Prime Ends and Dynamical Systems

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Abstract. These notes were written for a short course at CIMAT in Mexico. We make an introduction to the theory of prime ends with an application to the theory of dynamical systems. In order to avoid topological difficulties we work with homeomorphisms of the two sphere, though results are true in a much more general setting. I would like to thank the people from CIMAT, especially Gonzalo Contreras for the hospitality.

1. A problem in Dynamical Systems

Let S be the two dimensional sphere and $f : S \to S$ an orientation preserving homeomorphism of S.

Let p be a fixed point of f.

We say that p is of *saddle type* or simply a *saddle*, if there is an open ball B centered at p and continuous coordinates (u, v) on B taking p to the origin of $\mathbb{R}^2$ and in which $f(u, v) = (\lambda u, \mu v)$, with $0 < \mu < 1 < \lambda$. For C^1 diffeomorphisms this notion coincides with the usual one, when the fixed point p is hyperbolic and the eigenvalues of df_p are real and satisfy the same condition for λ and μ above.

The sets

$$W_p^u = \{x \in S; \lim_{n\to-\infty} f^n x = p\} \text{ and } W_p^s = \{x \in S; \lim_{n\to\infty} f^n x = p\}$$

are called the *unstable and stable invariant manifolds* of p, respectively. These are injectively immersed connected curves. The *stable branches* of p are the two components of $W_p^s - \{p\}$, if we consider the topology on W_p^s induced by a parametrization of $\mathbb{R}$. In the same way we define the *unstable branches* of p. Observe that branches and invariant manifolds are invariant sets under f.

A *connection* is a branch which is a stable branch of a saddle p and an unstable branch of a saddle q (p and q may coincide). When f is a C^1 diffeomorphism, these notions coincide with the usual ones, but the existence of invariant manifolds follows from different reasons.

We say that p is *elliptic* if there exist continuous coordinates around p in which f is differentiable at p and the eigenvalues of df_p are complex numbers of modulus one different from 1 and -1. If p is elliptic or of saddle type, we say that p is *nondegenerated*.

In these notes we will be working with area preserving homeomorphisms. We consider a finite Borel measure μ on S for which nonempty open sets have positive measure, for example, the measure given by the area form on the two sphere. We say that a homeomorphism $f : S \to S$ is *area preserving* if $\mu(fE) = \mu(E)$ for every Borel set $E \subset S$ and refer to $\mu(E)$ as the *area* of E.

For C^1 diffeomorphisms, it is not hard to prove the following. Let $D^1_\mu(S)$ be the space of all area preserving C^1 diffeomorphisms of S with the C^1 topology. $D^1_\mu(S)$ is a Baire space and contains a residual subset R, such that if $f \in R$ then every fixed point of f is nondegenerated and any two invariant manifolds intersect transversely. In particular there are no connections. Residual subsets of Baire spaces are topologically large, so in this sense, we say that for most C^1 diffeomorphisms every fixed point is nondegenerated, any two invariant manifolds intersect transversely and there are no connections.

We use the notation $cl_B A$ to denote the closure of A in B. The result we want to describe is the following.

Theorem 1. *Let f be an area preserving homeomorphism of S, p a fixed point of f of saddle type without connections. Assume that all fixed points of f in $cl_S(W_p^u \cup W_p^s)$ are nondegenerated. Then the closure of the four branches of p is the same set.*

Let p be a fixed saddle of f and B an open ball centered at p, where continuous coordinates (u, v) are defined and $f(u, v) = (\lambda u, \mu v)$, with $0 < \mu < 1 < \lambda$, in these coordinates. We define the *local invariant manifolds* of p as the set of points of B whose coordinates are of the form $(u, 0)$ or $(0, v)$. The four *local branches* of p are defined in the same way. Invariant manifolds and branches are obtained by the local corresponding objects by iteration in the right direction.

The complement in B of the union of local invariant manifolds is made of four open sets that we call *sectors of* p. Let L be a set which does not contain p, and Σ a sector of p. We say that L *accumulates on* Σ, if Σ contains points of L arbitrarily close to p. This definition does not depend on the radius of B.

In the rest of this section, we are going to show that theorem 1 follows from the following result.

Theorem 2. *Let f be an area preserving homeomorphism of S, p a fixed point of f of saddle type and L a branch of p. Assume that all fixed points of f in $cl_S L$ are nondegenerated. Then either L accumulates on both adjacent sectors or L is a connection.*

The proof of this result will be given at the end of these notes.

