

# The extended hypergeometric class of Lévy processes

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With a view to computing fluctuation identities related to stable processes, we review and extend the class of hypergeometric Lévy processes explored in Kuznetsov and Pardo [17]. We give the Wiener–Hopf factorisation of a process in the extended class, and characterise its exponential functional. Finally, we give three concrete examples arising from transformations of stable processes.

## 1 Introduction

The simple definition of a Lévy process—a stochastic process with stationary independent increments—has been sufficient to fuel a vast field of study for many decades, and Lévy processes have been employed in many successful applied models. However, until recently, there were relatively few classes of processes for which many functionals could be computed explicitly. In recent years, the field has seen a proliferation of examples which have proved to be more analytically tractable; in particular, we single out spectrally negative Lévy processes [21], Lamperti-stable processes [6, 9],  $\beta$ - and  $\theta$ -processes [15, 16], and finally the inspiration for this work, hypergeometric Lévy processes [17, 18, 26]. It is also worth mentioning that the close relationship which appears to hold between hypergeometric Lévy processes and stable processes has also allowed the computation of several identities for the latter [26, 17, 25].

In this work, we review the hypergeometric class of Lévy processes introduced by Kuznetsov and Pardo [17], and introduce a new class of extended hypergeometric processes which have many similar properties. In particular, for an extended hypergeometric process  $\xi$  we compute the Wiener–Hopf factorisation and find that its ladder height

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processes are related to Lamperti-stable subordinators; and we are able to characterise explicitly the distribution of the exponential functional of  $\xi/\delta$  for any  $\delta > 0$ . We also give three examples of processes connected via the Lamperti representation to  $\alpha$ -stable processes, which fall into the hypergeometric class when  $\alpha \leq 1$ , and into the extended hypergeometric class when  $\alpha > 1$ , and give some new identities for the stable process when  $\alpha > 1$ .

We will first discuss the results of Kuznetsov and Pardo [17]. A Lévy process  $\xi$  is a member of the *hypergeometric class* of Lévy processes<sup>1</sup> if it has Laplace exponent

$$\psi(z) = -\frac{\Gamma(1-\beta+\gamma-z)}{\Gamma(1-\beta-z)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(\hat{\beta}+z)},$$

in the sense that  $\mathbb{E}[e^{z\xi_1}] = e^{\psi(z)}$ , and the parameters  $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$  lie in the set

$$\mathcal{A}_{\text{HG}} = \{\beta \leq 1, \gamma \in (0, 1), \hat{\beta} \geq 0, \hat{\gamma} \in (0, 1)\}.$$

In [17], it is shown that for any choice of parameters in  $\mathcal{A}_{\text{HG}}$ , there is a Lévy process with Laplace exponent  $\psi$ , and they find its Wiener–Hopf factorisation.

The (*spatial*) *Wiener–Hopf factorisation* of a Lévy process  $\xi$  with Laplace exponent  $\psi$  consists of the equation

$$\psi(z) = -\kappa(-z)\hat{\kappa}(z),$$

where  $\kappa$  and  $\hat{\kappa}$  are the Laplace exponents of subordinators  $H$  and  $\hat{H}$ , respectively, this time in the sense that  $\mathbb{E}[e^{-\lambda H_1}] = e^{-\kappa(\lambda)}$ . The subordinators  $H$  and  $\hat{H}$  are known as the ascending and descending ladder heights, and are related via a time-change to the running maximum and running minimum of the process  $\xi$ . For more details, we refer the reader to [22, Chapter 6]. The insight into the structure of  $\xi$  given by the Wiener–Hopf factorisation allows one to simplify first passage problems for  $\xi$ ; see [22, Chapter 7] for a collection of results.

The paper [17] computes that

$$\kappa(z) = \frac{\Gamma(1-\beta+\gamma+z)}{\Gamma(1-\beta+z)}, \quad \hat{\kappa}(z) = \frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(\hat{\beta}+z)},$$

thus demonstrating that the ascending and descending ladder height processes are Lamperti-stable subordinators (see [6]).

Kuznetsov and Pardo also consider the exponential functional of a hypergeometric Lévy process  $\xi$ . For each  $\delta > 0$ , the random variable

$$I(\xi/\delta) = \int_0^\infty e^{-\xi t/\delta} dt$$

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<sup>1</sup> This is a somewhat ambiguous designation. A class of ‘hypergeometric Lévy processes’ was first defined in [26, §6.5], and expanded in [18, §3.2]. The class introduced by [17] is larger than that of [26] but smaller than that of [18]. Since this paper is focused on extending the class of [17], we reserve the term for the processes defined in that work.

is a.s. finite provided that  $\xi$  drifts to  $+\infty$ . This random variable is known as the *exponential functional* of the Lévy process  $\xi$ , and it has been studied extensively in general; the paper of Bertoin and Yor [3] gives a survey of the literature, and mentions, among other aspects, applications to diffusions in random environments, mathematical finance and fragmentation theory. In the context of self-similar Markov processes, the exponential functional appears in the entrance law of a pssMp started at zero (see, for example, Bertoin and Yor [2]), and Pardo [31] relates the exponential functional of a Lévy process to envelopes of its associated pssMp; furthermore, it is related to the hitting time of points for pssMps, and we shall make use of it in this capacity in our example of subsection 4.2.

For the purpose of characterising the distribution of  $I(\xi/\delta)$ , its Mellin transform

$$\mathcal{M}(s) = \mathbb{E}[I(\xi/\delta)^{s-1}],$$

is useful. The following expression for the Mellin transform holds in the case of a hypergeometric Lévy process with  $\hat{\beta} > 0$ :

$$\mathcal{M}(s) = C\Gamma(s) \frac{G((1-\beta)\delta + s; \delta)}{G((1-\beta+\gamma)\delta + s; \delta)} \frac{G((\hat{\beta} + \hat{\gamma})\delta + 1 - s; \delta)}{G(\hat{\beta}\delta + 1 - s; \delta)},$$

for  $\operatorname{Re} s \in (0, 1 + \hat{\beta}\delta)$ , where  $C$  is a normalising constant such that  $\mathcal{M}(1) = 1$ , and  $G$  is the double gamma function; see [17] for details.

We now give a brief outline of the main body of the paper. In section 2, we demonstrate that the parameter set  $\mathcal{A}_{\text{HG}}$  may be extended by changing the domains of the two parameters  $\beta$  and  $\hat{\beta}$ , and find the Wiener–Hopf factorisation of a process  $\xi$  in this new class, identifying explicitly the ladder height processes. In section 3, we find an expression for the Mellin transform  $\mathcal{M}$  in this new case, making use of an auxiliary hypergeometric Lévy process. In section 4, we cover three examples where the extended hypergeometric class is of use, on the way extending the result of [7] on the Wiener–Hopf factorisation of the Lamperti representation associated with the radial part of a stable process.

## 2 The extended hypergeometric class

We begin by defining the set of admissible parameters

$$\mathcal{A}_{\text{EHG}} = \{\beta \in [1, 2], \gamma, \hat{\gamma} \in (0, 1), \hat{\beta} \in [-1, 0]; 1 - \beta + \hat{\beta} + \gamma \geq 0, 1 - \beta + \hat{\beta} + \hat{\gamma} \geq 0\}.$$

We are interested in proving the existence of, and investigating the properties of, a Lévy process  $\xi$  whose Laplace exponent is given by the meromorphic function

$$\psi(z) = -\frac{\Gamma(1 - \beta + \gamma - z)}{\Gamma(1 - \beta - z)} \frac{\Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(\hat{\beta} + z)}, \quad z \in \mathbb{C},$$

when  $(\beta, \gamma, \hat{\beta}, \hat{\gamma}) \in \mathcal{A}_{\text{EHG}}$ .

