Self-similar Markov processes.

Juan Carlos Pardo [1] and Víctor Rivero [2]

Abstract: This note surveys some recent results on self-similar Markov processes. Since the research around the topic has been very rich during the last fifteen years we do not pretend to cover all the recent developments in the field, and hence we focus mainly in giving a panorama of the areas where the authors have made contributions.

Key words: Self-similar Markov process, Lévy processes, Markov additive processes, Lamperti representation, exponential functionals, entrance laws, quasi-stationary distributions, fluctuation theory, law of the iterated logarithm.

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1 Introduction

As the title shows the main object of study in this paper is the class of real valued self-similar Markov processes, and in fact much of the results that will be summarised here concern the class of positive self-similar Markov processes. Before going into the detail we lift the following definition from Lamperti's pioneering work [43].

Definition 1. A stochastic process $X = \{X_t, t \geq 0\}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in \mathbb{R}^d})$ and $\mathbb{R}^d$-valued is said *semi-stable*, now a days called *self-similar*, if there exists an $\alpha \in \mathbb{R}$, such that for any $c > 0$,

$$\mathrm{Law}\left(\{cX_{c^{-\alpha}t}, t \geq 0\}, \mathbb{P}_x\right) = \mathrm{Law}\left(\{X_t, t \geq 0\}, \mathbb{P}_{cx}\right)$$

that is, both processes have the same finite dimensional laws, viz. for any $0 < t_1 < t_2 < \cdots < t_n < \infty$

$$\begin{aligned}\mathbb{P}_x\left(cX_{c^{-\alpha}t_1} \in \mathrm{d}x_1, cX_{c^{-\alpha}t_2} \in \mathrm{d}x_2, \ldots, cX_{c^{-\alpha}t_n} \in \mathrm{d}x_n\right)&\\ = \mathbb{P}_{cx}\left(X_{t_1} \in \mathrm{d}x_1, X_{t_2} \in \mathrm{d}x_2, \ldots, X_{t_n} \in \mathrm{d}x_n\right),&\end{aligned} \tag{1.1}$$

where by $\mathbb{P}_x$ we understand the law of the process stating at x. Whenever $\alpha \neq 0$ we will say that the process is $1/\alpha$-self-similar.

[1,2]CIMAT A.C. Calle Jalisco s/n. C.P. 36240, Guanajuato, Mexico. E-mail: jcpardo@cimat.mx
[2]E-mail: rivero@cimat.mx

Lamperti's main contribution in [43] has been to fully answer the question: *which are all the stochastic processes that can be obtained as the weak limit of some process on which we have applied an infinite sequence of contractions of the scale of time and space?* This question has been motivated by some rather transcendental results about weak convergence of normalised processes as for instance the famous result by Donsker [23] about convergence of random walk towards a Brownian motion. Lamperti's [43] main results are summarised in the following theorem.

Theorem 1. *Let $Y = \{Y_t, t > 0\}$ be a stochastic process $\mathbb{R}^d$-valued and $f : \mathbb{R} \to \mathbb{R}$ a function such that the process Y^ζ defined by*

$$Y_t^\zeta = \frac{Y_{\zeta t}}{f(\zeta)}, \qquad t > 0,$$

converges in the sense of finite dimensional laws, towards a non degenerated process, X, that is for any $0 < t_1 < t_2 < \cdots < t_n < \infty$, the convergence in law between random vector holds

$$\left(Y_{t_1}^\zeta, Y_{t_2}^\zeta, \ldots, Y_{t_n}^\zeta\right) \xrightarrow[\zeta\to\infty]{W} (X_{t_1}, X_{t_2}, \ldots, X_{t_n}). \tag{1.2}$$

Then, the process X is self-similar with an index α, for some $\alpha \in \mathbb{R}$. The function f is regularly varying with index α, that is $f(\zeta) = \zeta^\alpha L(\zeta)$, with L a function such that

$$\lim_{\zeta\to\infty} \frac{L(c\zeta)}{L(\zeta)} = 1, \qquad \textit{for all } c > 0.$$

In this case it is nowadays said that X is the scaling limit of Y. Furthermore, any self-similar Markov process can be obtained this way.

Among the class of self-similar processes there are several important sub-families that permit a better understanding of these. Some of them are the self-similar Gaussian processes; the class of additive self-similar processes, that is those with independent increments, those with homogeneous increments, and, those which are of particular interest to us, which have the strong Markov property. For properties and references about the former classes of processes see for instance the thorough survey by Embrechts and Maejima [26]. In the sequel we will restrict ourselves to the class of real valued self-similar Markov processes.

