb{R}^*$. This defines an identification $\mathbb{R}^3 \times \mathbb{R}^* \rightarrow B$, $(x, y, p, s) \mapsto s \cdot \eta(x, y, p)$. Under this identification,

$$\omega^1 = \eta^1/s, \quad \omega^2 = s\eta^2, \quad \omega^3 = \eta^3. \quad (24)$$

The following proposition was proved in [22, equation (31)] by solving the full equivalence problem for path geometry.

Proposition 3.5. *The Fefferman metric on $B = J^1(\mathbb{R}, \mathbb{R}) \times \mathbb{R}^*$ is*

$$g = -dx \cdot (dp - f dx) + \frac{1}{6}(dy - p dx) \cdot [4f_p dx + f_{pp}(dy - p dx) - 4d\tau], \quad (25)$$

where $d\tau = ds/s$.

Proof. Solving equations (18)–(19), with ω^i given by equations (23)–(24), we obtain

$$\alpha = -f_p dx + d\tau, \quad K = f_{pp} \implies \sigma = \frac{2}{3}f_p dx + \frac{1}{6}f_{pp}(dy - p dx) - \frac{2}{3}d\tau.$$

Using this in equations (21)–(21) gives the claimed formula. \square

Proposition 3.6. *The chains of the path geometry corresponding to a 2nd order ODE $y'' = f(x, y, y')$ are the curves in $J^1(\mathbb{R}, \mathbb{R})$ which are the graphs of solutions $(y(x), p(x))$ of the system*

$$y'' = f + f_p \Delta + \frac{1}{2}f_{pp}\Delta^2 + \frac{1}{6}f_{ppp}\Delta^3 \quad (26)$$

$$p'' = -\frac{2(p' - f)^2}{\Delta} + f_p(3p' - 2f) + f_x + pf_y + [f_{pp}(p' - f) + 2f_y] \Delta \quad (27)$$

$$+ \frac{1}{6}[f_{ppp}(p' - 2f) - f_{xpp} + 4f_{yp} - pf_{ypp}] \Delta^2,$$

where $\Delta = y' - p$.

Proof. Using the metric (25), we write the geodesic equations on B ,

$$\begin{aligned} \ddot{x} &= \frac{1}{6} [(p\dot{x} - \dot{y})(f_{ppp}(\dot{y} - p\dot{x}) + 2\dot{x}f_{pp}) - 2\dot{x}^2 f_p - 4\dot{\tau}\dot{x}], \\ \ddot{y} &= \frac{1}{6} [2\dot{x}(pf_{pp}(p\dot{x} - \dot{y}) - p\dot{x}f_p - 2p\dot{\tau} + 3\dot{p}) - pf_{ppp}(\dot{y} - p\dot{x})^2], \\ \ddot{p} &= \frac{1}{6} [-p^3\dot{x}^2 f_{ypp} + 2p^2\dot{x}\dot{y}f_{ypp} + 4p^2\dot{x}^2 f_{yp} \\ &\quad - 2f((\dot{y} - p\dot{x})(f_{ppp}(\dot{y} - p\dot{x}) + 2\dot{x}f_{pp}) + 2\dot{x}^2 f_p + 4\dot{\tau}\dot{x}) \\ &\quad + 2\dot{p}f_{pp}(\dot{y} - p\dot{x}) - f_{xpp}(\dot{y} - p\dot{x})^2 - 8p\dot{x}\dot{y}f_{yp} - 6p\dot{x}^2 f_y \\ &\quad + 8p\dot{x}f_p - p\dot{y}^2 f_{ypp} + 12\dot{x}\dot{y}f_y + 6\dot{x}^2 f_x + 4\dot{y}^2 f_{yp} + 4\dot{p}\dot{\tau}]. \end{aligned}$$

(We do not need the τ equation.) Next use formula (25) and the nullity condition to eliminate $\dot{\tau}$ from the equations,

$$\dot{\tau} = \frac{1}{4} (f_{pp} (\dot{y} - p\dot{x}) + 4\dot{x}f_p) - \frac{3}{2} \frac{\dot{x}(\dot{p} - f\dot{x})}{(\dot{y} - p\dot{x})},$$

(τ itself does not appear explicitly, because of the \mathbb{R}^* -invariance of the metric). Then substitute into $y'' = (\dot{y}\dot{x} - \ddot{x}\dot{y})/\dot{x}^3$, $p'' = (\dot{p}\dot{x} - \ddot{x}\dot{p})/\dot{x}^3$ the expressions for $\ddot{x}, \dot{y}, \ddot{p}$ from the geodesic equations, and finally make the substitutions $\dot{y} = \dot{x}y', \dot{p} = \dot{x}p'$ to obtain the desired equations (all instances of \dot{x} cancel out because the geodesic equation is homogeneously quadratic in velocities). \square

3.2 Chains of projective path geometries

Here we prove Theorem 1, which was announced in the introduction. Recall that, by definition, a path geometry is projective if the paths are the (unparametrized) geodesics of a torsion-free affine connection.

Theorem 1. *A path geometry on a 2-dimensional manifold Σ is projective if and only if all chains on $\mathbb{P}T\Sigma$ project to the paths in Σ .*

Proof. This is a local statement so we can assume without loss of generality the situation studied in the previous subsection, i.e. the paths are given in the xy plane by graphs of solutions $y(x)$ of $y'' = f(x, y, y')$ for some smooth f , and the associated chains in xyp -space are the graphs of solutions $(y(x), p(x))$ to the chain equations (26)–(27) of Proposition 3.6. As is well known, such a path geometry is projective if and only if $f(x, y, p)$ is a polynomial in p of degree at most 3 (see [9], also Section 4 of [13]). The statement we are to prove therefore reduces to the following lemma:

Lemma 3.7. *Every solution $(y(x), p(x))$ of equations (26)–(27) satisfies $y'' = f(x, y, y')$ if and only if $f(x, y, p)$ is polynomial in p of degree at most 3.*

We proceed with the proof of the lemma. Assume $f(x, y, p)$ is polynomial in p of degree ≤ 3 . Then $f(x, y, y')$ is given by the cubic Taylor polynomial of f with respect to p :

$$f(x, y, y') = f + f_p(y' - p) + \frac{1}{2}f_{pp}(y' - p)^2 + \frac{1}{6}f_{ppp}(y' - p)^3, \quad (28)$$

where f and its derivatives on the right hand side are evaluated at (x, y, p) . Now the right hand side of the last equation, evaluated at $y = y(x), y' = y'(x), p = p(x)$, is the right hand side of the chain equation (26). It follows that if $(y(x), p(x))$ satisfy equations (26)–(27) then $y(x)$ satisfies $y''(x) = f(x, y(x), y'(x))$, as needed.

