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A brief introduction to self-similar processes.

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1 Introduction.

Self-similar processes are stochastic processes that are invariant in distribution under suitable scaling of time and space. These processes can be used to model many space-time scaling random phenomena that can be observed in physics, biology and other fields. To give just a few examples let us simply mention stellar fragments, growth and genealogy of populations, option pricing in finance, various areas of image processing, climatology, environmental science, ...

Self-similar processes appear in various parts of probability theory, such as in Lévy processes, branching processes, statistical physics, fragmentation theory, coalescent theory, random fields, ... Some well known examples are: stable Lévy process, fractional Brownian motion, Feller branching diffusion, Bessel processes, self-similar fragmentations, brownian sheet, ...

The first rigurous probabilistic study of self-similar processes is due to J. Lamperti [13] at the beggining of the sixties. In [13], the author was interested in characterize the following limit in distribution

$$\lim_{r\to\infty}\frac{Y_{rt}}{f(r)}, \tag{1.1}$$

where Y is a stochastic processes on $\mathbb{R}$ and f is an increassing real function. The main result in [13] affirms that if the law of X is obtained as a nontrivial limit as above, then there exists $\alpha > 0$ such that for any $k \geq 0$, X satisfies

$$(X_{kt}, t \geq 0) \overset{(\mathrm{d})}{=} \Big(k^{\alpha} X_t, t \geq 0\Big).$$

A stochastic process X satisfying the above condition is called self-similar process.

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The aim of this note is to survey some remarkable properties of self-similar processes. In particular, we are interested in self-similar processes with independent increments or self-similar additive processes. The relation that exist between such class of self-similar processes, generalized Ornstein-Uhlenbeck processes and a particular class of Lévy processes is proved in detail. We will assume that the reader is related with the theory of infinitely divisible distributions and Lévy processes.

2 Self-similar processes.

In this section, we introduce self-similar processes. Some remarkable properties of such processes are studied, as the main result in [13] which characterize the limit distribution (1.1), and the Lamperti representation which gives us a bijection between self-similar processes and stationary processes.

2.1 Definition and examples

In the sequel, we consider stochastic processes defined on $\mathcal{D}$, the space of Skorokhod of càdlàg paths on $[0,\infty)$ with values in $\mathbb{R}$. This space is endowed with the J_1-Skorokhod's topology. In particular $\mathcal{D}$ is a polish space (metric-complete and separable). We denote by $\mathcal{F}$ for the Borel σ-field of the open subsets of $\mathcal{D}$ and $\mathbb{P}$ will be our reference probability measure.
Recall that a stochastic process $(X_t, t \geq 0)$ is *trivial* if the law of X_t is a dirac measure for every $t > 0$.

Definition 1 *A nontrivial stochastic processes $X = (X_t, t \geq 0)$ taking values on $\mathbb{R}$ is said to be self-similar if for any $a \geq 0$, there exist $b(a) \in \mathbb{R}$ such that*

$$(X_{at}, t \geq 0) \stackrel{(d)}{=} \Big(b(a)X_t, t \geq 0\Big). \tag{2.2}$$

In [13], Lamperti proved that $b(a)$ can be expressed as $b(a) = a^\gamma$, where $\gamma \geq 0$. This fact is not difficult to see it: From the right-continuity at 0 of the process X and the self-similar property (2.2), it follows that the law of X_t is not a dirac measure for any $t > 0$. Now, we take $a, c \geq 0$ and from the self-similar property we get

$$X_{act} \stackrel{(d)}{=} b(ac)X_t \stackrel{(d)}{=} b(a)X_{ct} \stackrel{(d)}{=} b(a)b(c)X_t.$$

From the above equality in distribution, we get $X_{a^n} = b^n(a)X_1$, $n \geq 1$. Since the process is nontrivial, the function b is increasing which implies that there exists $\gamma \in \mathbb{R}_+$ such that $b(a) = a^\gamma$.
On the other hand, it is clear that if $\gamma > 0$, then $X_0 = 0$ a.s. When $\gamma = 0$ then X is a.s. constant, i.e. $X_t = X_0$ for all $t \geq 0$, a.s. Hence the definition of self-similar processes becomes:

Definition 1' *A nontrivial stochastic processes $X = (X_t, t \geq 0)$ taking values on $\mathbb{R}$ is said to be self-similar if for any $a \geq 0$, there exist $\gamma > 0$ such that*

$$(X_{at}, t \geq 0) \stackrel{(d)}{=} \Big(a^\gamma X_t, t \geq 0\Big).$$

The process X is said to be self-similar of index $\gamma = 0$ if it is a.s. constant.

In the recent literature, self-similar processes are defined as above but there is a more general way to define them. We refer to the Lectures Notes on self-similar processes of Chaumont [6] for a more general definition.
Before establishing the main results of this section, we first introduce some important examples of self-similar processes.

Example 1. Recall that a Lévy process is a stochastic process defined on $\mathcal{D}$ with independent and stationary increments, i.e. if $X = (X_t, t \geq 0)$ is a such process then for any $t \geq 0$ and $s \geq 0$,

$$X_{t+s} - X_t \quad \text{is independent of} \quad (X_u, 0 \leq u \leq t),$$

and has the same law as X_s. Also recall that the class of all Lévy processes is in one-to-one correspondence with the class of all infinitely divisible distributions.
In this example, we are interested in a particular sub-class of infinitely divisible distributions, the well-know stable distributions.

Definition 2 *Let μ be an infinitely divisible probability measure on $\mathbb{R}$ and $\hat{\mu}(z)$ denotes its characteristic function for $z \in \mathbb{R}$, i.e.*

$$\hat{\mu}(z) = \int_{\mathbb{R}^d} e^{i<z,x>} \mu(\mathrm{d}x), \qquad z \in \mathbb{R}.$$

The probability measure μ is said to be stable, if for any $a > 0$ there are $b > 0$ and $c \in \mathbb{R}$ such that

$$\big(\hat{\mu}(z)\big)^a = \hat{\mu}(bz)e^{i<c,z>}, \qquad \textit{for all } z \in \mathbb{R}.$$

Moreover, μ is said to be strictly stable, if for any a, there is $b > 0$ such that

$$\big(\hat{\mu}(z)\big)^a = \hat{\mu}(bz), \qquad \textit{for all } z \in \mathbb{R}.$$

According to Sato [18] (see theorems 14.1, 14.2 and 14.3) the constant $b = a^{1/\alpha}$, where $\alpha \in (0, 2]$. The index α is known as the stability index. In particular, when $\alpha = 2$ the law of μ corresponds to the Gaussian case.
Here, we are interested in strictly stable distributions and in all the sequel we may refer to them as α-stable. We say that X is a stable Lévy process if X is a Lévy process and the law of X_1 is α-stable. Note that when $\alpha = 2$, X is proportional to a Brownian motion.

