

CONTINUOUS-STATE BRANCHING PROCESSES AND SELF-SIMILARITY

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Abstract

In this paper we study the α -stable continuous-state branching processes (for $\alpha \in (1, 2]$) and the latter process conditioned never to become extinct in the light of positive self-similarity. Understanding the interaction of the Lamperti transformation for continuous state branching processes and the Lamperti transformation for positive self-similar Markov processes gives access to a number of explicit results concerning the paths of α -stable continuous-state branching processes and α -stable continuous-state branching processes conditioned never to become extinct.

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1. Introduction

This paper is principally concerned with positive self-similar Markov processes (pssMp) which are also either continuous-state branching processes or continuous-state branching processes with immigration. All of the aforementioned processes will be defined in more detail in the next section. However for the purpose of giving a brief sense of the main goals of this paper in this section, we may briefly recall the following.

A positive self-similar Markov process $X = (X_t, t \geq 0)$ has the defining property that it is a non-negative valued strong Markov process with probabilities $(\mathbb{Q}_x, x \geq 0)$ such that for each $k > 0$,

the law of $(kX_{k^{-\alpha}t}, t \geq 0)$ under \mathbb{Q}_x is given by \mathbb{Q}_{kx} ,

where $\alpha > 0$ is a constant known as the index of self-similarity. A continuous-state branching process (CB-process) on the other hand is a non-negative valued strong Markov process with probabilities $(\mathbb{P}_x, x \geq 0)$ such that for any $x, y \geq 0$, \mathbb{P}_{x+y} is equal in law to the convolution of \mathbb{P}_x and \mathbb{P}_y . Continuous-state branching processes may be thought of as the continuous (in time and space) analogues of classical Bienaymé-Galton-Watson branching processes. Associated with a continuous-state branching process (Y, \mathbb{P}_x) is its Malthusian parameter, $-\rho$, which characterizes the mean rate of growth in the sense that $\mathbb{E}_x(Y_t) = xe^{-\rho t}$ for all $t \geq 0$. The pssMp which are simultaneously CB-processes that we shall consider are critical in the sense that $\rho = 0$ and are closely related, via a path transformation, to spectrally positive α -stable Lévy processes for $\alpha \in (1, 2]$. Being critical processes, standard theory allows us to talk about CB-processes

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conditioned never to become extinct in an appropriate sense. Roughly speaking one may think of the the latter as the result of a Doob h -transform of the law of a continuous state branching process with $h(x) = x$. Thanks to recent works of Lambert [17; 18] it is known that in the case $\rho = 0$, a CB-process conditioned never to become extinct also corresponds, in an appropriate sense, to a CB-process with immigration (CBI-process). We shall see that when the underlying CB-process is a pssMp then so is the associated CBI-process.

The first result in this paper, in Section 3.2, plays upon the explicit nature of all of the aforementioned processes and their special link with α -stable process and specifies how one transforms between each of them with the help of either a Doob h -transform or one of two possible space-time changes which are commonly referred to as Lamperti-transformations. Some of these transformations are already known, however we provide a complete picture. Our goal is then to take advantage of some of these transforms and to provide a more detailed description of the dynamics of both the self-similar CB- and CBI-processes.

Specifically, for self-similar CB-processes, we are interested in a more detailed description of how the process becomes extinct. We do this by time-reversing its path from the moment of extinction. In particular we specify integral tests furnishing LIL type results of the time reversed process. For the case of self-similar CBI-processes we are interested in integral tests for its lower envelope at times zero and infinity when the process is issued from the origin. Understanding CB- and CBI-processes in the context of self-similarity also leads to some explicit fluctuation identities involving their last passage times. This we do in Section 4. As our work is closely related to [18] in spirit, we conclude the paper with some remarks about a different kind of conditioning considered there which results in quasi-stationary distributions.

2. Some processes revisited.

This section is dedicated to introducing more notation as well as providing more rigorous definitions of the stochastic processes that are of primary interest in this article.

2.1. Spectrally positive Lévy processes

Let $(\mathbb{P}_x, x \in \mathbb{R})$ be a family of probability measures on the space of cadlag mappings from $[0, \infty)$ to \mathbb{R} , denoted \mathcal{D} , such that for each $x \in \mathbb{R}$, the canonical process X is a Lévy process with no negative jumps issued from x . Set $\mathbb{P} := \mathbb{P}_0$, so \mathbb{P}_x is the law of $X + x$ under \mathbb{P} . The Laplace exponent $\psi : [0, \infty) \rightarrow (-\infty, \infty)$ of X is specified by $\mathbb{E}(e^{-\lambda X_t}) = e^{t\psi(\lambda)}$, for $\lambda \in \mathbb{R}$, and can be expressed in the form

$$\psi(\lambda) = a\lambda + \beta\lambda^2 + \int_{(0, \infty)} (e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x < 1\}}) \Pi(dx), \quad (2.1)$$

where $a \in \mathbb{R}$, $\beta \geq 0$ and Π is a σ -finite measure such that

$$\int_{(0, \infty)} (1 \wedge x^2) \Pi(dx) < \infty.$$

Henceforth, we shall assume that (X, \mathbb{P}) is not a subordinator (recall that a subordinator is a Lévy process with increasing sample paths). In that case, it is known that the Laplace exponent ψ is strictly convex and tends to ∞ as λ goes to ∞ . In this case, we define for $q \geq 0$

$$\Phi(q) = \inf \{ \lambda \geq 0 : \psi(\lambda) > q \}$$

the right inverse of ψ and then $\Phi(0)$ is the largest root of the equation $\psi(\lambda) = 0$. Theorem VII.1 in [1] implies that condition $\Phi(0) > 0$ holds if and only if the process drifts to ∞ . Moreover, almost surely, the paths of X drift to ∞ , oscillate or drift to $-\infty$ accordingly as $\psi'(0+) < 0$, $\psi'(0+) = 0$ or $\psi'(0+) > 0$.

2.2. Conditioning to stay positive

We also need to make use of Lévy processes conditioned to stay positive. The following commentary is taken from Chaumont and Doney [7] and Chapter VII of Bertoin [1] and is adapted to our setting where (X, \mathbb{P}) is a Lévy process with no negative jumps but not a subordinator. The process, X , conditioned to stay positive is the strong Markov process whose law is given by

$$\mathbb{P}_x^\uparrow(X_t \in dy) = \lim_{q \downarrow 0} \mathbb{P}_x(X_t \in dy, t < \mathbf{e}/q | \tau_0 > \mathbf{e}/q), \quad t \geq 0, \quad x, y > 0 \quad (2.2)$$

where \mathbf{e} is an independent and exponentially distributed random variable with mean 1 and

$$\tau_0 = \inf\{t > 0 : X_t \leq 0\}.$$

It turns out that the measure on the left hand side can also be constructed as the result of a Doob h -transform of X killed when it first exists $(0, \infty)$. In the special case that $\psi'(0+) \leq 0$, the resulting semi-group is thus given by

$$\mathbb{P}_x^\uparrow(X_t \in dy) = \frac{1 - e^{-\Phi(0)y}}{1 - e^{-\Phi(0)x}} \mathbb{P}_x(X_t \in dy, t < \tau_0), \quad t \geq 0, \quad x, y > 0 \quad (2.3)$$

where the ratio on the right hand side is understood as y/x in the case that $\Phi(0) = 0$ (i.e. the case that $\psi'(0+) = 0$). Moreover, the family of measures $(\mathbb{P}_x^\uparrow, x > 0)$ induced on \mathcal{D} are probability measures and when X has unbounded variation paths, the law \mathbb{P}_x^\uparrow converges weakly as $x \downarrow 0$ to a measure denoted by \mathbb{P}^\uparrow .