To show that theorem 1 follows from theorem 2, we need the following lemma.

Lemma 3. *Let f be an area preserving homeomorphism of S, L a branch, and K a compact connected invariant set. Then either $L \subset K$ or $L \cap K = \varnothing$.*

Proof. The proof is by contradiction, so we assume that K does not contain L and that K contains some points of L. Let U be a connected component of $S - K$ which contains points of L. Then U is simply connected, and since f is area preserving, we have that $f^k U = U$ for some k.

Take an arc $\alpha \subset L \cap U$ with end points in K. Since L is a branch, we have that $f^n \alpha \cap f^m \alpha = \varnothing$ if $n \neq m$. Besides that, α separates U into two simply connected open sets A_1 and A_2.

Since $f^k \times f^k$ is area preserving, we have that $\left(f^k \times f^k\right)^i (A_1 \times A_2) \cap (A_1 \times A_2) \neq \emptyset$ for some i, that is, $f^{ki}(A_1) \cap A_1 \neq \emptyset$ and $f^{ki}(A_2) \cap A_2 \neq \emptyset$.

We have $U = A_1 \cup \alpha \cup A_2 = f^{ki} A_1 \cup f^{ki} \alpha \cup f^{ki} A_2$, both disjoint unions. Since $f^{ki}\alpha$ and α are disjoint, $f^{ki}\alpha$ must be contained in either A_1 or A_2. Say $f^{ki}\alpha \subset A_1$.

One component of $U - f^{ki}\alpha$ is contained in A_1, and this component has area smaller than that of A_1. Such component is either $f^{ki} A_1$ or $f^{ki} A_2$. If $f^{ki} A_2 \subset A_1$ then $f^{ki} A_2 \cap A_2 = \varnothing$, a contradiction. If $f^{ki} A_1 \subset A_1$ then $\mu(f^{ki} A_1) < \mu(A_1)$, contradicting the fact that f is area preserving. ■

Now we show that theorem 1 follows from theorem 2. For this, let p be a fixed point of f of saddle type without connections, and such that all fixed points of f in $cl_S(W_p^u \cup W_p^s)$ are nondegenerated.

It follows that if L is a branch of p, then all fixed points of f in $cl_S L$ are nondegenerated. Therefore we can apply theorem 2 to the four branches of p. Since there are no connections starting at p, each of the four branches accumulates on both adjacent sectors.

It suffices to prove that if L_1 and L_2 are adjacent branches of p, then $L_2 \subset cl_S(L_1)$.

Let Σ be the sector between L_1 and L_2. Since L_1 accumulates on Σ, it is easy to see that L_1 accumulates on a point of L_2, and hence $L_2 \cap cl_S(L_1) \neq \varnothing$. By lemma 3 we have $L_2 \subset cl_S(L_1)$.

2. Prime ends

In this section we are going to describe a compactification of a simply connected open subset of the two dimensional sphere, by adding a circle of points and obtaining a compact set homeomorphic to the two dimensional closed disk.

We use the notation $fr_B A$ to denote the frontier of A in B.

Let U be a simply connected open subset of S such that $S - U$ has more than one point.

A *chain* is a sequence $V_1 \supset V_2 \supset \ldots$ of open connected subsets of U such that

(1) $fr_U(V_i)$ is nonempty and connected for every $i \geq 1$, and

(2) $cl_S(fr_U(V_i)) \cap cl_S(fr_U(V_j)) = \emptyset$ for $i \neq j$.

A chain (W_j) *divides* (V_i) if for every i there exists j such that $W_j \subset V_i$. Two chains are *equivalent* if each one divides the other. A chain is *prime* if any chain which divides it is equivalent to it.

A *prime point* is an equivalence class of prime chains. Let $x \in U$ and consider a family of closed balls centered at x, contained in U, and whose radii decrease to zero. The interiors of these balls make a prime chain which represents a prime point denoted by $\omega(x)$. A *prime end* is a prime point which is not of the form $\omega(x)$ for any $x \in U$.

The set of prime points of U is denoted by $\hat{U}$.