The following proposition shows us that stable Lévy processes are the only self-similar processes with independent and stationary increments.

Proposition 1 *Let X be a Lévy process with values in $\mathbb{R}$ and such that $X_0 = 0$ a.s. The law of X_1 is α-stable if and only if X is self-similar. Moreover, the scaling index is equal to $1/\alpha$.*

Proof: Let $\hat{\mu}_t(\lambda) = \mathbb{E}(e^{i\lambda X_t})$ and suppose that X is a self-similar process of index $1/\alpha > 0$. From definition 1', we have that $X_t = t^{1/\alpha} X_1$ for any $t > 0$. On the other hand, since X is a Lévy processes we known that

$$\hat{\mu}_t(\lambda) = \big(\hat{\mu}_1(\lambda)\big)^t, \qquad \text{for all} \quad \lambda \in \mathbb{R},$$

hence $\big(\hat{\mu}_1(\lambda)\big)^t = \hat{\mu}_1(t^{1/\alpha}\lambda)$, which implies that the law of X_1, denoted by μ_1, is α-stable.
Now, let us suppose that the law of X_1 is α-stable. Since X is a process with independent and stationary increments, it is enough to show that for any $t > 0$ and $a > 0$, $X_{at} \overset{\text{(d)}}{=} a^{1/\alpha} X_t$. However, $\hat{\mu}_{at}(\lambda) = \big(\hat{\mu}_1(\lambda)\big)^{at} = \hat{\mu}_1^t(a^{1/\alpha}\lambda) = \hat{\mu}_t(a^{1/\alpha}\lambda)$, which completes the proof. ■

It is important to note that stable Lévy processes also belong to the class of self-similar additive processes.

Example 2. Another important example is the fractional Brownian motion, which is a self-similar Gaussian process.

Definition 3 *Let $H \in (0, 1]$. The fractional Brownian motion is a mean-zero Gaussian process $B^{(H)} = (B_t^{(H)}, t \geq 0)$ which covariance is given by*

$$\mathbb{E}\Big(B_t^{(H)} B_s^{(H)}\Big) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})\mathbb{E}\Big(\big(B_1^{(H)}\big)^2\Big).$$

Note that when $H = 1/2$, the process $B^{(1/2)}$ is the standard Brownian motion.

Proposition 2 *The fractional Brownian motion $B^{(H)}$ is a self-similar process with scaling index H.*

Proof: From its covariance, it is clear

$$\begin{aligned}\mathbb{E}\Big(B_{at}^{(H)} B_{as}^{(H)}\Big) &= \frac{1}{2}\Big((at)^{2H} + (as)^{2H} - (a|t-s|)^{2H}\Big)\mathbb{E}\Big(\big(B_1^{(H)}\big)^2\Big) \\ &= a^{2H}\mathbb{E}\Big(B_t^{(H)} B_s^{(H)}\Big).\end{aligned}$$

Since $B^{(H)}$ is a Gaussian process, the above equality shows that the processes $a^H B^{(H)}$ and $(B_{at}^{(H)}, t \geq 0)$ have the same law. ■

The fractional Brownian motion (FBM for short) also have stationary increments, this can be easily seen using again its covariance and the fact that $B^{(H)}$ is a Gaussian process. We also note that when $H \neq 1/2$, the FBM is neither a Markov process and nor a semi-martingale. In particular, Ito's calculus does not work for the FBM. Recently many authors introduce different methods to define a stochastic integral with respect to the FBM, many of them use the Hölder continuity of the paths of $B^{(H)}$. In order to give some examples, we just mention: the Young integral, Skorokhod integral (via Malliavin Calculus), rough paths, ...

Self-similar processes with stationary increments is probably the most studied class of self-similar processes (thanks to the FBM). The stationarity of the increments allow us to obtain nice properties of their trajectories. In this note we will not study such class of processes, we refer to the monographe of Embrechts and Maejima [8] and Chaumont [6] for a deep discussion on this topic.

Example 3. Now let us mention an example of self-similar processes with independent increments (or self-similar additive processes) which does not have stationary increments, i.e. not a stable Lévy process.
Let X be a self-similar Markov process with no positive jumps taking values in $\mathbb{R}_+$ and starting from $x \geq 0$. From the Markov property, self-similarity and the absence of positive jumps, it is clear that the first passage time

$$y \longmapsto \inf \big\{ t : X_t = y \big\},$$

is an increasing self-similar process with independent increments. Self-similar processes with independent increments are also called as processes of class L and have an important connection with generalized Ornstein-Uhlenbeck processes and Lévy processes. Such connection will be studied in section 3.

2.2 Convergence theorems and "first" Lamperti representation.

Self-similar processes are obtained as weak limits of rescaled stochastic processes. This phenomena is well-known for stable Lévy processes and branching processes since long time ago (see for instance Lamperti [14, 15] and Grimvall [10]) and recently remarked for Lévy trees [7] and fragmentation processes [3, 4].
Here, we are interested in sequences (X_n) of random variables which converge in law after a change of location and scale. That is, there exist centring constants b_n and norming constants $a_n > 0$ such that

$$\frac{X_n - b_n}{a_n} \stackrel{\text{(d)}}{\longrightarrow} X, \tag{2.3}$$

where "$\stackrel{\text{(d)}}{\longrightarrow}$" means convergence in distribution or in law. Recall that two random variables X and Y are said to belong to the same type if they have the same law after change of location and scale, i.e. if

$$Y \stackrel{\text{(d)}}{=} uX + v, \qquad \text{for some } u > 0 \text{ and } v \in \mathbb{R}^d,$$

where "$\stackrel{\text{(d)}}{=}$" means equality in law.
The following result knonw as Convergence of Types Lemma means in particular that, to whitin type, the convergence (2.3) cannot hold in two differents ways. Such result will not be proved here but its proof can be found in Gnedenko and Kolmogorov [9] or in chapter 1 of [6].