Now, define $\widehat{X} := -X$, the dual process of X . Denote by $\widehat{\mathbb{P}}_x$ the law of \widehat{X} when issued from x so that $(X, \widehat{\mathbb{P}}_x) = (\widehat{X}, \mathbb{P}_{-x})$. The dual process conditioned to stay positive in the sense of (2.2) is again a Doob h -transform of $(X, \widehat{\mathbb{P}}_x)$ killed when it first exists $(0, \infty)$. In this case, assuming (conversely to \mathbb{P}_x^\uparrow) that $\psi'(0+) \geq 0$, one has

$$\widehat{\mathbb{P}}_x^\uparrow(X_t \in dy) = \frac{W(y)}{W(x)} \widehat{\mathbb{P}}_x(X_t \in dy, t < \tau_0), \quad t \geq 0, \quad x, y > 0,$$

where W is the so-called scale function for the process $-X$. The latter is the unique continuous function on $(0, \infty)$ with Laplace transform

$$\int_0^\infty e^{-\lambda x} W(x) dx = \frac{1}{\psi(\lambda)}, \quad \lambda \geq 0, \quad (2.4)$$

where we recall that ψ is the Laplace exponent of X , defined in the previous subsection. In that case the measure $\widehat{\mathbb{P}}_x^\uparrow$ is always a probability measure and there is always weak convergence as $x \downarrow 0$ to a probability measure which we denote by $\widehat{\mathbb{P}}^\uparrow$.

2.3. CB- and CBI-processes

Continuous state branching processes are the analogue of Bienaymé-Galton-Watson processes in continuous time and continuous state space. Such classes of processes have been introduced by Jirina [14] and studied by many authors included Bingham [3], Grey [11], Grimvall [12], Lamperti [19; 20], to name but a few. A continuous state branching process $Y = (Y_t, t \geq 0)$ is a Markov process taking values in $[0, \infty]$, where 0 and ∞ are two absorbing states. Moreover, Y satisfies the branching property; that is to say, the Laplace transform of Y_t satisfies

$$\mathbb{E}_x(e^{-\lambda Y_t}) = \exp\{-xu_t(\lambda)\}, \quad \text{for } \lambda \geq 0, \quad (2.5)$$

for some function $u_t(\lambda)$. According to Silverstein [29], the function $u_t(\lambda)$ is determined by the integral equation

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(u)} du = t \quad (2.6)$$

where ψ is the Laplace exponent of a spectrally positive Lévy process and is known as the branching mechanism of Y .

Lamperti [19] observed that continuous state branching processes are connected to Lévy processes with no negative jumps by a simple time-change. More precisely, consider the spectrally positive Lévy process (X, \mathbb{P}_x) started at $x > 0$ and with Laplace exponent ψ . Now, we introduce the clock

$$A_t = \int_0^t \frac{ds}{X_s}, \quad t \in [0, \tau_0).$$

and its inverse

$$\theta(t) = \inf\{s \geq 0 : A_s > t\}.$$

Then, the time changed process $Y = (X_{\theta(t)}, t \geq 0)$, under \mathbb{P}_x , is a continuous state branching process with initial population of size x . The transformation described above will henceforth be referred to as the *CB-Lamperti representation*.

In respective order, a CB-process is called supercritical, critical or subcritical accordingly as its associated Lévy process drifts to $+\infty$, oscillates or drifts to $-\infty$, in other words accordingly as $\psi'(0+) < 0$, $\psi'(0+) = 0$ or $\psi'(0+) > 0$. It is known that a CB-process Y with branching mechanism ψ has a finite time extinction almost surely if and only if

$$\int_1^{\infty} \frac{du}{\psi(u)} < \infty \quad \text{and} \quad \psi'(0+) \geq 0. \quad (2.7)$$

In this work, we are also interested in CB-processes with immigration. In the remainder of this subsection, we assume that the CB-process is critical, i.e. $\psi'(0+) = 0$. Recall that a CB-process with immigration (or CBI-process) is a strong Markov process taking values in $[0, \infty]$, where 0 is no longer absorbing. If $(Y_t^\uparrow : t \geq 0)$ is a process in this class, then its semi-group is characterized by

$$\mathbb{E}_x(e^{-\lambda Y_t^\uparrow}) = \exp\left\{-xu_t(\lambda) - \int_0^t \phi(u_{t-s}(\lambda)) ds\right\} \quad \text{for } \lambda \geq 0,$$

where ϕ is a Bernstein function satisfying $\phi(0) = 0$ and is referred to as the immigration mechanism. See for example Lambert [18] for a formal definition. Roelly and Rouault [27], and more recently Lambert [18], show that, if

$$T_0 = \inf\{t > 0 : Y_t = 0\},$$

then the limit

$$\lim_{s \uparrow \infty} \mathbb{P}_x(Y_t \in dy | T_0 > t + s), \quad t \geq 0, \quad x, y > 0 \quad (2.8)$$

exists and defines a semi-group which is that of a CBI-process having initial population size x and immigration mechanism

$$\phi(\lambda) = \psi'(\lambda), \quad \lambda \geq 0.$$

The limit (2.8) may be thought of as conditioning the CB-process to not become extinct.

Lambert [18] also proved an interesting connection between the conditioning (2.8) for CB-process and (2.2) for the underlying Lévy process. Specifically he showed that $(X_\theta, \mathbb{P}_x^\uparrow) = (Y^\uparrow, \mathbb{P}_x)$ where the latter process has immigration mechanism given by $\psi'(\lambda)$. Another way

of phrasing this is that the CBI-process obtained by conditioning a critical CB-process not to become extinct is equal in law to the underlying spectrally positive Lévy process conditioned to stay positive and then time changed with the CB-Lamperti representation. Moreover, Lambert also showed that when \mathbb{P}_x^\dagger is used to describe the law of Y , then it fulfills the following Doob h -transform

$$\mathbb{P}_x^\dagger(Y_t \in dy) = \frac{y}{x} \mathbb{P}_x(Y_t \in dy, t < T_0)$$

for $y > 0$ and $t \geq 0$.

3. Self-similarity and α -stable CB- and CBI-processes

Before we investigate self-similar CB- and CBI-processes, let us first address how self-similarity and the associated Lamperti transformation manifests itself for the case of a spectrally positive α -stable processes. This turns out to be key to understanding self-similarity of CB- and CBI-processes.

3.1. Stable processes and pssMp-Lamperti representation.

Stable Lévy processes with no negative jumps are Lévy processes with Laplace exponent of the type (2.1) which satisfy the scaling property for some index $\alpha > 0$. More precisely, there exists a constant $\alpha > 0$ such that for any $k > 0$,

$$\text{the law of } (kX_{k^{-\alpha}t}, t \geq 0) \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{kx}. \quad (3.9)$$

In this subsection, (X, \mathbb{P}_x) will denote a stable Lévy process with no negative jumps of index $\alpha \in (1, 2]$ starting at $x \in \mathbb{R}$, (see Chapter VII in Bertoin [1] for further discussion on stable Lévy processes). It is known, that the Laplace exponent of (X, \mathbb{P}_x) takes the form

$$\psi(\lambda) = c_+ \lambda^\alpha, \quad \lambda \geq 0, \quad \alpha \in (0, 2) \quad (3.10)$$

where c_+ is a nonnegative constant. The case $\alpha = 2$ corresponds the process (X, \mathbb{P}_x) being a multiple of standard Brownian motion. In the remainder of this work, when we consider the case $\alpha = 2$ we will refer to the Brownian motion, i.e. that we choose $c_+ = 1/2$.

Recall that the stable Lévy process killed at the first time that it enters the negative half-line is defined by

$$X_t^\dagger := X_t \mathbf{1}_{\{t < \tau_0\}}, \quad t \geq 0,$$

where $\tau_0 = \inf\{t \geq 0 : X_t \leq 0\}$. From the previous subsection, a stable Lévy process with no negative jumps conditioned to stay positive is tantamount to a Doob- h transform of the killed process where $h(x) = x$. According to Caballero and Chaumont [4], both the process X and its conditioned version belong to the class of positive self-similar Markov processes; that is to say positive Markov processes satisfying the property (3.9).