To allow for more concise expressions below, we also define

$$\eta = 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}.$$

We now give our main result on the existence and properties of  $\xi$ .

**Proposition 1.** *There exists a Lévy process  $\xi$  such that  $\mathbb{E}[e^{z\xi_1}] = e^{\psi(z)}$ . Its Wiener-Hopf factorisation may be expressed as*

$$\psi(z) = -(-\hat{\beta} - z) \frac{\Gamma(1 - \beta + \gamma - z)}{\Gamma(2 - \beta - z)} \times (\beta - 1 + z) \frac{\Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(1 + \hat{\beta} + z)}.$$

*Its Lévy measure possesses the density*

$$\pi(x) = \begin{cases} -\frac{\Gamma(\eta)}{\Gamma(\eta - \hat{\gamma})\Gamma(-\gamma)} e^{-(1-\beta+\gamma)x} {}_2F_1(1 + \gamma, \eta; \eta - \hat{\gamma}; e^{-x}), & x > 0, \\ -\frac{\Gamma(\eta)}{\Gamma(\eta - \gamma)\Gamma(-\hat{\gamma})} e^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1(1 + \hat{\gamma}, \eta; \eta - \gamma; e^x), & x < 0, \end{cases} \quad (1)$$

where  ${}_2F_1$  is the Gauss hypergeometric function.

If  $\beta \in (1, 2)$  and  $\hat{\beta} \in (-1, 0)$ , the process  $\xi$  is killed at rate

$$q = \frac{\Gamma(1 - \beta + \gamma)}{\Gamma(1 - \beta)} \frac{\Gamma(\hat{\beta} + \hat{\gamma})}{\Gamma(\hat{\beta})}.$$

Otherwise, the process has infinite lifetime and:

- (i)  $\xi$  drifts to  $+\infty$  if  $\beta > 1$ ,  $\hat{\beta} = 0$ .
- (ii)  $\xi$  drifts to  $-\infty$  if  $\beta = 1$ ,  $\hat{\beta} < 0$ .
- (iii)  $\xi$  oscillates if  $\beta = 1$ ,  $\hat{\beta} = 0$ . In this case,  $\xi$  is a hypergeometric Lévy process.

Furthermore, the process  $\xi$  has no Gaussian component, and is of bounded variation with zero drift when  $\gamma + \hat{\gamma} < 1$  and of unbounded variation when  $\gamma + \hat{\gamma} \geq 1$ .

*Proof.* We remark that there is nothing to do in case (iii) since such processes are analysed in [17]; however, the proof we give below also carries through in this case.

We will first identify the proposed ascending and descending ladder processes. Once we have shown that  $\psi$  really is the Laplace exponent of a Lévy process, this will be the proof of the Wiener-Hopf factorisation.

Before we begin, we must review the definitions of special subordinators and the  $\mathcal{T}$ -transformations of subordinators. Suppose that  $v$  is the Laplace exponent of a subordinator  $H$ , in the sense that  $\mathbb{E}[e^{-zH_1}] = e^{-v(z)}$ .  $H$  is said to be a *special subordinator*, and  $v$  a *special Bernstein function*, if the function

$$v^*(z) = z/v(z), \quad z \geq 0,$$

is also the Laplace exponent of a subordinator. The function  $v^*$  is said to be *conjugate* to  $v$ . Special Bernstein functions play an important role in potential theory; see, for example, [33] for more details.

Again taking  $v$  to be the Laplace exponent of a subordinator, not necessarily special, we define, for  $c \geq 0$ , the transformation

$$\mathcal{T}_c v(z) = \frac{z}{z+c} v(z+c), \quad z \geq 0.$$

It is then known (see [13, 23]) that  $\mathcal{T}_c v$  is the Laplace exponent of a subordinator. Furthermore, if  $v$  is in fact a special Bernstein function, then  $\mathcal{T}_c v$  is also a special Bernstein function.

We are now in a position to identify the ladder height processes in the Wiener–Hopf factorisation of  $\xi$ . Let the (proposed) ascending factor be given, for  $z \geq 0$ , by

$$\kappa(z) = (-\hat{\beta} + z) \frac{\Gamma(1 - \beta + \gamma + z)}{\Gamma(2 - \beta + z)}.$$

Then some simple algebraic manipulation shows that

$$\kappa(z) = (\mathcal{T}_{-\hat{\beta}} v)^*(z),$$

with

$$v(z) = \frac{\Gamma(2 - \beta + \hat{\beta} + z)}{\Gamma(1 - \beta + \hat{\beta} + \gamma + z)},$$

provided that  $v$  is a special Bernstein function. This follows immediately from Example 2 of Kyprianou and Rivero [24], under the constraint  $1 - \beta + \hat{\beta} + \gamma \geq 0$  which is included in the parameter set  $\mathcal{A}_{\text{EHG}}$ . Furthermore, we note that  $v$  is in fact the Laplace exponent of a Lamperti-stable subordinator (see [6]), although we will not use this fact.

Proceeding similarly for the descending factor, we obtain

$$\hat{\kappa}(z) = (\beta - 1 + z) \frac{\Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(1 + \hat{\beta} + z)} = (\mathcal{T}_{\beta-1} \hat{v})^*(z), \quad z \geq 0,$$

where

$$\hat{v}(z) = \frac{\Gamma(2 - \beta + \hat{\beta} + z)}{\Gamma(1 - \beta + \hat{\beta} + \hat{\gamma} + z)}, \quad z \geq 0,$$

and again the function  $\hat{v}$  is a special Bernstein function provided that  $1 - \beta + \hat{\beta} + \hat{\gamma} \geq 0$ . As before,  $\hat{v}$  is the Laplace exponent of a Lamperti-stable subordinator.

We have now shown that both  $\kappa$  and  $\hat{\kappa}$  are Laplace exponents of subordinators; we wish to show that the function

$$\psi(z) = -\kappa(-z)\hat{\kappa}(z)$$

is the Laplace exponent of a Lévy process. For this purpose we apply the theory of *philanthropy* developed by Vigon [34, chapter 7]. This states, in part, that it is sufficient

for both of the subordinators corresponding to  $\kappa$  and  $\hat{\kappa}$  to be ‘philanthropists’, which means that their Lévy measures possess decreasing densities.

We recall our discussion of  $\mathcal{T}$ -transforms and special Bernstein functions. We have already stated that when  $v$  is a special Bernstein function, then so is  $\mathcal{T}_c v$ ; furthermore, one may show that its conjugate satisfies

$$(\mathcal{T}_c v)^*(z) = \mathcal{E}_c v^*(z) + v^*(c), \quad z \geq 0,$$

where  $\mathcal{E}_c$  is the *Esscher transform*, given by

$$\mathcal{E}_c v^*(z) = v^*(z + c) - v^*(c), \quad z \geq 0.$$

The Esscher transform of the Laplace exponent of any subordinator is again the Laplace exponent of a subordinator; and if the subordinator corresponding to  $v^*$  possesses a Lévy density  $\pi_{v^*}$ , then the Lévy density of  $\mathcal{E}_c v^*$  is given by  $x \mapsto e^{-cx} \pi_{v^*}(x)$ , for  $x > 0$ .

Returning to our Wiener-Hopf factors, we have

$$\kappa(z) = (\mathcal{T}_{-\hat{\beta}} v)^*(z) = \mathcal{E}_{-\hat{\beta}} v^*(z) + v^*(-\hat{\beta}), \quad z \geq 0,$$

where  $v^*$  is the Laplace exponent conjugate to  $v$ . Now,  $v$  is precisely the type of special Bernstein function considered in [24, Example 2]. In that work, the authors even establish that the subordinator corresponding to  $v^*$  has a decreasing Lévy density  $\pi_{v^*}$ . Finally, the Lévy density of the subordinator corresponding to  $\kappa$  is  $x \mapsto e^{\hat{\beta}x} \pi_{v^*}(x)$ , and this is then clearly also decreasing.