Lemma 1 *Let X, Y and (X_n) a.s. finite random variables taking values in $\mathbb{R}$. We suppose that the law of Y is nontrivial. If there exist real sequences $a_n > 0$, $\alpha_n > 0$, $b_n \in \mathbb{R}$ and $\beta_n \in \mathbb{R}$ verifying*

$$\frac{X_n - b_n}{a_n} \xrightarrow{(d)} X \quad \text{and} \quad \frac{X_n - \beta_n}{\alpha_n} \xrightarrow{(d)} Y, \qquad \text{as } n \to \infty, \tag{2.4}$$

then

$$\frac{a_n}{\alpha_n} \to u \in (0, \infty) \quad \text{and} \quad \frac{b_n - \beta_n}{\alpha_n} \to v \in \mathbb{R}, \qquad \text{as } n \to \infty. \tag{2.5}$$

Moreover if the relations in (2.4) and (2.5) are satisfied, u and v are the unique constants such that $Y \stackrel{(d)}{=} uX + v$.

Regularly varying functions are so important in the theory of self-similar processes since they appear in their construction (see Theorem 2 below). Here, we will only recall their definition and some important properties which will be needed for the proof of Theorem 2. We refer to the monographe of Bingham et al. [5] for a deep study on this subject.

Definition 4 *A function $L : \mathbb{R}_+ \to \mathbb{R}$ is said to be a slowly varying function in $+\infty$, if for any $a > 0$,*

$$\lim_{x \to +\infty} \frac{L(ax)}{L(x)} = 1.$$

Moreover, a function $F : \mathbb{R}_+ \to \mathbb{R}$ is said to be a regularly varying function of index β in $+\infty$, if for any $a > 0$,

$$\lim_{x \to +\infty} \frac{F(ax)}{F(x)} = a^\beta.$$

It is not difficult to see that if $F : \mathbb{R}_+ \to \mathbb{R}$ is a regularly varying function of index β, then there exists L a slowly varying function such that

$$F(x) \sim x^\beta L(x), \qquad \text{as} \quad x \to +\infty.$$

The following result is an important characterization of regularly varying functions.

Theorem 1 *Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a Lebesgue measurable function. If F satisfies*

$$\lim_{x \to +\infty} \frac{F(ax)}{F(x)} \in (0, \infty), \qquad \text{for any } a > 0,$$

then F is a regularly varying function.

Proof: Let

$$G(a) = \lim_{x \to \infty} \frac{F(ax)}{F(x)}, \qquad \text{for} \quad a > 0.$$

For $b > 0$, we have

$$G(ab) = \lim_{x \to \infty} \frac{F(abx)}{F(x)} = \lim_{x \to \infty} \frac{F(abx)}{F(bx)} \frac{F(bx)}{F(x)} = G(a)G(b).$$

The function $G : (0,\infty) \to (0,\infty)$ is multiplicative. Since G is measurable and multiplicative then there exists some $\beta \in \mathbb{R}$ such that $G(a) = a^\beta$. Finally, let $L(x) = F(x)/x^\beta$. Therefore for every $\lambda > 0$,

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1,$$

and the result follows. ■

The next result, due to Lamperti [13], connects self-similar processes with limit theorems.

Theorem 2 *Let Y be a $\mathbb{R}$-valued stochastic process. If there exist a process X such that X_1 is nontrivial and f a Borel function satisfying that $\lim_{r\to\infty} f(r) = \infty$ and*

$$\frac{Y_{rt}}{f(r)} \longrightarrow X_t \qquad \text{as } r \to \infty, \tag{2.6}$$

holds in the sense of finite-dimensional distributions, then the process X is self-similar with index $\gamma > 0$. Moreover, f is a regularly varying functions of index γ.
Conversely, any self-similar process X of strictely positive index may be obtained as the limit in law (2.6).

Proof: Assume that (2.6) is satisfied and choose a real sequence (r_n) which diverge towards $+\infty$. We first prove that f is a regularly varying function. From (2.6), we deduce that for $t > 0$ and any $y \in \mathbb{R}$

$$\begin{aligned} \mathbb{P}\Big(Y_{r_n} \le f(r_n)y\Big) &\longrightarrow \mathbb{P}(X_1 \le y), \\ \mathbb{P}\Big(Y_{r_n} \le f(r_n t)y\Big) &\longrightarrow \mathbb{P}(X_{1/t} \le y), \end{aligned}$$

as n goes to $+\infty$. Hence from Lemma 1 and the fact that X_1 is nontrivial, the limit of $f(r_n t)/f(r_n)$, as n goes to $+\infty$, exists and is strictely positive. Note that for $t > 0$ the above limit holds for any sequence which diverge towards $+\infty$. Then the limit exists taking $r \to \infty$. By Theorem 1, the function f is regularly varying and since f diverge towards $+\infty$, its index γ is positive or zero. Since the function f diverges towards $+\infty$ and taking $t = 0$ in (2.6), it is clear that $X_0 = 0$ a.s.
Now, we will prove the self-similar property of the process X. Let $a > 0$, $(t_1, \ldots, t_n) \in \mathbb{R}_+^n$ and $(x_1, \ldots, x_n)$ a continuity point for the distribution function of $(X_{at_1}, \ldots, X_{at_n})$ such that $(a^{-\gamma}x_1, \ldots, a^{-\gamma}x_n)$ is a continuity point for the distribution function of $(X_{t_1}, \ldots, X_{t_n})$ as well. Then from (2.6) we have

$$\lim_{r\to+\infty} \mathbb{P}\left(\frac{Y_{rat_1}}{f(r)} \le x_1, \ldots, \frac{Y_{rat_n}}{f(r)} \le x_n\right) = \mathbb{P}(X_{at_1} \le x_1, \ldots, X_{at_n} \le x_n).$$

The distribution function of the left hand-side of the above equality can be written as

$$\mathbb{P}\left(\frac{Y_{rat_1}}{f(ar)} \le \frac{f(r)}{f(ar)}x_1, \ldots, \frac{Y_{rat_n}}{f(ar)} \le \frac{f(r)}{f(ar)}x_n\right).$$

Hence, taking the limit when r goes to $+\infty$ we obtain that the above distribution function converge towards

$$\mathbb{P}\Big(X_{t_1} \leq a^{-\gamma}x_1, \dots, X_{t_n} \leq a^{-\gamma}x_n\Big),$$

which proves that the process X is self-similar. The case where $\gamma = 0$ is not possible since we are assuming that the process X is nontrivial.
The second part of the proof is evident. Let X be a self-similar process of index $\gamma > 0$. In order to obtain (2.6), it is enough to take $Y = X$ and $f(r) = r^\gamma$ and then apply the scaling property of X. ■

There exists a version of this theorem for discrete-time stochastic processes. In order to establish such result, we first recall the definition of regularly varying sequence: a sequence $a(n)$ is regularly varying if there exists $\gamma \in \mathbb{R}$ such that for all $t > 0$

$$\lim_{n\to+\infty} \frac{a(nt)}{a(n)} = t^\gamma.$$

Theorem 3 *Let (Y_n) be a sequence of a.s. finites random variables taking values in $\mathbb{R}$. If there exist a process X a.s. finite such that X_1 is nontrivial and $a(n)$ a sequence satisfying that $\lim_{n\to+\infty} a(n) = +\infty$ and*

$$\frac{Y_{[nt]}}{a(n)} \longrightarrow X_t \qquad \text{as } n \to +\infty, \tag{2.7}$$

holds in the sense of finite-dimensional distributions, then the process X is self-similar with index $\gamma > 0$. Moreover, $a(n)$ is a regularly varying sequence of index γ.
Conversely, if a self-similar process X of strictly positive index may be obtained as the limit in law (2.7).