From Lamperti's work [20] it is known that the family of positive self-similar Markov processes up to its first hitting time of 0 may be expressed as the exponential of a Lévy process, time changed by the inverse of its exponential functional. More precisely, let (X, \mathbb{Q}_x) be a self-similar Markov process started from $x > 0$ that fulfills the scaling property for some $\alpha > 0$, then under \mathbb{Q}_x , there exists a Lévy process $\xi = (\xi_t, t \geq 0)$ possibly killed at an independent exponential time which does not depend on x and such that

$$X_t = x \exp \left\{ \xi_{\zeta(tx^{-\alpha})} \right\}, \quad 0 \leq t \leq x^\alpha I(\alpha\xi), \quad (3.11)$$

where

$$\zeta(t) = \inf \left\{ s \geq 0 : I_s(\alpha\xi) > t \right\}, \quad I_s(\alpha\xi) = \int_0^s \exp \{ \alpha \xi_u \} du \quad \text{and} \quad I(\alpha\xi) = \lim_{t \rightarrow +\infty} I_t(\alpha\xi).$$

We will refer to this transformation as *pssMp-Lamperti representation*.

In [8], it was shown that a stable Lévy process with index $\alpha \in (1, 2)$ killed at the first time that it enters the negative half-line has underlying Lévy process, ξ , whose Laplace exponent is given by

$$\Psi(\lambda) = m \frac{\Gamma(\lambda + \alpha)}{\Gamma(\lambda)\Gamma(\alpha)}, \quad \text{for } \lambda \geq 0, \quad (3.12)$$

where $m > 0$ is the mean of $-\xi$ which is finite. Note that this last fact implies that the process ξ drifts towards $-\infty$. In the Brownian case, i.e when $\alpha = 2$, we have that the Lévy process ξ is a Brownian motion with drift $a = -1/2$.

The Laplace exponent of the underlying Lévy process, denoted by ξ^* , of the stable Lévy process (with $\alpha \in (1, 2)$) conditioned to stay positive is also computed in [8]. It is given by

$$\Psi^*(\lambda) = m \frac{\Gamma(\lambda - 1 + \alpha)}{\Gamma(\lambda - 1)\Gamma(\alpha)} \quad \text{for } \lambda \geq 0. \quad (3.13)$$

Here we use the gamma function for values $x \in (-1, 0)$ via the relation $x\Gamma(x) = \Gamma(1 + x)$. In this case, the Lévy process ξ^* drifts towards $+\infty$. When $\alpha = 2$, it is not difficult to show that the process ξ^* is a Brownian motion with drift $a = 1/2$.

Next, we remark that under $\widehat{\mathbb{P}}_x$, the stable Lévy process X has no positive jumps. From Corollary 6 in [8], it is known that the underlying Levy process in the pssMp-Lamperti representation of the spectrally negative stable Lévy process conditioned to stay positive is $\widehat{\xi}$, the dual of ξ . Note that in the case $\alpha = 2$, the processes $\widehat{\xi}$ and ξ^* are the same.

There is a definitive relation between ξ and ξ^* which will be used later and hence we register it as a proposition below. Its proof can be found in [8]. In the sequel, P will be a reference probability measure on \mathcal{D} (with associated expectation operator E) under which ξ and ξ^* are Lévy processes whose respective laws are defined above.

Proposition 1. *For every, $t \geq 0$, and every bounded measurable function f ,*

$$E(f(\xi_t^*)) = E(\exp\{\xi_t\}f(\xi_t)).$$

In particular, the process $-\xi^$ and ξ satisfy Cramér's condition, i.e.*

$$E(\exp\{-\xi_1^*\}) = 1 \quad \text{and} \quad E(\exp\{\xi_1\}) = 1.$$

Finally we note that ξ and ξ^* are two examples of so called *Lamperti-stable processes* (see for instance [2; 4; 6; 10; 25] for related expositions and the formal definition of a Lamperti-stable process).

3.2. PssMp-Lamperti representation for CB- and CBI-processes

Suppose as in the previous section that (X, \mathbb{P}_x) is a spectrally positive α -stable process with index $\alpha \in (1, 2]$ starting from $x > 0$. We refer to Y , the associated continuous state branching process, as the α -stable CB-process. Moreover, when talking of the latter process conditioned to stay positive in the sense of the description in Section 2.3, in other words the processes (Y, \mathbb{P}_x^+) for $x > 0$, we shall refer to the associated CBI-process as the α -stable CBI-process.

We begin by showing that α -stable CB- and CBI-process are self-similar processes (positivity is obvious). To this end we state and prove a generic result which, in some sense, is well known folklore and will be useful throughout the remainder of this section. For the sake of completeness we include its proof.

Proposition 2. *Suppose that X is any positive self-similar Markov process issued from $x > 0$ with self-similarity index $\alpha > 1$ and let θ be the CB-Lamperti time change. Then X_θ is a positive self-similar Markov process issued from x with self-similarity index $\alpha - 1$ and with the same underlying Lévy process as X .*

Proof: Suppose that η is the underlying Lévy process for the process X . We first define,

$$A. = \int_0^\cdot \frac{ds}{X_s}, \quad I.(\alpha\eta) = \int_0^\cdot e^{\alpha\eta_s} ds \quad \text{and} \quad I.((\alpha-1)\eta) = \int_0^\cdot e^{(\alpha-1)\eta_s} ds.$$

Recall that $\zeta(\cdot)$ is the right-continuous inverse of $I.(\alpha\eta)$. From the pssMp-Lamperti transform of X and the change of variable $s = x^\alpha I_u(\alpha\eta)$, we get that

$$A_{x^\alpha I.(\alpha\eta)} = \int_0^{x^\alpha I.(\alpha\eta)} \frac{ds}{x \exp\{\eta_{\zeta(s/x^\alpha)}\}} = x^{\alpha-1} \int_0^\cdot \frac{e^{\alpha\eta_u}}{e^{\eta_u}} du = x^{\alpha-1} \int_0^\cdot e^{(\alpha-1)\eta_u} du.$$

On the other hand, the right-continuous inverse of $I.((\alpha-1)\eta)$ is defined by

$$h(t) = \inf \left\{ s \geq 0 : I_s((\alpha-1)\eta) > t \right\},$$

and recall that θ is the right-continuous inverse function of A . Hence, we have that for any $0 \leq t < x^{\alpha-1} I_\infty((\alpha-1)\eta)$,

$$\begin{aligned} h(t/x^{\alpha-1}) &= \inf \left\{ s \geq 0 : I_s((\alpha-1)\eta) > t/x^{\alpha-1} \right\} = \inf \left\{ s \geq 0 : A_{x^\alpha I_s(\alpha\eta)} > t \right\} \\ &= \inf \left\{ \zeta(u/x^\alpha) \geq 0 : A_u > t \right\} = \zeta(\theta(t)/x^\alpha). \end{aligned}$$

From the pssMp-Lamperti representation of X , we have that

$$\inf\{t \geq 0 : X_{\theta(t)} = 0\} = x^{\alpha-1} I_\infty((\alpha-1)\eta)$$

and for all $0 \leq t \leq x^{\alpha-1} I_\infty((\alpha-1)\eta)$,

$$X_{\theta(t)} = x \exp \left\{ \eta_{\zeta(\theta(t)/x^\alpha)} \right\} = x \exp \left\{ \eta_{h(t/x^{\alpha-1})} \right\}$$

thus completing the proof. ■

This leads immediately to the conclusion that the α -stable CB- and CBI-processes are self-similar with index $\alpha - 1$.

Corollary 1.

