PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 0, Fasc. 0 (0000), pp. 000–000

ON LAMPERTI STABLE PROCESSES

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Abstract.

In this paper, we consider a new family of $\mathbb{R}^d$-valued Lévy processes that we call Lamperti stable. One of the advantages of this class is that the law of many related functionals can be computed explicitly. In the one dimensional case we provide an explicit form for the characteristic exponent and other several useful properties of the class. This family of processes shares many tractable properties with the tempered stable and the layered stable processes, defined by Rosiński [33] and Houdré and Kawai [16] respectively. We also find a series representation which is used for sample path simulation, illustrated in the case $d = 1$. Finally we provide many examples, some of which appear in recent literature.

2000 AMS Mathematics Subject Classification: Primary: 60E07; Secondary: 60G51, 60G52.

Key words and phrases: Lamperti stable distributions and processes, stable processes, Lévy processes.

1. INTRODUCTION.

In recent years the interest in having more accurate models in various domains of applied probability has lead to an increasing attention paid to some special classes of Lévy processes related to the stable law, for example: the tempered stable and the layered stable processes introduced by Rosiński [33] and Houdré and Kawai [16], respectively. Both families of processes have nice structural and analytical properties, such as combining in small times the behavior of stable processes and in long times the behavior of a Brownian motion. They also have a series representation which may be used for sample paths simulation.

Lamperti [25] and more recently, Caballero and Chaumont [6] studied some type of Lévy processes which are related to the stable subordinator and to some conditioned stable processes via the Lamperti representation of positive self-similar Markov processes. In these papers, the authors obtain Lévy processes without Gaussian component whose Lévy measure has the following form

$$\pi(\mathrm{d}x) = e^{bx}\nu(e^x - 1)\mathrm{d}x,$$

where ν is the density of the stable Lévy measure and b is a positive parameter which depends on its characteristics. Lévy processes with this type of Lévy measure also appear in recent literature, see for instance Donati-Martin and Yor [11] (subordinators) and Patie [29, 30, 31] (spectrally one-sided).

These previous works lead us to investigate a generalization of the Lévy processes mentioned above and we will refer to them as Lamperti stable processes. The importance of this new family of processes comes from the fact that they have nice structural and analytical properties, and that in many cases, some tractable mathematical expressions can be explicitely computed (see for instance [6, 9, 23, 29, 30, 31]).

Section 2 is devoted to Lamperti stable distributions, which are multivariate infinitely divisible distributions with no Gaussian component and whose Lévy measure is characterized by a triplet (α, f, σ), more precisely an index $\alpha \in (0, 2)$, a function f with certain boundedness condition, and a finite measure σ, both defined on the unit sphere in $\mathbb{R}^d$. Then we obtain several properties of these distributions. In section 3, we formally introduce the Lamperti stable processes and study their properties with emphasis in the one dimensional case, where we obtain an explicit closed form for the characteristic exponent. Motivated by the works of Rosiński

[33] and Houdré and Kawai [16], we prove in section 4, 5 and 6 that Lamperti stable processes in small times behave like stable processes while in long times like the Brownian motion, that they are absolutely continuous with respect to its small time limiting stable process and that they admit a series representation that allows simulations of their paths, respectively.

In the last section, we illustrate with several examples the presence of Lamperti stable distributions in recent literature.

2. LAMPERTI STABLE DISTRIBUTIONS.

Recall that the Lévy measure Π of a stable distribution with index α on $\mathbb{R}^d$ in polar coordinates is of the form

$$\Pi(dr, d\xi) = r^{-(\alpha+1)}\mathrm{d}r\sigma(\mathrm{d}\xi),$$

where $\alpha \in (0,2)$ and σ is a finite measure on S^{d-1}, the unit sphere on $\mathbb{R}^d$. The measure σ is uniquely determined by Π. Conversely, for any non-zero finite measure σ on S^{d-1} and for any $\alpha \in (0,2)$ we can define a stable distribution with Lévy measure defined as above (see Theorem 14.3 in Sato [35]).

Motivated by the form of the Lévy measure of the processes mentioned in the introduction, we define a new family of infinitely divisible distributions that we call Lamperti stable.

DEFINITION 2.1. Let μ be an infinitely divisible probability measure on $\mathbb{R}^d$ without Gaussian component. Then, μ is called Lamperti stable if its Lévy measure on $\mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}$ is given by

$$\nu_\sigma^{\alpha,f}(B) = \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_0^\infty \mathbb{I}_B(r\xi) e^{rf(\xi)}(e^r - 1)^{-(\alpha+1)} dr, \qquad B \in \mathcal{B}(\mathbb{R}^d_0), \tag{2.1}$$

where $\alpha \in (0,2)$, σ is non-zero finite measure on S^{d-1}, and $f : S^{d-1} \to \mathbb{R}$ is a measurable function such that $\gamma := \sup_{\xi\in S^{d-1}} f(\xi) < \alpha + 1$.

In the one dimensional case f as well as σ take only two values, since $S^0 = \{-1, 1\}$. In the sequel, we denote these values by $f(1) := \beta$ and $f(-1) := \delta$, $\sigma(\{1\}) = c_+$ and $\sigma(\{-1\}) = c_-$. Then, the measure $\nu_\sigma^{\alpha,f}$ has a density given by

$$c_+ \frac{e^{\beta x}}{(e^x - 1)^{\alpha+1}} \mathbb{I}_{\{x>0\}} + c_- \frac{e^{-\delta x}}{(e^{-x} - 1)^{\alpha+1}} \mathbb{I}_{\{x<0\}}.$$

Note that Lamperti stable distributions satisfy the divergence condition, i.e.

$$\int_0^\infty e^{rf(\xi)}(e^r - 1)^{-(\alpha+1)} dr = \infty \qquad \text{for any} \qquad \xi \in S^{d-1}.$$

Thus from Theorem 27.10 in [35], we deduce that they are absolutely continuous with respect to the Lebesgue measure. Also note that the class of Lamperti stable distributions and that of layered stable distributions (see [16]) are disjoint. This follows from the following estimate

$$\frac{e^{rf(\xi)}}{(e^r - 1)^{\alpha+1}} \sim e^{-(\alpha+1-f(\xi))r} \qquad \text{as} \qquad r \to \infty.$$

Lamperti stable distributions do not belong in general to the class of tempered stable distributions. For instance, fix $\xi \in S^{d-1}$ and take $f(\xi) \in ((\alpha+1)/2, \alpha+1)$. It is not difficult to see that the first derivative of the function

$$q(r, \xi) = \frac{e^{rf(\xi)}}{(e^r - 1)^{\alpha+1}} r^{1+\alpha},$$

is positive for $r \in (0, 2 - (\alpha + 1)/c)$, which implies that $q(r, \xi)$ is not completely monotone.

PROPOSITION 2.1. *Let μ be a Lamperti stable distribution with Lévy measure $\nu_\sigma^{\alpha,f}$ given by (2.1). If $\zeta < \alpha + 1 - \gamma$, then*

$$\int_{\mathbb{R}^d} e^{\zeta\|x\|} \mu(dx) < \infty,$$

In particular, for $\kappa < \alpha + 1$ and if $f \equiv \kappa$, we have

$$\int_{\mathbb{R}^d} e^{\zeta\|x\|} \mu(dx) < \infty \qquad \text{if and only if} \qquad \zeta < \alpha + 1 - \kappa.$$

P r o o f. Consider

$$\int_{\{\|x\|>1\}} e^{\zeta\|x\|}\nu_\sigma^{\alpha,f}(dx) \leqslant \sigma(S^{d-1})(1-e^{-1})^{-(\alpha+1)}\int_1^\infty e^{r(\zeta+\gamma-(\alpha+1))}dr,$$

which is finite since $\zeta < \alpha+1-\gamma$. Hence by Theorem 25.3 in [35], we obtain the desired result.
Next, we suppose that $f \equiv \kappa$. The former arguments imply that for $\zeta < \alpha+1-\kappa$, the Lamperti stable distribution μ has a finite exponential moment of order ζ. In a similar way, it is clear that

$$\int_{\{\|x\|>1\}} e^{\zeta\|x\|}\nu_\sigma^{\alpha,\kappa}(dx) \geqslant \sigma(S^{d-1})\int_1^\infty e^{r(\zeta+\kappa-(\alpha+1))}dr.$$

This implies that $\int_{\{\|x\|>1\}} e^{\zeta\|x\|}\nu_\sigma^{\alpha,\kappa}(dx)$ is finite if and only if $\zeta < \alpha+1-\kappa$. ■

COROLLARY 2.1. *Let μ be be a Lamperti stable distribution. Then*

$$\int_{\mathbb{R}^d} \|x\|^p\mu(dx) < \infty \quad \textit{for all} \quad p > 0.$$

Our next result shows that Lamperti stable distributions belong to the Jurek class and that in some cases they are self-decomposable. We recall briefly these definitions. The class of infinitely divisible distributions for which the Lévy measure ν takes the following form

$$\nu(B) = \int_{S^{d-1}} \sigma(\mathrm{d}\xi)\int_0^\infty \mathbb{I}_B(r\xi)\ell(\xi,r)dr, \qquad \text{for} \quad B \in \mathcal{B}(\mathbb{R}_0^d),$$

is called:

(1) Self-decomposable if $r\ell(\xi,r)$ is non negative, measurable in $\xi \in S^{d-1}$ and decreasing in $r \in (0,\infty)$.

(2) Jurek class if $\ell(\xi,r)$ is measurable in $\xi \in S^{d-1}$, and decreasing in $r \in (0,\infty)$.