Hence, we have shown that the subordinator whose Laplace exponent is  $\kappa$  is a philanthropist. By a very similar argument, the subordinator corresponding to  $\hat{\kappa}$  is also a philanthropist. As we have stated, the theory developed by Vigon now shows that the function  $\psi$  really is the Laplace exponent of a Lévy process  $\xi$ , with the Wiener-Hopf factorisation claimed.

We now proceed to calculate the Lévy measure of  $\xi$ . A fairly simple way to do this is to make use of the theory of ‘meromorphic Lévy processes’, as developed in Kuznetsov et al. [19]. We first show that  $\xi$  is in the meromorphic class. Initially suppose that

$$1 - \beta + \hat{\beta} + \gamma > 0, \quad 1 - \beta + \hat{\beta} + \hat{\gamma} > 0; \quad (2)$$

we will relax this assumption later. Looking at the expression  $\psi$ , we see that it has zeroes  $(\zeta_n)_{n \geq 1}$ ,  $(-\hat{\zeta}_n)_{n \geq 1}$  and (simple) poles  $(\rho_n)_{n \geq 1}$ ,  $(-\hat{\rho}_n)_{n \geq 1}$  given as follows:

$$\begin{aligned} \zeta_1 &= -\hat{\beta}, & \zeta_n &= n - \beta, & n &\geq 2, \\ \hat{\zeta}_1 &= \beta - 1, & \rho_n &= n - \beta + \gamma, & n &\geq 1, \\ & & \hat{\zeta}_n &= \hat{\beta} + n - 1, & n &\geq 2, \\ & & \hat{\rho}_n &= \hat{\beta} + \hat{\gamma} + n - 1, & n &\geq 1, \end{aligned}$$

and that they satisfy the interlacing condition

$$\cdots < -\hat{\rho}_2 < -\hat{\zeta}_2 < -\hat{\rho}_1 < -\hat{\zeta}_1 < 0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \cdots.$$

To show that  $\xi$  belongs to the meromorphic class, one applies [19, Theorem 1(v)] when  $\xi$  is killed, and [19, Corollary 2] in the unkilld case. The proof is a routine calculation using the Weierstrass representation [14, 8.322] to expand  $\kappa$  and  $\hat{\kappa}$  as infinite products, and we omit it for the sake of brevity.

We now calculate the Lévy density. For a process in the meromorphic class, it is known that the Lévy measure has a density of the form

$$\pi(x) = \mathbb{1}_{\{x>0\}} \sum_{n \geq 1} a_n \rho_n e^{-\rho_n x} + \mathbb{1}_{\{x<0\}} \sum_{n \geq 1} \hat{a}_n \hat{\rho}_n e^{\hat{\rho}_n x}, \quad (3)$$

for some coefficients  $(a_n)_{n \geq 1}$ ,  $(\hat{a}_n)_{n \geq 1}$ , where the  $\rho_n$  and  $\hat{\rho}_n$  are as above. Furthermore, from [19, equation (8)], we see that

$$a_n \rho_n = -\text{Res}(\psi(z) : z = \rho_n),$$

and correspondingly for  $\hat{a}_n \hat{\rho}_n$ . (This remark is made on p. 1111 of [19].) From here it is simple to compute

$$a_n \rho_n = -\frac{(-1)^{n-1}}{(n-1)!} \frac{1}{\Gamma(1-\gamma-n)} \frac{\Gamma(\eta+n-1)}{\Gamma(\eta-\hat{\gamma}+n-1)}, \quad n \geq 1,$$

and similarly for  $\hat{a}_n \hat{\rho}_n$ . The expression (1) follows by substituting in (3) and using the series definition of the hypergeometric function.

Thus far we have been working under the assumption that (2) holds. Suppose now that this fails and we have, say,  $1-\beta+\hat{\beta}+\hat{\gamma}=0$ . Then  $\zeta_1 = \rho_1$ , which is to say the first zero-pole pair to the right of the origin is removed. It is clear that  $\xi$  still falls into the meromorphic class, and indeed, our expression for  $\pi$  remains valid: although the initial pole  $\rho_1$  no longer exists, the corresponding coefficient  $a_1 \rho_1$  vanishes as well. Similarly, we may allow  $1-\beta+\hat{\beta}+\gamma=0$ , in which case the zero-pole pair to the left of the origin is removed; or we may allow both expressions to be zero, in which case both pairs are removed. The proof carries through in all cases.

The claim about the large time behaviour of  $\xi$  follows from the Wiener-Hopf factorisation:  $\kappa(0) = 0$  if and only if the range of  $\xi$  is a.s. unbounded above, and  $\hat{\kappa}(0) = 0$  if and only if the range of  $\xi$  is a.s. unbounded below; so we need only examine the values of  $\kappa(0)$ ,  $\hat{\kappa}(0)$  in each of the four parameter regimes.

Finally, we prove the claim about the variation and Gaussian component of  $\xi$ . This proof proceeds along the same lines as that in [17]. Firstly, using [14, formulas 9.131.1 and 9.122.2] one sees that

$$\pi(x) = O(|x|^{-(1+\gamma+\hat{\gamma})}), \quad x \rightarrow 0,$$

and together with the necessary and sufficient condition  $\int_{\mathbb{R}} (1 \wedge |x|) \pi(x) dx < \infty$  for bounded variation, this is sufficient to give the claim about the variation. To prove the claims about the drift coefficient and the Gaussian component, we observe using [14, formula 8.328.1] that

$$\psi(i\theta) = O(|\theta|^{\gamma+\hat{\gamma}}), \quad |\theta| \rightarrow \infty.$$

Then an application of [1, Proposition I.2] completes the proof.  $\square$

We propose to call this the *extended hypergeometric class* of Lévy processes.

*Remark 2.* If  $\xi$  is a process in the extended hypergeometric class, with parameters  $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ , then the dual process  $-\xi$  also lies in this class, and has parameters  $(1 - \hat{\beta}, \hat{\gamma}, 1 - \beta, \gamma)$ .

*Remark 3.* We remark here that one may instead extend the parameter range  $\mathcal{A}_{\text{HG}}$  by moving only  $\beta$ , or only  $\hat{\beta}$ . To be precise, both

$$\mathcal{A}_{\text{EHG}}^{\beta} = \{\beta \in [1, 2], \gamma, \hat{\gamma} \in (0, 1), \hat{\beta} \geq 0; 1 - \beta + \hat{\beta} + \gamma \leq 0, 1 - \beta + \hat{\beta} + \hat{\gamma} \geq 0\}$$

and

$$\mathcal{A}_{\text{EHG}}^{\hat{\beta}} = \{\beta \leq 1, \gamma, \hat{\gamma} \in (0, 1), \hat{\beta} \in [-1, 0]; 1 - \beta + \hat{\beta} + \gamma \geq 0, 1 - \beta + \hat{\beta} + \hat{\gamma} \leq 0\}$$

are suitable parameter regimes, and one may develop a similar theory for such processes; for instance, for parameters in  $\mathcal{A}_{\text{EHG}}^{\beta}$ , one has the Wiener–Hopf factors

$$\kappa(z) = \frac{\Gamma(2 - \beta + \gamma + z)}{\Gamma(2 - \beta + z)}, \quad \hat{\kappa}(z) = \frac{\beta - 1 + z}{\beta - 1 - \gamma + z} \frac{\Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(\hat{\beta} + z)}.$$

However, we are not aware of any examples of processes in these classes.

### 3 The exponential functional

Suppose that  $\xi$  is a Lévy process in the extended hypergeometric class with  $\beta > 1$ , which is to say either  $\xi$  is killed or it drifts to  $+\infty$ .