A real valued self-similar Markov process $X^{(x)}$, starting from x is a càdlàg strong Markov process which fulfills the above described *scaling property*. Real valued self-similar processes often arise in various parts of probability theory as limit of re-scaled processes. These processes are involved for instance in branching processes, Lévy processes, coalescent processes and fragmentation theory. Some particularly well known examples are Brownian motion, Bessel processes, stable subordinators, stable processes, stable Lévy processes conditioned to stay positive, etc.

Our main purpose in this paper is to give a panorama of properties of rssMp that have been obtained since the early sixties under the impulse of Lamperti's work [44], where the study of the case of positive self-similar Markov processes is initiated, and we will put particular emphasis in topics where the authors of this note have contributed.

2 Positive self-similar Markov processes

Throughout this paper we will assume that the self-similarity index, say $1/\alpha$, is strictly positive so $\alpha > 0$. When we restrict self-similar Markov processes to take values on the positive half-line we have an interesting relationship between this class and $\mathbb{R}\cup\{-\infty\}$-valued Lévy processes, such relation was obtained by Lamperti [44] and it is now known as the Lamperti representation of positive valued self-similar Markov processes, **pssMp** for short.

To state a precise result we recall that an $\mathbb{R}\cup\{-\infty\}$-valued stochastic process $\xi = (\xi_t, t \geq 0)$ is a Lévy process if its paths are càdlàg, the state $\{-\infty\}$ is an absorbing point, and it has stationary and independent increments. The state $\{-\infty\}$ is understood as an isolated point and hence the process hits this state and dies at an independent exponential time ζ, with some parameter $q \geq 0$, the case $q = 0$ is included to allow this time to be infinite a.s. The law of ξ is characterized completely by its Lévy-Khintchine exponent Ψ, which takes the following form

$$\log \mathbf{E}\left[e^{z\xi_1}, 1 < \zeta\right] = \Psi(z) = -q + bz + \frac{\sigma^2}{2}z^2 + \int_{-\infty}^{\infty}\left(e^{zy} - 1 - zy\mathbb{I}_{\{|y|<1\}}\right)\Pi(dy), \quad (2.3)$$

for any $z \in i\mathbb{R}$, where $\sigma, b \in \mathbb{R}$ and Π is a Lévy measure satisfying the condition $\int_{\mathbb{R}}(y^2 \wedge 1)\Pi(dy) < \infty$. For background about Lévy processes see [4], [40], [57].

Hereafter, for $x > 0$ the measure $\mathbb{P}_x$ denotes the law of a pssMp issued from x, and to refer to a pssMp issued from $x > 0$ we will use indistinctly $(X, \mathbb{P}_x)$, $(X^{(x)}, \mathbb{P}_x)$ and $X^{(x)}$.

Lamperti representation of self-similar $\mathbb{R}_+$–valued Markov processes killed at their first hitting time of 0, enables us to construct the paths of any such process starting from a strictly positive point from those of a Lévy process, and viceversa. More precisely, Lamperti [44] found the representation

$$X_t^{(x)} = \begin{cases} x\exp\xi_{\tau(tx^{-\alpha})}, & 0 \leq t \leq x^{\alpha}I(\alpha\xi), \\ 0, & t \geq x^{\alpha}I(\alpha\xi), \end{cases} \quad (2.4)$$

under $\mathbb{P}_x$, for $x > 0$, where

$$\tau(t) = \inf\{s > 0 : I_s(\alpha\xi) \geq t\}, \quad I_s(\alpha\xi) = \int_0^s \exp\alpha\xi_u\, du, \quad I(\alpha\xi) = \lim_{t\to\zeta} I_t(\alpha\xi),$$

and where ξ is a $\mathbb{R}\cup\{-\infty\}$–Lévy process with law $\mathbf{P}$. Note that for $t < I(\alpha\xi)$, we have the equality $\tau_t = \int_0^t \left(X_s\right)^{-\alpha} ds$, so that (2.4) is invertible. Indeed, any $\mathbb{R}\cup\{-\infty\}$-valued Lévy process ξ can be represented as

$$\xi_t = \begin{cases} \log\left(\frac{X_{\gamma_t}}{X_0}\right), & 0 \leq t < \int_0^{T_0-} X_s^{-\alpha}ds, \\ -\infty, & \int_0^{T_0-} X_s^{-\alpha}ds \leq t, \end{cases}$$

where X is some $1/\alpha$-pssMp, $\{\gamma_t, t \geq 0\}$ is the inverse of the additive functional

$$\int_0^t X_s^{-\alpha}\mathrm{d}s, \quad 0 \leq t < T_0 = \inf\{u > 0 : X_u = 0\}.$$

Observe that the process ξ does not depend on the starting point of X. Hence, we will denote the law of ξ by $\mathbf{P}$, and it is obtained as the image measure of $\mathbb{P}_x$ under the latter transformation, independently of the starting point $x > 0$. Reciprocally given a Lévy process $(\xi, \mathbf{P})$ using the former transformation we construct a family of Markovian measures $(\mathbb{P}_x)_{x>0}$, sharing the same semigroup. Hence Lamperti's transformation yields a one to one relation between the class of pssMp killed at their first hitting time of 0 and the one of Lévy processes. Unless otherwise stated, in the sequel we will denote a $1/\alpha$-pssMp by $(X, (\mathbb{P}_x)_{x>0})$, and by $(\xi, \mathbf{P})$ the Lévy process associated to it.