Conversely, suppose $f(x, y, p)$ is not polynomial in p of degree ≤ 3 . Then there is a neighborhood $U \subset \mathbb{R}^3$ such that for all $(x, y, p), (x, y, y') \in U$, with $y' \neq p$, equation (28) does not hold. It follows that the chains in this neighborhood do not project to solutions of $y'' = f(x, y, y')$. \square

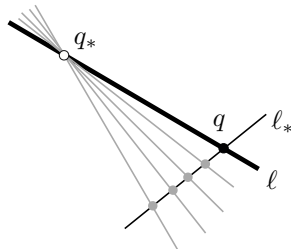


Figure 2. Chains of the flat path geometry (straight lines).

Remark 3.8. One should also be able to prove Theorem 1 using the general machinery of parabolic geometry concerning correspondence spaces [6, Section 4.4] and canonical curves [6, Section 5.3] in a way that may be readily generalizable to other types of parabolic geometries and families of curves. Such a proof would take us too far afield here, so we will take up this approach elsewhere.

4 Examples of path geometries and their chains

In this section we illustrate the general theory of the previous section by determining explicitly the chains of some *homogeneous* path geometries. First, the flat path geometry on $\mathbb{R}P^2$, admitting an 8-dimensional symmetry group, then 3 of the items of Tresse’s classification [24] of ‘submaximal’ path geometries, i.e. those admitting a 3-dimensional group of symmetries. In each case we exploit the symmetry to reduce the chain equations to determining null geodesics on a group with respect to a left-invariant sub-Riemannian metric. Then a well-known procedure reduces the equation to the Euler equations on the dual of the Lie algebra of the group and are integrable.

4.1 The flat path geometry

Here $M \subset \mathbb{R}P^2 \times (\mathbb{R}P^2)^*$ is the set of incident point-line pairs (q, ℓ) (equivalently, the manifold $F_{1,2}$ of full flags in \mathbb{R}^3) and $L_1, L_2 \subset TM$ are the tangents to the fibers of the projections onto the first and second factor (respectively).

Proposition 4.1. *For each non-incident pair $(q_*, \ell_*) \in \mathbb{R}P^2 \times (\mathbb{R}P^2)^* \setminus M$ consider the set of incident pairs $(q, \ell) \in M$ such that $q \in \ell_*, q_* \in \ell$. This is a chain in M and all chains in M are of this form. See Figure 2.*

To prove it, note that $GL_3(\mathbb{R})$ acts naturally on (M, L_1, L_2) . We look for a 3-dimensional subgroup $G \subset GL_3(\mathbb{R})$ acting on M with an open orbit. Fixing a point $m_0 = (q_0, \ell_0) \in M$ yields we get two left-invariant line fields $L_1, L_2 \subset TG$, given by their value $(L_1)_{id}, (L_2)_{id} \subset \mathfrak{g}$, the Lie algebras of the stabilizers of q_0, ℓ_0 (resp.). It is then easy to find left-invariant adapted coframes on G describing L_1, L_2 and the associated Fefferman metric. We consider two such G : the Heisenberg group and the Euclidean group.

4.1.1 First proof of Proposition 4.1: via the Heisenberg group

Let H be the set of matrices of the form

$$\begin{pmatrix} 1 & z & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R}. \quad (29)$$

Its Lie algebra \mathfrak{h} consists of matrices of the form

$$\begin{pmatrix} 0 & x^1 & x^3 \\ 0 & 0 & x^2 \\ 0 & 0 & 0 \end{pmatrix}, \quad x^i \in \mathbb{R}. \quad (30)$$

Let θ^i be the left-invariant 1-form on H whose value at $\text{id} \in H$ is x^i , $i = 1, 2, 3$. Then

$$\Theta := \begin{pmatrix} 0 & \theta^1 & \theta^3 \\ 0 & 0 & \theta^2 \\ 0 & 0 & 0 \end{pmatrix}$$

is the left-invariant Maurer–Cartan form on H , satisfying $d\Theta = -\Theta \wedge \Theta$, from which we get

$$d\theta^1 = d\theta^2 = 0, \quad d\theta^3 = -\theta^1 \wedge \theta^2. \quad (31)$$

Identify \mathbb{R}^2 with an affine plane in \mathbb{R}^3 , $(x, y) \mapsto (y, x, 1)$. It is H -invariant, and the resulting affine action on \mathbb{R}^2 is $(x_0, y_0) \mapsto (x_0 + x, y_0 + y + zx_0)$. This action is transitive on \mathbb{R}^2 and transitive and free on the set M of incident pairs (q, ℓ) , where $q \in \mathbb{R}^2$ and $\ell \subset \mathbb{R}^2$ is a non-vertical line through q . There are H -invariant line fields $L_1, L_2 \subset TM$, where L_1 (resp. L_2) is tangent to the fibers of the projection $(q, \ell) \mapsto q$ (resp. $(q, \ell) \mapsto \ell$).

Let $q_0 = (0, 0)$, $\ell_0 = \{y = 0\}$ (the real axis). Let (X_1, X_2, X_3) the (left-invariant) frame on H dual to $(\theta^1, \theta^2, \theta^3)$. Then the Lie algebras $(L_1)_{\text{id}}, (L_2)_{\text{id}}$ of the stabilizers of q_0, ℓ_0 are spanned by X_1, X_2 (resp.). Thus,

$$D = L_1 \oplus L_2 = (\theta^3)^\perp, \quad L_1 = \{\theta^2, \theta^3\}^\perp, \quad L_2 = \{\theta^1, \theta^3\}^\perp,$$

with an adapted coframe

$$\eta^1 := \theta^1, \quad \eta^2 := \theta^2, \quad \eta^3 := -\theta^3.$$

Solving the structure equations (18)–(19), we get $\alpha = \theta^4$, $K = 0$, where $\theta^4 = (ds)/s$ (the Maurer–Cartan form on \mathbb{R}^*), which gives, using equations (20)–(21), $\sigma = -(2/3)\theta^4$ and

$$g = \theta^1 \cdot \theta^2 + \frac{2}{3}\theta^3 \cdot \theta^4. \quad (32)$$

Lemma 4.2. *Null geodesics of (32), projected to H and passing through $\text{id} \in H$ at $t = 0$, are of the form*

$$x = b(1 - e^{-ct}), \quad y = -ab(1 - e^{-ct}), \quad z = a(e^{ct} - 1), \quad a, b, c \in \mathbb{R}.$$

They correspond to chains $(q_t, \ell_t) \in M$, passing through (q_0, ℓ_0) at $t = 0$, where q_t moves along the line ℓ_* through q_0 of slope $-a$, and ℓ_t is a line through q_t and $q_* = (b, 0)$.