We are then interested in the *exponential functional* of the process, given by

$$I(\xi/\delta) = \int_0^\infty e^{-\xi_t/\delta} dt,$$

for  $\delta > 0$ . This is an a.s. finite random variable under the conditions we have just outlined.

It will emerge that the best way to characterise the distribution of  $I(\xi/\delta)$  is via its *Mellin transform*,

$$\mathcal{M}(s) = \mathbb{E}[I(\xi/\delta)^{s-1}],$$

whose domain of definition will be a vertical strip in the complex plane to be determined. As we discussed in the introduction, when  $\xi$  is a hypergeometric Lévy process, an expression for  $\mathcal{M}$  can be given in terms of gamma functions and double gamma functions. Our goal in this section is the following result, which does the same for the extended hypergeometric class.

**Proposition 4.** *Suppose that  $\xi$  is a Lévy process in the extended hypergeometric class with  $\beta > 1$ . Define  $\theta = \delta(\beta - 1)$ .*



Then, the Mellin transform  $\mathcal{M}$  of  $I(\xi/\delta)$  is given by

$$\mathcal{M}(s) = c\widetilde{\mathcal{M}}(s) \frac{\Gamma(\delta(1 - \beta + \gamma) + s)}{\Gamma(-\delta\hat{\beta} + s)} \frac{\Gamma(\delta(\beta - 1) + 1 - s)}{\Gamma(\delta(\hat{\beta} + \hat{\gamma}) + 1 - s)}, \quad \operatorname{Re} s \in (0, 1 + \theta), \quad (4)$$

where  $\widetilde{\mathcal{M}}$  is the Mellin transform of  $I(\zeta/\delta)$ , and  $\zeta$  is an auxiliary Lévy process in the hypergeometric class, with parameters  $(\beta - 1, \gamma, \hat{\beta} + 1, \hat{\gamma})$ . The constant  $c$  is such that  $\mathcal{M}(1) = 1$ .

*Proof.* The process  $\xi/\delta$  has Laplace exponent  $\psi_\delta$  given by  $\psi_\delta(z) = \psi(z/\delta)$ . The relation-ship with  $\zeta$  arises from the following calculation:

$$\begin{aligned} \psi_\delta(z) &= \frac{-\hat{\beta} - z/\delta}{1 - \beta + \gamma - z/\delta} \frac{\beta - 1 + z/\delta}{\hat{\beta} + \hat{\gamma} + z/\delta} \frac{\Gamma(2 - \beta + \gamma - z/\delta)}{\Gamma(2 - \beta - z/\delta)} \frac{\Gamma(1 + \hat{\beta} + \hat{\gamma} + z/\delta)}{\Gamma(1 + \hat{\beta} + z/\delta)} \\ &= \frac{-\hat{\beta} - z/\delta}{1 - \beta + \gamma - z/\delta} \frac{\beta - 1 + z/\delta}{\hat{\beta} + \hat{\gamma} + z/\delta} \tilde{\psi}_\delta(z), \end{aligned} \quad (5)$$

where  $\tilde{\psi}_\delta$  is the Laplace exponent of a Lévy process  $\zeta/\delta$ , with  $\zeta$  as in the statement of the theorem.

Denote by  $f(s)$  the right-hand side of (4). The proof now proceeds via the ‘verification result’ [17, Proposition 2].

Recall that a Lévy process with Laplace exponent  $\phi$  is said to satisfy the *Cramér condition with Cramér number*  $\theta$  if there exists  $z_0 < 0$  and  $\theta \in (0, -z_0)$  such that  $\phi(z)$  is defined for all  $z \in (z_0, 0)$  and  $\phi(-\theta) = 0$ .

Inspecting the Laplace exponent  $\psi_\delta$  reveals that  $\xi/\delta$  satisfies the Cramér condition with Cramér number  $\theta = \delta(\beta - 1)$ .

Furthermore,  $\zeta/\delta$  satisfies the Cramér condition with Cramér number  $\tilde{\theta} = \delta(\hat{\beta} + 1)$ . It follows that  $\widetilde{\mathcal{M}}(s)$  is analytic and zero-free in the strip  $\operatorname{Re} s \in (0, 1 + \tilde{\theta})$ . The constraints in the parameter set  $\mathcal{A}_{\text{EHG}}$  ensure that  $\tilde{\theta} \geq \theta$ ; this, together with inspecting the right-hand side of (4) and comparing again with the conditions in  $\mathcal{A}_{\text{EHG}}$ , demonstrates that  $\mathcal{M}(s)$  is analytic and zero-free in the strip  $\operatorname{Re} s \in (0, 1 + \theta)$ .

We must then check the functional equation  $f(s + 1) = -sf(s)/\psi_\delta(-s)$ , for  $s \in (0, \theta)$ . Apply (5) to write

$$\begin{aligned} -\frac{s}{\psi_\delta(-s)} &= -\frac{s}{\tilde{\psi}_\delta(-s)} \frac{1 - \beta + \gamma + s/\delta}{-\hat{\beta} + s/\delta} \frac{\hat{\beta} + \hat{\gamma} - s/\delta}{\beta - 1 - s/\delta} \\ &= \frac{\widetilde{\mathcal{M}}(s + 1)}{\widetilde{\mathcal{M}}(s)} \frac{\delta(1 - \beta + \gamma) + s}{-\delta\hat{\beta} + s} \frac{\delta(\hat{\beta} + \hat{\gamma}) - s}{\delta(\beta - 1) - s} \\ &= \frac{\widetilde{\mathcal{M}}(s + 1)}{\widetilde{\mathcal{M}}(s)} \frac{\Gamma(-\delta\hat{\beta} + s)}{\Gamma(-\delta\hat{\beta} + s + 1)} \frac{\Gamma(\delta(1 - \beta + \gamma) + s + 1)}{\Gamma(\delta(1 - \beta + \gamma) + s)} \\ &\quad \times \frac{\Gamma(\delta(\hat{\beta} + \hat{\gamma}) + 1 - s)}{\Gamma(\delta(\hat{\beta} + \hat{\gamma}) - s)} \frac{\Gamma(\delta(\beta - 1) - s)}{\Gamma(\delta(\beta - 1) + 1 - s)}, \end{aligned}$$

making use of the same functional equation for the Mellin transform  $\widetilde{\mathcal{M}}$ . It is then clear that the right-hand side is equal to  $f(s+1)/f(s)$ .

Finally, it remains to check that  $|f(s)|^{-1} = o(\exp(2\pi|\operatorname{Im}(s)|))$ , as  $|\operatorname{Im} s| \rightarrow \infty$ , uniformly in  $\operatorname{Re} s \in (0, 1 + \theta)$ . The following asymptotic relation may be derived from Stirling's asymptotic formula for the gamma function:

$$\log \Gamma(z) = z \log z - z + O(\log z), \quad (6)$$

and since Stirling's asymptotic formula is uniform in  $|\arg(z)| < \pi - \omega$  for any choice of  $\omega > 0$ , it follows that (6) holds uniformly in the strip  $\operatorname{Re} s \in (0, 1 + \theta)$ ; see [29, Chapter 8, §4]. We thus obtain

$$\log \left| \frac{\Gamma(\delta(1 - \beta + \gamma) + s)}{\Gamma(-\delta\hat{\beta} + s)} \frac{\Gamma(\delta(\beta - 1) + 1 - s)}{\Gamma(\delta(\hat{\beta} + \hat{\gamma}) + 1 - s)} \right|^{-1} = O(\log s) = o(\operatorname{Im} s),$$

and comparing this with the proof of [17, Theorem 2], where the asymptotic behaviour of  $\widetilde{\mathcal{M}}(s)$  is given, we see that this is sufficient for our purposes.