PROPOSITION 2.2. *Let μ be a Lamperti-stable distribution on $\mathbb{R}^d$ with Lévy measure $\nu_\sigma^{\alpha,f}$ given by (2.1), then μ belongs to the Jurek class. Moreover, μ is selfdecomposable if $f(\xi) \leqslant \alpha+1/2$, for all $\xi \in S^{d-1}$ and $\alpha \in (0,2)$.*

P r o o f. In the case of a Lamperti stable distribution, we have

$$\ell(\xi, r) = \frac{e^{f(\xi)r}}{(e^r - 1)^{\alpha+1}},$$

so the measurability of $r\ell(\xi, r)$ and $\ell(\xi, r)$ is clear.
In order to prove that ℓ is decreasing in $r > 0$, we fix $\xi \in S^{d-1}$ and consider the derivative of $\ell_1(\cdot) = \ell(\xi, \cdot)$, i.e.

$$\ell_1'(r) = \frac{e^{f(\xi)r}}{(e^r - 1)^{\alpha+2}} \big(e^r(f(\xi) - \alpha - 1) - f(\xi)\big).$$

Hence $\ell_1'(r) < 0$ for $r > 0$, since $f(\xi) \leqslant \alpha + 1$. This implies that μ is in the Jurek class.

For the second part of the Proposition, we take $k(\xi, r) = r\ell(\xi, r)$. Note that the derivative of $k(\xi, r)$ with respect to r, can be written as

$$\frac{e^{f(\xi)r}}{(e^r - 1)^{\alpha+2}} \big(e^r \left[(1 + f(\xi)r - (\alpha + 1)r\big) - f(\xi)\right],$$

Elementary calculations prove that k is decreasing for $r > 0$, if $f(\xi) \leqslant \alpha + 1/2$ for all $\xi \in S^{d-1}$ and $\alpha \in (0, 2)$. We leave the details to the reader. ∎

We note that we can find $\alpha \in (0, 2)$ such that if $f(\xi) > \alpha + 1/2$, the Lamperti stable distribution μ is not self-decomposable.

The last part of this section is devoted to some properties of Lamperti stable distributions defined on $\mathbb{R}$. The first of which says in particular that the density of any Lamperti stable distribution belongs to C^∞.

PROPOSITION 2.3. *Let μ be a Lamperti stable distribution on $\mathbb{R}$, then μ has a C^∞ density and all the derivatives of the density tend to* 0 *as $|x|$ tends to ∞.*

P r o o f. Recall that the function f takes two values, $\beta = f(1)$ and $\delta = f(-1)$ as usual. According to [28] it is enough to prove that

$$g(r) = \int_0^r x^2 \frac{e^{\beta x}}{(e^x - 1)^{\alpha+1}} \mathrm{d}x, \quad \text{verifies that} \quad \liminf_{r \to 0} \frac{g(r)}{r^{2-a}} > 0, \tag{2.2}$$

for some $a \in (0, 2)$. But this is immediate since for r sufficiently small, we have

$$\int_0^r x^2 \frac{e^{\beta x}}{(e^x - 1)^{\alpha+1}} \mathrm{d}x \geqslant K \int_0^r \frac{x^2}{x^{\alpha+1}} \mathrm{d}x = K r^{2-\alpha},$$

where $K > 0$, which implies that (2.2) is satisfied for $a = \alpha$. Then the statement follows. ■

We now recall the definition of a particular class of distributions which is important in risk theory (see for instance [12] and [21]).

DEFINITION 2.2 (Class $\mathcal{L}^{(q)}$). Take a parameter $q \geqslant 0$. We shall say that a distribution function G on $[0, \infty)$ with tail $\overline{G} := 1 - G$ belongs to class $\mathcal{L}^{(q)}$ if $\overline{G}(x) > 0$ for each $x \geqslant 0$ and

$$\lim_{u\to\infty} \frac{\overline{G}(u-x)}{\overline{G}(u)} = e^{qx} \quad \text{for each } x \in \mathbb{R}.$$

The tail of any (Lévy or other) measure, finite and non-zero on (x_0, ∞) for some $x_0 > 0$, can be renormalised to be the tail of a distribution function and by extension, then is said to be in $\mathcal{L}^{(q)}$, if the associated distribution function is in $\mathcal{L}^{(q)}$.

PROPOSITION 2.4. *Let μ be a Lamperti-stable distribution on $\mathbb{R}$, then the tail of its Lévy measure belongs to the class $\mathcal{L}^{(\alpha+1-\beta)}$. In particular when μ is defined on $\mathbb{R}_+$, we have that μ belongs to the class $\mathcal{L}^{(\alpha+1-\beta)}$.*

Proof. First, we define

$$\nu(u) = \frac{1}{K_1}\int_1^u \frac{e^{\beta r}}{(e^r-1)^{\alpha+1}}\mathrm{d}r, \qquad u \geqslant 1,$$

where $K_1 = \int_1^\infty \frac{e^{\beta r}}{(e^r-1)^{\alpha+1}}\mathrm{d}r$. Note that ν corresponds to the distribution function associated to the tail of the Lévy measure of a Lamperti stable distribution.
We carry on the calculations and obtain,

$$\begin{aligned}\frac{\overline{\nu}(u-x)}{\overline{\nu}(u)} &= \int_{u-x}^\infty \frac{e^{\beta r}}{(e^r-1)^{\alpha+1}}\mathrm{d}r \left(\int_u^\infty \frac{e^{\beta r}}{(e^r-1)^{\alpha+1}}\mathrm{d}r\right)^{-1} \\ &\leqslant e^{(\alpha+1-\beta)x}\big(1-e^{-(u-x)}\big)^{-\alpha-1}.\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\overline{\nu}(u-x)}{\overline{\nu}(u)} &\geqslant (1-e^{-u})^{\alpha+1}(\alpha+1-\beta)e^{u(\alpha+1-\beta)}\int_{u-x}^\infty e^{(\beta-(\alpha+1))r}\mathrm{d}r \\ &= e^{(\alpha+1-\beta)x}(1-e^{-u})^{\alpha+1}.\end{aligned}$$

Therefore taking the limit as u tends to ∞, we deduce that $\nu \in \mathcal{L}^{(\alpha+1-\beta)}$. The case when μ is defined in $\mathbb{R}_+$ follows from Proposition 3.4 in Kuppelberg et al. [21]. ■

3. LAMPERTI STABLE LÉVY PROCESSES.

Let us define the Lévy processes associated to a Lamperti stable distribution.

DEFINITION 3.1. A Lévy process without gaussian component, and linear term θ, is called Lamperti stable with characteristics $L = (\alpha, f, \sigma, \theta)$ if its Lévy measure is given by (2.1).

In the sequel, we denote the Lamperti stable Lévy process with characteristics $L = (\alpha, f, \sigma, \theta)$ by $X^L = (X^L_t, t \geqslant 0)$. Its characteristic exponent Ψ defined by $E[\exp(i\langle y, X^L_t\rangle)] = \exp(-t\Psi(y))$ for $t \geqslant 0$, $y \in \mathbb{R}^d$, has the form

$$\Psi(y) = i\langle y, \theta\rangle + \int_{\mathbb{R}^d_0} \big(1 - e^{i\langle y,x\rangle} + i\langle y, x\rangle \mathbb{I}_{\{\|x\|<1\}}\big)\nu^{\alpha,f}_\sigma(\mathrm{d}x), \tag{3.1}$$

where the measure $\nu^{\alpha,f}_\sigma$ is given in (2.1) and $\theta \in \mathbb{R}^d$.

We first study the p-th variation of Lamperti stable processes.

PROPOSITION 3.1. *Let X^L be a Lamperti stable process with characteristics $L = (\alpha, f, \sigma, \theta)$.*

i) If $\alpha \in (1, 2)$, the process X^L is a.s. of finite p-variation in every finite interval if and only if $p \in (\alpha, 2)$.

ii) The process X^L is a.s. of finite variation in every finite interval if and only if $\alpha \in (0, 1)$.

Proof. (i) From Theorem III in Bretagnolle [5], we have that for $p \in (1, 2)$, the process X^L is a.s. of finite p-th variation on every finite interval if and only if

$$\int_{\{\|x\|\leqslant 1\}} \|x\|^p \nu^{\alpha,f}_\sigma(\mathrm{d}x) < \infty.$$

Recall that $\gamma := \sup_{\xi\in S^{d-1}} f(\xi)$. From the form of the Lévy measure $\nu^{\alpha,f}_\sigma$ we have

$$\int_{\{\|x\|\leqslant 1\}} \|x\|^p \nu^{\alpha,f}_\sigma(\mathrm{d}x) \leqslant \sigma(S^{d-1})e^\gamma \int_0^1 r^{p-(\alpha+1)}\mathrm{d}r. \tag{3.2}$$

On the other hand, we have

(3.3)

$$\begin{aligned}
\int\limits_{\{\|x\|\leqslant 1\}} \|x\|^p \nu_\sigma^{\alpha,f}(\mathrm{d}x) &\geqslant \sigma\big(\{\xi\in S^{d-1}: f(\xi)\geqslant 0\}\big)\int\limits_0^1 \frac{r^p}{(e^r-1)^{\alpha+1}}\mathrm{d}r\\
&+ \int\limits_{S^{d-1}} \mathbb{I}_{\{f(\xi)<0\}} e^{f(\xi)}\sigma(\mathrm{d}x)\int\limits_0^1 \frac{r^p}{(e^r-1)^{\alpha+1}}\mathrm{d}r\\
&\geqslant K\Big(\sigma\big(\{\xi\in S^{d-1}: f(\xi)\geqslant 0\}\big) + \int\limits_{S^{d-1}} \mathbb{I}_{\{f(\xi)<0\}} e^{f(\xi)}\sigma(\mathrm{d}x)\Big)\int\limits_0^1 r^{p-(\alpha+1)}\mathrm{d}r,
\end{aligned}$$

for some $K>0$. Therefore X^L is of finite p-th variation on every finite interval if and only if $p>\alpha$.