The idea of the proof of the above theorem is basically the same as that of the continuous version, but note that it requires additional results on regularly varying sequences. We refer to section 8.5 of the monographe of Bingham, Goldie and Teugels [5] for its proof.

Now, we introduce the "first" Lamperti representation. In this direction, we recall the definition of stationary process.

Definition 5 *A stochastic process $(Y_t, t \in \mathbb{R})$ is said to be stationary if its law remains equal regardless of any shift, θ_t for any t, in time, i.e. if it verifies the following identity in law*

$$(Y_s, s \in \mathbb{R}) \overset{(d)}{=} (Y_{t+s}, s \in \mathbb{R}) \overset{(def)}{=} Y \circ \theta_t, \qquad \text{for all} \quad t \in \mathbb{R}.$$

Proposition 3 ("First" Lamperti representation) *If $(Y_t, t \in \mathbb{R})$ is an stationary process, then for all $\gamma > 0$ the process defined by*

$$X_t \overset{(def)}{=} t^\gamma Y_{\log t}, \qquad \text{for} \quad t > 0 \quad \text{and} \quad X_0 = 0,$$

is a self-similar process of index $\gamma > 0$. Conversely, if the process X is self-similar of index $\gamma > 0$ and such that $X_0 = 0$, then the processus defined by

$$(Y_t, t \in \mathbb{R}) \stackrel{(def)}{=} \left(e^{-\gamma t} X_{e^t}, t \in \mathbb{R}\right),$$

is stationary.

Proof: Let $n \geq 1$ and $\beta_1, \dots, \beta_n$ real numbers. Let us suppose that the process Y is stationary, then for any $a, \gamma > 0$ and $t_1, ..., t_p \in \mathbb{R}_+$,

$$\sum_{j=1}^{n} \beta_j X_{at_j} = \sum_{j=1}^{n} \beta_j a^\gamma t_j^\gamma Y_{\log a + \log t_j} \stackrel{\text{(d)}}{=} \sum_{j=1}^{n} \beta_j a^\gamma t_j^\gamma Y_{\log t_j} = \sum_{j=1}^{n} \beta_j a^\gamma X_{t_j},$$

proving that X is self-similar of index γ.
Conversely, if X is self-similar of index $\gamma > 0$, then for any $h > 0$ and $t_1, ..., t_p \in \mathbb{R}_+$,

$$\sum_{j=1}^{n} \beta_j Y_{t_j + h} = \sum_{j=1}^{n} \beta_j e^{-\gamma t_j} e^{-\gamma h} X_{e^{t_j} e^h} \stackrel{\text{(d)}}{=} \sum_{j=1}^{n} \beta_j e^{-\gamma t_j} X_{e^{t_j}}) = \sum_{j=1}^{n} \beta_j Y_{t_j},$$

proving that Y is stationary. ∎

The stationary process related with a self-similar process X by the "first" Lamperti representation is called some times as the Ornstein-Uhlenbeck process associated to X. This name comes from the known Ornstein-Uhlenbeck process which is defined by $Y = (e^{-t/2} B^{(1/2)}_{e^t}, t \in \mathbb{R})$ where $B^{(1/2)}$ is the standard Brownian motion.

It is important to note that the self-similar (or scaling) property by its self does not allow us to determine the distributions of self-similar processes. Actually it was noted in [1] that even in the case when self-similar processes have stationary increments there is no simple characterization of the possible families of their marginal distributions. This is not the case for self-similar processes with independent increments. We will see in the next section that in fact self-similar additive processes is related to the class of self-decomposable distribution as Lévy processes is related to the class of infinitely divisible distributions.

3 Self-similar additive processes.

This section introduce the laws of class L and self-similar additive processes (ssap for short), i.e. self-similar processes with independent increments. The connection of the laws of class L and ssap is described. Self-similar additive processes are related to a particular class of Lévy processes, such connection is also described here. In particular we will obtain that the stationary process Y related with a ssap by the firs Lamperti representation is the solution of the Ornstein-Uhlenbeck equation associated with a Background Driving Lévy process ξ

$$\mathrm{d}Y_t = -\gamma Y_t \,\mathrm{d}t + \mathrm{d}\xi_t,$$

with initial condition $Y_0 = H$ is an independent r.v. of ξ.

3.1 Laws of Class L and self-decomposable laws.

Roughly speaking a random variable has a distribution of class L if the random variable has the same distribution as the limit of some normalized sequence of sums of independent random variables. More precisely, consider a sequence $(X_n, n \geq 1)$ of independent random variables on $\mathbb{R}$ and let $S_n = \sum_{k=1}^{n} X_k$ denote their sum. Suppose that there exist two sequences $c(n) \in \mathbb{R}$ and $b(n) > 0$ such that for every $\epsilon > 0$

$$\lim_{n\to\infty} \max_{1\leq k\leq n} \mathbb{P}(b_n|X_k| > \epsilon) = 0, \tag{3.8}$$

and the distribution of $b(n)S_n + c(n)$ converges to the distribution of some random variable X on $\mathbb{R}$. The random variable X is said to be on the class L.
From Theorem 9.3 in Sato [18] and condition (3.8), we deduce that the distribution of any random variable in the class L is infinitely divisible. The class L have been studied by many authors, the oldest of these is due to Levy [16]. Khintchine [12] seems to be the first to use the name “Class L”. Well-known examples of laws of class L are the Gaussian and stable distributions (see Theorem 5 below).

The following definition is an extension of the notion of stable distributions.