A first implication of Lamperti's transformation is that the first hitting time of 0, for a $1/\alpha$–pssMp and the exponential functional of a Lévy process, $I(\alpha\xi)$, are equal in law, more precisely

$$(T_0, \mathbb{P}_x) \stackrel{\text{Law}}{=} (x^\alpha I(\alpha\xi), \mathbf{P}).$$

Another useful consequence is the following classification of pssMp's:

(LC1) $\zeta < \infty$ $\mathbf{P}$–a.s. if and only if

$$\mathbb{P}_x(T_0 < \infty,\ X_{T_0-} > 0,\ X_{T_0+s} = 0,\ \forall s \geq 0) = 1, \tag{2.5}$$

for all $x > 0$.

(LC2) $\zeta = \infty$ and $\lim_{t\to\infty} \alpha\xi_t = -\infty$ $\mathbf{P}$–a.s. if and only if

$$\mathbb{P}_x\left(T_0 < \infty,\ X_{T_0-} = 0,\ X_{T_0+s} = 0,\ \forall s \geq 0\right) = 1, \tag{2.6}$$

for all $x > 0$.

(LC3) $\zeta = \infty$ and $\limsup_{t\to\infty} \alpha\xi_t = \infty$ $\mathbf{P}$–a.s. if and only if

$$\mathbb{P}_x(T_0 = \infty) = 1, \qquad \text{for all } x > 0. \tag{2.7}$$

Furthermore, a useful way to characterise the pssMp associated to a Lévy process ξ is via its infinitesimal generator. Indeed, Volkonskii's Theorem allow us to ensure that the infinitesimal generator of X, say $\mathcal{L}$, evaluated in a function $f : \mathbb{R}^+ \to \mathbb{R}$, such that $\tilde{f}(\cdot) = f(e^{\cdot})$ is in the domain of the infinitesimal generator of ξ, that we denote $\mathcal{A}$, takes the form

$$\begin{aligned}
\mathcal{L}f(x) &= x^{-\alpha}\mathcal{A}\tilde{f}(\log x) \\
&= -qx^{-\alpha}f(x) + x^{1-\alpha}(-b + \frac{1}{2}\sigma^2)f'(x) + x^{2-\alpha}\frac{1}{2}\sigma^2 f''(x) \\
&\quad + x^{-\alpha}\int_{\mathbb{R}} \left(f(xe^y) - f(x) - yxf'(x)1_{\{|y|<1\}}\right)\Pi(dy),
\end{aligned}$$

where (b, σ, Π) is the characteristic triple of ξ, and q is the rate at which it is killed. Using this characterisation we can easily obtain the following examples.

Example 1 (Continuous pssMp and Bessel processes)**.** Given that the totality of Lévy processes with continuous paths is necessarily of the form $\xi_t = \epsilon B_t + \mu t$, $t \geq 0$, with $(B_t, t \geq 0)$ a standard Brownian motion and some $\epsilon, \mu \in \mathbb{R}$, we get that the totality of pssMp with continuous paths killed at its first hitting time of zero is obtained as a

Lamperti transformation of a process of the latter form, with ϵ, μ and the self-similarity index appropriately chosen. For instance, when X is a standard Brownian motion killed at its first hitting time of 0, the self-similarity index is $1/2$, and the Lévy process is $\xi_t = B_t - \frac{t}{2}$, $t \geq 0$. Furthermore, taking $\epsilon = 1$, and $\mu = \frac{d}{2} - 1$, with $d > 0$, and $\alpha = 2$ we obtain a d-dimensional Bessel process. It is also an interesting exercise to prove this assertion using stochastic calculus.

Example 2 (Stable subordinators)**.** Let X be an α-stable Lévy process with non-decreasing paths, $0 < \alpha < 1$. X is a $1/\alpha$-pssMp. Its infinitesimal generator is

$$\widetilde{\mathcal{A}}f(x) = \int_0^\infty \Big(f(x+y) - f(x)\Big)\alpha c \frac{\mathrm{d}x}{x^{1+\alpha}}, \qquad c > 0.$$

By a change of variables

$$\widetilde{\mathcal{A}}f(x) = x^{-\alpha} \int_0^\infty \left(f(xe^z) - f(x)\right) \frac{c\alpha e^z}{(e^x - 1)^{1+\alpha}} \mathrm{d}x,$$

and by Volkonskii's formula we obtain that the underlying Lévy process has jump measure

$$\Pi(\mathrm{d}x) = \frac{c\alpha e^z}{(e^x - 1)^{1+\alpha}} \mathrm{d}x.$$

A family of processes associated to stable processes will be described in Section 2.2.