The proof of part (ii) is very similar. According to Theorem 3 of Gikhman and Skorokhod [14], it is enough to prove that

$$\int\limits_{\{\|x\|\leqslant 1\}} \|x\|\nu_\sigma^{\alpha,\gamma}(dx) < \infty,$$

if and only if $\alpha\in(0,1)$. But this follows from (3.2) and (3.3) taking $p=1$, which concludes the proof. ■

Recall that the characteristic exponent of a Lévy process has a simpler expression when its sample paths have a.s. finite variation in every finite interval. In this case, Ψ_L takes the form

$$\Psi_L(y) = -i\langle \mathbf{d}, y\rangle + \int\limits_{\mathbb{R}_0^d} \big(1-e^{i\langle y,x\rangle}\big)\nu_\sigma^{\alpha,f}(\mathrm{d}x),$$

where

$$\mathbf{d} = -\theta - \int\limits_{\{\|x\|\leqslant 1\}} x\nu_\sigma^{\alpha,f}(\mathrm{d}x),$$

is known as the drift coefficient.

In what follows, we deal with real case. Define for each $x\geqslant 0$, the first passage time of a Lévy process X

$$\tau_x^+ = \inf\big\{t>0: X_t > x\big\},$$

with the convention $\inf \emptyset = \infty$. We say that X *creeps upwards* if for some $x \geqslant 0$, $\mathbf{P}_0(X_{\tau_x^+} = x) > 0$. If $-X$ creeps upwards, we say that X *creeps downwards*. Recall that if creeping occurs at just one x then creeping occurs at all x.

PROPOSITION 3.2. *Let X^L be a Lamperti stable process with characteristics $L = (\alpha, f, \sigma, \theta)$.*

i) If $\alpha \in (0,1)$ and $\mathbf{d} > 0$, the process X^L creeps upwards.

ii) If $\alpha \in [1,2)$ and $c_+ = 0$, the process X^L creeps upwards.

iii) If $\alpha \in [1,2)$ and $c_+ > 0$, the process X^L does not creeps upwards.

Proof. The first part of our statement follows directly from part (i) of Theorem 8 in [22].

From Proposition 3.1, for $\alpha \in [1,2)$ the process X^L is of unbounded variation. In this case, a result due to Vigon [37] says that X^L creeps upwards if and only if the following integral converges,

$$\int_0^1 \frac{x\nu_\sigma^{\alpha,f}\big([x,\infty)\big)}{H(x)}\mathrm{d}x, \quad \text{where} \quad H(x) = \int_{-x}^{0}\int_{-1}^{y} \nu_\sigma^{\alpha,f}\big((-\infty,u]\big)\mathrm{d}u\mathrm{d}y. \tag{3.4}$$

If $c_+ = 0$, it is clear that the above integral is equal to 0 which implies part (ii). In order to prove part (iii), we first study the case when $c_+ > 0$ and $c_- > 0$; in this case we have

$$|u|^\alpha \nu_\sigma^{\alpha,f}((-\infty,u]) = c_-|u|^\alpha \int_{-\infty}^{u} \frac{e^{-\delta x}}{(e^{-x}-1)^{\alpha+1}}\mathrm{d}x \sim \frac{c_-}{\alpha} \qquad \text{as} \qquad u \uparrow 0. \tag{3.5}$$

Then, it is not difficult to deduce that

$$x^{\alpha-2}H(x) \sim \frac{c_-}{(2-\alpha)(\alpha-1)\alpha} \qquad \text{as} \qquad x \downarrow 0.$$

Similar arguments as those used in (3.5) allows us to write

$$x^\alpha \nu_\sigma^{\alpha,f}\big([x,\infty)\big) \sim \frac{c_+}{\alpha} \qquad \text{as} \qquad x \downarrow 0.$$

Therefore,

$$\frac{x\nu_\sigma^{\alpha,f}\big([x,\infty)\big)}{H(x)} \sim \frac{(2-\alpha)(\alpha-1)c_+}{c_-}\frac{1}{x} \qquad \text{as} \qquad x \downarrow 0,$$

which implies that the integral K in (3.4) diverges.

Finally, if $c_+ > 0$ and $c_- = 0$ the integral (3.4) obviously diverges. The proof is now complete. ∎

We recall that for a Lévy process X a point $x \in \mathbb{R}$ is regular for $(0,\infty)$ if

$$\mathbf{P}_x(\tau^{(0,\infty)} = 0) = 1,$$

where $\tau^{(0,\infty)} = \inf\{t > 0 : X_t \in (0,\infty\}$.

PROPOSITION 3.3. *For a Lamperti stable process X^L with characteristics $(\alpha, f, \sigma, \theta)$, the point 0 is regular for $(0,\infty)$ if one of these three conditions hold:*

i) $\alpha \in [1,2)$.

ii) $\alpha \in (0,1)$ *and* $\mathbf{d} > 0$.

iii) $\alpha \in (0,1)$, $\mathbf{d} = 0$ *and* $c_+ > 0$.

P r o o f. (i) Recall from Proposition 3.1 that X^L has unbounded variation for $\alpha \in [1,2)$. Hence from Theorem 11 in [22], we deduce that 0 is regular for $(0,\infty)$.

Now we prove parts (ii) and (iii). Suppose that $\alpha \in (0,1)$. In this case X^L has bounded variation and again from Theorem 11 in [22], we know that the point 0 is regular for $(0,\infty)$ if the drift coefficient $\mathbf{d} > 0$ or if $\mathbf{d} = 0$ and the following condition holds

$$\int_0^1 \frac{x\nu_\sigma^{\alpha,f}(\mathrm{d}x)}{H_1(x)} = \infty, \quad \text{where} \quad H_1(x) = \int_0^x \nu_\sigma^{\alpha,f}(-\infty,-y)\mathrm{d}y. \tag{3.6}$$

Then, it is enough to prove that (3.6) holds when $\mathbf{d} = 0$ and $c_+ > 0$ to conclude our proof. The case when $c_+ > 0$ and $c_- = 0$ is immediate. For the second case, i.e. when $c_+ > 0$ and $c_- > 0$, we first recall from (3.5) that

$$y^\alpha \nu_\sigma^{\alpha,f}((-\infty,-y]) \sim \frac{c_-}{\alpha} \qquad \text{as} \quad y \downarrow 0,$$

which implies that

$$x^{\alpha-1}H_1(x) \sim \frac{c_-}{\alpha(1-\alpha)} \qquad \text{as} \quad x \downarrow 0.$$

We observe then, that

$$\frac{x^2 e^{\beta x}(e^x - 1)^{-(\alpha+1)}}{H(x)} \sim \frac{\alpha(1-\alpha)}{c_-}, \qquad \text{as} \quad x \downarrow 0,$$

which implies (3.6). ∎

Our next result deals with the computation of the characteristic exponents of Lamperti stable processes. Denote by

$$(z)_\alpha = \frac{\Gamma(z+\alpha)}{\Gamma(z)}, \qquad \text{for} \quad z \in \mathbb{C},$$

which is known as the Pochhammer symbol.

THEOREM 3.1. *Let X^L be a Lamperti stable process with characteristics $L = (\alpha, f, \sigma, \theta)$.*

i) If $\alpha \in (0,1) \cup (1,2)$, the characteristic exponent of X^L is given by

$$\begin{aligned}\Psi_L(\lambda) = i\lambda\tilde{\theta} - c_+\Gamma(-\alpha)\left((-i\lambda + 1 - \beta)_\alpha - (1-\beta)_\alpha\right)\\ - c_-\Gamma(-\alpha)\left((i\lambda + 1 - \delta)_\alpha - (1-\delta)_\alpha\right), \qquad \lambda \in \mathbb{R}.\end{aligned}$$

ii) If $\alpha = 1$, the characteristic exponent of X^L is given by

$$\begin{aligned}\Psi_L(\lambda) = i\lambda\tilde{\theta} - c_+\Big((-i\lambda + 1 - \beta)\psi(-i\lambda + 2 - \beta) - (1-\beta)\psi(2-\beta)\Big)\\ - c_-\Big((i\lambda + 1 - \delta)\psi(i\lambda + 2 - \delta) - (1-\delta)\psi(2-\delta)\Big), \qquad \lambda \in \mathbb{R}.\end{aligned}$$

Where ψ is the Digamma function, $\tilde{\theta}$ is given by

$$\tilde{\theta} = \begin{cases} -\mathbf{d} & \text{if } \alpha \in (0,1), \\ \theta - \Big(c_+\tilde{a}_\beta - c_-\tilde{b}_\delta + (c_+ - c_-)(1-\mathcal{C})\Big) & \text{if } \alpha = 1, \\ \theta - \Big(c_+\tilde{a}_\beta - c_-\tilde{b}_\delta + \frac{c_+ - c_-}{\alpha - 1}\Big) & \text{if } \alpha \in (1,2), \end{cases} \tag{3.7}$$

where $\mathcal{C}$ is the Euler constant and $\tilde{a}_\beta, \tilde{b}_\delta$ are given by:

$$\tilde{a}_\beta = \int_0^1 \frac{xe^{-x}(1 - e^{-(\alpha-\beta)x})}{(1-e^{-x})^{\alpha+1}}dx + \int_0^1 e^{-x}\frac{1 - x - e^{-x}}{(1-e^{-x})^{\alpha+1}}dx + \int_1^\infty \frac{e^{-x}}{(1-e^{-x})^\alpha}dx,$$

$$\tilde{b}_\delta = \int_0^1 \frac{xe^{-x}(1 - e^{-(\alpha-\delta)x})}{(1-e^{-x})^{\alpha+1}}dx + \int_0^1 e^{-x}\frac{1 - x - e^{-x}}{(1-e^{-x})^{\alpha+1}}dx + \int_1^\infty \frac{e^{-x}}{(1-e^{-x})^\alpha}dx,$$

for all $\beta, \delta < \alpha + 1$.