Hence,  $\mathcal{M}(s) = f(s)$  when  $\operatorname{Re} s \in (0, 1 + \theta)$ .  $\square$

This Mellin transform may be inverted to give an expression for the density of  $I(\xi/\delta)$  in terms of series whose terms are defined iteratively, but we do not pursue this here. For details of this approach, see [17, §4].

## 4 Three examples

It is well-known that hypergeometric Lévy processes appear as the Lamperti transforms of stable processes killed passing below zero, conditioned to stay positive and conditioned to hit zero continuously; see [17, Theorem 1]. In this section we briefly present three additional examples in which the extended hypergeometric class comes into play. The examples may all be obtained in the same way: one begins with a stable process, modifies its path in some way to obtain a positive, self-similar Markov process, and then applies the Lamperti transform to obtain a new Lévy process. We therefore begin by presenting a short description of these concepts.

We work with the (strictly) stable process  $X$  with scaling parameter  $\alpha$  and positivity parameter  $\rho$ , which is defined as follows. For  $(\alpha, \rho)$  in the set

$$\begin{aligned} \mathcal{A}_{\text{st}} = & \{(\alpha, \rho) : \alpha \in (0, 1), \rho \in (0, 1)\} \cup \{(\alpha, \rho) = (1, 1/2)\} \\ & \cup \{(\alpha, \rho) : \alpha \in (1, 2), \rho \in (1 - 1/\alpha, 1/\alpha)\}, \end{aligned}$$

and let  $X$ , with probability laws  $(P_x)_{x \in \mathbb{R}}$ , be the Lévy process with characteristic exponent

$$\Psi(\theta) = \begin{cases} c|\theta|^\alpha(1 - i\beta \tan \frac{\pi\alpha}{2} \operatorname{sgn} \theta) & \alpha \in (0, 2) \setminus \{1\}, \\ c|\theta| & \alpha = 1, \end{cases} \quad \theta \in \mathbb{R},$$

where  $c = \cos(\pi\alpha(\rho - 1/2))$  and  $\beta = \tan(\pi\alpha(\rho - 1/2))/\tan(\pi\alpha/2)$ ; by this we mean that  $E_0[e^{i\theta X_1}] = e^{-\Psi(\theta)}$ . This Lévy process has absolutely continuous Lévy measure with density

$$c_+ x^{-(\alpha+1)} \mathbb{1}_{\{x>0\}} + c_- |x|^{-(\alpha+1)} \mathbb{1}_{\{x<0\}}, \quad x \in \mathbb{R},$$

where

$$c_+ = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)}, \quad c_- = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})}$$

and  $\hat{\rho} = 1 - \rho$ .

The parameter set  $\mathcal{A}_{\text{st}}$  and the characteristic exponent  $\Psi$  represent, up a multiplicative constant in  $\Psi$ , all (strictly) stable processes which jump in both directions, except for Brownian motion and the symmetric Cauchy processes with non-zero drift.

The choice of  $\alpha$  and  $\rho$  as parameters is explained as follows.  $X$  satisfies the  $\alpha$ -scaling property, that

$$\text{under } P_x, \text{ the law of } (cX_{tc^{-\alpha}})_{t \geq 0} \text{ is } P_{cx}, \quad (7)$$

for all  $x \in \mathbb{R}$ ,  $c > 0$ . The second parameter satisfies  $\rho = P_0(X_t > 0)$ .

A *positive self-similar Markov process (pssMp)* with *self-similarity index*  $\alpha > 0$  is a standard Markov process  $Y = (Y_t)_{t \geq 0}$  with filtration  $(\mathcal{G}_t)_{t \geq 0}$  and probability laws  $(P_x)_{x>0}$ , on  $[0, \infty)$ , which has 0 as an absorbing state and which satisfies the scaling property (7) (with  $Y$  in place of  $X$ ). Here, we mean “standard” in the sense of [4], which is to say,  $(\mathcal{G}_t)_{t \geq 0}$  is a complete, right-continuous filtration, and  $Y$  has càdlàg paths and is strong Markov and quasi-left-continuous.

In the seminal paper [27], Lamperti describes a one to one correspondence between pssMps and Lévy processes, which we now outline. It may be worth noting that we have presented a slightly different definition of pssMp from Lamperti; for the connection, see [35, §0].

Let  $S(t) = \int_0^t (Y_u)^{-\alpha} du$ . This process is continuous and strictly increasing until  $Y$  reaches zero. Let  $(T(s))_{s \geq 0}$  be its inverse, and define

$$\xi_s = \log Y_{T(s)} \quad s \geq 0.$$

Then  $\xi := (\xi_s)_{s \geq 0}$  is a Lévy process started at  $\log x$ , possibly killed at an independent exponential time; the law of the Lévy process and the rate of killing do not depend on the value of  $x$ . The real-valued process  $\xi$  with probability laws  $(\mathbb{P}_y)_{y \in \mathbb{R}}$  is called the *Lévy process associated to  $Y$* , or the *Lamperti transform of  $Y$* .

An equivalent definition of  $S$  and  $T$ , in terms of  $\xi$  instead of  $Y$ , is given by taking  $T(s) = \int_0^s \exp(\alpha \xi_u) du$  and  $S$  as its inverse. Then,

$$Y_t = \exp(\xi_{S(t)}) \quad (8)$$

for all  $t \geq 0$ , and this shows that the Lamperti transform is a bijection.

## 4.1 The path-censored stable process

Let  $X$  be the stable process defined in section 4. In [25], the present authors considered a ‘path-censored’ version of the stable process, formed by erasing the time spent in the negative half-line. To be precise, define

$$A_t = \int_0^t \mathbb{1}_{\{X_s > 0\}} ds, \quad t \geq 0,$$

and let  $\gamma(t) = \inf\{s \geq 0 : A_s > t\}$  be its right-continuous inverse. Also define

$$T_0 = \inf\{t \geq 0 : X_t = 0\},$$

which is finite or infinite a.s. accordingly as  $\alpha > 1$  or  $\alpha \leq 1$ . Then, the process

$$Y_t = X_{\gamma(t)} \mathbb{1}_{\{t < T_0\}}, \quad t \geq 0,$$

is a pssMp, called the *path-censored stable process*.

In Theorems 5.3 and 5.5 of [25], it is shown that the Laplace exponent  $\psi^Y$  of the Lamperti transform  $\xi^Y$  associated to  $Y$  is given by

$$\psi^Y(z) = \frac{\Gamma(\alpha\rho - z)}{\Gamma(-z)} \frac{\Gamma(1 - \alpha\rho + z)}{\Gamma(1 - \alpha + z)},$$

and there it is remarked that when  $\alpha \leq 1$ , this process is in the hypergeometric class with parameters

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = (1, \alpha\rho, 1 - \alpha, \alpha\hat{\rho}).$$

It is readily seen from our definition that, when  $\alpha > 1$ , the process  $\xi^Y$  is in the extended hypergeometric class, with the same set of parameters.

From the Lamperti transform we know that

$$I(-\alpha\xi^Y) = \inf\{u \geq 0 : Y_u = 0\} = \int_0^{T_0} \mathbb{1}_{\{X_t > 0\}} dt,$$

where the latter is the occupation time of  $(0, \infty)$  up to first hitting zero for the stable process. This motivates the following proposition, whose proof is a direct application of Proposition 4.

**Proposition 5.** *The Mellin transform of the random variable  $I(-\alpha\xi^Y)$  is given by*

$$\mathcal{M}(s) = c \frac{G(2/\alpha - 1 + s; 1/\alpha)}{G(2/\alpha - \rho + s; 1/\alpha)} \frac{G(1/\alpha + \rho + 1 - s; 1/\alpha)}{G(1/\alpha + 1 - s; 1/\alpha)} \frac{\Gamma(1/\alpha - \rho + s)}{\Gamma(\rho + 1 - s)} \Gamma(2 - 1/\alpha - s),$$

where  $c$  is a normalising constant such that  $\mathcal{M}(1) = 1$ .