Proof. The metric (32) is a left-invariant metric on $G = \mathbb{H} \times \mathbb{R}^*$, with an inertia operator $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$

$$A = \frac{1}{6} \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

The geodesic flow on T^*G projects via left translation to the Euler equations on \mathfrak{g}^* , $\dot{P} = \text{ad}_{A^{-1}P}^* P$, where $\text{ad}_X^* = (\text{ad}_X)^t \in \text{End}(\mathfrak{g}^*)$, $X \in \mathfrak{g}$, $P \in \mathfrak{g}^*$ and $\text{ad}_X Y = [X, Y]$. These are the Hamiltonian equations $\dot{P} = \{H, P\}$ with respect to the standard Lie-Poisson structure on \mathfrak{g}^* , where $H = \frac{1}{2}(P, A^{-1}P)$. See [2, page 66]. Equivalently, $\dot{X} = A^{-1} \text{ad}_X^* A X$. To write these down explicitly with respect to our bases, we first represent $X \in \mathfrak{g}$ and $\text{ad}_X^* \in \text{End}(\mathfrak{g}^*)$ by 4×4 matrices

$$X = \begin{pmatrix} 0 & x^1 & x^3 & 0 \\ 0 & 0 & x^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^4 \end{pmatrix}, \quad \text{ad}_X^* = (\text{ad}_X)^t = \begin{pmatrix} 0 & 0 & -x^2 & 0 \\ 0 & 0 & x^1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so $\dot{X} = A^{-1} \text{ad}_X^* A X$ becomes

$$\dot{x}^1 = \frac{2}{3} x^1 x^4, \quad \dot{x}^2 = -\frac{2}{3} x^2 x^4, \quad \dot{x}^3 = \dot{x}^4 = 0. \quad (33)$$

The general solution, with $H = (AX, X)/2 = x^1 x^2 / 2 + x^3 x^4 / 3 = 0$ (we are interested in the zero level set because we are computing the null geodesics), is

$$x^1 = a e^{ct}, \quad x^2 = b e^{-ct}, \quad x^3 = -\frac{ab}{c}, \quad x^4 = \frac{3c}{2}, \quad (34)$$

where $a, b, c \in \mathbb{R}$, $c \neq 0$. (In addition to these solutions there are some fixed points, which we now ignore).

Now let $g(t) \in \mathbb{H} \times \mathbb{R}^*$ be a null geodesic, with

$$g(t) = \begin{pmatrix} 1 & z & y & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{pmatrix}, \quad x, y, z, s \in \mathbb{R}, \quad s \neq 0.$$

Then $X = g^{-1} \dot{g} \in \mathfrak{g}$ is given by (34). Explicitly,

$$\dot{x} = x^2 = b e^{-ct}, \quad \dot{y} - z \dot{x} = x^3 = -\frac{ab}{c}, \quad \dot{z} = x^1 = a e^{ct} \quad (35)$$

(we do not need the s equation). Change the time variable to $\tau = ct$, denoting derivative with respect to τ by $()'$ and renaming the constants, $a \mapsto a/c, b \mapsto b/c$, we get

$$x' = be^{-\tau}, \quad y' - zx' = -ab, \quad z' = ae^\tau. \quad (36)$$

Consider chains through $\text{id} \in \mathbb{H}$, i.e. $x_0 = y_0 = z_0 = 0$. Then $z = a(e^\tau - 1)$, hence $y' = zx' - ab = -abe^{-\tau}$. The solution of (36) is then

$$x = b(1 - e^{-\tau}), \quad y = -ab(1 - e^{-\tau}), \quad z = a(e^\tau - 1).$$

Thus (x, y) traces a line of slope $-a$ through the origin, and each line of slope z through (x, y) passes through $(b, 0)$. \square

4.1.2 Second proof of Proposition 4.1: via the Euclidean group

Here $M = \mathbb{P}(T\mathbb{R}^2) = \mathbb{R}^2 \times \mathbb{P}(\mathbb{R}^2)$ is the set of pairs (q, ℓ) with ℓ a line through q , $L_1 \subset TM$ is tangent to the fibers of the projection onto the first factor and similarly for L_2 . The group SE_2 of orientation-preserving isometries of \mathbb{R}^2 acts transitively on M , with stabilizer \mathbb{Z}_2 (reflection about a point), preserving L_1, L_2 . Fixing a point $(q_0, \ell_0) \in M$ identifies M with SE_2/\mathbb{Z}_2 , and hence equips SE_2 with left-invariant line fields L_1, L_2 given by a pair of 1-dimensional subspaces $(L_1)_{\text{id}}, (L_2)_{\text{id}}$, the Lie algebras of the stabilizers of q_0, ℓ_0 (resp.).

Identify $\mathbb{R}^2 = \mathbb{C}$ with the affine plane $\mathbf{z} = 1$ in \mathbb{C}^2 , $\mathbf{z} \mapsto (\mathbf{z}, 1)$; then SE_2 is identified with the subgroup of $\text{GL}_2(\mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} e^{i\theta} & \mathbf{z} \\ 0 & 1 \end{pmatrix}, \quad \mathbf{z} \in \mathbb{C}, \theta \in \mathbb{R}. \quad (37)$$

Its Lie algebra \mathfrak{se}_2 consists of matrices of the form

$$\begin{pmatrix} ix^1 & x^2 + ix^3 \\ 0 & 0 \end{pmatrix}, \quad x^i \in \mathbb{R}. \quad (38)$$

Let θ^j be the left-invariant 1-form on SE_2 whose value at id is x^j , $j = 1, 2, 3$. Then

$$\Theta := \begin{pmatrix} i\theta^1 & \theta^2 + i\theta^3 \\ 0 & 0 \end{pmatrix}$$

is the left-invariant Maurer–Cartan form on SE_2 , satisfying $d\Theta = -\Theta \wedge \Theta$, from which we get

$$d\theta^1 = 0, \quad d\theta^2 = \theta^1 \wedge \theta^3, \quad d\theta^3 = -\theta^1 \wedge \theta^2. \quad (39)$$

Let X_1, X_2, X_3 the left-invariant vector fields on SE_2 dual to $\theta^1, \theta^2, \theta^3$. Let $q_0 = 0, \ell_0 = \mathbb{R}$ (the real axis). Then the Lie algebras of the stabilizers of q_0, ℓ_0 are spanned by X_1, X_2 (resp.). Thus

$$D = L_1 \oplus L_2 = (\theta^3)^\perp, \quad L_1 = \{\theta^2, \theta^3\}^\perp, \quad L_2 = \{\theta^1, \theta^3\}^\perp,$$

with an adapted coframe

$$\eta^1 := \theta^1, \quad \eta^2 := \theta^2, \quad \eta^3 := -\theta^3.$$

Solving the structure equations (18)–(19), we get $\alpha = \theta^4$, $K = 0$, where $\theta^4 = (ds)/s$ (the MC form on \mathbb{R}^*), which gives, using equations (20)–(21), $\sigma = -(2/3)\theta^4$ and

$$\mathfrak{g} = \theta^1 \cdot \theta^2 + \frac{2}{3}\theta^3 \cdot \theta^4. \quad (40)$$