- (i) *The process (Y, \mathbb{P}_x) is a positive self similar Markov process with self-similarity index $\alpha - 1$. Moreover, its pssMp-Lamperti representation under \mathbb{P}_x is given by*

$$Y_t = x \exp \left\{ \xi_{h(tx^{-(\alpha-1)})} \right\}, \quad 0 \leq t \leq x^{\alpha-1} \int_0^\infty \exp \left\{ (\alpha-1)\xi_u \right\} du,$$

where

$$h(t) = \inf \left\{ s \geq 0 : \int_0^s \exp \left\{ (\alpha-1)\xi_u \right\} du > t \right\}.$$

- (ii) The process $(Y, \mathbb{P}_x^\dagger)$ is a positive self-similar Markov process with index of self-similarity $(\alpha - 1)$. Moreover, its pssMp-Lamperti representation under \mathbb{P}_x^\dagger is given by

$$Y_t = x \exp\{\xi_{\zeta^*(tx^{-(\alpha-1)})}^*\}, \quad t \geq 0.$$

where

$$\zeta^*(t) = \inf \left\{ s \geq 0 : \int_0^s \exp\{(\alpha - 1)\xi_u^*\} du > t \right\}.$$

This last corollary, Proposition 1 and the discussion in the previous sections give rise to the following flow of transformations. Let $h_1(y) = e^y$, $h_2(y) = e^{-y}$, $h_3(y) = y$ and $h_4 = 1/y$, then

$$\begin{array}{ccccccc}
 \xi & \begin{array}{c} \text{pssMp-Lamp} \\ \leftarrow \text{---} \rightarrow \end{array} & (X^\dagger, \mathbb{P}_x) & \begin{array}{c} \text{CB-Lamp} \\ \leftarrow \text{---} \rightarrow \end{array} & (Y, \mathbb{P}_x) & \begin{array}{c} \text{pssMp-Lamp} \\ \leftarrow \text{---} \rightarrow \end{array} & \xi \\
 \uparrow h_2 & & h_4 \uparrow & & \uparrow h_4 & & h_2 \uparrow \\
 & \alpha & & & & \alpha - 1 & \\
 \downarrow h_1 & & h_3 \downarrow & & \downarrow h_3 & & h_1 \downarrow \\
 \xi^* & \begin{array}{c} \text{pssMp-Lamp} \\ \leftarrow \text{---} \rightarrow \end{array} & (X, \mathbb{P}_x^\dagger) & \begin{array}{c} \text{CB-Lamp} \\ \leftarrow \text{---} \rightarrow \end{array} & (Y, \mathbb{P}_x^\dagger) & \begin{array}{c} \text{pssMp-Lamp} \\ \leftarrow \text{---} \rightarrow \end{array} & \xi^*
 \end{array}$$

where the vertical arrows are the result of a Doob h -transform with the h -function indicated in each direction and the parameters α and $\alpha - 1$ are the index of self-similarity on the pssMp-Lamperti representation.

4. Some path properties of α -stable CB- and CBI-processes

In this section we state the remainder of our main results leaving the proofs to the next section. Throughout we shall always be assuming that X is a spectrally positive α -stable process unless otherwise stated.

4.1. Entrance laws

We begin by introducing

$$\sigma_x = \sup\{t > 0 : X_t \leq x\},$$

the last passage time of X below $x \in \mathbb{R}$, for the next theorem which gives us self-similarity of the α -stable CB-process when time-reversed from extinction. Recall the notation $\widehat{\mathbb{P}}^\dagger$ for the law of \widehat{X} conditioned to stay positive. We remark that, under $\widehat{\mathbb{P}}^\dagger$, the canonical process X drifts towards ∞ and also that $X_t > 0$ for $t > 0$.

Theorem 1. For each $x > 0$

$$\left\{ (Y_{(T_0-t)^-} : t < T_0), \mathbb{P}_x \right\} \stackrel{d}{=} \left\{ (X_{\theta(t)}, 0 \leq t < A_{\sigma_x}), \widehat{\mathbb{P}}^\dagger \right\}. \quad (4.14)$$

Moreover, the process $X_\theta := (X_{\theta(t)}, t \geq 0)$, under $\widehat{\mathbb{P}}^\dagger$ is a positive self-similar Markov process with index $\alpha - 1$, starting from 0, with the same semigroup as the processes $(X_\theta, \widehat{\mathbb{P}}_y^\dagger)$ for $y > 0$, and with entrance law given by

$$\widehat{\mathbb{E}}^\dagger(f(X_{\theta(t)})) = \frac{c_\alpha}{m} \int_0^\infty f(t^{1/(\alpha-1)}x) x^{\alpha-1} e^{-c_\alpha x} dx, \quad (4.15)$$

where $c_\alpha = (c_+(\alpha - 1))^{-1/(\alpha-1)}$, $t > 0$ and f is a positive measurable function.

Remark 1. It is important to note that when $\alpha = 2$, the process $(X_{\theta(t)}, t \geq 0)$ under $\widehat{\mathbb{P}}^\uparrow$ is in fact the CB-process with immigration. This follows from the remark made in Section 3.1 that in this particular case, we have that $\widehat{\xi} = \xi^*$.

Remark 2. As we shall see in the next section when we prove this theorem, the identity in law (4.14) is in fact true for any CB-process (not just α -stable CB-processes) which becomes extinct almost surely (and thus conditions (2.7) are in force).

In the case of an α -stable CBI-process, we have thus far only discussed the case that it is issued from $x > 0$. The following theorem tells us that the α -stable CBI-process is still well defined as a self-similar process when issued from the origin. Moreover, the theorem also specifies the entrance law.

Theorem 2. *The process $(Y, \mathbb{P}_x^\uparrow)$ converges weakly with respect to the Skorokhod topology as x tends to 0 towards (Y, \mathbb{P}^\uparrow) , a pssMp starting from 0 with same semigroup as $(Y, \mathbb{P}_x^\uparrow)$, for $x > 0$, and with entrance law given by*

$$\mathbb{E}^\uparrow \left(e^{-\lambda Y_t} \right) = \left(1 + c_+(\alpha - 1)t\lambda^{\alpha-1} \right)^{-\frac{\alpha}{\alpha-1}}. \quad (4.16)$$

4.2. Asymptotic results

Theorems 3, 4, 5 below each study the asymptotic behaviour for the α -stable CB-process towards its moment of extinction. We start by stating the integral test for the lower envelope of $(Y_{(T_0-t)^-}, 0 \leq t \leq T_0)$, under \mathbb{P}_x , at 0.

Theorem 3. *Let f be an increasing function such that $\lim_{t \rightarrow 0} f(t)/t = 0$, then for every $x > 0$*

$$\mathbb{P}_x \left(Y_{(T_0-t)^-} < f^{1/(\alpha-1)}(t), \text{ i.o., as } t \rightarrow 0 \right) = 0 \text{ or } 1,$$

accordingly as

$$\int_{0+} f^{1/(\alpha-1)}(t) t^{-\alpha/(\alpha-1)} dt \quad \text{is finite or infinite.}$$

In particular,

$$\liminf_{t \rightarrow 0} \frac{Y_{(T_0-t)^-}}{t^\kappa} = \begin{cases} 0 & \text{if } \kappa < \frac{1}{\alpha-1} \\ +\infty & \text{if } \kappa \geq \frac{1}{\alpha-1} \end{cases} \quad \mathbb{P}_x - \text{a.s.}$$

Next introduce \mathcal{H}_0 the class of increasing functions $f : (0, +\infty) \mapsto [0, +\infty)$ such that

i) $f(0) = 0$ and

ii) there exists a $\beta \in (0, 1)$ such that $\sup_{t < \beta} \frac{t}{f(t)} < \infty$.

We also denote by \underline{Y}_t the infimum of the CB-process (Y, \mathbb{P}_x) over $[0, t]$. The upper envelope of the process $(\underline{Y}_{(T_0-t)^-}, 0 \leq t \leq T_0)$, under \mathbb{P}_x , at 0 is described by the integral test in the following theorem.

Theorem 4. *Let $f \in \mathcal{H}_0$, then for every $x > 0$*

i) *If*

$$\int_{0+} \exp \left\{ - (c_+(\alpha - 1)t/f(t))^{-1/(\alpha-1)} \right\} \frac{dt}{t} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}_x \left(\underline{Y}_{(T_0-t)^-} > (1 + \epsilon)f^{1/(\alpha-1)}(t), \text{ i.o., as } t \rightarrow 0 \right) = 0.$$

ii) If

$$\int_{0+} \exp \left\{ - (c_+(\alpha-1)t/f(t))^{-1/(\alpha-1)} \right\} \frac{dt}{t} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}_x \left(\underline{Y}_{(T_0-t)-} > (1-\epsilon)f^{1/\alpha-1}(t), \text{ i.o., as } t \rightarrow 0 \right) = 1.$$

In particular, we have the following law of the iterated logarithm

$$\limsup_{t \rightarrow 0} \frac{\underline{Y}_{(T_0-t)-}}{t^{1/(\alpha-1)}(\log \log(1/t))^{1-\alpha}} = (c_+(\alpha-1))^{1/(\alpha-1)}, \quad \mathbb{P}_x - \text{a.s.}$$

Finally, still within the framework of α -stable CB-processes, the following result describes the upper envelope of the time reversed processes $(Y_{(T_0-t)-}, 0 \leq t \leq T_0)$ and $((Y - \underline{Y})_{(T_0-t)-}, 0 \leq t \leq T_0)$, in the form of a law of the iterated logarithm.