Definition 6 *Let μ be a probability measure on $\mathbb{R}$ and $\hat{\mu}$ denotes its characteristic function. We say that μ is self-decomposable if for any $b > 1$, there is a probability measure ρ_b on $\mathbb{R}$ such that*

$$\hat{\mu}(z) = \hat{\mu}(bz)\hat{\rho}(z), \qquad z \in \mathbb{R},$$

where $\hat{\rho}$ denotes the characteristic function of ρ_b.

In terms of random variables this definition becomes: a random variable X is said to be self-decomposable if for any constant $c \in (0,1)$ there exists an independent random variable, say $X^{(c)}$ such that

$$X \overset{(d)}{=} cX + X^{(c)}.$$

In other words, a random varialbe is self-decomposable if it has the same distribution as the sum of a scaled down version of itself and an independent residual random variable. An important example of such class of distributions are stable distribution.

The following theorem shows that a random variable has a distribution of class L if and only if the law of the random variable is self-decomposable. The proof of this theorem can be found in Sato [18] (Theorem 15.3).

Theorem 4 *Let $(X_n, n \geq 1)$ be a sequence of independent random variables on $\mathbb{R}$ and S_n denote their sum as above. Let X be a random variable with values on $\mathbb{R}$ which has a distribution of class L, then X is self-decomposable.*
Conversely, for any self-decomposable random variable X on $\mathbb{R}$ we can find a sequence $(X_n, n \geq 1)$ of independent random variables on $\mathbb{R}$ and two sequences $a(n) \in \mathbb{R}$ and $b(n) > 0$ satisfying that the distribution of $b(n)S_n + a(n)$, where S_n is defined as before, converge to the distribution of X and (3.8).

The above result give us automatically that the class of self-decomposable distributions is a sub-class of infinitely divisible distributions. In particular, if μ is self-decomposable its Lévy measure has the following structure

$$\Pi(\mathrm{d}x) = \frac{k(x)}{|x|}\mathrm{d}x,$$

where k is a positive measurable function increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$, i.e that the Levy measure Π is unimodal with mode 0. Yamazato [20] proved that self-decomposable laws in $\mathbb{R}$ have the unimodal property, i.e. that their densities are unimodal.

The following limit theorem gives a characterization of stable distributions as in the case of self-decomposable distributions. In particular, we obtain that stable distributions are self-decomposable and unimodal.

Theorem 5 *A probability measure μ on $\mathbb{R}$ is stable if and only if there are a random walk $(S_n, n \geq 1)$ and a sequence $b(n) > 0$ such that the distribution of $b(n)S_n$ converges to μ, as n goes to ∞.*

Proof: Suppose that the distribution of $b(n)S_n$ converges to μ and that μ is nontrivial. For any $k \in \mathbb{N}$, consider

$$b(n)S_{kn} = \sum_{j=0}^{k-1} \Big(b(n)\big(S_{(j+1)n} - S_{jn}\big)\Big).$$

The distribution of the right-hand side tends to μ^{*n} as n goes to ∞, while the distribution of $b(kn)S_{kn}$ converges to μ. From Lemma 1, there is $b > 0$ such that $\hat{\mu}(z)^k = \hat{\mu}(bz)$. This implies that μ is stable.

Conversely, let μ be stable. Choose a random walk $(S_n, n \geq 1)$ with step distribution equal to μ. The stability implies that $b(n)S_n$ has distribution μ if $b_n > 0$ is suitable chosen. ■

It is known that a probability measure μ with support in $\mathbb{R}_+$ is infinitely divisible if and only if its Laplace transform has the form

$$\int_0^\infty e^{-ux}\mu(\mathrm{d}x) = \exp\left\{ -au - \int_0^\infty \left(1 - e^{-ux}\right)\nu(\mathrm{d}x)\right\},$$

where $a \geq 0$ and the measure ν is defined on $(0, \infty)$ and satisfies

$$\int_0^\infty (1 \wedge x)\nu(\mathrm{d}x) < \infty.$$

Then the Lévy measure of a self-decomposable distribution with support in $\mathbb{R}_+$ has the form $\nu(\mathrm{d}x) = k(x)/x$. The asymptotic of the densities of self-decomposable distributions with support in $\mathbb{R}_+$ at 0 satisfy a particular behaviour when the drift is 0. In particular such densities are regularly varying at 0.

Theorem 6 *Let μ be a self-decomposable distribution on $\mathbb{R}_+$ with k-function $k(x)$ satisfying $c = k(0+) < \infty$, and let*

$$\int_0^\infty e^{-ux}\mu(\mathrm{d}x) = \exp\left\{-\int_0^\infty \left(1-e^{-ux}\right)\frac{k(x)}{x}\mathrm{d}x\right\}.$$

Define

$$L(x) = \exp\left\{\int_x^1 (c-k(y))\frac{\mathrm{d}y}{y}\right\}, \quad x>0.$$

Then the density $f(x)$ of μ satisfies

$$f(x) \sim \frac{K}{\Gamma(c)}x^{c-1}L(x), \qquad \text{as } x\to 0,$$

where K is the constant given by

$$K = \exp\left\{c\int_0^1 (e^{-x}-1)\frac{\mathrm{d}x}{x} + \int_1^\infty (ce^{-x}-k(x))\frac{\mathrm{d}x}{x}\right\}.$$

Proof: First we prove that L is slowly varying function at 0 for any $\lambda \in (0,1)$,

$$\frac{L(\lambda x)}{L(x)} = \exp\left\{\int_{\lambda x}^x (c-k(y))\frac{\mathrm{d}y}{y}\right\} = \exp\left\{\int_\lambda^1 (c-k(xy))\frac{\mathrm{d}y}{y}\right\},$$

which goes to 1 as x tends to 0.
Now, let us show that

$$\int_0^\infty e^{-ux}\mu(\mathrm{d}x) \sim Ku^{-c}L(1/u), \qquad \text{as} \quad u\to\infty.$$

We first observe that

$$\int_1^\infty \left(e^{-ux}-1\right)\frac{k(x)}{x}\mathrm{d}x \to -\int_1^\infty \frac{k(x)}{x}\mathrm{d}x,$$

and that

$$\int_0^{1/u} \left(e^{-ux}-1\right)\frac{k(x)}{x}\mathrm{d}x = \int_0^1 \left(e^{-x}-1\right)\frac{k(x/u)}{x}\mathrm{d}x \to -c\int_1^\infty \left(e^{-x}-1\right)\frac{\mathrm{d}x}{x}$$

as u goes to ∞. On the other hand,

$$\begin{aligned}\int_{1/u}^1 \left(e^{-ux}-1\right)\frac{k(x)}{x}\mathrm{d}x + c\log u - \log L(1/u) &= \int_{1/u}^1 e^{-ux}\frac{k(x)}{x}\mathrm{d}x \\ &= \int_1^u e^{-x}\frac{k(x/u)}{x}\mathrm{d}x \to c\int_1^\infty e^{-x}\frac{\mathrm{d}x}{x}, \quad \text{as} \quad u\to\infty.\end{aligned}$$