This is a left-invariant metric on $G = \text{SE}_2 \times \mathbb{R}^*$, with an inertia operator $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$

$$A = \frac{1}{6} \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

The geodesic flow on T^*G projects via left translation to the Euler equations on \mathfrak{g}^* , $\dot{P} = \text{ad}_{A^{-1}P}^* P$, where $\text{ad}_X^* = -(\text{ad}_X)^t \in \text{End}(\mathfrak{g}^*)$. These are the Hamiltonian equations $\dot{P} = \{H, P\}$ with respect to the standard Lie-Poisson structure on \mathfrak{g}^* , where $H = \frac{1}{2}(P, A^{-1}P)$. See [2, p. 66]. To write these down explicitly with respect to our bases, we first represent $X \in \mathfrak{g}$ and $\text{ad}_X^* \in \text{End}(\mathfrak{g}^*)$ by 4×4 matrices

$$X = \begin{pmatrix} 0 & -x^1 & x^2 & 0 \\ x^1 & 0 & x^3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^4 \end{pmatrix}, \quad \text{ad}_X^* = (\text{ad}_X)^t = \begin{pmatrix} 0 & x^3 & -x^2 & 0 \\ 0 & 0 & x^1 & 0 \\ 0 & -x^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so $\dot{P} = \text{ad}_{A^{-1}P}^* P$ becomes

$$\dot{P}_1 = -2P_1P_3 + 3P_2P_4, \quad \dot{P}_2 = 2P_2P_3, \quad \dot{P}_3 = -2P_2^2, \quad \dot{P}_4 = 0. \quad (41)$$

with constants of motion (in addition to P_4),

$$H = \frac{1}{2}(P, A^{-1}P) = 2P_1P_2 + 3P_3P_4 = 0, \quad k = (P_2)^2 + (P_3)^2.$$

Let us use polar coordinates in the P_2P_3 -plane:

$$P_2 = r \cos \phi, \quad P_3 = r \sin \phi.$$

Then

$$\dot{\phi} = -2r \cos \phi, \quad P_1 = c \tan \phi, \quad P_4 = -2c/3, \quad c = \text{const.}, \quad r = \text{const.}$$

Now let $g(t) \in \text{SE}_2 \times \mathbb{R}^*$ be a null geodesic, with

$$g(t) = \begin{pmatrix} e^{i\theta} & \mathbf{z} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{pmatrix}, \quad \mathbf{z} \in \mathbb{C}, \quad \theta, s \in \mathbb{R}^*.$$

Let $X = g^{-1}\dot{g} \in \mathfrak{g}$. Then $P = AX$ satisfies Euler equations (41). Explicitly,

$$\dot{\theta} = x^1 = 2P_2 = -\dot{\phi}, \quad \dot{\mathbf{z}} = e^{i\theta}(x^2 + ix^3) = e^{i\theta}(2P_1 + i3P_4) = 2ce^{i\theta}(\tan \phi - i).$$

(The s equation is omitted; it will not be used). Assume, without loss of generality, that $g(0) = \text{id}$, i.e. $\theta(0) = 0$ and $\mathbf{z}(0) = 0$, so $\theta = \phi_0 - \phi$. We reparametrize $g(t)$ by ϕ , denote derivative with respect to ϕ by $(\)'$, and get $\mathbf{z}' = i(c/r)e^{i\phi_0}\dot{\mathbf{z}}/\dot{\phi} = i(c/r)e^{i\phi_0}\sec^2\phi$. Integrating yields $\mathbf{z} = i(c/r)e^{i\phi_0}\tan\phi$. Now we rotate the chain by $-\phi_0$, reflect about the x -axis and rename c , so that

$$\mathbf{z} = ict \tan \phi, \quad \theta = \phi, \quad c \in \mathbb{R}. \quad (42)$$

This corresponds to a chain (q_ϕ, ℓ_ϕ) , where q_ϕ moves along ℓ^* =the y axis and ℓ_ϕ is the line connecting $q_* = (-c, 0)$ with q_ϕ .

4.2 Circles of fixed radius

Here $M \subset \mathbb{C} \times \mathbb{C}$ is the set of pairs of points (p, q) with $|p - q| = 1$, $L_1, L_2 \subset TM$ are tangent to the fibers of the projection onto the first (resp. second) factor. The first projection maps the fibers of the second projection to the set of plane circles of radius 1. The group SE_2 of orientation preserving isometries of $\mathbb{C} = \mathbb{R}^2$ acts transitively and freely on M , preserving L_1, L_2 . We use the same notation for this group as in Section 4.1.2. Let $p_0 = 0, q_0 = 1$. Then the Lie algebras $(L_1)_{\text{id}}, (L_2)_{\text{id}}$ of the stabilizers of these points are spanned by $X_1, X_1 - X_3$ (resp.). Thus

$$D = L_1 \oplus L_2 = (\theta^2)^\perp, \quad L_1 = \{\theta^2, \theta^3\}^\perp, \quad L_2 = \{\theta^2, \theta^1 + \theta^3\}^\perp,$$

with an adapted coframe

$$\eta^1 := \theta^3, \quad \eta^2 := \theta^1 + \theta^3, \quad \eta^3 := -\theta^2.$$

Solving the structure equations (18)–(19), we get $\alpha = \theta^2 + \theta^4$, $K = -1$, where $\theta^4 = (ds)/s$ (the Maurer–Cartan form on \mathbb{R}^*), which gives, using equations (20)–(21), $\sigma = -\theta^2/2 - 2\theta^4/3$ and

$$\mathfrak{g} = \frac{1}{2}(\theta^2)^2 + (\theta^3)^2 + \theta^1 \cdot \theta^3 + \frac{2}{3}\theta^2 \cdot \theta^4. \quad (43)$$

This is a left-invariant metric on $G = \text{SE}_2 \times \mathbb{R}^*$, with an inertia operator $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$

$$A = \frac{1}{6} \begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 3 & 0 & 6 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

The geodesic flow on T^*G projects via left translation to the Euler equations on \mathfrak{g}^* , $\dot{P} = \text{ad}_{A^{-1}P}^*P$. To write these down explicitly with respect to our bases, we first represent $X \in \mathfrak{g}$ and $\text{ad}_X^* \in \text{End}(\mathfrak{g}^*)$ by 4×4 matrices

$$X = \begin{pmatrix} 0 & -x^1 & x^2 & 0 \\ x^1 & 0 & x^3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^4 \end{pmatrix}, \quad \text{ad}_X^* = (\text{ad}_X)^t = \begin{pmatrix} 0 & x^3 & -x^2 & 0 \\ 0 & 0 & x^1 & 0 \\ 0 & -x^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so $\dot{P} = \text{ad}_{A^{-1}P}^* P$ become

$$\begin{aligned}\dot{P}_1 &= 2P_1P_2 - 3P_3P_4, & \dot{P}_2 &= 2P_3(P_3 - 2P_1), \\ \dot{P}_3 &= -2P_2(P_3 - 2P_1), & \dot{P}_4 &= 0,\end{aligned}\tag{44}$$

with constants of motion (in addition to P_4),

$$\begin{aligned}H &= \frac{1}{2}(P, A^{-1}P) = -2P_1^2 + 2P_1P_3 + 3P_2P_4 - \frac{9}{4}M_4^2 \\ r^2 &= (P_2)^2 + (P_3)^2.\end{aligned}$$

We make the following change of variables:

$$y = 4P_1 - 2P_3, \quad P_2 = r \cos \phi, \quad P_3 = -r \sin \phi, \quad P_4 = c/3.\tag{45}$$

Then (44) reduces to

$$\dot{\phi} = -y, \quad \dot{y} = 4r(c - r \cos \phi) \sin \phi, \quad c, r \in \mathbb{R}, \quad r \geq 0,\tag{46}$$

and the nullity condition $H = 0$ becomes

$$y^2 = 8cr \cos \phi + 4r^2 \sin^2 \phi - 2c^2.\tag{47}$$

Remark 4.3. Equations (46) can be written as a single Newton type second order ODE, $\ddot{\phi} = f(\phi)$, where $f(\phi) = 4r(r \cos \phi - c) \sin \phi$. As usual, one writes $f(\phi) = -U'(\phi)$, say $U(\phi) = -4cr \cos \phi - 2r^2 \sin^2 \phi$. Then $\dot{\phi}^2/2 + U(\phi)$ is constant along solutions of $\ddot{\phi} = f(\phi)$ ('conservation of energy'). Equation (47) fixes the value of this energy.