*Remark 6.* When  $X$  is in the class  $\mathcal{C}_{k,l}$  introduced by Doney [12], which is to say

$$\rho + k = l/\alpha,$$

for  $k, l \in \mathbb{Z}$ , simpler equivalent expressions may be found via repeated application of certain identities of the double-gamma function; see, for example, [17, equations (19) and (20)].

For example, when  $k, l \geq 0$ , one has

$$\begin{aligned} \mathcal{M}(s) &= c(-1)^l (2\pi)^{l(1/\alpha-1)} (1/\alpha)^{l(1-2/\alpha)} \frac{\Gamma(\frac{1-l}{\alpha} + k + s)}{\Gamma(\frac{l}{\alpha} + 1 - k - s)} \frac{\Gamma(2 - 1/\alpha - s)}{\Gamma(2 - l - \alpha - \alpha s)} \\ &\quad \times \prod_{j=1}^l \Gamma(j/\alpha + 1 - s) \Gamma(2/\alpha - (j/\alpha + 1 - s)) \prod_{i=0}^{k-1} \frac{\sin(\pi\alpha(s+i))}{\pi}, \end{aligned}$$

and when  $k < 0, l \geq 0$ ,

$$\begin{aligned} \mathcal{M}(s) &= c(-1)^l (2\pi)^{l(1/\alpha-1)} (1/\alpha)^{l(1-2/\alpha)} \\ &\quad \times \frac{\Gamma(\frac{1-l}{\alpha} + k + s) \Gamma(2 - 1/\alpha - s) \Gamma(l + 1 + \alpha - \alpha s) \Gamma(2 - l + \alpha k + \alpha s)}{\Gamma(\frac{l}{\alpha} + 1 - k - s)} \\ &\quad \times \prod_{j=1}^l \Gamma(j/\alpha + 1 - s) \Gamma(2/\alpha - (j/\alpha + 1 - s)) \prod_{i=2}^{-k-1} \frac{\pi}{\sin(\pi\alpha(s-i))}. \end{aligned}$$

Similar expressions may be obtained when  $k \geq 0, l < 0$  and  $k, l < 0$ .

## 4.2 The radial part of the symmetric stable process

If  $X$  is a symmetric stable process—that is,  $\rho = 1/2$ —then the process

$$R_t = |X_t|, \quad t \geq 0,$$

is a pssMp, which we call the *radial part* of  $X$ . The Lamperti transform,  $\xi^R$ , of this process was studied by Caballero et al. [7] in dimension  $d$ ; these authors computed the Wiener–Hopf factorisation of  $\xi^R$  under the assumption  $\alpha < d$ , finding that the process is a hypergeometric Lévy process. Using the extended hypergeometric class, we extend this result, in one dimension, by finding the Wiener–Hopf factorisation when  $\alpha > 1$ .

In Kuznetsov et al. [20] the following theorem is proved using the work of Caballero et al. [7].

**Theorem 7** (Laplace exponent). *The Laplace exponent of the Lévy process  $2\xi^R$  is given by*

$$\psi^R(2z) = 2^\alpha \frac{\Gamma(\alpha/2 - z)}{\Gamma(-z)} \frac{\Gamma(1/2 + z)}{\Gamma((1 - \alpha)/2 + z)}. \quad (9)$$

We now identify the Wiener–Hopf factorisation of  $\xi^R$ , which will depend on the value of  $\alpha$ . However, note the factor  $2^\alpha$  in (9). In the context of the Wiener–Hopf factorisation, we could ignore this factor by picking an appropriate normalisation of local time; however, another approach is as follows.

Let us write  $R' = \frac{1}{2}R$ , and denote by  $\xi^{R'}$  the Lamperti transform of  $R'$ . Then the scaling of space on the level of the self-similar process is converted by the Lamperti transform into a scaling of time, so that  $\xi_s^R = \log 2 + \xi_{s2^\alpha}^{R'}$ . In particular, if we write  $\psi'$  for the characteristic exponent of  $\xi^{R'}$ , it follows that  $\psi' = 2^{-\alpha}\psi^R$ . This allows us to disregard the inconvenient constant factor in (9), if we work with  $\xi^{R'}$  instead of  $\xi^R$ .

The following corollary is now simple when we bear in mind the hypergeometric class of Lévy processes introduced in section 2. We emphasise that this Wiener–Hopf factorisation was derived by different methods in [7, Theorem 7] for  $\alpha < 1$ , though not  $\alpha = 1$ .

**Corollary 8** (Wiener–Hopf factorisation,  $\alpha \in (0, 1]$ ). *The Wiener–Hopf factorisation of  $2\xi^{R'}$  when  $\alpha \in (0, 1]$  is given by*

$$\psi'(2z) = \frac{\Gamma(\alpha/2 - z)}{\Gamma(-z)} \times \frac{\Gamma(1/2 + z)}{\Gamma(1 - \alpha)/2 + z}$$

and  $2\xi^{R'}$  is a Lévy process of the hypergeometric class with parameters

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = (1, \alpha/2, (1 - \alpha)/2, \alpha/2).$$

*Proof.* It suffices to compare the characteristic exponent with that of a hypergeometric Lévy process.  $\square$

When  $\alpha > 1$ , the process  $\xi^{R'}$  is not a hypergeometric Lévy process; however, it is in the extended hypergeometric class, and we therefore have the following result, which is new.

**Theorem 9** (Wiener–Hopf factorisation,  $\alpha \in (1, 2)$ ). *The Wiener–Hopf factorisation of  $2\xi^{R'}$  when  $\alpha \in (1, 2)$  is given by*

$$\psi'(2z) = \left( \frac{\alpha - 1}{2} - z \right) \frac{\Gamma(\alpha/2 - z)}{\Gamma(1 - z)} \times z \frac{\Gamma(1/2 + z)}{\Gamma((3 - \alpha)/2 + z)} \quad (10)$$

and  $2\xi^{R'}$  is a Lévy process in the extended hypergeometric class, with parameters

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = (1, \alpha/2, (1 - \alpha)/2, \alpha/2).$$

*Proof.* Simply use Theorem 7; using the formula  $x\Gamma(x) = \Gamma(x + 1)$  yields (10). That this is indeed the Wiener–Hopf factorisation follows once we recognise  $2\xi^{R'}$  as a process in the extended hypergeometric class, and apply Proposition 1.  $\square$

As an illustration of the utility of the extended hypergeometric class, we will now derive an expression for the Mellin transform of the exponential functional for the dual process  $-\xi^{R'}$ . This quantity is linked by the Lamperti representation to the hitting time of zero for  $X$ ; see section 4. In particular, if

$$T_0 = \inf\{t \geq 0 : X_t = 0\},$$

we have that

$$T_0 = \int_0^\infty e^{\alpha \xi_t^R} dt = \int_0^\infty e^{\alpha \xi_{2^\alpha t}^{R'}} dt = 2^{-\alpha} \int_0^\infty e^{\alpha \xi_s^{R'}} ds = 2^{-\alpha} I(-\alpha \xi^{R'}). \quad (11)$$

Since  $-2\xi^{R'}$  is an extended hypergeometric Lévy process with parameters  $(\frac{\alpha+1}{2}, \frac{\alpha}{2}, 0, \frac{\alpha}{2})$ , which drifts to  $+\infty$ , we may apply the theory just developed to compute the Mellin transform of  $I(-\alpha \xi^{R'})$ . Denote this by  $\mathcal{M}$ ; that is,

$$\mathcal{M}(s) = \mathbb{E}[I(-\alpha \xi^{R'})^{s-1}],$$

for some range of  $s \in \mathbb{C}$  to be determined.