Theorem 5. For every $x > 0$, we have

$$\limsup_{t \rightarrow 0} \frac{Y_{(T_0-t)-}}{t^{1/(\alpha-1)}(\log \log(1/t))^{1-\alpha}} = (c_+(\alpha-1))^{1/(\alpha-1)}, \quad \mathbb{P}_x - \text{a.s.}, \quad (4.17)$$

and

$$\limsup_{t \rightarrow 0} \frac{(Y - \underline{Y})_{(T_0-t)-}}{t^{1/(\alpha-1)}(\log \log(1/t))^{1-\alpha}} = (c_+(\alpha-1))^{1/(\alpha-1)}, \quad \mathbb{P}_x - \text{a.s.} \quad (4.18)$$

One should note that when $\alpha = 2$, the upper envelope of the time reversed process $\{(Y_{(T_0-t)-}, 0 \leq t \leq T_0), \mathbb{P}_x\}$ is described by the Kolmogorov-Dvoretzky-Erdős integral test since the latter process has the same law as the square of a Bessel process of dimension $d = 3$ killed at its last passage time below the level $x > 0$. (see for instance Itô and McKean [13])

Next we turn to the integral test which describes the lower envelope at zero and at infinity of the α -stable CBI-process.

Theorem 6. Let f be an increasing function such that $\lim_{t \rightarrow 0} f(t)/t = 0$, then

$$\mathbb{P}^\uparrow (Y_t < f^{1/(\alpha-1)}(t), \text{ i.o., as } t \rightarrow 0) = 0 \text{ or } 1,$$

accordingly as

$$\int_{0+} f^{1/(\alpha-1)}(t)t^{-\alpha/(\alpha-1)} dt \quad \text{is finite or infinite.}$$

In particular,

$$\liminf_{t \rightarrow 0} \frac{Y_t}{t^\kappa} = \begin{cases} 0 & \text{if } \kappa < \frac{1}{\alpha-1} \\ +\infty & \text{if } \kappa \geq \frac{1}{\alpha-1} \end{cases} \quad \mathbb{P}^\uparrow - \text{a.s.}$$

Let g be an increasing function such that $\lim_{n \rightarrow \infty} g(t)/t = 0$, then for all $x \geq 0$:

$$\mathbb{P}_x^\uparrow (Y_t < g^{1/(\alpha-1)}(t), \text{ i.o., as } t \rightarrow \infty) = 0 \text{ or } 1,$$

accordingly as

$$\int^\infty g^{1/(\alpha-1)}(t)t^{-\alpha/(\alpha-1)} dt \quad \text{is finite or infinite.}$$

In particular, for any $x \geq 0$

$$\liminf_{t \rightarrow \infty} \frac{Y_t}{t^\kappa} = \begin{cases} 0 & \text{if } \kappa \geq \frac{1}{\alpha-1} \\ +\infty & \text{if } \kappa < \frac{1}{\alpha-1} \end{cases} \quad \mathbb{P}_x - \text{a.s.}$$

4.3. Fluctuation identities at last passage

Define, the last passage time above x of ξ by

$$D_x = \sup \{t \geq 0 : \xi_t \geq x\}.$$

Theorem 7. *Suppose that the Lévy process (X, \mathbb{P}) does not drift towards $+\infty$. Then for every $x > 0$ and $0 < y \leq x$,*

$$\mathbb{P}_x \left(\inf_{0 \leq t \leq U_y} Y_t \geq z \right) = \frac{W(y-z)}{W(y)} \mathbf{1}_{\{z \leq y\}},$$

where $U_y = \sup\{t > 0 : Y_t \geq y\}$ and the scale function W satisfies (2.4). In particular, in the case that X is a spectrally positive α -stable process with $\alpha \in (1, 2]$

$$\mathbb{P}_x \left(\inf_{0 \leq t \leq U_y} Y_t \geq z \right) = P \left(\inf_{0 \leq t \leq D_u} \xi_t \geq v \right) = (1 - e^{v-u})^{\alpha-1}$$

where $v = \log(z/x)$ and $u = \log(y/x)$ and $z \leq y$.

Next we establish a similar result but for the supremum at last passage for the CBI-process (Y, \mathbb{P}^\uparrow) . For this we define

$$U_y^- = \sup\{t \geq 0 : Y_t \leq y\}.$$

Theorem 8. *Let $z \geq y > 0$, then*

$$\mathbb{P}^\uparrow \left(\sup_{0 \leq s < U_y^-} Y_s \leq z \right) = 1 - \frac{y}{m^* z}, \quad (4.19)$$

where m^* is the mean of ξ_1^* .

Now, we define the following exponential functional of ξ^* ,

$$I^* := \int_0^\infty e^{-\alpha \xi_s^*} ds.$$

The exponential functional I^* was studied by Chaumont et al. [8]. In particular the authors in [8] found that

$$P(1/I^* \in dy) = \alpha m^* q_1(y) dy,$$

where q_1 is the density of the entrance law of the excursion measure under \mathbb{P} of the reflected process, $(X_t - \underline{X}_t, t \geq 0)$, away from 0.

The time reversal property of (Y, \mathbb{P}^\uparrow) at its last passage time (see Proposition 1 in [9]) combined with the CB-Lamperti representation and the pssMp-Lamperti representation gives us the following result for the total progeny of the self-similar CB-process with immigration.

Proposition 3. *For $y > 0$, the total progeny of (Y, \mathbb{P}^\uparrow) up to time U_y^- , $\int_0^{U_y^-} Y_s ds$, and the last passage time of (X, \mathbb{P}^\uparrow) below y , σ_y , are both equal in law to $\Gamma^\alpha I^*$, where $\Gamma = Y_{U_y^-}$ and is independent of I^* .*

5. Proofs for Section 4

5.1. Proofs of Theorem 1 and 2

We start by establishing some preliminary results needed for the Proof of Theorem 1.

Proposition 4. *Suppose that X is any spectrally positive Lévy process. If condition (2.7) is satisfied, then for every $y > 0$*

$$\left\{ (Y_{(T_0-t)-}, 0 \leq t < T_0), \mathbb{P}_y \right\} \stackrel{d}{=} \left\{ (X_{\theta(t)}, 0 \leq t < A_{\sigma_y}), \widehat{\mathbb{P}}^\uparrow \right\},$$

where $\stackrel{d}{=}$ denotes equality in law or distribution.

Proof: We first prove that the CB-Lamperti representation is well defined for the process $(X, \widehat{\mathbb{P}}^\uparrow)$. In order to do so, it is enough to prove that the map $s \mapsto 1/X_s$ is $\widehat{\mathbb{P}}^\uparrow$ -almost surely integrable in a neighbourhood of 0. To this end, recall that (2.7) implies that $\mathbb{P}_y(\tau_0 < \infty) = 1$ for all $y \geq 0$. Hence by Theorem VII.18 of Bertoin [1], we know that for $y > 0$

$$\left\{ (X_t, 0 \leq t < \sigma_y), \widehat{\mathbb{P}}^\uparrow \right\} \stackrel{d}{=} \left\{ (X_{(\tau_0-t)-}, 0 \leq t < \tau_0), \mathbb{P}_y \right\}, \quad (5.20)$$

which in turn can be used to deduce that

$$\int_0^{\sigma_y} \frac{1}{X_s} ds \quad \text{under } \widehat{\mathbb{P}}^\uparrow \quad \text{is equal in law to} \quad \int_0^{\tau_0} \frac{1}{X_s} ds \quad \text{under } \mathbb{P}_y. \quad (5.21)$$

However the latter integral is equal to T_0 which is \mathbb{P}_y -almost surely finite given the assumption (2.7).