We consider a topology on $\hat{U}$ defined as follows. Let A be an open subset of U and denote by A' the set of prime points of U whose representing chains are eventually contained in A. It is easy to see that $(A_1 \cap A_2)' = A_1' \cap A_2'$, and therefore $\{A'$ such that A is an open subset of $U\}$ is a basis for a topology on $\hat{U}$. The function $\omega : U \to \hat{U}$ is a homeomorphism from U onto an open subset of $\hat{U}$, and we simply identify U with $\omega(U)$.

The topological structure of $\hat{U}$ is the following.

Theorem 4. *$\hat{U}$ homeomorphic to the two dimensional closed disk and $\hat{U} - U$ is homeomorphic to a circle.*

Therefore $\partial\hat{U} = \hat{U} - U$ (or more carefully $\partial\hat{U} = \hat{U} - \omega(U)$) is the set of prime ends of U, and it is usually called the *Caratheodory circle of U*.

2.1. Accessibility.

Let e be a prime end of U.

We say that $p \in S$ is a *principal point* of e if there is a chain (V_i) which represents e and for which $fr_U(V_i) \to p$. The set of principal points of e is called the *principal set* of e and is denoted by $X(e)$. It is easy to see that $X(e) \subset fr_S U$.

Some topological properties of $X(e)$ are the following.

Theorem 5. *$X(e)$ is non-empty, compact and connected.*

Assume that $X(e) = \{p\}$ for some $p \in fr_S U$. Then it is possible to prove the existence of a continuous path $\beta : (0,1] \to U$ such that $\lim_{t\to 0} \beta(t) = p$ and $\lim_{t\to 0} \beta(t) = e$ in $\hat{U}$, (that is, $\lim_{t\to 0} \omega(\beta(t)) = e$). β is called an *end path.*

Conversely, if $p \in fr_S U$ and there exists a continuous path $\beta : (0,1] \to U$ such that $\lim_{t\to 0} \beta(t) = p$, then there exists a prime end e such that $\lim_{t\to 0} \beta(t) = e$ in $\hat{U}$, and $X(e) = \{p\}$.

When $X(e) = \{p\}$ for some $p \in fr_S U$, we say that p and e are accessible**.**

2.2. The extension of f to $\hat{U}$ and a fixed point theorem. Let f_U be the restriction of f to U.

Since f_U maps irreducible chains to irreducible chains and equivalent chains to equivalent chains, it is easy to see that f_U also extends to a homeomorphism $f_{\hat{U}} : \hat{U} \to \hat{U}$. When there is no ambiguity, we simply write $\widehat{f} : \hat{U} \to \hat{U}$. The restriction of $\widehat{f}$ to $\partial\hat{U}$ is a orientation preserving homeomorphism of the circle, which has a very well understood dynamics. If the rotation number of $\widehat{f}$ is $\frac{p}{q}$, then there exists periodic orbits and they all have period q. If the rotation number of $\widehat{f}$ is irrational, then either $\widehat{f}$ is minimal (all orbits are dense), or there is a minimal invariant Cantor set K and wandering intervals that by iteration accumulate on K both in the past and the future.

Now we head towards a result due to Cartwright and Littlewood [3], which provides the existence fixed points of f in $fr_S U$, in terms of fixed prime ends.

Firstly, we need a couple of technical lemmas.

Lemma 6. *Let $A_1$and A_2 open connected subsets of U, both non-empty and different from U.*

Suppose that

1) $A_1 \cap A_2 \neq \emptyset$.
2) $A_1 \cap (U - A_2) \neq \emptyset$.
3) $A_2 \cap (U - A_1) \neq \emptyset$.
4) $fr_U(A_1)$ and $fr_U(A_2)$ are connected and disjoint.

Then we have the following facts:

a) $fr_U(A_1) \subset A_2$ and $fr_U(A_2) \subset A_1$.
b) $U = A_1 \cup A_2$.

Proof. Since $fr_U(A_1)$ is connected and does not intersect $fr_U(A_2)$, we must have either $fr_U(A_1) \subset A_2$ or $fr_U(A_1) \subset U - A_2$. But A_2 is a connected set that intersects both A_1 and $U - A_1$, and therefore we have $A_2 \cap fr_U(A_1) \neq \emptyset$. This implies that $fr_U(A_1) \subset A_2$. Similarly $fr_U(A_2) \subset A_1$, which proves part a).