P r o o f. We first consider the case where $\alpha \in (0,1)$ and without loss of generality, we will assume that $\mathbf{d} = 0$. In this case X^L has finite variation (see Proposition 3.1), so it can be seen as the difference of two independent Lamperti stable subordinators with characteristics (α, β, c_+) and (α, δ, c_-) respectively. Then in order to obtain $\Psi_L(\lambda)$, it is enough to compute the characteristic exponent of the former, i.e.

$$\Psi_\beta(\lambda) = -\left(c_+ \int_0^\infty (e^{i\lambda x} - 1)\frac{e^{\beta x}}{(e^x - 1)^{\alpha+1}} dx\right).$$

Since all the computations involved are valid for all $\lambda \in \mathbb{R}$, we center our attention in the variable β and define in the set $U = \{z \in \mathbb{C} : \Re(z) < \alpha + 1\}$, the following function $F : U \to \mathbb{C}$, given by

$$F(z) := \int_0^\infty (e^{i\lambda x} - 1)\frac{e^{zx}}{(e^x - 1)^{\alpha+1}} dx = \int_0^1 (u^{-i\lambda} - 1)u^{z_1 - 1}(1-u)^{-(\alpha+1)} du,$$

where $z_1 = \alpha + 1 - z$. Integration by parts and the integral representation for the Beta function allows us to write for $\Re(z_1) > 1$

$$\int_0^1 (u^{-i\lambda} - 1)u^{z_1-1}(1-u)^{-(\alpha+1)} du = \frac{-(i\lambda + z_1 - 1)}{\alpha}\frac{\Gamma(-i\lambda + z_1 - 1)\Gamma(1-\alpha)}{\Gamma(-i\lambda + z_1 - \alpha)} + \frac{(z_1 - 1)}{\alpha}\frac{\Gamma(z_1 - 1)\Gamma(1-\alpha)}{\Gamma(z_1 - \alpha)}.$$

Now, from the recurrence relation for the Gamma function $\Gamma(x+1) = x\Gamma(x)$, we obtain

$$F(z) = \Gamma(-\alpha)\left(\frac{\Gamma(-i\lambda + \alpha + 1 - z)}{\Gamma(-i\lambda + 1 - z)} - \frac{\Gamma(\alpha + 1 - z)}{\Gamma(1-z)}\right),$$

for $\Re(z) < \alpha$, which implies our result for $\beta < \alpha$. In order to obtain the above identity for $\beta \in [\alpha, \alpha+1)$ we use analytic extension arguments.

Using a series expansion, it is not difficult to see that $F(z)$ is analytic on the disk $D_{\alpha+1} = \{z \in \mathbb{C} : \|z\| < \alpha + 1\}$. On the other hand, we note that for $\|z\| < \alpha + 1$, we have $\Re(-i\lambda + \alpha + 1 - z) > 0$ and $\Re(\alpha + 1 - z) > 0$. Therefore the function $G : U \to \mathbb{C}$, defined by

$$G(z) := \Gamma(-\alpha)\left(\frac{\Gamma(-i\lambda + \alpha + 1 - z)}{\Gamma(-i\lambda + 1 - z)} - \frac{\Gamma(\alpha + 1 - z)}{\Gamma(1-z)}\right),$$

is also analytic in $D_{\alpha+1}$. Since the functions F and G coincide in $D_\alpha = \{z \in \mathbb{C} : \|z\| < \alpha\}$ and they are both analytic in $D_{\alpha+1}$, we conclude that $F \equiv G$ in $D_{\alpha+1}$. In particular, we have that $F(\beta) = G(\beta)$ for $\beta \leqslant \alpha + 1$, and our result follows.

Next, we will compute the characteristic exponent for $\alpha = 1$. In the following we assume that $\theta = 0$ and that $c_+ = c_- = 1$. Similarly as the previous case, we can see from the form of the Lévy measure of X^L, that the latter can be written as the sum of two independent one-sided Lamperti stable processes, one spectrally positive and with characteristics $(1, \beta, 1)$ and the second one spectrally negative and with characteristics $(1, \delta, 1)$. Hence, in order to get our result it is enough to compute

$$\Psi_\beta(\lambda) = -\int_0^\infty \left(e^{i\lambda x} - 1 - i\lambda x \mathbb{1}_{\{x<1\}}\right) \frac{e^{\beta x}}{(e^x - 1)^2} dx.$$

In this case, let $U := \{z \in \mathbb{C} : \Re(z) < 2\}$ and define the function $G_1 : U \to \mathbb{C}$ by

$$G_1(z) := \int_0^\infty \left(e^{i\lambda x} - 1 - i\lambda x \mathbb{1}_{\{x<1\}}\right) \frac{e^{zx}}{(e^x - 1)^2} dx = i\lambda\tilde{a} + i\lambda I(z) + J(z)$$

where

$$\tilde{a} := \int_0^1 e^{-x} \frac{1 - x - e^{-x}}{(1 - e^{-x})^2} dx + \int_1^\infty \frac{e^{-x}}{1 - e^{-x}} dx, \quad I(z) := \int_0^1 \frac{x e^{-x}(1 - e^{-(1-z)x})}{(1 - e^{-x})^2} dx,$$

$$J(z) := \int_0^1 \frac{(u^{-i\lambda} - 1)u^{z_1 - 1} + i\lambda(u - 1)}{(1 - u)^2} du$$

and $z_1 = 2 - z$. Note that $\tilde{a}$ is finite and that $I(z)$ is well-defined in U and analytic in $D_2 = \{z \in \mathbb{C} : \|z\| < 2\}$. Then the main issue is to compute $J(z)$. Making an integration by parts in the region $\Re(z_1) > 1$ we obtain,

$$\begin{aligned} J(z) =& i\lambda - (z_1 - 1) \int_0^1 \frac{(u^{-i\lambda} - 1)u^{z_1 - 2}}{(1 - u)} du + i\lambda \int_0^1 \frac{(u^{-i\lambda + z_1 - 2} - 1)}{(1 - u)} du. \\ &= i\lambda - (z_1 - 1)A(z_1) + i\lambda B(z_1). \end{aligned}$$

The integrals $A(z_1), B(z_1)$ can be written in terms of the Digamma function ψ, using the following integral representation of the Digamma function

$$\psi(z) = \int_0^1 \frac{t^{z-1} - 1}{z - 1} dt - \mathcal{C}, \qquad \text{for} \quad z \in \mathbb{C},$$

(where $\mathcal{C}$ is the Euler constant), and the recurrence relation $\psi(z+1) = \psi(z) + z^{-1}$ (see for instance [15]). So for $\Re(z_1) > 1$, we have

$$A(z_1) = \psi(z_1) - \frac{1}{z_1 - 1} - \psi(-i\lambda + z_1) + \frac{1}{-i\lambda + z_1 - 1},$$

and for $B(z_1)$, we get

$$B(z_1) = -\psi(-i\lambda + z_1 - 1) - \mathcal{C} = \frac{1}{-i\lambda + z_1 - 1} - \psi(-i\lambda + z_1) - \mathcal{C}.$$

Recalling that $z_1 = 2 - z$, we deduce

$$J(z) = i\lambda(1 - \mathcal{C}) + (-i\lambda + 1 - z)\psi(-i\lambda + 2 - z) - (1 - z)\psi(2 - z), \tag{3.8}$$

for $\Re(z) < 1$. This proves our result for $\beta < 1$. Again, in order to extend the above formula to the case $\beta \in [1, 2)$ we will use analytic extension arguments.

Using a series expansion, we may prove that $G_1(z)$ is analytic in D_2. Moreover, it is not difficult to see that the right-hand side of (3.8) is analytic in the same region. This implies that $G_1 \equiv i\lambda\tilde{a} + i\lambda I + J$ in D_2. In particular, we have

$$\Psi_L(\lambda) = G(\beta) = i\lambda(\tilde{a}_\beta + 1 - \mathcal{C}) + (-i\lambda + 1 - \beta)\psi(-i\lambda + 2 - \beta) - (1 - \beta)\psi(2 - \beta),$$

where

$$\tilde{a}_\beta = \int_0^1 \frac{xe^{-x}(1 - e^{-(1-\beta)x})}{(1 - e^{-x})^2} dx + \int_0^1 e^{-x} \frac{1 - x - e^{-x}}{(1 - e^{-x})^2} dx + \int_1^\infty \frac{e^{-x}}{(1 - e^{-x})} dx,$$

for all $\beta < 2$.

Finally, we study the case $\alpha \in (1, 2)$. Similarly as in the previous cases, it is enough to compute

$$\Psi_\beta(\lambda) = -c_+ \int_0^\infty \left(e^{i\lambda x} - 1 - i\lambda x \mathbb{1}_{\{x<1\}}\right) \frac{e^{\beta x}}{(e^x - 1)^{\alpha+1}} dx.$$

We omit the calculations of this integral since the main ideas are similar as those used in the case $\alpha = 1$ and the case $\alpha \in (0, 1)$. We leave the details to the reader. ■

It is important to note that the computation of the characteristic exponent for the case $\alpha \in (1, 2)$ can also be obtained from the paper [31] where the author gets the Laplace exponent of a spectrallly negative Lamperti stable process with characteristics (α, δ, c_-). These computations deal with hypergeometric functions.

Now we turn our attention to another group of properties. Let $H = (H_t, t \geqslant 0)$ be the increasing ladder height process of X^L (see chapter VI in [2]) and $\widehat{H} = (\widehat{H}_t, t \geqslant 0)$, its decreasing ladder height process. We denote by k and $\hat{k}$ the characteristic exponents of H and $\widehat{H}$, which are subordinators, and we assume that X^L

drifts to $-\infty$ and $\nu_\sigma^{\alpha,f}(0,\infty) > 0$. Under these hypotheses, the process H is a killed subordinator and we denote by Π_H its Lévy measure. With these notations we get the following relation between $\nu_\sigma^{\alpha,f}$ and Π_H.

PROPOSITION 3.4. *Let X^L be a Lamperti stable process with positive jumps and characteristics (α, f) such that it drifts to $-\infty$. Then, the tail of the Lévy measure of H belongs to $\mathcal{L}^{(\alpha+1-\beta)}$ and*

$$\nu_\sigma^{\alpha,f}(u,\infty) \sim \hat{k}(-i(\alpha+1-\beta))\Pi_H(u,\infty) \qquad \textit{as} \quad u \to \infty.$$

P r o o f. The proof follows directly from Proposition 5.3 in [21] and 2.4. ∎

We finish this section with some properties of Lamperti stable processes with no positive jumps.