**Proposition 10.** *For  $\operatorname{Re} s \in (-1/\alpha, 2 - 1/\alpha)$ ,*

$$\begin{aligned} \mathbb{E}_1[T_0^{s-1}] &= 2^{-\alpha(s-1)} \mathcal{M}(s) \\ &= 2^{-\alpha(s-1)} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{\alpha})\Gamma(1-\frac{1}{\alpha})} \frac{\Gamma(1+\frac{\alpha}{2}-\frac{\alpha s}{2})}{\Gamma(\frac{1-\alpha}{2}+\frac{\alpha s}{2})} \Gamma(\frac{1}{\alpha}-1+s) \frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)}. \end{aligned} \quad (12)$$

*Proof.* Let  $\zeta$  be a hypergeometric Lévy process with parameters  $(\frac{\alpha-1}{2}, \frac{\alpha}{2}, 1, \frac{\alpha}{2})$ , and denote by  $\widetilde{\mathcal{M}}$  the Mellin transform of the exponential functional  $I(\alpha/2 \cdot \zeta)$ , which is known to be defined for  $\operatorname{Re} s \in (0, 1 + 2/\alpha)$ .

We can then use Proposition 4 to make the following calculation, provided that  $\operatorname{Re} s \in (0, 2 - 1/\alpha)$ . Here  $G$  is the double gamma function, as defined in [17, Section 3], and we use [17, equation (19)] in the third line and the identity  $x\Gamma(x) = \Gamma(x+1)$  in the final line. For normalisation constants  $C$  (and  $C'$ ) to be determined, we have

$$\begin{aligned} \mathcal{M}(s) &= C \widetilde{\mathcal{M}}(s) \frac{\Gamma(\frac{1}{\alpha}+s)}{\Gamma(s)} \frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)} \\ &= C \frac{G(\frac{3}{\alpha}-1+s; \frac{2}{\alpha})}{G(\frac{3}{\alpha}+s; \frac{2}{\alpha})} \frac{G(\frac{2}{\alpha}+2-s; \frac{2}{\alpha})}{G(\frac{2}{\alpha}+1-s; \frac{2}{\alpha})} \Gamma(s) \frac{\Gamma(\frac{1}{\alpha}+s)}{\Gamma(s)} \frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)} \\ &= C \frac{\Gamma(1+\frac{\alpha}{2}-\frac{\alpha s}{2})}{\Gamma(\frac{3-\alpha}{2}+\frac{\alpha s}{2})} \Gamma(\frac{1}{\alpha}+s) \frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)} \\ &= C' \frac{\Gamma(1+\frac{\alpha}{2}-\frac{\alpha s}{2})}{\Gamma(\frac{1-\alpha}{2}+\frac{\alpha s}{2})} \Gamma(\frac{1}{\alpha}-1+s) \frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)}. \end{aligned}$$

The condition  $\mathcal{M}(1) = 1$  means that we can calculate

$$C' = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{\alpha})\Gamma(1-\frac{1}{\alpha})}, \quad (13)$$

and this gives the Mellin transform explicitly, for  $\operatorname{Re} s \in (0, 2 - 1/\alpha)$ .

We now expand the domain of  $\mathcal{M}$ . Note that, in contrast to the general case of Proposition 4, the right-hand side of (12) is well-defined when  $\operatorname{Re} s \in (-1/\alpha, 2 - 1/\alpha)$ , and is indeed analytic in this region. (The reason for this difference is the cancellation of a simple pole and zero at the point 0.) Theorem 2 of [28] shows that, if the Mellin transform of a probability measure is analytic in a neighbourhood of the point  $1 \in \mathbb{C}$ , then it is analytic in a strip  $\operatorname{Re} s \in (a, b)$ , where  $-\infty \leq a < b \leq \infty$ ; and furthermore, the function has singularities at  $a$  and  $b$ , if they are finite. It then follows that the right-hand side of (12) must actually be equal to  $\mathcal{M}$  in all of  $\operatorname{Re} s \in (-1/\alpha, 2 - 1/\alpha)$ , and this completes the proof.  $\square$

We remark that the distribution of  $T_0$  has been characterised previously by Yano et al. [36] and Cordero [11], using rather different methods; and the Mellin transform above was also obtained, again via the Lamperti transform but without the extended hypergeometric class, in Kuznetsov et al. [20].

It is also fairly straightforward to produce the following hitting distribution. Define

$$\sigma_{-1}^1 = \inf\{t \geq 0 : X_t \notin [-1, 1]\},$$

the *first exit time* of  $[-1, 1]$  for  $X$ . We give the distribution of the position of the symmetric stable process  $X$  at time  $\sigma_{-1}^1$ , provided this occurs before  $X$  hits zero. Note that when  $\alpha \in (0, 1]$ , the process does not hit zero, so the distribution is simply that found by Rogozin [32].

**Proposition 11.** *Let  $X$  be the symmetric stable process with  $\alpha \in (1, 2)$ . Then, for  $|x| < 1$ ,  $y > 1$ ,*

$$\begin{aligned} P_x(|X_{\sigma_{-1}^1}| \in dy; \sigma_{-1}^1 < T_0) \\ = \frac{\sin(\pi\alpha/2)}{\pi} |x|(1 - |x|)^{\alpha/2} y^{-1} (y - 1)^{-\alpha/2} (y - |x|)^{-1} \\ + \frac{1}{2} \frac{\sin(\pi\alpha/2)}{\pi} y^{-1} (y - 1)^{-\alpha/2} |x|^{(\alpha-1)/2} \int_0^{1-|x|} t^{\alpha/2-1} (1-t)^{-(\alpha-1)/2} dt. \end{aligned}$$

*Proof.* The starting point of the proof is the ‘second factorisation identity’ [22, Exercise 6.7],

$$\int_0^\infty e^{-qz} \mathbb{E}\left[e^{-\beta(\xi_{S_z^+} - z)}; S_z^+ < \infty\right] dx = \frac{\kappa(q) - \kappa(\beta)}{(q - \beta)\kappa(q)}, \quad q, \beta > 0,$$

where

$$S_z^+ = \inf\{t \geq 0 : \xi_t > z\}.$$

We now invert in  $q$  and  $z$ , in that order; this is a lengthy but routine calculation, and we omit it. We then apply the Lamperti transform: if  $g(z, \cdot)$  is the density of the measure  $\mathbb{P}(\xi_{S_z^+} - z \in \cdot; S_z^+ < \infty)$ , then

$$P_x(|X_{\sigma_{-1}^1}| \in dy; \sigma_{-1}^1 < T_0) = y^{-1} g(\log|x|^{-1}, \log y),$$

and this completes the proof.  $\square$



Integrating in the above proposition gives the following hitting probability.

**Corollary 12.** *For  $|x| < 1$ ,*

$$P_x(T_0 < \sigma_{-1}^1) = (1 - |x|)^{\alpha/2} - \frac{1}{2}|x|^{(\alpha-1)/2} \int_0^{1-|x|} t^{\alpha/2-1} (1-t)^{-(\alpha-1)/2} dt.$$

Finally, it is not difficult to produce the following slightly more general result. Applying the Markov property at time  $T_0$  gives

$$\begin{aligned} P_x(X_{\sigma_{-1}^1} \in dy; \sigma_{-1}^1) &= P_x(X_{\sigma_{-1}^1} \in dy) - P_x(X_{\sigma_{-1}^1} \in dy; T_0 < \sigma_{-1}^1) \\ &= P_x(X_{\sigma_{-1}^1} \in dy) - P_x(T_0 < \sigma_{-1}^1) P_0(X_{\sigma_{-1}^1} \in dy); \end{aligned}$$

the hitting distributions on the right-hand side were found by Rogozin [32], and substituting yields the following corollary.