In order to prove b), we are going to show that $A_1 \cup A_2$ is open and closed in U. It is obviously open. Let (a_n) be a sequence in $A_1 \cup A_2$ converging to a point a. We may assume that $a_n \in A_1$ for infinitely many values of n, and therefore $a \in Cl_U(A_1)$. If $a \in A_1$ we are done. If not, we have $a \in fr_U(A_1)$, and from part a) $a \in A_2$. ■

We use the notation $int_B A$ to denote the interior of A in B.

Lemma 7. *Let e be a fixed prime end of U, p a principal point of e and (V_i) a chain defining e such that $fr_U(V_i) \to p$. Then there exists i_0 such that $fr_U(V_i) \cap fr_U(fV_i) \neq \varnothing$ for every $i \geq i_0$.*

Proof. (fV_i) is a chain that represents $\widehat{f}(e)$. Since e is fixed by $\widehat{f}$, we have that (V_i) and (fV_i) are equivalent.

Let i_0 be such that $fV_i \subset V_1$ for $i \geq i_0$. We are going to apply lemma 6 to V_i and fV_i to show that $fr_U(V_i) \cap fr_U(fV_i) \neq \varnothing$ for $i \geq i_0$.

Assume by contradiction that $fr_U(V_i) \cap fr_U(fV_i) = \varnothing$ for some $i \geq i_0$.

The equivalence of (V_i) and (fV_i) implies that $V_i \cap fV_i \neq \varnothing$ for every $i \geq 1$, and condition 1) holds.

Now, we want to prove that 2) and 3) hold. We have

$$U = V_i \cup fr_U(V_i) \cup int_U\,(U - V_i) = fV_i \cup fr_U(fV_i) \cup int_U\,(U - fV_i)\,,$$

both unions disjoint.

Suppose we had $fV_i \subset V_i$. Since $fr_U(fV_i)$ is connected and disjoint from $fr_U(V_i)$, we would have $fr_U(fV_i) \subset V_i$ or $fr_U(fV_i) \subset int_U\,(U - V_i)$.

If $fr_U(fV_i) \subset V_i$, then for any $x \in fr_U(fV_i)$ there would exist a ball $B_x(\delta)$ centered at x and contained in V_i. Since $x \in fr_U(fV_i)$, $B_x(\delta)$ would contain a point y of $int_U\,(U - fV_i)$, and $B_x(\delta)$ would contain a ball $B_y(\delta_1)$ centered at y. Therefore $B_y(\delta_1) \subset V_i$. But $y \in int_U\,(U - fV_i)$, and decreasing δ_1 if necessary, we would have $B_y(\delta_1) \subset int_U\,(U - fV_i)$, implying that $B_y(\delta_1) \cap fV_i = \varnothing$. Since $\mu\,(B_y(\delta_1)) > 0$, we would have $\mu\,(fV_i) < \mu\,(V_i)$, contradicting the fact that f is area preserving.

If $fr_U(fV_i) \subset int_U\,(U - V_i)$, then for any $x \in fr_U(fV_i)$ a ball $B_x(\delta)$ would be contained in $int_U\,(U - V_i)$, and $B_x(\delta) \cap V_i = \varnothing$. But $x \in fr_U(fV_i)$ would imply that $\varnothing \neq B_x(\delta) \cap fV_i \subset B_x(\delta) \cap V_i$, a contradiction.

Hence, we can not have $fV_i \subset V_i$, which proves 2). In the same way we can not have $V_i \subset fV_i$, which proves 3).

Recall that $fV_i \subset V_1$. Since $V_i \subset V_1$ we have $V_i \cup fV_i \subset V_1$, and b) does not hold. We conclude that 4) of lemma 6 does not hold as well, that is, $fr_U(V_i) \cap fr_U(fV_i) \neq \varnothing$, a contradiction. ∎

Now we state Cartwright and Littlewood's result [3].

Theorem 8. *Let e be a fixed prime end of U and p a principal point of e. Then $f(p) = p$.*

Proof. Let (V_i) be a chain defining e such that $fr_U(V_i) \to p$. From lemma 7, if $i \geq i_0$ there exists a point $x_i \in fr_U(V_i)$ such that $f(x_i) \in fr_U(V_i)$. Since $fr_U(V_i) \to p$, we have $x_i \to p$, $f(x_i) \to p$ and $f(p) = p$. ∎

It is easy to make examples where the previous theorem is false if f is not area preserving.