Hence,

$$\frac{\int_0^\infty e^{-ux}\mu(\mathrm{d}x)}{u^c L(1/u)} \to K, \qquad \text{as} \quad u\to\infty.$$

From Karamata's Tauberian Theorem (see for instance Theorem 1.7.1 in [5]), we obtain that

$$\mu\big([0,x)\big) \sim \frac{K}{\Gamma(c+1)}x^c L(x) \qquad \text{as} \quad x\to 0.$$

Since the density of μ is monotone in a right neighborhood of the origin, the monotone density theorem (see for instance Theorem 1.7.2 in [5]), give us the desired result. ■

3.2 Additive processes and self-decomposable laws.

Recall that a stochastic process $X = (X_t, t \geq 0)$ on $\mathbb{R}$ is stochastically continuous if for every $t \geq 0$ and $\epsilon > 0$,

$$\lim_{s \to t} \mathbb{P}(|X_s - X_t| > \epsilon) = 0.$$

Definition 7 *A stochastic process $X = (X_t, t \geq 0)$ on $\mathbb{R}$ is said to be an additive process if X is stochastically continuous with càdlàg paths, starting from 0 and with independent increments, i.e. for any sequence $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the random variables $X_{t_n} - X_{t_{n-1}}, \dots, X_{t_2} - X_{t_1}$ are independent.*

Note that additive processes satisfy the Markov property (but they are not necessarily homogeneous). If we suppose that the increments are also time homogeneous, then they are Lévy processes.

The following result, due to Sato, connects self-decomposable laws and self-similar additive processes. Some times these processes are also called Sato processes.

Proposition 4 *Let X be an additive self-similar Markov process taking values in $\mathbb{R}$, then for any $t \geq 0$ the law of X_t is self-decomposable. Conversely, if μ is a self-decomposable measure on $\mathbb{R}$, then for any $\gamma > 0$ there exist an additive self-similar process X such that μ is the distribution of X_1.*

Proof: We first suppose that X is a self-similar additive process. For $t \geq s > 0$, let μ_t and $\mu_{s,t}$ be the laws of X_t and $X_t - X_s$, respectively. The self-similar (or scaling) property of X give us

$$\hat{\mu}_{kt}(u) = \hat{\mu}_t(k^\gamma u), \qquad u \in \mathbb{R},$$

for any $k > 0$. Now, let us take $k \in (0,1)$, the above equality and the independence of the increments allow us to obtain

$$\hat{\mu}_t(u) = \hat{\mu}_{bt}(u)\hat{\mu}_{bt,t}(u) = \hat{\mu}_t(b^\gamma u)\hat{\mu}_{bt,t}(u).$$

Conversely, let μ be a nontrivial self-decomposable distribution on $\mathbb{R}$ and $\gamma > 0$. From the definition 7, we know that for any $b > 1$ there is a probability measure ρ such that $\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\hat{\rho}_b(z)$. For any $t > s > 0$, we define μ_t and $\mu_{s,t}$ by

$$\hat{\mu}_t(z) = \hat{\mu}(t^\gamma z) \qquad \text{and} \qquad \hat{\mu}_{s,t}(z) = \hat{\rho}_{(s/t)^\gamma}(t^\gamma z), \qquad \text{for} \quad z \in \mathbb{R}.$$

Hence,

$$\hat{\mu}_t(z) = \hat{\mu}\big((s/t)^\gamma t^\gamma z\big)\hat{\rho}_{(s/t)^\gamma}(t^\gamma z) = \hat{\mu}_s(z)\hat{\mu}_{s,t}(z).$$

From the above equality, we deduce that for $s \leq t \leq u$

$$\mu_t = \mu_s * \mu_{s,t} \qquad \text{and} \qquad \mu_{s,t} * \mu_{t,u} = \mu_{s,u}.$$

Then an application of the Kolmogorov's extension theorem tell us that μ_t is the law at time t of an additive process X such that $X_t - X_s$ is distributes as $\mu_{s,t}$. On the other hand, it is clear that $X_{at} \stackrel{(d)}{=} a^\gamma X_t$, for every $t \geq 0$ and $a > 0$. Since both processes $(X_{at}, t \geq 0)$ and $(a^\gamma X_t, t \geq 0)$ are additive, the above equality in law implies that

$$(X_{at}, t \geq 0) \stackrel{(d)}{=} (a^\gamma X_t, t \geq 0).$$

Finally, since X_t is unique in law, X is unique in law. ■

3.3 Generalized Ornstein-Uhlenbeck processes.

The aim of this section is to establish a representation of self-similar additive processes with respect to a particular class of Lévy processes. In order to do so, we first introduce some properties of stochastic integration of continuous functions of locally bounded variation with respect to a given Lévy process.

Recall that for all $t > 0$, a right continuous function g which is of bounded variation induces a signed measure μ on $\mu([0,t], \mathcal{B}([0,t]))$:

$$\mu((a,b]) = g(b) - g(a) \quad \text{for} \quad 0 \le a < b \le t \quad \text{and} \quad \mu(\{0\}) = 0.$$

Note that the variation $|\mu|$ of μ is the measure associated with the variation $|g|$ of g. If $f \in \mathcal{L}(|\mu|)$, then the Lebesgue-Stieltjes integral of f with respect to g over $[0,t]$, for all $t > 0$, is defined by

$$\int_{[0,t]} f(s)\mathrm{d}g(s) \overset{\text{(def)}}{=} \int_{[0,t]} f\mathrm{d}\mu \quad \text{and} \quad \left| \int_{[0,t]} f(s)\mathrm{d}g(s) \right| \le \int_{[0,t]} |f|\mathrm{d}|\mu|.$$

If f is a continuous function, then the Riemman-Stieltjes integral of f with respect to g on $[0,t]$ is well defined and equals the Lebesgue-Stieltjes integral, i.e.,

$$\int_{[0,t]} f(s)\mathrm{d}g(s) = \lim_{n\to\infty} \sum_{k=1}^{p_n} f(t^n_{i-1})\big(g(t^n_i) - g(t^n_{i-1})\big), \tag{3.9}$$

for any sequence of partitions $0 = t^n_0 < t^n_1 < \cdots < t^n_{p_n} = t$ of $[0,t]$ where $\max |t^n_k - t^n_{k-1}|$ tends to 0 as n goes to ∞. On the other hand, if g is continuous as well as being of bounded variation then the Lebesgue-Stieltjes integral of f with respect to g is also given by (3.9) when f is a càdlàg function. This last property will give us the following lemma.

