The monotone class theorem

In this section, we will discuss the monotone class theorem in the form we find most useful for application to our course (and also to probability theory).

DEFINITION 1. *Let Ω be a set and $\mathcal{S}$ a collection of subsets of Ω; then*

i) *$\mathcal{S}$ is a π-system (on Ω) if $\mathcal{S}$ is closed under finite intersections;*

ii) *$\mathcal{S}$ is a δ-system (on Ω) if*

a) *$\Omega \in \mathcal{S}$,*

b) *if $A, B \in \mathcal{S}$, $A \subset B$, then $B \setminus A \in \mathcal{S}$,*

c) *if $(A_n, n \geq 1)$ is an increasing sequence of elements of $\mathcal{S}$, then $\cup A_n \in \mathcal{S}$.*

Obviously any σ-field is both a δ-system and π-system. Conversely, one easily verifies that if $\mathcal{S}$ is both a δ-system and π-system, then $\mathcal{S}$ is a σ-field. If $\mathcal{M}$ is any collection of subsets of Ω, then we define $\delta(\mathcal{M})$ to be the smallest δ-system on Ω containing $\mathcal{M}$. The existence of $\delta(\mathcal{M})$ is clear since the intersection of an arbitrary number of δ-systems is again a δ-system. We will say that $\delta(\mathcal{M})$ is the δ-system generated by $\mathcal{M}$.

THEOREM 1. *Let $\mathcal{S}$ be a π-system on Ω; then $\delta(\mathcal{S}) = \sigma(\mathcal{S})$.*

Proof: Since $\delta(\mathcal{S}) \subset \sigma(\mathcal{S})$ it is enought to show that $\delta(\mathcal{S})$ is a σ-field (since $\sigma(\mathcal{S})$ is the smallest σ-field containing $\mathcal{S}$), and for this it suffices to show that $\delta(\mathcal{S})$ is a π-system.
Lets define

$$\mathcal{D}_1 = \Big\{ B \in \delta(\mathcal{S}) : B \cap A \in \delta(\mathcal{S}) \text{ for all } A \in \mathcal{S} \Big\}.$$

We first prove that $\mathcal{S} \subset \mathcal{D}_1$. This is clear, since $\mathcal{S}$ is a π-system. More precisely, we have that for $B \in \mathcal{S}$, $B \cap A \in \mathcal{S}$ for all $A \in \mathcal{S}$ which implies that $B \in \mathcal{D}_1$.
Next, we verify that $\mathcal{D}_1$ is a δ-system. Conditions (a) and (c) are easy to verify. Hence we just need to show that condition (b) holds. Suppose that $B, C \in \mathcal{D}_1$ such that $B \subset C$, hence for any $A \in \mathcal{S}$, we have $B \cap A, C \cap A \in \delta(\mathcal{S})$ and $B \cap A \subset C \cap A$ which implies that $C \cap A \setminus B \cap A \in \delta(\mathcal{S})$ but note that $(C \setminus B) \cap A = C \cap A \setminus B \cap A$, therefore $C \setminus B \in \mathcal{D}_1$. Since $\delta(\mathcal{S})$ is the smallest δ-system containing $\mathcal{S}$, we have that $\delta(\mathcal{S}) \subset \mathcal{D}_1$ but by definition $\mathcal{D}_1 \subset \delta(\mathcal{S})$ and so $\delta(\mathcal{S}) = \mathcal{D}_1$. Next define

$$\mathcal{D}_2 = \Big\{ B \in \delta(\mathcal{S}) : B \cap A \in \delta(\mathcal{S}) \text{ for all } A \in \delta(\mathcal{S}) \Big\}.$$

Again one shows without difficulty that $\mathcal{D}_2$ is a δ-system (in fact we use the same arguments as above). If $A \in \mathcal{S}$, then for all $B \in \mathcal{D}_1$ (which is $\delta(\mathcal{S})$) we have that $B \cap A \in \delta(\mathcal{S})$, and consequently $\mathcal{S} \subset \mathcal{D}_2$. Hence $\mathcal{D}_2 = \delta(\mathcal{S})$, and this is just the statement that $\delta(\mathcal{S})$ is closed under finite intersections. Thus Theorem 1 is established. ∎

We give a version of the previous Theorem that deals with functions rather than sets.