2.3. The main theorem.

In this section we prove the following result.

Theorem 9. *Let f be an area preserving homeomorphism of S and U a simply connected open invariant set of f, such that all fixed points of f in $fr_S U$ are nondegenerated. Assume that U has a fixed prime end e. Then we know the following:*

1) e is accessible and if p is its unique principal point, then $f(p) = p$.

2) p can not be elliptic.

3) if p is a saddle, then $fr_S U$ contains a connection starting at p.

The proof of 1) is simple.

Since nondegenerated fixed points are isolated, by theorem 8 we have that all points of $X(e)$ are isolated. But by theorem 5 $X(e)$ is connected. It follows that $X(e) = \{p\}$ for some $p \in fr_S U$ and $f(p) = p$.

The proof of 2) is long, and it has been written in [4]. The main idea is the following. The derivative of f at p is a rotation. If $\gamma_i = fr_U(V_i)$ is very close to p, then the iterates $f^n(\gamma_i)$ turn around p. As before, $f(\gamma_i) \cap \gamma_i \neq \emptyset$, implying that $f^{k+1}(\gamma_i) \cap f^k(\gamma_i) \neq \emptyset$ for $k \geq 0$. The crucial point is to prove that for some large n we have $f^n(\gamma_i) \cap \gamma_i \neq \emptyset$. This means that the iterates $\gamma_i,\ f\gamma_i,\ ...\ f^n(\gamma_i)$ turn around p, one intersecting the next, implying that their union contains a simply closed curve that bounds a disk which contains p. This curve is contained in U, and since U is simply connected, it must contain p, a contradiction.

It remains to prove 3).

Recall that p is a saddle, when there is an open ball B centered at p and continuous coordinates (u, v) on B taking p to the origin of $\mathbb{R}^2$, such that $f(u, v) = (\lambda u, \mu v)$, with $0 < \mu < 1 < \lambda$.

Let (V_i) be a chain which represents e and for which $fr_U(V_i) \to p$.

We need the following lemma.

Lemma 10. *There exists a neighborhood W of p with the following properties:*

1) $W \subset B$.

2) $U \cap W$ contains a component W_0, which has one of the following forms:

a) the intersection of W with one sector of p,

b) the intersection of W with two adjacent sectors and the local branch in between,

c) the intersection of W with three adjacent sectors and the local branches in between,

d) the complement in W of p and one local branch.

Proof. We consider only the case when U does not intersect any of the four branches of p, and show the existence of W for which W_0 satisfies 2a).

By taking a subsequence if necessary, we may assume all the sets $fr_U(V_i)$ belong to one of the sectors. Call this sector Σ. We also assume that Σ is the sector with coordinates $u > 0$ and $v > 0$.

For $\delta > 0$, consider the arc

$$\Gamma_\delta = \{(u, \delta) \text{ such that } 0 \leq u \leq \delta\} \cup \{(\delta, v) \text{ such that } 0 \leq v \leq \delta\}.$$

Γ_δ is made of two sides of a square S_δ, for which the other sides are contained in the u and v axis.

For δ_0 small enough, we have $S_{\delta_0} \cap fr_U(V_1) = \varnothing$.

Let $\delta < \delta_0$. If n is large enough, then $fr_U(V_n) \subset int_S(S_\delta)$. Therefore U contains points of S_δ and points of $S - S_\delta$. Being connected, U must intersect $fr_S(S_\delta)$, and since two sides of $fr_S(S_\delta)$ are contained in branches, U must intersect Γ_δ.

We have that $U - \Gamma_\delta$ is disconnected, and $fr_U(V_1)$ and $fr_U(V_n)$ belong to different components of $U - \Gamma_\delta$. The connected components of $U \cap \Gamma_\delta$ are open arcs contained in U and with end points not in U. One of these arcs separates $fr_U(V_1)$ and $fr_U(V_n)$ in U. Let us denote this arc by β.

Let Z be the component of $U - \beta$ which contains $fr_U(V_n)$. Then $\beta = fr_U Z$ and $f(Z) \cap Z \neq \varnothing$. With an argument similar to that of lemma 7, one shows that $f\beta \cap \beta \neq \varnothing$.