Now let $g(t) \in \text{SE}_2 \times \mathbb{R}^*$ be a null geodesic, with

$$g(t) = \begin{pmatrix} e^{i\theta} & \mathbf{z} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{pmatrix}, \quad \mathbf{z} \in \mathbb{C}, \quad \theta, s \in \mathbb{R}^*.$$

Let $X = g^{-1}\dot{g} \in \mathfrak{g}$. Then $P = AX$ satisfies (44). Explicitly,

$$\dot{\mathbf{z}} = e^{i\theta}(x^2 + ix^3) = e^{i\theta}(3P_4 + i2P_1), \quad \dot{\theta} = x^1 = -4P_1 + 2P_3.$$

Using the change of variables (45), we get

$$\dot{\mathbf{z}} = e^{i\theta}[c - i(\dot{\phi}/2 + r \sin \phi)], \quad \dot{\theta} = \dot{\phi},$$

where $\phi(t)$ satisfies equations (46)–(47). For a fixed $\phi(t)$ these equations are invariant under rigid motions (adding a constant angle to θ , rotating \mathbf{z} by this angle and translating \mathbf{z} by some constant vector). So we can assume, without loss of generality, that $\theta = \phi$. Hence

$$\dot{\mathbf{z}} = e^{i\theta}[c - i(\dot{\theta}/2 + r \sin \theta)], \quad \dot{\theta}^2 = 8cr \cos \theta + 4r^2 \sin^2 \theta - 2c^2.$$

Next we use the scaling invariance, $t \mapsto \lambda t$, $c \mapsto \lambda c$, $r \mapsto \lambda r$, to assume $r = 1$. We can also use the reflection symmetry $t \mapsto -t$, $\theta \mapsto \theta + \pi$, $\mathbf{z} \mapsto -\mathbf{z}$, $c \mapsto -c$ to assume that $c \geq 0$. Thus every chain, up to a rigid motion and reparametrization, is a solution to

$$\dot{\mathbf{z}} = e^{i\theta}[c - i(\dot{\theta}/2 + \sin\theta)], \quad (48)$$

$$(\dot{\theta})^2 = 8c \cos\theta + 4 \sin^2\theta - 2c^2, \quad (49)$$

with $c \in \mathbb{R}$, $c \geq 0$.

Lemma 4.4. *Let $F(\theta, c) = 8c \cos\theta + 4 \sin^2\theta - 2c^2$ (the right-hand side of equation (49)). Then $F \geq 0$ has a solution if and only if $|c| \leq 4$. For every $c \in [0, 4]$ the set of $\theta \in [-\pi, \pi]$ such that $F(\theta, c) \geq 0$ is an interval $[-\theta_{max}, \theta_{max}]$, where $\theta_{max} = \cos^{-1}(c - \sqrt{c^2/2 + 1}) \in [0, \pi]$. For $c \in (0, 4)$ every solution $\theta(t)$ of (49) oscillates between $-\theta_{max}$ and θ_{max} . If $c = 0$ then $\lim\theta$ is 0 or π as $t \rightarrow \pm\infty$. If $c = 4$ then $\theta \equiv 0$.*

Proof. We write $F = -4x^2 + 8cx + 4 - 2c^2$, where $x = \cos\theta$. The roots of this polynomial are $x_{\pm} = c \pm \sqrt{1 + c^2/2}$ and $F > 0$ in the interval (x_-, x_+) . To be able to solve for θ we need $[x_-, x_+]$ to intersect the interval $[-1, 1]$. It is elementary to show that this occurs if and only if $|c| \leq 4$. \square

Proposition 4.5. *Every chain of the path geometry of circles of radius 1 in the Euclidean plane, up to an affine reparametrization and rigid motion, is given by a unique solution of equations (48)–(49) with $c \in [0, 4)$, $\mathbf{z}(0) = \theta(0) = 0$ (for $c = 0$ one should take $\theta(0) \neq 0, \pi$).*

See Figure 3. The projection of the chains on the Euclidean plane (the curves $\mathbf{z}(t)$) look like inflectional elastica, but they are not (checked numerically).

Further properties/questions about these chains:

1. From the pictures, $\mathbf{z}(t) + e^{i\theta(t)}$ (the red curve) is obtained from $\mathbf{z}(t)$ by translation and parameter shift. Presumably, this comes out of equations (48)–(49). Is this a manifestation of the self-duality of this path geometry? How exactly?
2. One should be able to write explicit solutions of equations (48)–(49) using elliptic functions. See [23].

Note. One can write down an explicit general solution for the case $c = 0$ without any special functions, and one can verify analytically that the arcs are semicircles. So, as embedded submanifolds they are C^1 but not C^2 at inflection points. In particular, the chain ODE is not satisfied at these inflection points.