**Corollary 13.** *For  $|x| < 1$ ,  $|y| > 1$ ,*

$$\begin{aligned} P_x(X_{\sigma_{-1}^1} \in dy; \sigma_{-1}^1 < T_0) &= \frac{\sin(\pi\alpha/2)}{\pi} (1-x)^{\alpha/2} (1+x)^{\alpha/2} (y-1)^{-\alpha/2} (y+1)^{-\alpha/2} (y-x)^{-1} \\ &\quad - \left[ (1-|x|)^{\alpha/2} - \frac{1}{2}|x|^{(\alpha-1)/2} \int_0^{1-|x|} t^{\alpha/2-1} (1-t)^{-(\alpha-1)/2} dt \right] \\ &\quad \times \frac{\sin(\pi\alpha/2)}{\pi} (y-1)^{-\alpha/2} (y+1)^{-\alpha/2} y^{-1}. \end{aligned}$$

### 4.3 The radial part of the symmetric stable process conditioned to avoid zero

Above, we computed the Lamperti transform of the pssMp  $R' = \frac{1}{2}|X|$ , where  $X$  was a symmetric stable process, and called it  $\xi^{R'}$ . In this section we consider instead the symmetric stable process conditioned to avoid zero, and obtain its Lamperti transform.

In [30], Pantí shows (among many other things!) that the function

$$h(x) = \begin{cases} -\Gamma(1-\alpha) \frac{\sin(\pi\alpha\hat{\rho})}{\pi} x^{\alpha-1}, & x > 0, \\ -\Gamma(1-\alpha) \frac{\sin(\pi\alpha\rho)}{\pi} x^{\alpha-1}, & x < 0, \end{cases}$$

is invariant for the stable process killed upon hitting zero, and defines the family of measures  $(P_x^\dagger)_{x \neq 0}$  given via the Doob  $h$ -transform:

$$P_x^\dagger(\Lambda) = \frac{1}{h(x)} E_x[h(X_t) \mathbb{1}_\Lambda; t < T_0], \quad x \neq 0,$$

for  $\Lambda \in \mathcal{F}_t = \sigma(X_s, s \leq t)$ . In [30] it is also shown that the laws  $P_x^\dagger$  arise as limits of the stable process conditioned not to have hit zero up to an exponential time of rate  $q$ ,

as  $q \downarrow 0$ . The canonical process associated with the laws  $(P_x^\dagger)_{x \neq 0}$  is therefore called the *stable process conditioned to avoid zero*, and we shall denote it by  $X^\dagger$ .

Consider now the process  $R^\dagger = \frac{1}{2}|X^\dagger|$ . This is a pssMp, and we may consider its Lamperti transform, which we will denote by  $\xi^\dagger$ . The characteristics of the generalised Lamperti representation of  $X^\dagger$  have been computed explicitly in [10], and the Laplace exponent,  $\psi^\dagger$ , of  $\xi^\dagger$  could be computed from this information; however, the harmonic transform gives us the following straightforward relationship between Laplace exponents:

$$\psi^\dagger(z) = \psi'(z + \alpha - 1).$$

This allows us to calculate

$$\psi^\dagger(2z) = \frac{\Gamma(1/2 - z)}{\Gamma((1 - \alpha)/2 - z)} \frac{\Gamma(\alpha/2 + z)}{\Gamma(z)},$$

which demonstrates that  $2\xi^\dagger$  is a process in the extended hypergeometric class, with parameters

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = ((\alpha + 1)/2, \alpha/2, 0, \alpha/2).$$

The present authors and A. Kuznetsov previously computed  $\psi^\dagger$  in [20], where we also observed that the process  $\xi^\dagger$  is the dual Lévy process to  $\xi^{R'}$ , and remarked that this implies a certain time-reversal relation between  $R$  and  $R^\dagger$ ; see [8, §2].

## 5 Concluding remarks

In this section, we offer some comments on how our approach may be adapted in order to offer new insight on an existing class of processes, the Lamperti-stable processes. These were defined in general in the work of Caballero et al. [6], and form a proper subclass of the  $\beta$ -class of Lévy processes of Kuznetsov [15]. It was observed in [17] that there is an intersection between the hypergeometric class and the Lamperti-stable class. In particular, the Lamperti representations of killed and conditioned stable processes (see [5]) fall within the hypergeometric class; and generally speaking, setting  $\beta = \hat{\beta}$  in the hypergeometric class and choosing  $\gamma, \hat{\gamma}$  as desired, one obtains a Lamperti-stable process.

However, not all Lamperti-stable processes may be obtained in this way, and we now outline how the ideas developed in this work can be used to characterise another subset of the Lamperti-stable processes.

Define the set of parameters

$$\mathcal{A}_{\text{EHL}} = \{\beta \in [1, 2], \gamma \in (1, 2), \hat{\gamma} \in (-1, 0)\}$$

and for  $(\beta, \gamma, \hat{\gamma}) \in \mathcal{A}_{\text{EHL}}$ , let

$$\psi(z) = \frac{\Gamma(1 - \beta + \gamma - z)}{\Gamma(2 - \beta - z)} \frac{\Gamma(\beta + \hat{\gamma} + z)}{\Gamma(\beta + z)}.$$

Note that this is the negative of the usual hypergeometric Laplace exponent, with  $\beta = \hat{\beta}$ . We claim that the following proposition holds.

**Proposition 14.** *There exists a Lévy process  $\xi$  with Laplace exponent  $\psi$ . Its Wiener–Hopf factorisation  $\psi(z) = -\kappa(-z)\hat{\kappa}(z)$  is given by the components*

$$\kappa(z) = \frac{\Gamma(1 - \beta + \gamma + z)}{\Gamma(2 - \beta + z)}, \quad \hat{\kappa}(z) = (\beta - 1 + z) \frac{\Gamma(\beta + \hat{\gamma} + z)}{\Gamma(\beta + z)}.$$

*The ascending ladder height process is a Lamperti-stable subordinator, and the descending factor satisfies*

$$\hat{\kappa}(z) = (\mathcal{T}_{\beta-1}v)^*(z), \quad v(z) = \frac{\Gamma(1 + z)}{\Gamma(1 + \hat{\gamma} + z)}.$$

*Here  $v$  is a Lamperti-stable subordinator.*

*The process  $\xi$  has Lévy density given by*

$$\pi(x) = \begin{cases} \frac{\Gamma(\gamma + \hat{\gamma} + 1)}{\Gamma(1 + \gamma)\Gamma(-\gamma)} e^{(\beta + \hat{\gamma})x} (e^x - 1)^{-(\gamma + \hat{\gamma} + 1)}, & x > 0, \\ \frac{\Gamma(\gamma + \hat{\gamma} + 1)}{\Gamma(1 + \hat{\gamma})\Gamma(-\hat{\gamma})} e^{-(1 - \beta + \gamma)x} (e^{-x} - 1)^{-(\gamma + \hat{\gamma} + 1)}, & x < 0. \end{cases}$$

*Thus,  $\xi$  falls within the Lamperti-stable class; in particular, using the notation of [6], a subset of its parameters is given by*

$$(\alpha, \beta, \delta) = (\gamma + \hat{\gamma}, \beta + \hat{\gamma}, 1 - \beta + \gamma).$$

The proposition may be proved in much the same way as Proposition 1, first using the theory of philanthropy to prove existence, and then the theory of meromorphic Lévy processes to deduce the Lévy measure.

We have thus provided an explicit spatial Wiener–Hopf factorisation of a subclass of Lamperti-stable processes disjoint from that given by the hypergeometric processes.

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