Next from the definition of Y , under \mathbb{P}_y , we have

$$(Y_{(T_0-t)-}, 0 \leq t < T_0) = (X_{\theta(A_{\tau_0-t}-)}, 0 \leq t < A_{\tau_0}). \quad (5.22)$$

Define

$$\theta'(t) = \inf\{s > 0 : B_s > t\} \quad \text{where} \quad B_s = \int_0^s \frac{1}{X_{\tau_0-u}} du.$$

Setting $t = B_s$, we have

$$A_{\tau_0} - B_s = \int_0^{\tau_0} \frac{1}{X_u} du - \int_0^s \frac{1}{X_{\tau_0-u}} du = \int_0^{\tau_0-s} \frac{1}{X_u} du$$

and hence

$$X_{\theta(A_{\tau_0-t}-)} = X_{\theta(A_{\tau_0-s}-)} = X_{(\tau_0-s)-} = X_{(\tau_0-\theta'(t))--}.$$

As noted earlier $T_0 = A_{\tau_0} = B_{\tau_0}$. Now, it follows from (5.20), (5.21) and (5.22) that

$$\left\{ (Y_{(T_0-t)-}, 0 \leq t < T_0), \mathbb{P}_y \right\} \stackrel{d}{=} \left\{ (X_{\theta(t)}, 0 \leq t < A_{\sigma_y}), \widehat{\mathbb{P}}^\uparrow \right\}$$

as required. ■

We need now the following auxiliary lemma which says in particular that the distribution of $I_\infty((\alpha - 1)\xi) = \int_0^\infty \exp\{(\alpha - 1)\xi_t\} dt$ has a Fréchet distribution and moreover that the Fréchet distribution is self-decomposable.

Lemma 1. *The distribution of $I := I_\infty((\alpha - 1)\xi)$ is given by*

$$P(I \leq t) = \exp \left\{ - [c_+(\alpha - 1)t]^{-1/(\alpha-1)} \right\}. \quad (5.23)$$

Moreover, I is self-decomposable and thus has a completely monotone density with respect to the Lebesgue measure.

Proof: From the pssMp-Lamperti representation of (Y, \mathbb{P}_x) and the previous proposition, we deduce that $T_0 = x^{\alpha-1}I$. From Bingham [3], it is known that

$$\mathbb{P}_x(T_0 \leq t) = e^{-xu_t(\infty)},$$

where $u_t(\infty)$ solves

$$\int_{u_t(\infty)}^{\infty} \frac{1}{c_+v^\alpha} dv = t.$$

Therefore $u_t(\infty) = [c_+(\alpha - 1)t]^{-1/(\alpha-1)}$ and hence

$$\mathbb{P}_x(T_0 \leq t) = P(I \leq t/x^{\alpha-1}) = \exp \left\{ - x[c_+(\alpha - 1)t]^{-1/(\alpha-1)} \right\},$$

which implies (5.23).

Let $a < 0$, then

$$I = \int_0^\infty e^{(\alpha-1)\xi_u} du = \int_0^{S_a} e^{(\alpha-1)\xi_u} du + e^{(\alpha-1)a} \int_0^\infty e^{(\alpha-1)\xi'_u} du,$$

where $\xi' = (\xi_{S_a+t} - a, t \geq 0)$ and $S_a = \inf\{t \geq 0 : \xi_t \leq a\}$. Then, self-decomposability follows from the independence of $(\xi_t, 0 \leq t \leq \tau_a)$ and ξ' . Self-decomposable distributions on \mathbb{R}_+ are unimodal (see for instance Chapter 10 in Sato [28]), i.e. they have a completely monotone density on $(0, \infty)$, with respect to the Lebesgue measure. \blacksquare

Proof of Theorem 1: The time reversal property (4.14) follows from Proposition 4. The pssMp-Lamperti representation of the process $(X_{\theta(t)}, t \geq 0)$ under $\widehat{\mathbb{P}}_y^\uparrow$ when issued from $y > 0$ follows from Proposition 2, noting in particular that $(X, \widehat{\mathbb{P}}_y^\uparrow)$ is a spectrally positive stable process conditioned to stay positive which is a positive self-similar process with index α .

The Lévy process $\widehat{\xi}$ is not arithmetic and recall that $E(\widehat{\xi}_1) = m > 0$ which means that $\widehat{\xi}$ satisfies the conditions of Theorems 1 and 2 in [5]. Hence, the family of processes $(X_{\theta(t)}, t \geq 0)$ under $\widehat{\mathbb{P}}_y^\uparrow$, for $y > 0$, converges weakly with respect to the Skorohod topology, as $y \downarrow 0$, towards a pssMp starting from 0 which is $(X_{\theta(t)}, t \geq 0)$ under $\widehat{\mathbb{P}}^\uparrow$. Moreover, according to the aforementioned results, the latter process has the same semigroup as $(X_\theta, \widehat{\mathbb{P}}_y^\uparrow)$ for $y > 0$.

From Theorem 1 in [2], the entrance law of $(X_\theta, \widehat{\mathbb{P}}^\uparrow)$ is given by

$$\widehat{\mathbb{E}}^\uparrow \left(f(X_{\theta(t)}) \right) = \frac{1}{(\alpha - 1)m} E \left(I^{-1} f((tI^{-1})^{1/(\alpha-1)}) \right),$$

for every $t > 0$ and every f positive and measurable function. Therefore, from the distribution of I given in the previous Lemma and some basic calculations, we get (4.15). \blacksquare

Proof of Theorem 2: The entrance law of the process (Y, \mathbb{P}^\uparrow) can again be constructed from general considerations found in Theorem 1 in Caballero and Chaumont [5]. However, we give a more direct construction appealing to properties of the CBI branching process.

The pssMp-Lamperti representation follows from Proposition 2 where now the underlying Lévy process is ξ^* . Now, note that the Lévy process ξ^* satisfies the conditions of Theorem 1 and 2 in [5], i.e. that ξ is not arithmetic and $m^* = E(\xi_1^*) > 0$. Hence the family of processes $\{(Y_t, t \geq 0), \mathbb{P}_x^\uparrow\}$, for $x > 0$, converges weakly with respect to the Skorohod topology, as x goes to 0, towards a pssMp starting from 0 which is (Y, \mathbb{P}^\uparrow) . It is well-known (see for instance [16]) that its entrance law is of the form

$$\mathbb{E}^\uparrow \left(e^{-\lambda Y_t} \right) = \exp \left\{ - \int_0^t \phi(u_s(\lambda)) ds \right\}.$$

Solving (2.6) explicitly we find that

$$u_t(\lambda) = [c_+(\alpha - 1)t + \lambda^{-(\alpha-1)}]^{-\frac{1}{(\alpha-1)}}.$$

We obtain (4.16) from straightforward calculations, recalling that $\phi(\lambda) = c_+\alpha\lambda^{\alpha-1}$ thus completing the proof. \blacksquare

Remark 3. Referring again to Proposition 3 in [5], which says in our particular case that

$$\mathbb{E}^\uparrow \left(e^{-\lambda Y_1} \right) = \frac{1}{m^*(\alpha - 1)} E \left(e^{-\lambda(I')^{-1}} (I')^{-1} \right), \quad \text{for } \lambda \geq 0$$

where I' is the exponential functional of $(\alpha - 1)\xi^*$ and is given by

$$I' := \int_0^\infty e^{-(\alpha-1)\xi_s^*} ds$$

It follows that we may characterize the law of I' via the relation

$$E \left(e^{-\lambda(I')^{-1}} (I')^{-1} \right) = m^*(\alpha - 1) (1 + c_+(\alpha - 1)\lambda^{\alpha-1})^{-\alpha/(\alpha-1)}, \quad \text{for } \lambda \geq 0.$$

5.2. Proofs of Theorems 3, 4, 5 and 6

Proof of Theorem 3: Fix $x > 0$. From Theorem 1, we deduce that

$$\mathbb{P}_x \left(Y_{(T_0-t)^-} < f^{1/(\alpha-1)}(t), \text{ i.o., as } t \rightarrow 0 \right) = \widehat{\mathbb{P}}^\uparrow \left(X_{\theta(t)} < f^{1/(\alpha-1)}(t), \text{ i.o., as } t \rightarrow 0 \right)$$

Since $(X_\theta, \widehat{\mathbb{P}}^\uparrow)$ is a pssMp with index $\alpha - 1$ starting from 0, it is then clear that the process $(X_\theta^{\alpha-1}, \widehat{\mathbb{P}}^\uparrow)$ is a pssMp with index 1. Then from Theorem 3 in [9], the above probability is equal to 0 or 1 accordingly as

$$\int_{0^+} P \left(I > t/f(t) \right) \frac{dt}{t} \quad \text{is finite or infinite.}$$