PROPOSITION 3.5. *Let X^L be a Lamperti stable process with no positive jumps and characteristics $(\alpha, \delta, \sigma, \theta)$, such that $\tilde{\theta} = 0$ in (3.7). Then,*

i) there exist $\delta_0 \in (1,2)$ such that X_L drifts to ∞, oscillates or drifts to $-\infty$ according as $\delta \in (-\infty, \delta_0)$, $\delta = \delta_0$ or $\delta \in (\delta_0, \alpha+1)$.

ii) for $\delta \in (\delta_0, \alpha+1)$, we have that there exist $\lambda > 0$ such that

$$\mathbf{P}_0\big(S_\infty^L > x\big) \sim \frac{c}{\lambda k} e^{-\lambda x}, \quad \textit{as} \quad x \to \infty, \tag{3.9}$$

where $S_\infty^L = \sup_{t \geqslant 0} X_t^L$, $c = -\log \mathbf{P}_0(H_1 < \infty)$, $k = \mathbf{E}_0(H_1 e^{\lambda H_1}; H_1 < \infty)$ and H is the increasing ladder height process.

iii) for $\delta \in (\delta_0, \alpha+1)$, we have that there exist $\lambda > 0$ such that

$$\mathbf{P}_0\big(I(X^L) > x\big) \sim K_2 x^{-\lambda}, \quad \textit{as} \quad x \to \infty, \tag{3.10}$$

where K_2 is a positive constant and

$$I(X^L) = \int_0^\infty \exp\big\{X_t^L\big\} \mathrm{d}t.$$

iv) the probability that the process X^L has increase times is 1. [1]

v) the process X^L satisfies the Spitzer's condition at ∞, i.e.

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{P}(X^L_s \geqslant 0)\mathrm{d}s = 1/\alpha \quad \textit{as} \quad x \to \infty.$$

vi) the process X^L satisfies the following law of the iterated logarithm

$$\limsup_{x\to 0} \frac{X^L_t \Phi_L^{-1}(t^{-1}\log|\log t|)}{\log|\log t|} = \alpha^{-\alpha}(1-\alpha)^{\alpha-1} \qquad a.s., \tag{3.11}$$

where Φ_L^{-1} denotes the right-continuous inverse of Φ_L^{-1}.

P r o o f. (i) We know that in this case $\alpha \in (1,2)$, so from Corollary VII.2 in [2], the process X^L drifts to $+\infty$, oscillates or drifts to $-\infty$ according as $\Phi_L'(0^+)$ is positive, zero or negative. Hence, from the Laplace exponent of X^L we have, using the recursion formula for the Gamma and Digamma functions, the following

$$\begin{aligned}
\Phi_L'(0^+) &= c_-\Gamma(-\alpha)(1+\alpha-\delta)_\alpha(\psi(1-\delta+\alpha)-\psi(1-\delta)),\\
&= c_-\Gamma(-\alpha)\frac{\Gamma(1+\alpha-\delta)}{\Gamma(3-\delta)}\\
&\cdot\Big((2-\delta)(1-\delta)((\psi(1-\delta+\alpha)-\psi(1-\delta))+3-2\delta\Big),\\
&= g(\delta). \qquad (3.12)
\end{aligned}$$

We have from (3.12) that $g(1) < 0$, and $g(2) > 0$. On the other hand, in the interval $(1,2)$, the function g is continuous and decreasing which implies that there exist $\delta_0 \in (1,2)$ such that $g(\delta_0) = 0$. Thus, we deduce that X^L drifts to ∞, oscillates or drifts to $-\infty$ according as $\delta \in (-\infty, \delta_0)$, $\delta = \delta_0$ or $\delta \in (\delta_0, \alpha+1)$.
(ii) Any Lévy process with no positive jumps which drifts to $-\infty$ has the property that its Laplace exponent has a strictly positive root. Hence for a Lamperti stable process with no positive jumps and with $\delta \in (\delta_0, \alpha+1)$, there exists $\lambda > 0$ such that

$$\mathbf{E}_0\big(\exp\{\lambda X^L_1\}\big) = 1,$$

i.e. that X^L satisfies the Cramér condition. Thus, the main result in [3] gives us the sharp estimate in (3.9).

---

[1] Recall that an instant $t > 0$ is an increase time for a path ω if for some $\epsilon > 0$,

$$\omega(t') \leqslant \omega(t) \leqslant \omega(t'') \qquad \text{for all } t \in [t-\epsilon, t] \text{ and } t'' \in [t, t+\epsilon].$$

(iii) First note that X^L is not arithmetic and that under our assumptions the Cramér condition is satisfied for some $\lambda > 0$. Hence from Lemma 4 in [32], we get the sharp estimate (3.10) for the exponential functional $I(X^L)$.
(iv) Here, we need the following estimate of the Pochhammer symbol (see for instance [26]),

$$(\lambda + 1 - \delta)_\alpha \sim \lambda^\alpha \qquad \text{as} \quad \lambda \to \infty. \tag{3.13}$$

From Corollary VII.9 and Proposition VII.10 in [2] we know that X^L has increase times if

$$\int^\infty \lambda^{-3}\Phi_L(\lambda)\mathrm{d}\lambda < \infty,$$

which in our case is satisfied since from (3.13), we have

$$\Phi_L(\lambda) \sim c_-\Gamma(-\alpha)\lambda^\alpha \qquad \text{as} \quad \lambda \to \infty. \tag{3.14}$$

(v) From (3.14), we see that Φ_L is regularly varying at ∞ with index α. Hence, the statement follows from Proposition VII.6 in [2].
(vi) Since Φ_L is regularly varying at ∞ with index α, we have that its right-continuous inverse Φ_L^{-1} is regularly varying at ∞ with index $1/\alpha$ which corresponds to the Laplace exponent of the first passage time of X^L (which is a subordinator). Therefore, from Theorem III.11 in [2] we deduce the law of the iterated logarithm (3.11). ■

4. SMALL AND LONG TIME BEHAVIOUR.

Motivated by the works of Rosiński [33] and Houdré and Kawai [16], we study the small and long time behaviour of Lamperti stable processes. In particular, we will show that this class of processes share with the tempered and layered stable processes, the peculiarity that in small time they behave like stable processes. The convergence in distribution of processes, considered in this section, is in the functional sense, i.e. in the sense of the weak convergence of the laws of the processes in the Skorokhod space and will be denoted by " $\stackrel{d}{\to}$ ".

PROPOSITION 4.1. *Let X^L be a Lamperti stable process with characteristics*

$(\alpha, f, \sigma, 0)$ *and*

$$\eta_\alpha = \begin{cases} 0 & \textit{if } \alpha = 1, \\ \int\limits_{S^{d-1}} \xi\sigma(\mathrm{d}\xi) \int\limits_0^1 r e^{f(\xi)r}(e^r - 1)^{-(\alpha+1)}\mathrm{d}r & \textit{if } \alpha \in (0,1), \\ \int\limits_{S^{d-1}} \xi\sigma(\mathrm{d}\xi) \int\limits_1^\infty r e^{f(\xi)r}(e^r - 1)^{-(\alpha+1)}\mathrm{d}r & \textit{if } \alpha \in (1,2). \end{cases}$$

Then,

$$\big(h^{-1/\alpha}\big(X^L_{ht} - ht\eta_\alpha\big), t > 0\big) \xrightarrow{d} (X_t, t > 0) \quad as \quad h \to 0,$$

where $(X_t, t > 0)$ *is a stable process of index* α*.*

Proof. The proof is similar to that of the small time behaviour of layered stable process (see Theorem 3.1 in [16]), since for each $\xi \in S^{d-1}$

$$e^{f(\xi)r}(e^r - 1)^{-(\alpha+1)} \sim r^{-(\alpha+1)} \quad as \quad r \to 0.$$

We leave the details to the reader. ■

THEOREM 4.1. *Let* X^L_t *be a Lamperti stable process with characteristics* $(\alpha, f, \sigma, 0)$ *and*

$$\eta_\alpha = -\int\limits_{S^{d-1}} \xi\sigma(\mathrm{d}\xi) \int\limits_1^\infty r e^{f(\xi)r}(e^r - 1)^{-(\alpha+1)}\mathrm{d}r.$$

Then,

$$\big(h^{-1/2}\big(X^L_{ht} - ht\eta_\alpha\big), t > 0\big) \xrightarrow{d} (W_t, t > 0) \quad as \quad h \to \infty, \tag{4.1}$$

where $(W_t, t > 0)$ *is a Brownian motion with covariance matrix* $\int\limits_{\mathbb{R}^d_0} xx'\nu^{\alpha,f}_\sigma(\mathrm{d}x)$*.*

Proof. According to a standard result on the convergence of processes with independent increments due to Skorokhod (see for instance Theorem 15.17 of Kallenberg [19]), the functional convergence (4.1) holds if and only if

$$h^{-1/2}\big(X^L_h - h\eta_\alpha\big) \xrightarrow{d} W_1 \quad as \quad h \to \infty.$$

Now, we introduce the following transform for positive measures, for any $r > 0$

$$(T_r\nu)(B) = \nu(r^{-1}B) \qquad \text{for} \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Note that the random variable $h^{-1/2}X_h^L$ is infinitely divisible and since it has finite first moment, we may rewrite its characteristic exponent as follows;

$$
\begin{aligned}
& ih \int_{\mathbb{R}_0^d} \langle y, x\rangle \mathbb{I}_{\{\|x\|\geqslant 1\}} (T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x) \\
& \qquad - h \int_{\mathbb{R}_0^d} \left(e^{i\langle y,x\rangle} - 1 - i\langle y, x\rangle \mathbb{I}_{\{\|x\|\leqslant 1\}}\right)(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x).
\end{aligned}
\tag{4.2}
$$