We have that $f(\Gamma_\delta) \cap \Gamma_\delta$ consists of one point, say q. Therefore $f\beta \cap \beta = \{q\}$, implying that β contains the closed arc joining q and $f^{-1}q$. Let us denote this arc by Λ_δ, and consider the set $\Lambda = \bigcup_{n \in \mathbb{Z}} f^n \left(\bigcup_{0 < \delta < \delta_0} \Lambda_\delta \right)$. We have that $\Lambda \subset U$, and taking W as the open ball of radius δ_0 centered at p, it follows that $W_0 = W \cap \Lambda$ is a sector of p. ■

Now we finish the proof. We will only consider the case when all fixed prime ends are such that its principal points are fixed saddles, and U intersects exactly one sector of each saddle (case 2a) of lemma 10. The other cases are analogous.

Let e be a fixed prime end with $X(e) = \{p\}$. We are assuming that U contains a sector Σ, and if K and L are the adjacent branches to Σ, then $U \cap K = U \cap L = \varnothing$. $\Sigma\prime$ is a neighborhood of e in $\hat{U}$, and $J = \Sigma\prime \cap \partial\hat{U}$ is an arc in $\partial\hat{U}$ which contains e in its interior. To each x belonging to the local branches of K and L, there is a unique prime end $e_x \in J$ such that $X(e_x) = \{x\}$.

Let I be the largest neighborhood of e in $\partial\hat{U}$ without fixed points. I is an open arc whose end points are fixed and $J \subset I$. Given $e\prime \in I$, there exists $n \in \mathbb{Z}$ such that $\widehat{f}^n(e\prime) \in J$. If $X(\widehat{f}^n(e\prime)) = \{x\}$ then $X(e\prime) = f^{-n}(x)$, and we have a bijection $X : I \to K \cup L$.

Let $e_1$and e_2 be prime ends that bound an arc I_{12} in $\partial\hat{U}$ without fixed points. Let $X_1 : I_1 \to K_1 \cup L_1$ and $X_2 : I_2 \to K_2 \cup L_2$ be the corresponding mappings as defined above. We have $I_{12} = I_1 \cap I_2$ implying that $X_1|_{I_{12}} = X_2|_{I_{12}}$, and therefore a branch of p_1 coincides with a branch of p_2.

3. The proof of theorem 2

Assume that L does not accumulate on one adjacent sector Σ.

Let U be the connected component of $S - cl_S L$ which contains Σ. Since $fr_S U \subset cl_S L$, all fixed points of f contained in $fr_S U$ are nondegenerated. We have that $cl_S L$ is compact and connected, implying that U is simply connected. Since $f(\Sigma) \cap \Sigma \neq \varnothing$, we have that $f(U) = U$.

We need the following lemma.

Lemma 11. *If U contains one point of a branch K, then K and its adjacent sectors are contained in U.*

The proof of the lemma is simple, and we leave it to the reader.

Returning to the proof of the theorem, let L_1 be the other branch adjacent to Σ. From the previous lemma, either $L_1 \cap U = \varnothing$ or L_1 and its adjacent sectors are contained in U.

Assume $L_1 \cap U = \varnothing$. By choosing arcs with end points in L_1 and L, we can define a prime chain whose equivalence class gives a prime end e such that $X(e) = \{p\}$. By theorem 9 (and its proof), L_1 and L are connections.

If $L_1 \cap U \neq \varnothing$, then by lemma 11, L_1 and its adjacent sectors are contained in U. Therefore U contains L_1 and the adjacent sectors to L_1, Σ and say Σ_1. U does not contain L, and may or may not contain the other branch adjacent to Σ_1, say L_2.

Assume $L_2 \cap U = \varnothing$. By choosing arcs with end points in L_2 and L, we can define a prime chain whose equivalence class gives a prime end e such that $X(e) = \{p\}$. By theorem 9, L_2 and L are connections.

If $L_2 \cap U \neq \varnothing$, we proceed in the same way to show that U contains three adjacent sectors Σ, Σ_1, Σ_2 and the branches in between L_1 and L_2. As before, we show that either p has two connections, or that U contains a neighborhood of p minus the local branch L and p, and in this case L is the only connection of p.

4. Some comments

The theory of prime ends was originally developed by Caratheodory [1] and [2]. At that time the notion of a topological space had not yet been defined! He used tools of complex analysis and stated all results in terms of convergent sequences.

A purely topological approach to this theory was developed by Mather [6] and [7]. Our exposition follows this work, and that of Franks Le Calvez [5].

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