In order to get our result, it is enough to show that

$$P(I > t) \sim (c_+(\alpha - 1)t)^{-1/(\alpha-1)} \quad \text{as } t \rightarrow +\infty. \quad (5.24)$$

From Lemma 1 and with the change of variable $h = (c_+(\alpha - 1)t)^{-1/(\alpha-1)}$, we have that

$$\lim_{t \rightarrow +\infty} \frac{P(I > t)}{(c_+(\alpha - 1)t)^{-1/(\alpha-1)}} = \lim_{h \rightarrow 0} \frac{1 - e^{-h}}{h} = 1,$$

which establishes (5.24) and hence completes the proof. \blacksquare

Proof of Theorem 4: Here, we will apply Theorem 1 in [23] to the process $(X_\theta^{\alpha-1}, \widehat{\mathbb{P}}^\uparrow)$. First, we note again that from Theorem 1, we have the following equality

$$\begin{aligned} \mathbb{P}_x \left(Y_{(T_0-t)^-} > (1 + \epsilon) f^{1/(\alpha-1)}(t), \text{ i.o., as } t \rightarrow 0 \right) \\ = \widehat{\mathbb{P}}^\uparrow \left(X_{\theta(t)} > (1 + \epsilon) f^{1/(\alpha-1)}(t), \text{ i.o., as } t \rightarrow 0 \right). \end{aligned}$$

Hence, according to part *i*) in Theorem 1 in [23] and noting that the process $(X_\theta, \widehat{\mathbb{P}}^\uparrow)$ has no positive jumps, the right-hand side of the above equality is equal 0, for all $\epsilon > 0$, if

$$\int_{0+} \exp \left\{ - (c_+(\alpha - 1)t/f(t))^{-1/(\alpha-1)} \right\} \frac{dt}{t} < \infty.$$

In order to prove part *ii*), we note that from Theorem 1, we have

$$\begin{aligned} \mathbb{P}_x \left(Y_{(T_0-t)^-} > (1 - \epsilon) f^{1/(\alpha-1)}(t), \text{ i.o., as } t \rightarrow 0 \right) \\ = \widehat{\mathbb{P}}^\uparrow \left(X_{\theta(t)} > (1 - \epsilon) f^{1/(\alpha-1)}(t), \text{ i.o., as } t \rightarrow 0 \right). \end{aligned}$$

Hence applying part *ii*) of Theorem 1 in [23], we obtain that the above probability is equal 1, for all $\epsilon > 0$, if

$$\int_{0+} \exp \left\{ - (c_+(\alpha - 1)t/f(t))^{-1/(\alpha-1)} \right\} \frac{dt}{t} = \infty,$$

and the proof is complete. \blacksquare

Proof of Theorem 5: From Lemma 1, we have

$$-\log P(I \leq t) = \left(c_+(\alpha - 1)t \right)^{-1/(\alpha-1)}.$$

This fulfills condition (6.19) of Theorem 6 in [24] and hence applying directly the aforementioned result, we deduce the LIL (4.17). Now, the time reversed process $(Y_{(T_0-t)^-}, 0 \leq t \leq T_0)$, under \mathbb{P}_x , is a positive self-similar Markov process starting from 0, with no positive jumps and its upper envelope is described by (4.17); then from Theorem 8 in [24], the reflected process $((Y - \underline{Y})_{(T_0-t)^-}, 0 \leq t \leq T_0)$ also satisfies the same law of the iterated logarithm (4.18). \blacksquare

The following result is crucial for the proof of Theorem 6 and can be seen as a corollary of Proposition 1. Recall the definition

$$I' := \int_0^\infty e^{-(\alpha-1)\xi_s^*} ds.$$

Corollary 2. *There is a positive constant C which only depends on α , such that*

$$P(I' > t) \sim Ct^{-1/(\alpha-1)} \quad \text{as } t \rightarrow \infty \quad (5.25)$$

Proof: According to Lemma 4 in Rivero [26], if η is a non arithmetic Lévy process which drifts towards $-\infty$ and satisfying Cramér's condition for some $\theta > 0$, i.e. $E(\exp \theta \eta_1) = 1$ implies that for $\beta > 0$

$$P\left(\int_0^\infty e^{\beta \eta_s} ds > t\right) \sim Kt^{-\theta/\beta} \quad \text{as } t \rightarrow \infty,$$

where K is a nonnegative constant which depends on β . Hence, recalling Proposition 1, one may apply Lemma 4 in [26] for the process $-\xi^*$ and get

$$P(I' > t) \sim Ct^{-1/(\alpha-1)} \quad \text{as } t \rightarrow \infty,$$

where C is a nonnegative constant which depends on α . ■

Proof of Theorem 6: From to Theorem 3 in [9], we have that

$$\mathbb{P}^\uparrow(Y_t < f^{1/(\alpha-1)}(t), \text{ i.o., as } t \rightarrow 0) = 0 \text{ or } 1,$$

accordingly as

$$\int_{0+} P(I' > t/f(t)) \frac{dt}{t} \text{ is finite or infinite.}$$

Hence from corollary 2, we get the desired result.

The integral test at $+\infty$ is proven through the same way. ■

5.3. Proofs of Theorems 7 and 8

Proof of Theorem 7: From Theorem 1 and since X has no negative jumps, it is clear that

$$\left\{(Y_{(T_0-t)-}, T_0 - U_y \leq t \leq T_0), \mathbb{P}_x\right\} \stackrel{d}{=} \left\{(X_{\theta(t)}, A_{\tau_y^+} \leq t < A_{\sigma_x}), \widehat{\mathbb{P}}^\uparrow\right\} \quad (5.26)$$

where $\tau_y^+ = \inf\{t > 0 : X_t > y\}$. On the other hand, by Theorem 1 in [7], we have that for $z \leq y$

$$\widehat{\mathbb{P}}_y^\uparrow\left(\inf_{t \geq 0} X_t \geq z\right) = \widehat{\mathbb{P}}_y^\uparrow\left(\inf_{0 \leq t \leq \sigma_x} X_t \geq z\right) = \frac{W(y-z)}{W(y)}.$$

Hence from (5.26), the above formula and the Markov property of $(X, \widehat{\mathbb{P}}^\uparrow)$, the first statement of the theorem follows.

Next we remark that it is known for spectrally negative stable processes of index $\alpha \in (1, 2]$ that the scale function $W(x)$ is proportional to $x^{\alpha-1}$. Secondly let $U_y = \sup\{t \geq 0 : Y_t \geq y\}$ and note, again using the pssMp-Lamperti representation of (Y, \mathbb{P}_x) , that

$$P\left(\inf_{0 \leq t \leq D_u} \xi_t \geq v\right) = \mathbb{P}_x\left(\inf_{0 \leq t \leq U_y} Y_t \geq z\right),$$

where $v = \log(z/x)$ and $u = \log(y/x)$ and $z \leq y$. Hence, from Corollary 1, we have

$$P\left(\inf_{0 \leq t \leq D_u} \xi_t \geq v\right) = (1 - e^{v-u})^{\alpha-1},$$

which establishes the conclusion. \blacksquare

Proof of Theorem 8: We start by defining the family of positive self-similar Markov process $\widehat{X}^{(x)}$ whose pssMp-Lamperti representation is given by

$$\widehat{X}^{(x)} = \left(x \exp\left\{\widehat{\xi}_{\widehat{\zeta}^*(t/x^{\alpha-1})}^*\right\}, 0 \leq t \leq x^{\alpha-1}I'\right), \quad x > 0,$$

where $\widehat{\xi}^* = -\xi^*$ and

$$\widehat{\zeta}^*(t) = \inf\left\{t : \int_0^s \exp\left\{(\alpha-1)\widehat{\xi}_u^*\right\} du > t\right\}.$$

Note that the random variable $x^{\alpha-1}I'$ corresponds to the first time at which the process $\widehat{X}^{(x)}$ hits 0, moreover for each $x > 0$, the process $\widehat{X}^{(x)}$ hits 0 continuously.