Lemma 2 *Let $\xi = (\xi_t, t \ge 0)$ be a càdlàg process on $\mathbb{R}$ and $A = (A_s, s \ge 0)$ a continuous process of bounded variation. Then the stochastic integral of A with respect to ξ on $(t_0, t]$ defined by*

$$\int_{t_0}^t A_s \mathrm{d}\xi_s \overset{(def)}{=} A_t\xi_t - A_{t_0}\xi_{t_0} - \int_{t_0}^t \xi_{s-}\mathrm{d}A_s,$$

makes sense.

Now, we suppose that the process A is a deterministic function of bounded variation, hence the following corollary follows from the definition of the Riemman-Stieltjes integral (3.9).

Corollary 1 *Let $\xi = (\xi_t, t \ge 0)$ be an adapted càdlàg process on $\mathbb{R}$ and f a continuous function of bounded variation. Then, the process $t \mapsto \int_{t_0}^t f(s)\mathrm{d}\xi_s$ is càdlàg, progressively measurable and adapted with respect to the filtration of ξ.*
If ξ is an additive process, then the process $t \mapsto \int_{t_0}^t f(s)\mathrm{d}\xi_s$ is also additive.

If ξ is a Lévy process with characteristic exponent Φ, then $\int_{t_0}^t f(s)\mathrm{d}\xi_s$ is an infinitely divisible random variable with characteristic exponent Ψ

$$\Psi(t) = \int_{t_0}^t \Phi(tf(s))\mathrm{d}s, \qquad t \in \mathbb{R}.$$

In particular, if Π is the Lévy measure of ξ, the Lévy mesure $M_t(\mathrm{d}x)$ of $\int_0^t e^{-s}\mathrm{d}\xi_s$ is given by

$$M_t(\mathrm{d}x) = \int_0^t \Pi(e^s\mathrm{d}x)\mathrm{d}s.$$

Let Hbe a random variable on $\mathbb{R}$ and ξ a Lévy process which is independent of H. Given $c \in \mathbb{R}$, consider the equation

$$Y_t = H + \xi_t - c\int_0^t Y_s\mathrm{d}s, \qquad t \geq 0. \tag{3.10}$$

A stochastic process $Y = (Y_t, t \geq 0)$ is said to be a solution of (3.10) if Y is càdlàg and satisfies (3.10) a.s.

Proposition 5 *The equation (3.10) has an almost surely unique solution Y and, almost surely,*

$$Y_t = e^{-ct}H + e^{-ct}\int_0^t e^{cs}\mathrm{d}\xi_s, \qquad t \geq 0. \tag{3.11}$$

Proof: From lemma 2 and (3.10), we have

$$\begin{aligned} e^{ct}Y_t &= H + c\int_0^t e^{cs}Y_s\mathrm{d}s + \int_0^t e^{cs}\mathrm{d}Y_s \\ &= H + c\int_0^t e^{cs}Y_s\mathrm{d}s + \int_0^t e^{cs}\mathrm{d}\xi_s - c\int_0^t e^{cs}Y_s\mathrm{d}s \\ &= H + \int_0^t e^{cs}\mathrm{d}\xi_s, \end{aligned}$$

which proves that the process Y defined by (3.11) is a unique solution of (3.10). ■

The next result shows that the solution of (3.10) is an homogeneous Markov process.

Proposition 6 *The process Y defined by (3.11) is an homogeneous Markov process starting from x with semi-group $P_t(z, \mathrm{d}y)$, infinitely divisible and satisfying*

$$\int_{-\infty}^{\infty} e^{i\lambda y}P_t(z, \mathrm{d}y) = \exp\left\{ie^{-ct}\lambda z + \int_0^t \Phi(e^{-cs}\lambda)\mathrm{d}s\right\},$$

where Φ is the characteristic exponent of the Levy process ξ in (3.11).

Proof: From (3.11) we have for every $s \in [0, t]$,

$$Y_t = e^{-c(t-s)}Y_s + e^{-ct}\int_s^t e^{cu}\mathrm{d}\xi_u.$$

Corollary 1 give us that the integral of the right-hand side of the above equality is independent of $(Y_u, 0 \le u \le s)$ and therefore the Markov property follows.
Again from corollary 1 is clear that the law of Y_t is infinitely divisible and with characteristic exponent

$$\Psi(\lambda) = i\lambda e^{-ct}x + \int_0^t \Phi(\lambda e^{-c(t-u)})\mathrm{d}u = i\lambda e^{-ct}x + \int_0^t \Phi(\lambda e^{-cs})\mathrm{d}s,$$

which concludes the proof. ■

The process Y defined by (3.11) is called Ornstein-Uhlenbeck process driven by ξ with initial state H and parameter $c \in \mathbb{R}$. The following result deals with limit distributions of Ornstein-Uhlenbeck processes as t goes to 0. In particular, it shows that an Ornstein-Uhlenbeck process has a limit distributions when the background driving Lévy process does not have big jumps.

Theorem 7 *Let $c > 0$ be fixed.*

i) Let ξ be the background driving Lévy of Y the Ornstein-Uhlenbeck process defined by (3.11) starting from x. Assume that

$$\int_{\{|x|>2\}} \log|x|\Pi(\mathrm{d}x) < \infty, \tag{3.12}$$

where Π is the Lévy measure of ξ. Then the law of Y_t converge towards μ a self-decomposable law as t goes to ∞. Moreover, the characteristic function of μ is given by

$$\hat{\mu}(\lambda) = \exp\left\{\int_0^\infty \Phi(e^{-cs}\lambda)\mathrm{d}s\right\}, \tag{3.13}$$

where Φ is the characteristic exponent of ξ_1.

ii) Let μ be a self-decomposable, then there exist a Lévy process ξ which Lévy measure satisfies (3.12) such that μ satisfies (3.13) for the Ornstein-Uhlembeck process generated by ξ and a constant c.

iii) If we assume that

$$\int_{\{|x|>2\}} \log|x|\Pi(\mathrm{d}x) = \infty.$$

Then the law of Y_t does not tend to any distribution as t goes to ∞.