Hence, from Theorem 15.14 of Kallenberg [19] we only need to verify the following convergences as h increases:

(a) $h(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})$ converges vaguely towards 0 on $\mathbb{R}_0^d$,

(b) for each $k > 0$, $\qquad h \int_{\|x\|\leqslant\kappa} xx'(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x) \to \int_{\mathbb{R}_0^d} xx'\nu_\sigma^{\alpha,f}(\mathrm{d}x),$

(c) for each $k > 0$, $\qquad h \int_{\|x\|\geqslant k} x(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x) \to 0.$

We first prove (a) or equivalently

$$
\lim_{h\to\infty} \int_{\mathbb{R}_0^d} g(x)h(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x) = 0,
\tag{4.3}
$$

for all bounded continuous functions $g : \mathbb{R}_0^d \to \mathbb{R}$ vanishing in a neighborhood of the origin. Let g be such a function satisfying that $|g| \leqslant C$, and that for some $\delta > 0$, $g(x) \equiv 0$ on $\{x \in \mathbb{R}_0^d : \|x\| < \delta\}$. Let $\gamma := \sup_{\xi\in S^{d-1}} f(\xi)$, then we have

$$
\begin{aligned}
& \left| h \int_{\mathbb{R}_0^d} g(x)\,(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x) \right| \\
& \qquad \leqslant h^{1+1/2} \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_0^\infty |g(r\xi)| e^{rf(\xi)h^{1/2}} (e^{rh^{1/2}} - 1)^{-(\alpha+1)} \mathrm{d}r \\
& \qquad \leqslant \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_\delta^\infty |g(r\xi)| \frac{(rh^{1/2})^3}{r^3} e^{rh^{1/2}\gamma} (e^{rh^{1/2}} - 1)^{-(\alpha+1)} \mathrm{d}r.
\end{aligned}
\tag{4.4}
$$

On the other hand, since $\gamma < \alpha + 1$ it follows

$$
\lim_{r\to\infty} r^3 \frac{e^{r\gamma}}{(e^r - 1)^{\alpha+1}} = 0,
$$

then for $\epsilon > 0$ sufficiently small, there exist $M > 0$ such that for all $r \geqslant M$

$$
r^3 e^{rf(\xi)} (e^r - 1)^{-(\alpha+1)} < \epsilon.
$$

Since $r > \delta$, we may take $h > \left(\frac{M}{\delta}\right)^2$ in (4.4) and obtain

$$\left| h \int\limits_{\mathbb{R}_0^d} g(x)\,(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x) \right| < \epsilon \int\limits_{S^{d-1}} \sigma(d\xi) \int\limits_\delta^\infty |g(r\xi)| \frac{1}{r^3} \mathrm{d}r$$
$$\leqslant \epsilon C \int\limits_{S^{d-1}} \sigma(d\xi) \int\limits_\delta^\infty \frac{1}{r^3} \mathrm{d}r.$$

Note that the last integral in the right-hand side of the above inequality is finite and therefore the convergence (4.3) follows.

Next, we prove part (b). First note that $\int\limits_{\mathbb{R}_0^d} \|x\|^2 \nu_\sigma^{\alpha,f}(\mathrm{d}x)$ is finite. This implies that the integral $\int\limits_{\mathbb{R}_0^d} xx' \nu_\sigma^{\alpha,f}(\mathrm{d}x)$ is well defined. Now take $k > 0$ fixed, and note that

$$h \int\limits_{\{\|x\| \leqslant k\}} xx'(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x) = \int\limits_{\{\|x\| \leqslant h^{1/2}k\}} xx'\nu_\sigma^{\alpha,f}(\mathrm{d}x) \to \int\limits_{\mathbb{R}_0^d} xx'\nu_\sigma^{\alpha,f}(\mathrm{d}x),$$

as h goes to ∞, which proves part (b).

Finally we prove (c), we consider $k > 0$ and recall that $\gamma = \sup_{\xi \in S^{d-1}} f(\xi)$, then

$$\left\| h \int\limits_{\{\|x\| \geqslant k\}} z(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}z) \right\|$$
$$= \left\| h^{1+1/2} \int\limits_{S^{d-1}} \xi\sigma(\mathrm{d}\xi) \int\limits_k^\infty r e^{rf(\xi)h^{1/2}} (e^{rh^{1/2}} - 1)^{-(\alpha+1)} \mathrm{d}r \right\|$$
$$\leqslant (1 - e^{-kh^{1/2}})^{-(\alpha+1)} \left\| h^{1+1/2} \int\limits_{S^{d-1}} \xi\sigma(\mathrm{d}\xi) \int\limits_k^\infty r e^{rh^{1/2}(\gamma-(\alpha+1))} \mathrm{d}r \right\|$$
$$= \frac{e^{-kh^{1/2}(\alpha+1-\gamma)}}{(1 - e^{-kh^{1/2}})^{\alpha+1}} \left(\frac{hk}{\alpha+1-\gamma} - \frac{h^{1/2}}{(\alpha+1-\gamma)^2} \right) \left\| \int\limits_{S^{d-1}} \xi\sigma(d\xi) \right\|,$$

which goes to 0 as $h \to \infty$ since $\gamma < \alpha + 1$. This completes the proof. ∎

5. ABSOLUTE CONTINUITY WITH RESPECT TO STABLE PROCESSES

We showed that for small times a Lamperti-stable process behaves like a stable process, now following Rosiński [33] we will relate the law of both processes.

THEOREM 5.1. *Let P and Q be two probability measures on $(\Omega, \mathcal{F})$ and such that under P the canonical process $(X_t, t \geqslant 0)$ is a Lamperti stable pro-*

cess with characteristics (α, f, σ, a)*, while under* Q *it is a stable process with index* α *with linear term* b*. Let* $(\mathcal{F}_t)$ *be the canonical filtration, and assume that* $f \in L^2(S^{d-1}, \mathbb{B}(S^{d-1}), \sigma)$*. Then*

i) $P|_{\mathcal{F}_t}$ *and* $Q|_{\mathcal{F}_t}$ *are mutually absolutely continuous for every* $t > 0$ *if and only if*

$$a - b = \begin{cases} \int\limits_{S^{d-1}} \xi\sigma(\mathrm{d}\xi) \int\limits_0^1 r e^{rf(\xi)}(e^r - 1)^{-(\alpha+1)}\mathrm{d}r, & \text{if } \alpha \in (0,1), \\ \int\limits_{S^{d-1}} \xi\sigma(\mathrm{d}\xi) \int\limits_0^1 r(e^{rf(\xi)}(e^r - 1)^{-(\alpha+1)} - r^{-(\alpha+1)})\mathrm{d}r, & \text{if } \alpha = 1, \\ \int\limits_{S^{d-1}} \xi\sigma(\mathrm{d}\xi) \int\limits_0^1 r(e^{rf(\xi)}(e^r - 1)^{-(\alpha+1)} - r^{-(\alpha+1)})\mathrm{d}r & \\ - \int\limits_{S^{d-1}} \xi\sigma(\mathrm{d}\xi) \int\limits_1^\infty r^{-(\alpha+1)}\mathrm{d}r, & \text{if } \alpha \in (1,2). \end{cases}$$

ii) For each $t > 0$*,*

$$\frac{\mathrm{d}Q}{\mathrm{d}P}\Big|_{\mathcal{F}_t} = e^{U_t},$$

where $(U_t, t \geqslant 0)$ *is a Lévy process defined on* $(\Omega, \mathcal{F}, P)$ *by*

$$U_t = \lim_{\epsilon \downarrow 0} \sum_{\{s \in (0,t] : \|\Delta X_s\| > \epsilon\}} \Big[\big(e^{\|\Delta X_s\| f(\Delta X_s)} (e^{\|\Delta X_s\|} - 1)^{-(\alpha+1)} \|\Delta X_s\|^{\alpha+1} \big) - t(\nu_\sigma^{\alpha,f} - \Pi)\big(\{z \in \mathbb{R}_0^d : \|z\| > \epsilon\}\big) \Big].$$

In the above right hand side, the convergence holds P*-a.s. uniformly in* t *on every interval of positive length.*

P r o o f. From Theorem 33.2 in Sato [35], we only need to verify that

$$\int\limits_{\mathbb{R}_0^d} (e^{\varphi(x)/2} - 1)^2 \Pi(\mathrm{d}x) < \infty,$$

where $\varphi : \mathbb{R}_0^d \to \mathbb{R}$ is defined by $\frac{\mathrm{d}\nu_\sigma^{\alpha,f}}{\mathrm{d}\Pi}(x) = e^{\varphi(x)}$. In particular, we have that $\varphi(r\xi) = \log\big(e^{rf(\xi)}(e^r - 1)^{-(\alpha+1)} r^{\alpha+1}\big)$. Thus, we need to check

$$\text{(5.1)} \qquad \int\limits_{S^{d-1}} \sigma(\mathrm{d}\xi) \int\limits_0^\infty \left[\left(\frac{e^{rf(\xi)} r^{(1+\alpha)}}{(e^r - 1)^{(\alpha+1)}} \right)^{1/2} - 1 \right]^2 \frac{1}{r^{1+\alpha}} \mathrm{d}r < \infty.$$

By Taylor expansion and the Lagrange form for the remainder, we have $(e^r - 1) = re^{r\theta_r}$, where $\theta_r \in (0,1)$. This implies

$$\frac{e^{rf(\xi)}r^{(1+\alpha)}}{(e^r-1)^{(\alpha+1)}} = e^{r(f(\xi)-\theta_r(\alpha+1))}. \tag{5.2}$$