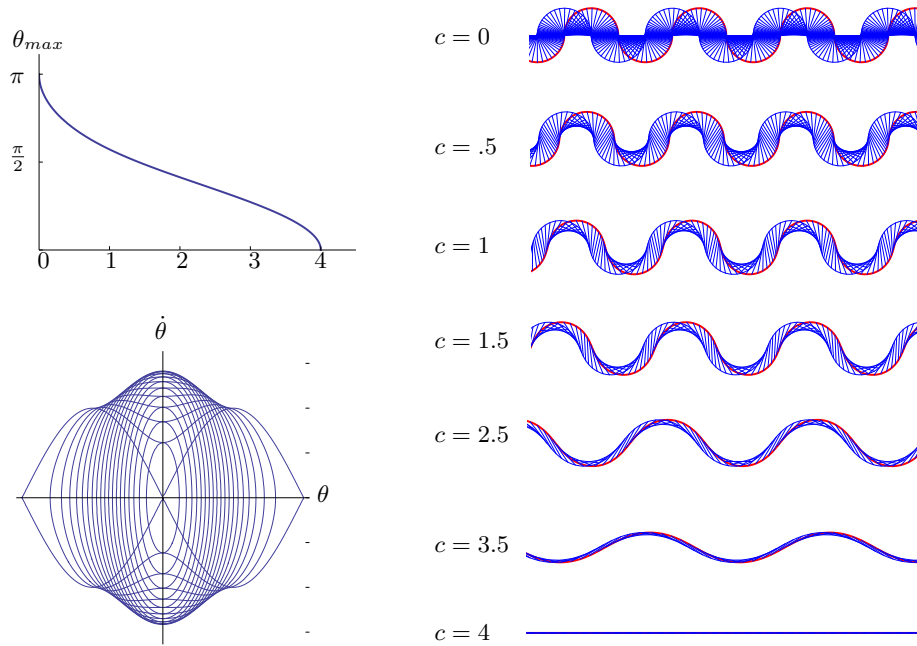


Figure 3. Chains of circles path geometry (solutions of equations (48)–(49)). Top left: plot of the maximum amplitude of oscillation of θ as a function of the chain parameter $c \in [0, 4]$. Bottom left: phase curves of equation (49) for various c values. Right: each red curve is the projection of the chain on the Euclidean plane. The blue curve represents the projection of the chain on the dual plane; it is formed by joining the tips of the unit vectors in the direction θ at each point \mathbf{z} of the red curve (the thin blue lines).

3. Equation (49) is the equation of a pendulum under a strange force law: $f(\theta) = 4(\cos \theta - c) \sin \theta$, with special initial conditions: $\theta(0) = 0, \dot{\theta}(0) = 8c - c^2$ (for $c = 0$ it is the homoclinic solution of the pendulum equation $\ddot{\theta} = 2 \sin 2\theta$). Is there a good mechanical/geometrical interpretation of this motion?
4. The chains of this geometry project to a 1-parameter family of curves in \mathbb{R}^2 (up to rigid motion). Is there a simple geometric description of this family? Our first guess was elastica but it is not the case.
5. In the pictures, there are points along $\mathbf{z}(t)$ at which $\theta(t)$ is the direction of the tangent $\dot{\mathbf{z}}(t)$ (the inflection points of the red curves on the right of Figure 3). Is this phenomenon unavoidable?

4.3 Hooke ellipses of fixed area

The manifold

$$M = \{(x, y, E, F, G) \in \mathbb{R}^5 \mid Ex^2 + 2Fxy + Gz^2 = 1, EG - F^2 = 1, E > 0\}$$

parametrizes the set of incident pairs $(\mathbf{r}, \mathcal{E})$, where $\mathbf{r} = (x, y)^t \in \mathbb{R}^2 \setminus \{0\}$ and \mathcal{E} is an ellipse centered at the origin (a ‘Hooke ellipse’) of area π .

Proposition 4.6. *The path geometry in $\mathbb{R}^2 \setminus \{0\}$ of Hooke ellipses of fixed area is projective (the paths are the unparametrized geodesics of a torsion-free affine connection).*

Proof. As mentioned before, this is equivalent to showing that the associated ODE $y'' = f(x, y, y')$ is cubic in y' . Let $\mathbb{H} = \{(E, F, G) \mid EG - F^2 = 1, E > 0\}$ be the path space. We parametrize \mathbb{H} by the upper half-plane $\mathbb{R}_+^2 = \{(a, b) \mid b > 0\}$,

$$(a, b) \mapsto \frac{1}{b}(1, -a, a^2 + b^2). \quad (50)$$

Hooke ellipses of area π are then given by equations of the form

$$x^2 - 2axy + (a^2 + b^2)y^2 = b, \quad a, b \in \mathbb{R}, b > 0. \quad (51)$$

Assuming $y = y(x)$ in this equation and taking two derivatives with respect to x , we get

$$\begin{aligned} x - a(y + xy') + (a^2 + b^2)yy' &= 0, \\ 1 - a(2y' + xy'') + (a^2 + b^2)[(y')^2 + yy''] &= 0. \end{aligned}$$

Eliminating a, b from the last 3 equations and solving for y'' , we obtain

$$y'' = (xy' - y)^3.$$

Another proof, more direct, consists of showing that Hooke ellipses of area π are the (unparametrized) geodesics of a Riemannian metric in $\mathbb{R}^2 \setminus \{0\}$, given in polar coordinates by $ds^2 = dr^2/\Delta^2 + r^2d\theta^2/\Delta$, $\Delta = 1 + cr^2 + r^4$, $c \in \mathbb{R}$.

See [3] for yet another proof, via equivalence with the path geometry of Kepler ellipses of fixed major axis, which is projective since these are geodesics of the Jacobi-Maupertuis metric of the Kepler problem. \square

Fefferman metric. Let $L_1 \subset TM$ be the tangents to the fibers of the projection on the first component, $(q, \mathcal{E}) \mapsto q$, and similarly for L_2 . The group $\mathrm{SL}_2(\mathbb{R})$ acts transitively and freely on M via its standard linear action on \mathbb{R}^2 , preserving L_1, L_2 . Fixing a point $(q_0, \mathcal{E}_0) \in M$ identifies M with $\mathrm{SL}_2(\mathbb{R})$, and L_1, L_2 with two left-invariant line fields on $\mathrm{SL}_2(\mathbb{R})$, given at $\mathrm{id} \in \mathrm{SL}_2(\mathbb{R})$ by the Lie algebras of the stabilizers of q_0, \mathcal{E}_0 , respectively.

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ consists of matrices of the form

$$\begin{pmatrix} x^1 & x^2 \\ x^3 & -x^1 \end{pmatrix}, \quad x^i \in \mathbb{R}.$$

The left-invariant $\mathfrak{sl}_2(\mathbb{R})$ -valued Maurer–Cartan form on $\mathrm{SL}_2(\mathbb{R})$ is

$$\Theta = g^{-1}dg = \begin{pmatrix} \theta^1 & \theta^2 \\ \theta^3 & -\theta^1 \end{pmatrix}. \quad (52)$$

The Maurer–Cartan equation $d\Theta = -\Theta \wedge \Theta$ gives

$$d\theta^1 = -\theta^2 \wedge \theta^3, \quad d\theta^2 = -2\theta^1 \wedge \theta^2, \quad d\theta^3 = 2\theta^1 \wedge \theta^3. \quad (53)$$

Fix $q_0 := (1, 0)^t$, $\mathcal{E}_0 := \{x^2 + y^2 = 1\}$. Then

$$(L_1)_{\mathrm{id}} = \mathrm{Span} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (L_2)_{\mathrm{id}} = \mathrm{Span} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad D = L_1 \oplus L_2 = \mathrm{Ker}(\theta^1).$$

An adapted coframe is thus

$$\eta^1 = \theta^2 + \theta^3, \quad \eta^2 = \theta^3, \quad \eta^3 := -\theta^1.$$

We use this coframe to trivialize the associated \mathbb{R}^* -structure $B \simeq \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}^*$ and put the standard coordinate s on the \mathbb{R}^* factor. The associated 1-forms on B are

$$\omega^1 = \frac{1}{s}(\theta^2 + \theta^3), \quad \omega^2 = s\theta^3, \quad \omega^3 = -\theta^1.$$