We will not prove the above result. We refer to Theorem 17.5 in Sato [18] for a proof. The limit distribution μ of an Ornstein-Uhlenbeck process is self-decomposable and from our previous sub-section, we know that there is a self-similar additive process related to μ then a natural question is: could we identify one process from another? The following theorem will give us an affirmative answer to this question.
Before establishing our next result, we recall that Wolfe [19] and Jurek and Vervaat [11] showed that the distribution of a random variable X_1 is self-decomposable if and only if

$$X_1 \stackrel{(\mathrm{d})}{=} \int_0^\infty e^{-s}\mathrm{d}\xi_s, \tag{3.14}$$

where $\xi = (\xi_t, t \geq 0)$ is a Lévy process satisfying that $\mathbb{E}\big(\log(1 \vee |\xi_s|)\big) < \infty$ for all s. The process is called the background driving Lévy process of X_1. We denote by Υ the class of Levy processes ξ satisfying that

$$\mathbb{E}\Big(\log(1 \vee |\xi_s|)\Big) < \infty \qquad \text{for all} \quad s \geq 0.$$

Note that from theorem 7, the distribution of X_1 is the limit distribution of a Ornstein-Uhlenbeck process driven by ξ and with parameter 1.

Theorem 8 *Let X be a self-similar additive process with index $\gamma > 0$. Then, the following formulas*

$$\xi_t^{(-)} \stackrel{(def)}{=} \int_{e^{-t}}^1 \frac{\mathrm{d}X_r}{r^\gamma} \qquad \text{and} \qquad \xi_t^{(+)} \stackrel{(def)}{=} \int_1^{e^t} \frac{\mathrm{d}X_r}{r^\gamma}$$

define two independent and identically distributed Lévy process $\xi^{(-)} = (\xi_t^{(-)}, t \geq 0)$ and $\xi^{(+)} = (\xi_t^{(+)}, t \geq 0)$ which belongs to Υ and from which X can be recovered by

$$X_r = \begin{cases} \displaystyle\int_{\log(1/r)}^\infty e^{-t\gamma}\mathrm{d}\xi_t^{(-)} & \text{if } 0 \leq r \leq 1, \\ \displaystyle X_1 + \int_0^{\log(r)} e^{t\gamma}\mathrm{d}\xi_t^{(+)} & \text{if } r \geq 1, \end{cases} \tag{3.15}$$

where the law of X_1 is a self-decomposable distribution. In particular the BDLP of X_1 is $(\xi_{t/\gamma}^{(-)}, t \geq 0)$. Conversely, given a BDLP $(\xi_t, t \geq 0)$ of the class Υ associated with a self-decomposable distribution of X_1 via (3.14), a corresponding self-similar additive process can be constructed by (3.15) from two independent copies $(\xi_t^{(-)}, t \geq 0)$ and $(\xi_t^{(+)}, t \geq 0)$ of $(\xi_{t\gamma}, t \geq 0)$.

Proof: From Corollary 1, it is clear that the processes $(\xi_t^{(-)}, t \geq 0)$ and $(\xi_t^{(+)}, t \geq 0)$ are independent and that each of these processes has independent increments. In order to show that both processes are Lévy processes, it just remains to check that they have stationary increments. Let $t, h \geq 0$, from the scaling property of X, we get

$$\xi_{t+h}^{(-)} - \xi_t^{(-)} = \int_{e^{-(t+h)}}^{e^{-t}} \frac{\mathrm{d}X_r}{r^\gamma} = \int_{e^{-h}}^1 \frac{\mathrm{d}_v X_{e^{-t}v}}{(e^{-t}v)^\gamma} \stackrel{(\mathrm{d})}{=} \int_{e^{-h}}^1 \frac{\mathrm{d}X_v}{v^\gamma} = \xi_h^{(-)}.$$

The corresponding result for $\xi^{(+)}$ follows from similar arguments, or by writing

$$\xi_t^{(+)} = \int_{e^{-t}}^{1} \frac{\mathrm{d}(-X_{1/v})}{v^{-\gamma}},$$

and appealing to the previous case with X_v replaced by $-X_{1/v}$. Since both $\xi^{(-)}$ and $\xi^{(+)}$ have independent increments, to show that they are identically distributed it suffices to show that they have the same one-dimensional distributions. But for each fixed t and the scaling property of X,

$$\int_{e^{-t}}^{1} \frac{\mathrm{d}X_r}{r^\gamma} = \int_1^{e^t} \frac{\mathrm{d}_v X_{e^{-t}r}}{(e^{-t}r)^\gamma} \stackrel{\text{(d)}}{=} \int_1^{e^t} \frac{\mathrm{d}X_r}{r^\gamma}.$$

In order to obtain (3.15), write

$$\xi_t^{(-)} = -\int_0^t \frac{\mathrm{d}_v X_{e^{-v}}}{e^{-v\gamma}},$$

so that

$$\int_0^\infty e^{-v\gamma} \xi_v^{(-)} = -\int_0^\infty \mathrm{d}_v X_{e^{-v}} = X_1$$

This is (3.15) for $r = 1$ and the general case is obtained by a similar claculation. Finally, the converse assertion is easily checked. ■

Recall that from the "first" Lamperti representation, we know that there exist a stationary process $(Y_t, t \in \mathbb{R})$ such that

$$Y_u = e^{-u\gamma} X_{e^u}.$$

If we compare this representation to (3.15) of a self-similar additive process of index $\gamma > 0$, in terms of the Lévy process $\xi^{(+)}$, we see that for $r \geq 1$

$$r^\gamma Y_{\log(r)} = Y_0 + \int_0^{\log(r)} e^{t\gamma} \mathrm{d}\xi_t^{(+)},$$

so that, with $r = e^u$ for $u \geq 0$,

$$Y_u = e^{-u\gamma} \left(Y_0 + \int_0^u e^{t\gamma} \mathrm{d}\xi_t^{(+)} \right).$$

Togheter with similar considerations for $(Y_{-u}, u \geq 0)$, we deduce the following:

Corollary 2 *The "first" Lamperti representation $(Y_u, u \in \mathbb{R})$ of a self-similar additive process X with index $\gamma > 0$ is such that for the two independent Lévy processes $\xi^{(+)}$ and $\xi^{(-)}$ introduced in Theorem 8:*

- *i) $(Y_u, u \geq 0)$ is the Ornstein-Uhlenbeck process driven by $\xi^{(+)}$ with initial state X_1 and parameter $c = \gamma$, and*
- *i) $(Y_{-u}, u \geq 0)$ is the Ornstein-Uhlenbeck process driven by $-\xi^{(-)}$ with initial state X_1 and parameter $c = -\gamma$.*

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