Now, noting that $f(\xi) - (\alpha+1) \leqslant f(\xi) - \theta_r(\alpha+1) \leqslant f(\xi)$, it follows

$$e^{r(f(\xi)-(\alpha+1))/2} - 1 \leqslant e^{r(f(\xi)-\theta_r(\alpha+1))/2} - 1 \leqslant e^{rf(\xi)/2} - 1,$$

and since $f(\xi) \leqslant \gamma = \sup_{\xi \in S^{d-1}} f(\xi)$, we have

$$\left(e^{r(f(\xi)-\theta_r(\alpha+1))/2} - 1\right)^2 \leqslant \left(e^{r(f(\xi)-(\alpha+1))/2} - 1\right)^2 \vee \left(e^{rf(\xi)/2} - 1\right)^2 \tag{5.3}$$

Using a Taylor expansion again and (5.3), it is clear that there exists a constant $R > 0$ such that if $r < R$, then

$$\left(e^{r(f(\xi)-\theta_r(\alpha+1))/2} - 1\right)^2 \leqslant K_3(f^2(\xi)+1)r^2, \tag{5.4}$$

where K_3 is a positive constant. Hence from (5.2) and (5.4), it follows that

$$\begin{aligned}\int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_0^R & \left[\left(\frac{e^{rf(\xi)}r^{(1+\alpha)}}{(e^r-1)^{(\alpha+1)}}\right)^{1/2} - 1\right]^2 \frac{1}{r^{1+\alpha}}\mathrm{d}r \\ & \leqslant K_3\left(\sigma(S^{d-1}) + \int_{S^{d-1}} f^2(\xi)d\xi\right)\int_0^R \frac{r^2}{r^{1+\alpha}}\mathrm{d}r,\end{aligned}$$

which is finite because $\alpha \in (0,2)$ and $f \in L^2(S^{d-1}, \mathbb{B}(S^{d-1}), \sigma)$. In the case when $r > R$, we have

$$\begin{aligned}&\int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_R^\infty \left[\left(\frac{e^{rf(\xi)}r^{(1+\alpha)}}{(e^r-1)^{(\alpha+1)}}\right)^{1/2} - 1\right]^2 \frac{1}{r^{1+\alpha}}\mathrm{d}r \\ &\quad \leqslant 4\left((1-e^{-R})^{-(\alpha+1)} \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_R^\infty e^{r(f(\xi)-(\alpha+1))}\mathrm{d}r + \sigma(S^{d-1}) \int_R^\infty \frac{1}{r^{1+\alpha}}\mathrm{d}r\right) \\ &\quad \leqslant 4\sigma(S^{d-1})\left((1-e^{-R})^{-(\alpha+1)} \int_R^\infty e^{r(\gamma-(\alpha+1))}\mathrm{d}r + \int_R^\infty \frac{1}{r^{1+\alpha}}\mathrm{d}r\right),\end{aligned}$$

which is also finite because $\gamma < \alpha + 1$. Therefore (5.1) follows.

The proof of the second statement of the Theorem follows directly from Theorem 33.2 of Sato [35]. ∎

Note that under the conditions of Theorem 4.1 in [16], if R is another probability measure on $(\Omega, \mathcal{F})$ under which the canonical process $X = (X_t, t \geqslant 0)$ is a layered stable process, we have that $R|_{\mathcal{F}_t}$ and $Q|_{\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$. From our previous result, we obtain the corresponding result for Lamperti stable processes, i.e. that $R|_{\mathcal{F}_t}$ and $P|_{\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$. Similar result holds for the tempered stable processes, see Theorem 4.1 in [33].

6. SERIES REPRESENTATIONS OF LAMPERTI STABLE PROCESS

In this section, we establish a series representation for Lamperti stable processes which allow us to generate some of their sample paths. To this end, we will use the LePage's method found in [27]. We first introduce the following sequences of mutually independent random variables defined in $[0, T]$. Let $\{\Gamma_i\}_{i\geqslant 1}$ be a sequence of of partial sums of iid standard exponential random variables, $\{U_i\}_{i\geqslant 1}$ be a sequence of uniform random variables on $[0, T]$, and let $\{V_i\}_{i\geqslant 1}$ be a sequence of iid random variables in S^{d-1} with common distribution $\sigma(d\xi)/\sigma(S^{d-1})$. In order to use the LePage's method, we consider the following function δ^{-1} : $(0, \infty) \times S^{d-1} \to \mathbb{R}_+$ given by

$$\delta^{-1}(u, \xi) := \inf \big\{ x > 0 : \delta([x, \infty), \xi) < u \big\},$$

where

$$\delta([x, \infty), \xi) = \int_x^\infty e^{f(\xi) r} (e^r - 1)^{-(\alpha+1)} \mathrm{d}r.$$

Now, let $\{c_i\}_{i\geqslant 1}$ be a sequence of constants defined as follows,

$$c_i = \int_{i-1}^{i} \mathbf{E}\big(\delta^{-1}(s/T, V_1) V_1 \mathbb{1}_{\{\delta^{-1}(s/T, V_1) \leqslant 1\}}\big) ds.$$

Then from Theorem 5.1 in [34], the process

$$\Big(\sum_{i=1}^{\infty} \big(\delta^{-1}(\Gamma_i/T, V_i) V_i \mathbb{1}_{\{U_i \leqslant t\}} - c_i \frac{t}{T} \big), t \in [0, T] \Big),$$

converges uniformly a.s. towards a Lamperti stable process with characteristics (α, f, σ) and linear term $\theta = 0$ (in the Lévy-Khintchine formula). In particular when $f(\xi) = 1$, we have that

$$\delta^{-1}(u, \xi) = \ln(1 + (\alpha u)^{-1/\alpha}),$$

hence the series representation for a Lamperti stable process X^L with characteristics $(\alpha, 1)$, is

$$X_t^L \overset{d}{=} \sum_{i=1}^{\infty} \Big(\ln\Big(1 + \Big(\frac{\alpha\Gamma_i}{T}\Big)^{-1/\alpha}\Big) V_i \mathbb{I}_{\{U_i \leqslant t\}} - c_i \frac{t}{T} \Big),$$

where

$$c_i = \mathbf{E}(V_1) \int_{i-1}^{i} \ln\Big(1 + \Big(\frac{\alpha s}{T}\Big)^{-1/\alpha}\Big) \mathbb{I}_{\{\ln(1+(\alpha s T^{-1})^{-1/\alpha}) \leqslant 1\}} \mathrm{d}s.$$

The figures 1,2,3,4 and 5 illustrate some cases of the sample paths generated via the series representation.

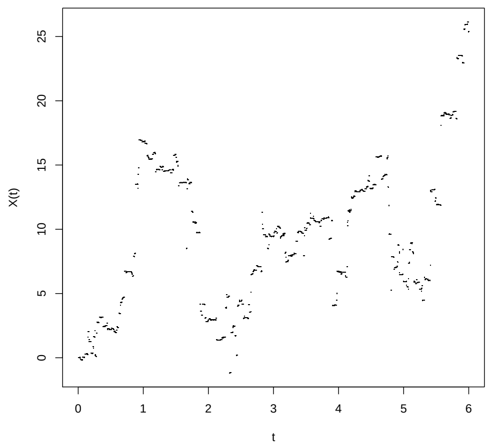

FIGURE 1. $\alpha = 0.5$, $f = 1$, $\sigma(1) = \sigma(-1) = 1$.

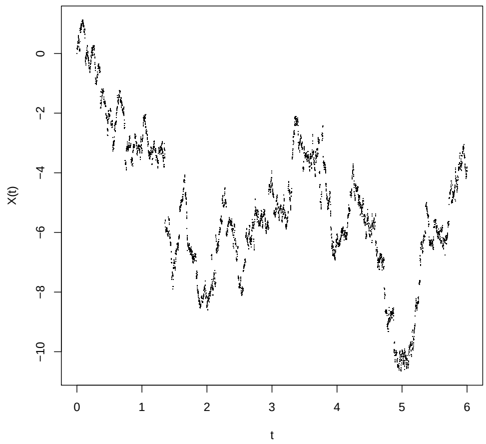

FIGURE 2. $\alpha = 1.5, f = 1, \sigma(1) = \sigma(-1) = 1.$

7. EXAMPLES.

Examples of Lamperti stable processes appear in the literature at least in the papers mentioned in the introduction ([6, 9, 11, 23, 25, 29, 30, 31]) but they also appear (in a hidden way) in many other recent works. We will give a quick overview of some of them, not pretending to be exhaustive in this list.

7.1.- As we mentioned in the introduction the starting point of our work are the processes studied in [25] and [6]. Here we make the link between these proceses and our definition of Lamperti stable processes.

In [6, 25], the Lévy processes are introduced via the Lamperti transformation of positive self-simliar Markov processes (PSSMP). The example treated in [25] is the case when the PSSMP is a stable subordinator of index $\alpha \in (0, 1)$ starting from a positive position. According to our definition, its associated Lévy process is a Lamperti stable subordinator with characteristics $\alpha, \beta = 1$ and $c_- = 0$.

In [6], the first PSSM to be considered is the stable Lévy processes killed when it first exits from the positive half-line, here denoted by $(X^*, \mathbf{P}_x)$. The second class

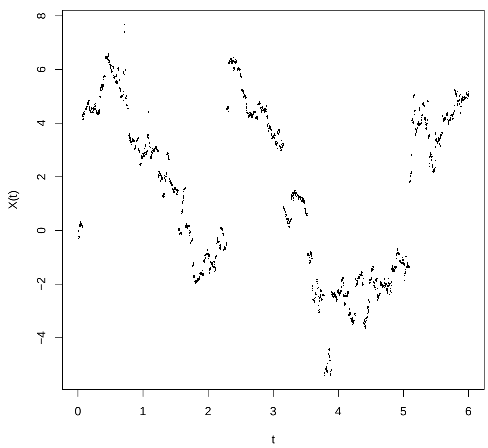

FIGURE 3. $\alpha = 1, f = 1, \sigma(1) = \sigma(-1) = 1$.

corresponds to that of stable processes conditioned to stay positive (see [8, 10]), that we denote by $(X^{\uparrow}, \mathbf{P}_x)$. Finally, the third class of PSSMP is that of stable processes conditioned to hit 0 continuously (see [8]), here denoted by $(X^{\downarrow}, \mathbf{P}_x)$. Their corresponding Lévy processes under the Lamperti transformation are denoted by $\xi^*, \xi^{\uparrow}$ and $\xi^{\downarrow}$, respectively. These three classes of Lévy processes are Lamperti stable processes and the characteristics of their Lévy measure are as follows:

i) for ξ^*, we have that $\beta = 1$ and $\delta = \alpha$,

ii) for $\xi^{\uparrow}$, we have that $\beta = \alpha\rho + 1$ and $\delta = \alpha(1-\rho)$,

iii) for $\xi^{\downarrow}$, we have that $\beta = \alpha\rho$ and $\delta = \alpha(1-\rho) + 1$,

where ρ is the negativity parameter defined by $\rho = \mathbf{P}(X_1 < 0)$.