Solving the structure equations (18)–(19), we get $\alpha = 2\theta^1 + \theta^4$, $a_1 = 4/s^2$, $a_2 = 0$, $K = -2$, where $\theta^4 = (ds)/s$ (the MC form on \mathbb{R}^*), which gives, using equations (20)–(21), $\sigma = -\theta^1 - (2/3)\theta^4$ and

$$g = (\theta^1)^2 + (\theta^3)^2 + \theta^2\theta^3 + \frac{2}{3}\theta^1\theta^4. \quad (54)$$

Hooke chains (null geodesics of the Fefferman metric). The pseudo-Riemannian metric (54) is a left-invariant metric on the Lie group $G := \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}^*$. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \times \mathbb{R}$ be its Lie algebra and $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$ the ‘inertia’ operator corresponding to the quadratic form (54); that is, $g(X, Y) = (AX)Y$, $X, Y \in \mathfrak{g}$. Then

$$A = \frac{1}{6} \begin{pmatrix} 6 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

(with respect to the basis $\{\theta^i\}$ and its dual). As in previous examples, the geodesic flow on T^*G projects to $\dot{P} = \text{ad}_{A^{-1}P}^*P$ on \mathfrak{g}^* , the Hamiltonian equations with respect to the standard Lie-Poisson structure on \mathfrak{g}^* with Hamiltonian $H = \frac{1}{2}(P, A^{-1}P)$. To write these down explicitly, we first represent $X \in \mathfrak{g}$ and $\text{ad}_X^* \in \text{End}(\mathfrak{g}^*)$ by the matrices

$$X = \begin{pmatrix} x^1 & x^2 & 0 \\ x^3 & -x^1 & 0 \\ 0 & 0 & x^4 \end{pmatrix}, \quad \text{ad}_X^* = \begin{pmatrix} 0 & -2x^2 & 2x^3 & 0 \\ -x^3 & 2x^1 & 0 & 0 \\ x^2 & 0 & -2x^1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so $\dot{P} = \text{ad}_{A^{-1}P}^*P$ becomes

$$\begin{aligned} \dot{P}_1 &= 8(P_2)^2, & \dot{P}_2 &= 2P_2(3P_4 - P_1), \\ \dot{P}_3 &= 2P_1(P_3 - 2P_2) - 6P_3P_4, & \dot{P}_4 &= 0, \end{aligned} \quad (55)$$

with constants of motion P_4, k, H , where

$$k = (P_1)^2 + 4P_2P_3, \quad H = 2P_2P_3 - 2(P_2)^2 + 3P_1P_4 - 9P_4^2/2 = 0. \quad (56)$$

Note that k is a Casimir of \mathfrak{g}^* coming from the Killing form of $\mathfrak{sl}_2(\mathbb{R})$. We set $H = 0$ since we are looking for null geodesics. Next we make the following change of variables

$$P_1 = b(c + \sin \phi), \quad P_2 = \frac{b}{2} \cos \phi, \quad P_3 = b \left(\cos \phi - \frac{p}{2} \right), \quad P_4 = \frac{bc}{3}.$$

We have $k - 2H = b^2$, hence b is constant. Since $P_4 = bc/2$ is constant c is constant as well. Equations (55)-(56) then reduce to

$$\dot{\phi} = 2b \cos \phi, \quad \dot{p} = \cos \phi + c(c + 2 \sin \phi) \sec \phi. \quad (57)$$

Next let $g(t) \in \text{SL}_2(\mathbb{R}) \times \mathbb{R}^*$ be a null geodesic, with

$$g(t) = \begin{pmatrix} x & z & 0 \\ y & w & 0 \\ 0 & 0 & s \end{pmatrix}, \quad x, y, z, w, s \in \mathbb{R}, \quad s \neq 0, \quad xw - yz = 1.$$

Let $X = g^{-1}\dot{g} \in \mathfrak{g}$. Then $P = AX$ satisfies equations (55). Explicitly,

$$\begin{aligned} \dot{x} &= x^1x + x^3z = b[cx + (\cos \phi)z], & \dot{z} &= x^2x - x^1z = -b[px + cz], \\ \dot{y} &= x^1y + x^3w = b[cy + (\cos \phi)w], & \dot{w} &= x^2y - x^1w = -b[py + cw], \end{aligned}$$

where p, ϕ are given by equation (57). Denote $\mathbf{r} := (x, y)$, $\mathbf{h} := (z, w) \in \mathbb{R}^2$, then the last system is

$$\dot{\mathbf{r}} = b[\mathbf{c}\mathbf{r} + (\cos \phi)\mathbf{h}], \quad \dot{\mathbf{h}} = -b[\mathbf{p}\mathbf{r} + \mathbf{c}\mathbf{h}]. \quad (58)$$

Lemma 4.7. ϕ is twice the centro-affine arclength of the projection of the chain to the Hooke plane (the \mathbf{r} plane).

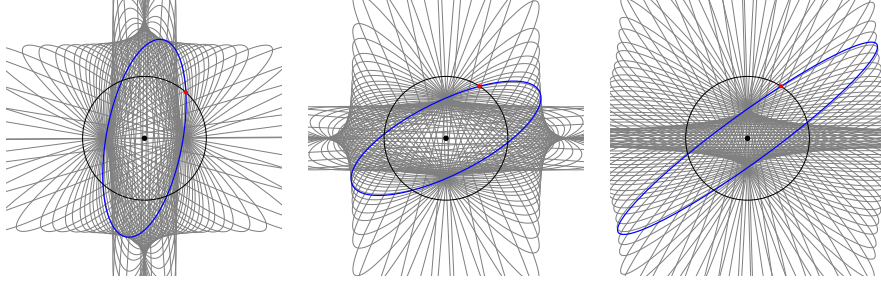


Figure 4. Hooke's chains, given by Proposition 4.9, for $c = -1, 1, 2$.

Proof. \mathbf{r}, \mathbf{h} are the columns of a matrix in $\mathrm{SL}_2(\mathbb{R})$, hence $[\mathbf{r}, \mathbf{h}] = 1$. It then follows from equations (58) that $[\mathbf{r}, d\mathbf{r}/d\phi] = [\mathbf{r}, \dot{\mathbf{r}}/\dot{\phi}] = [\mathbf{r}, \mathbf{h}/2] = 1/2$. \square

Let us reparametrize the chains by $\tau := \phi/2$ (the centro-affine arclength) and denote derivative with respect to τ by $(\)'$. Equations (58) now become

$$\begin{aligned} \mathbf{r}' &= c(\sec 2\tau)\mathbf{r} + \mathbf{h}, \\ \mathbf{h}' &= -[1 + c(c + 2(\sin 2\tau))\sec^2 2\tau]\mathbf{r} - c(\sec 2\tau)\mathbf{h}. \end{aligned} \quad (59)$$

Lemma 4.8.

$$\mathbf{r}'' = -\mathbf{r}.$$

Proof. Straightforward calculation from equations (59). \square

Thus, combined with $[\mathbf{r}, \mathbf{r}'] = 1$ (Lemma 4.7), each Hooke chain projects to a Hooke ellipse of area π in the \mathbf{r} plane, as expected from Proposition 4.6 and Theorem 1.