THEOREM 2. *Let Ω be a set and $\mathcal{S}$ a π-system on Ω. Let $\mathcal{H}$ be a vector space of real-valued functions on Ω satisfying:*

i) $1 \in \mathcal{H}$ *and* $\mathbb{I}_A \in \mathcal{H}$ *for all* $A \in \mathcal{S}$*;*
ii) *if* $(f_n, n \geq 1)$ *is an increasing sequence of nonnegative functions in* $\mathcal{H}$ *such that* $f = \sup_n f_n$ *is finite (bounded), then* $f \in \mathcal{H}$*.*

Under these assumptions $\mathcal{H}$ *contains all real-valued (bounded) functions on* Ω *that are* $\sigma(\mathcal{S})$ *measurable.*

Proof: Let $\mathcal{D} = \{A : \mathbb{I}_A \in \mathcal{H}\}$. According to assumption (i), $\Omega \in \mathcal{H}$ and $\mathcal{S} \subset \mathcal{D}$. If $A_1 \subset A_2$ are in $\mathcal{D}$ then $\mathbb{I}_{A_2 \setminus A_1} = \mathbb{I}_{A_2} - \mathbb{I}_{A_1} \in \mathcal{H}$ since $\mathcal{H}$ is a vector space, and so $A_2 \setminus A_1 \in \mathcal{D}$. Finally, if $(A_n, n \geq 1)$ is an increasing sequence of sets in $\mathcal{D}$ then $\mathbb{I}_{\cup_n A_n} = \sup_n \mathbb{I}_{A_n}$ which is in $\mathcal{H}$ by (ii) . Thus $\mathcal{D}$ is a δ-system on Ω containing $\mathcal{S}$ and hence $\sigma(\mathcal{S}) \subset \mathcal{D}$ by Theorem 1.
If $f \in \sigma(\mathcal{S})$ is real-valued function then $f = f^+ - f^-$ with f^+ and f^- being nonnegative, real-valued, and $\sigma(\mathcal{S})$ measurable. On the other hand, if $f \in \sigma(\mathcal{S})$ is nonnegative then f is an increasing limit of simple fucntions $f_n = \sum_{i=1}^n a_i^{(n)} \mathbb{I}_{A_i^n}$ with each $A_i^n \in \sigma(\mathcal{S})$. Hence each $f_n \in \mathcal{H}$ and using (ii) the result is now immediate. ■

Theorems 1 and 2, will be used in the following form: Let Ω be a set and $(E_i, \mathcal{E}_i)_{i \in I}$ be a family of measurable spaces indexed by an arbitrary set I. For each $i \in I$, let $\mathcal{S}_i$ be a π-system generating $\mathcal{E}_i$ and let f_i be a map from Ω to E_i. Using this set up we stat the following two propositions.

PROPOSITION 1. *Let* $\mathcal{S}$ *consist of all sets of the form* $\cap_{i \in J} f^{-1}(A_i)$ *where* $A_i \in \mathcal{S}_i$ *for* $i \in J$ *and* J *ranges over all finite subset of* I*. Then* $\mathcal{S}$ *is a* π*-system on* Ω *and* $\sigma(\mathcal{S}) = \sigma(f_i, i \in I)$*.*

PROPOSITION 2. *Let* $\mathcal{H}$ *be a vector space of real-valued functions on* Ω *satisfying:*

i) $1 \in \mathcal{H}$ *and containing all the functions* $\mathbb{I}_A$ *for* $A \in \mathcal{S}$*.*
ii) *if* $(f_n, n \geq 1)$ *is an increasing sequence of nonnegative functions in* $\mathcal{H}$ *such that* $f = \sup_n f_n$ *is bounded, then* $f \in \mathcal{H}$*.*

Under these assumptions $\mathcal{H}$ *contains all bounded, real-valued functions on* Ω *that are* $\sigma(\mathcal{S})$ *measurable.*

Proposition 1 is immediate and Proposition 2 follows from Proposition 1 and Theorem 3. In the sequel any of the above results will be referred to as the Monotone Class Theorem.