Let us apply the results if section 4 to these examples $\xi^*, \xi^{\uparrow}$ and $\xi^{\downarrow}$.We start with a stable process $(X, \mathbf{P}_x), x > 0$, of index α and we obtain,

$$X \xrightarrow{kill} X^* \xrightarrow{LT} X^L \xrightarrow{norm} X_h^L \xrightarrow{d} X \qquad \text{as} \quad h \to 0$$
$$X \xrightarrow{kill} X^* \xrightarrow{DT} X^C \xrightarrow{LT} X^L \xrightarrow{norm} X_h^L \xrightarrow{d} X \quad \text{as} \quad h \to 0$$

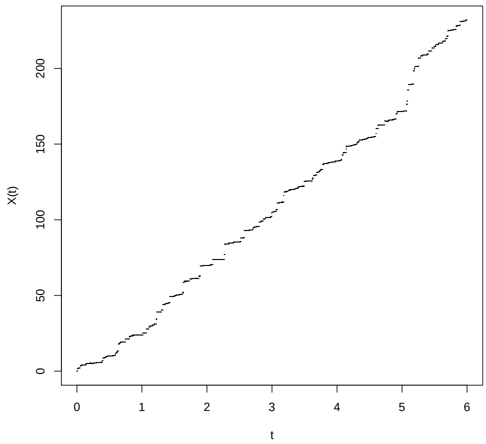

FIGURE 4. $\alpha = 0.5$, $f = 1$, $\sigma(1) = 1$, $\sigma(-1) = 0$.

where $kill$, LT, DT and $norm$ means killing , the Lamperti representation of pssMp, Doob-transform or conditioning, and normalization of a given process, respectively. Moreover X^C is the conditioned process (to be positive or to hit 0 continuously), X^L stands for any of the Lamperti stable processes $\xi^{\uparrow}, \xi^{\downarrow}$ and ξ^*, and X_h^L is the normalization of each of them given in Proposition 4.1. In the same spirit we could also write, using Theorem 4.1,

$$X \xrightarrow{kill} X^* \xrightarrow{LT} X^L \xrightarrow{norm} X_h^L \xrightarrow{d} W \qquad \text{as} \quad h \to \infty,$$
$$X \xrightarrow{kill} X^* \xrightarrow{DT} X^C \xrightarrow{LT} X^L \xrightarrow{norm} X_h^L \xrightarrow{d} W \quad \text{as} \quad h \to \infty,$$

where W is a centered brownian motion.

7.2.- In [29, 30, 31], the author deals with the class of Lévy processes with no positive jumps which turn out to be Lamperti stable processes with characteristics $\delta = 1 - \vartheta$ and $c_+ = 0$.

7.3.- In [4], there are two examples related to the factorization

$$\mathbf{e} \stackrel{law}{=} \mathbf{e}^{\alpha} \tau_{\alpha}^{-\alpha}.$$

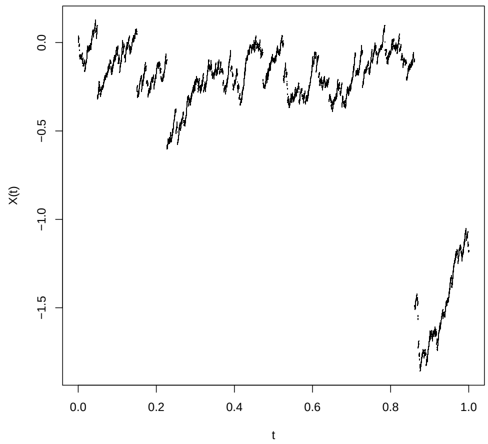

FIGURE 5. $\alpha = 1.5$, $f = 1$, $\sigma(1) = 0$, $\sigma(-1) = 1$.

where $\mathbf{e}$ is an exponential variable independent of the α-stable variable τ_α. The first of which is related with the exponential functional of a killed subordinator Z^1 whose Laplace exponent is given by

$$\phi_1(\lambda) = \frac{\Gamma(\alpha\lambda + 1)}{\Gamma(\alpha(\lambda - 1) + 1)}.$$

It is easy to see that Z^1 is a Lamperti stable subordinator with characteristics $(\alpha, \alpha, \sigma, \theta)$, $\sigma(\{1\}) = \alpha/\Gamma(1-\alpha)$, with no drift and killing rate $1/\Gamma(1-\alpha)$. The Laplace exponent of the second subordinator, here denoted by Z^2, is given

$$\phi_2(\lambda) = \lambda\frac{\Gamma(\alpha(\lambda - 1) + 1)}{\Gamma(\alpha\lambda + 1)},$$

and can be expressed in terms of the Laplace exponent Φ_L of a Lamperti stable subordinator X^L with characteristics $(1-\alpha, 1, \sigma, \theta)$, $\sigma(\{1\}) = \alpha/\Gamma(1-\alpha)$ and with no drift. The relation between them is $\phi_2(\lambda) = \alpha\Phi_{L,2}(\alpha\lambda)$.

7.4.- There is another example in [4] which is related to the factorization

$$\mathbf{e} \stackrel{law}{=} \gamma_s^\alpha J_s^{(\gamma)},$$

where $s \geqslant \alpha$, γ_s is a Gamma r.v. with parameter s and $J_s^{(\gamma)}$ denotes a certain r.v which is independent of γ_s. In this case, the killed subordinator related to the exponential functional which has the same moments as the γ_s, can be expressed as the sum of two independent Lamperti stable procesess.

7.5.- In the paper [36] in section 5.3, the authors found the Lévy measure of the inverse of the local time at 0 of an Ornstein Uhlenbeck process driven by a standard Brownian motion and parameter $\gamma > 0$. This measure is given by

$$\nu(t) = \frac{\gamma^{3/2} e^{\gamma t/2}}{\sqrt{2\pi}(\sinh(\gamma t))^{3/2}} = \frac{(2\gamma)^{3/2} e^{2\gamma t}}{\sqrt{2\pi}(e^{2\gamma t} - 1)^{3/2}}.$$

Its corresponding Laplace exponent is computed in [36] and it turns out to be a Lamperti stable distribution with characteristics $(1/2, 1, \sqrt{\gamma/\pi})$.

This computation as well as the three former examples can be carried out by recognizing that behind those measures there is a related Lamperti stable distribution and applying our Theorem 3.1 to calculate the corresponding Laplace exponent.

7.6.- Kyprianou and Rivero [24] constructed Lévy processes with no positive jumps around a given possibly killed subordinator which plays the role of the descending ladder height process. The Example 2 in [24] is related to the Lamperti stable subordinator X^L with characteristics $(\alpha, \beta, \sigma, \theta)$ with zero drift and killing rate given by $K = c_+\Gamma(-\alpha)\Gamma(1-\beta+\alpha)/\Gamma(1-\beta)$.
They prove that there is a subordinator, here denoted by Y, with no drift and no killing rate. The Laplace exponent of Y satisfies

$$\phi_Y(\lambda) = \frac{\lambda}{\phi_L(\lambda)}, \qquad \text{for} \quad \lambda \geqslant 0,$$

where ϕ_L is the Laplace exponent of X^L. Thus, its parent process Y^P has Laplace exponent

$$\psi_{Y^P}(\lambda) = \frac{\lambda^2\Gamma(1-\beta+\lambda)}{\Gamma(1-\beta+\lambda+\alpha)}.$$

According to Kyprianou and Rivero and the following identity

$$\int_x^\infty \frac{e^{\beta x}}{(e^x-1)^{\alpha+1}}\mathrm{d}x = e^{-x(\alpha+1-\beta)}\sum_{n\geqslant 0}\frac{(\alpha+1)_n(\alpha-\beta)_n}{(n!\alpha+1-\beta)_n}e^{-n},$$

its associated scale function is given by

$$W_{Y^P}(x) = -Kx + c_+\sum_{n\geqslant 0}\frac{(\alpha+1)_n(\alpha-\beta)_n}{n!(\alpha+2-\beta)_n}\big(1-e^{-(\alpha+2-\beta+n)x}\big), \quad x\geqslant 0.$$

Now $Y^{*,P}$, the parent process of the Lamperti subordinator X^L with killing rate K, is a spectrally negative Levy process which drifts to ∞ and whose Laplace exponent is given

$$\psi_{Y^{P,*}}(\lambda) = \frac{c_+\Gamma(-\alpha)\lambda\Gamma(\lambda+1-\beta+\alpha)}{\Gamma(\lambda+1-\beta)}.$$

It is important to note that the processes Y^P and $Y^{P,*}$ are the sum of two independent Lamperti stable processes and that they have been recently used for the risk neutral stock price model by Eberlein and Madan [13].

7.7.- In the papers [9] , [23], [31] the main processes in study are Lamperti stable processes. All these papers share the property that many useful explicit calculations are be carried out.

Acknowledgements.. This research was supported by EPSRC grant EP/D045-460/1, CONACYT grant (200419), and the project PAPIITT-IN120605. We thank Andreas Kyprianou, Victor Rivero, and Gerónimo Uribe for many fruitful discussions. We also thank the anonymous referees for useful observations. All of them helped to improve the paper.

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Received on 10.03.2008;
last revised version on xx.xx.xxxx