Proposition 4.9. *Every chain in $\mathrm{SL}_2(\mathbb{R})$ of the path geometry of Hooke ellipses of area π , up to left translation, is of the form*

$$\mathbf{r} = e^{i\tau}, \quad \mathbf{h} = e^{i\tau}(-c\sec(2\tau) + i)$$

(using complex notation), for some $c \in \mathbb{R}$, $c \neq 0$. See Figure 4.

Proof. $\mathrm{SL}_2(\mathbb{R})$ acts transitively on Hooke ellipses of area π , hence the projection of the chain to the \mathbf{r} plane can be brought to the unit circle. Parametrized by centro affine arc length, it is $\mathbf{r} = e^{i\tau}$. Then the 1st equation of (59) implies the formula for $\mathbf{h}(\tau)$. For $c = 0$ this formula produces a curve tangent to the contact distribution, which is excluded. \square

4.4 Horocycles in the hyperbolic plane

The space of Hooke ellipses is $\mathbb{H} = \{(E, F, G) \mid EG - F^2 = 1, E > 0\}$, the hyperboloid model of the hyperbolic plane. The curves in \mathbb{H} of constant (hyperbolic) curvature 1 are called *horocycles* and are the sections of \mathbb{H} by planes parallel to a generator of the cone $EG - F^2 = 0$. In the upper half-plane model these are (Euclidean) circles tangent to the real axis.

Lemma 4.10. *For each fixed $(x, y) \in \mathbb{R}^2 \setminus \{0\}$, the set of Hooke ellipses passing through (x, y) is a horocycle in \mathbb{H} . This defines a bijection between the punctured plane $\mathbb{R}^2 \setminus \{0\}$ and the space of horocycles in \mathbb{H} .*

Proof. For each $(x, y) \in \mathbb{R}^2 \setminus 0$, equation (51),

$$x^2 - 2axy + (a^2 + b^2)y^2 = b,$$

defines in the upper half-plane $\{(a, b) \mid b > 0\}$ either the circle of radius $\frac{1}{2y^2}$ centered at $(\frac{x}{y}, \frac{1}{2y^2})$ if $y \neq 0$, or the horizontal line $b = x^2$ if $y = 0$. These are precisely all the horocycles of the upper half plane model of the hyperbolic plane. \square

It follows that the horocycle path geometry in \mathbb{H} is dual to the path geometry in $\mathbb{R}^2 \setminus \{0\}$ of Hooke ellipses of fixed area. Thus we can use the analysis of the previous section to determine the projection of the chains to \mathbb{H} .

Proposition 4.11. *Each chain of the horocycle path geometry, up to the action of $\mathrm{SL}_2(\mathbb{R})$, projects to a curve in the hyperbolic plane, given in the upper half-plane model $\{(x, y) \mid y > 0\}$ by*

$$(x^2 + y^2)^2 - [4cx + (c^2 + 4)y](x^2 + y^2) + (6c^2 - 2)x^2 + 2c^3xy + 6y^2 - 4c(c^2 - 1)x - (c^4 - 3c^2 + 4)y + (c^2 - 1)^2 = 0 \quad (60)$$

where $c \neq 0$. See Figure 5. This curve is the projection of a chain in $\mathrm{SL}_2(\mathbb{R})$, the solution to equations (59) that passes through $\mathrm{id} \in \mathrm{SL}_2(\mathbb{R})$. The projection of this chain to the Hooke plane is the Hooke ellipse $(x - cy)^2 + y^2 = 1$. The horocycles along this chain, in the upper half plane model, all pass through $(c, 1)$, the point corresponding to this Hooke ellipse. The chains corresponding to c and $-c$ are congruent via an outer automorphism of $\mathrm{SL}_2(\mathbb{R})$ (conjugation by $\mathrm{diag}(-1, 1) \in \mathrm{GL}_2(\mathbb{R})$), acting by reflection about the y -axis.

Proof. Using $\mathbf{r}'' = -\mathbf{r}$ and $\mathbf{r}' = c(\sec 2\tau)\mathbf{r} + \mathbf{h}$ (Lemma 4.8 and equation (59)), the chain $g(\tau)$ in $\mathrm{SL}_2(\mathbb{R})$ with $g(0) = \mathrm{id}$ is

$$g(\tau) = \begin{pmatrix} \cos \tau + c \sin \tau & -\sin \tau (\sec(2\tau)c^2 + \tan(2\tau)c + 1) \\ \sin \tau & \cos \tau - c \sec(2\tau) \sin \tau \end{pmatrix}. \quad (61)$$

The projection of this chain to \mathbb{H} is obtained by acting by $g(\tau)$ on the point in \mathbb{H} corresponding to the Hooke ellipse $\mathcal{E}_0 = \{x^2 + y^2 = 1\}$. One can check that

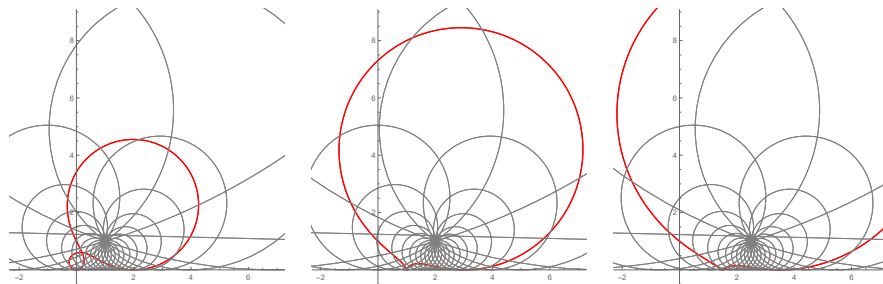


Figure 5. Horocycle chains, given by Proposition 4.11, projected to the hyperbolic plane (the upper half plane model), are rational bicircular quartics. Left: *crunodal* (one node), $|c| < 2$. Middle: *cuspidal* (one cusp), $c = 2$. Right: *acnodal* (smooth), $|c| > 2$.

the parametrization of \mathbb{H} by the upper half-plane in equation (50) is $\mathrm{SL}_2(\mathbb{R})$ -equivariant, so one can act instead by $g(\tau)$ via fractional linear transformations on $(0, 1)$, the point in the upper half-plane corresponding to \mathcal{E}_0 . Reverting to $\phi = 2\tau$, the outcome is

$$(x, y) = \frac{(c^2 [(c + 2 \sin \phi) \cos \phi - c - \sin \phi], -2 \cos^2 \phi)}{-2 \cos^2 \phi + c(c + 2 \sin \phi) \cos \phi - c^2}.$$

Eliminating ϕ in the above equation (we used Maple for this), one obtains equation (60). \square

Remark 4.12. The curves of Proposition 4.11 are examples of *bicircular quartics*, a remarkable class of plane curves introduced by J. Casey in 1871 [8]. They have many equivalent geometric and algebraic definitions, the simplest being *the inversion of a conic* (with respect to a circle).

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