We now set

$$U_y^- = \sup\{t \geq 0 : Y_t \leq y\} \quad \text{and} \quad \Gamma = Y_{U_y^-}.$$

According to Proposition 1 in [9], the law of the process $\widehat{X}^{(x)}$ killed when hitting 0 is a regular version of the law of the process $\{(Y_{(U_y^- - t)^-}, 0 \leq t \leq U_y^-), \mathbb{P}^\dagger\}$ conditionally on $\{\Gamma = x\}$, $x \in [0, y]$. Hence, the latter process is equal in law to

$$\left(\Gamma \exp\left\{\widehat{\xi}_{\widehat{\zeta}^*(t/\Gamma^{\alpha-1})}^*\right\}, 0 \leq t \leq \Gamma^{\alpha-1}I'\right),$$

and $\widehat{\xi}^*$ is independent of Γ . We deduce that

$$\mathbb{P}^\dagger\left(\sup_{0 \leq s < U_y^-} Y_s \leq z\right) = P\left(\sup_{s \geq 0} \widehat{\xi}_s^* \leq \log(z/\Gamma)\right).$$

On the one hand, it is a well established fact that the all-time supremum of a spectrally negative Lévy process which drifts to $-\infty$ is exponentially distributed with parameter equal to the largest root of its Laplace exponent. In particular, for $x \geq 0$,

$$P\left(\sup_{s \geq 0} \widehat{\xi}_s^* \leq x\right) = 1 - e^{-x}.$$

Note that by inspection of $\Psi^*(\theta)$ the largest root is clearly $\theta = 1$ (there are at most two and one of them is always $\theta = 0$). On the other hand, from the above discussion, the random variables $\widehat{\xi}^*$ and Γ are independent. Hence

$$\mathbb{P}^\dagger\left(\sup_{0 \leq s \leq U_y^-} Y_s \leq z\right) = E\left(1 - \frac{\Gamma}{z}\right).$$

Therefore, in order to complete the proof, it is enough to show that $E(\Gamma) = \frac{y}{m^*}$.

We thus momentarily turn our attention to describing the law of Γ . Let $\overline{H} = (H_t, t \geq 0)$ be the ascending ladder height process associated to ξ^* (see Chapter VI in [1] for a formal definition)

and denote by ν its Lévy measure. According to Lemma 1 in [9], the law of Γ is characterized as follows

$$\log(y^{-1}\Gamma) \stackrel{(d)}{=} -\mathcal{U}\mathcal{Z},$$

where \mathcal{U} and \mathcal{Z} are independent r.v.'s, \mathcal{U} is uniformly distributed over $[0, 1]$ and the law of \mathcal{Z} is given by

$$P(\mathcal{Z} > u) = E(H_1)^{-1} \int_{(u, \infty)} s\nu(ds), \quad u \geq 0.$$

We may now compute

$$\begin{aligned} E(\Gamma) &= y \int_{(0, \infty)} \int_0^1 e^{-uz} du P(\mathcal{Z} \in dz) \\ &= y \int_{(0, \infty)} \frac{1}{z} (1 - e^{-z}) P(\mathcal{Z} \in dz) \\ &= \frac{y}{E(H_1)} \int_{(0, \infty)} (1 - e^{-z}) \nu(dz). \end{aligned} \tag{5.27}$$

Next note that since $\Psi^*(\theta)$ has its largest root at $\theta = 1$, the Wiener-Hopf factorization for the process $\widehat{\xi}^*$ must necessarily take the form $\Psi^*(\theta) = (\theta - 1)\phi(\theta)$ for $\theta \geq 0$, where $\phi(\theta)$ is the Laplace exponent of the descending ladder height process of $\widehat{\xi}^*$. Note that ϕ has no killing term (i.e. $\phi(0) = 0$) as $\widehat{\xi}^*$ drifts to $-\infty$. Moreover, ϕ has no drift term as $\widehat{\xi}^*$ has no Gaussian component (cf. p175 [8]). The latter two observations imply that

$$\int_{(0, \infty)} (1 - e^{-z}) \nu(dz) = \phi(1) = \frac{\Gamma(\theta - 1 + \alpha)}{(\theta - 1)\Gamma(\theta - 1)\Gamma(\alpha)} \Big|_{\theta=1} = 1.$$

Note also that $-m^* = \mathbb{E}(\widehat{\xi}_1^*) = \Psi^{*'}(0+) = -\phi'(0+) = -E(H_1)$. Putting the pieces together in (5.27) completes the proof. \blacksquare

6. Concluding remarks on quasi-stationarity

We conclude this paper with some brief remarks on a different kind of conditioning of CB-processes to (2.8) which results in a so-called quasi-stationary distribution for the special case of the self-similar CB-process. Specifically we are interested in establishing the existence of normalization constants $\{c_t : t \geq 0\}$ such that the weak limit

$$\lim_{t \uparrow \infty} \mathbb{P}_x(Y_t/c_t \in dz | T_0 > t)$$

exists for $x > 0$ and $z \geq 0$.

Results of this kind have been established for CB-processes for which the underlying spectrally positive Lévy process has a second moment in [18]; see also [21]. In the more general setting, [22] formulates conditions for the existence of such a limit and characterizes the resulting quasi-stationary distribution. The result below shows that in the self-similar case we consider in this paper, an explicit formulation of the normalization sequence $\{c_t : t \geq 0\}$ and the limiting distribution is possible.

Lemma 2. Fix $\alpha \in (1, 2]$. For all $x \geq 0$, with $c_t = [c_+(\alpha - 1)t]^{1/(\alpha-1)}$

$$\lim_{t \uparrow \infty} \mathbb{E}_x(e^{-\lambda Y_t/c_t} | T_0 > t) = 1 - \frac{1}{[1 + \lambda^{-(\alpha-1)}]^{1/(\alpha-1)}}.$$

Proof: The proof pursues a similar line of reasoning to the the aforementioned references [18; 21; 22]. From (2.5) it is straightforward to deduce that

$$\lim_{t \uparrow \infty} \mathbb{E}_x(1 - e^{-\lambda Y_t/c_t} | T_0 > t) = \lim_{t \uparrow \infty} \frac{u_t(\lambda/c_t)}{u_t(\infty)}$$

if the limit on the right hand side exists. However, since $\psi(\lambda) = \lambda^\alpha$ it is easily deduced from (2.6) that

$$u_t(\lambda) = [c_+(\alpha - 1)t + \lambda^{-(\alpha-1)}]^{-1/(\alpha-1)}$$

and the result follows after a straightforward calculation. ■

Although quasi-stationarity in the sense of ‘conditioning to stay positive’ does not make sense in the case of the CBI-process $(Y, \mathbb{P}_x^\uparrow)$, it appears that the normalizing constants $\{c_t : t \geq 0\}$ serve a purpose to obtain the convergence in distribution below. A similar result is obtained in [18] for CBI-processes whose underlying Lévy process has finite variance.

Lemma 3. *Fix $\alpha \in (1, 2]$. For all $x \geq 0$, with $c_t = [c_+(\alpha - 1)t]^{1/(\alpha-1)}$*

$$\lim_{t \uparrow \infty} \mathbb{E}_x^\uparrow(e^{-\lambda Y_t/c_t}) = \frac{1}{[\lambda^{(\alpha-1)} + 1]^{\alpha/(\alpha-1)}}.$$

Proof: We follow ideas found in Lambert [18]. In the latter paper, it is shown that $(Y, \mathbb{P}_x^\uparrow)$ may also be obtained as the Doob h -transform of the process (Y, \mathbb{P}) with $h(x) = x$. That is to say

$$\mathbb{E}_x^\uparrow(e^{-\lambda Y_t}) = \mathbb{E}_x\left(\frac{Y_t}{x} e^{-\lambda Y_t}\right).$$

Differentiating (2.5) this implies that

$$\mathbb{E}_x^\uparrow(e^{-\lambda Y_t}) = e^{-x u_t(\lambda)} \frac{\psi(u_t(\lambda))}{\psi(\lambda)}.$$

Plugging in the necessary expressions for ψ and $u_t(\lambda)$ as well as replacing λ by λ/c_t in the previous formula, the result follows directly. ■

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