Remark 1. Although this will not be used in what follows, it is worth pointing out that the assumption made at the beginning of this section asking that the self-similarity index $1/\alpha$ is strictly positive is not essential. Indeed, if in Lamperti's transformation we take $\alpha < 0$, in order to make things consistent, we should just change the absorbing state of the pssMp to $\{\infty\}$, which will be reached in a finite time when the Lévy process $\alpha\xi$ will either jump in a finite time or drift towards $-\infty$. So, for a general self-similarity index $1/\alpha$ we should consider $(0, \infty) \cup \{\Delta\}$-valued pssMp where Δ is a cemetery state, that it is interpreted as 0 if $\alpha > 0$, and as ∞ if $\alpha < 0$.

An useful property of pssMp is that the self-similarity property remains valid when we take powers or we make time changes with power functions. More precisely, given a $1/\alpha$-pssMp and a $\beta \in \mathbb{R} \setminus \{0\}$, the process Y defined by

$$Y_t := (X_t)^\beta, \qquad t \geq 0,$$

with $1/0$ taken as ∞, is a α/β-pssMp and it is the Lamperti transform of the Lévy process $\beta\xi$. This makes that in most of the cases there is no loss of generality in assuming that the self-similarity index equals 1. Now, for a $\gamma \neq -\alpha$, define a time change

$$D_t^\gamma = \inf\{u > 0 : \int_0^u (X_s)^\gamma \, \mathrm{d}s > t\}, \qquad t \geq 0.$$

The process W defined by

$$W_t = X_{D_t^\gamma}, t \geq 0,$$

is a $1/(\alpha + \gamma)$-pMasp and the underlying Lévy process remains ξ. The proof of this result is an easy consequence of the fact that time changes with respect to additive functionals preserve the strong Markov property and that, in this case, the time change also preserves the scaling property of the process X. This can also be easily seen by using Lamperti's transform and understanding the composition of time changes. These properties, together with some duality properties, where studied in [60], see also [41]. Other duality properties where obtained in [8].

2.1 Defining a positive self-similar Markov process starting at 0.

In his seminal paper, Lamperti [44] studied the forms in which a self-similar diffusion could be started from the state 0. Lamperti findings lead to the following question

> *Given a positive $1/\alpha$-self-similar Markov process, $(X, (\mathbb{P}_x)_{x>0})$, constructed via the Lamperti transformation of some Lévy process ξ, when does there exists a pssMp that behaves like X when it is in $(0, \infty)$ and that it is not trivially started from 0?*

In the case where the pssMp $(X, (\mathbb{P}_x)_{x>0})$ never hits zero this question has been answered in full generality by Bertoin and Caballero [5], Bertoin and Yor [8], Caballero and Chaumont [11], and Chaumont, Kyprianou, Pardo and Rivero [16], by providing necessary and sufficient conditions for the existence of a probability measure $\mathbb{P}_{0+}$, that can be obtained as the weak limit of $\mathbb{P}_x$ as $x \downarrow 0+$, and under which the canonical process has the same transition semigroup as the one associated to $(X, (\mathbb{P}_x)_{x>0})$. Equivalently, we may ask under which conditions there exists a non-degenerate process $X^{(0)}$ that is the weak limit of $X^{(x)}$ as $x \to 0$.

Besides, when $(X, (\mathbb{P}_x)_{x>0})$ hits zero in a finite time to answer the above posed question one should look for all the recurrent extensions of it, that is the totality of positive self-similar Markov process that behave like the latter process before hitting zero for the first time but for which 0 is a recurrent and regular state. This problem has been studied by Lamperti [44] and Vuolle-Apiala [59], and solved in whole generality by Rivero [54, 55] and Fitzsimmons [27].

The main contributions of the papers quoted above will be summarized below.

2.1.1 Entrance laws

Bertoin and Caballero [5] and Bertoin and Yor [8] proved that, whenever the process drifts towards ∞, $\lim_{t\to\infty} X_t^{(x)} = \infty$, $\mathbb{P}_x$-a.s., the family of processes $X^{(x)}$ converges, as $x \downarrow 0$, in the sense of finite dimensional distributions towards $X^{(0)}$ if and only if the underlying Lévy process ξ in the Lamperti representation is such that

$$(H) \qquad \xi \text{ is non lattice and } \quad 0 < m =: \mathbf{E}(\xi_1) \leq \mathbf{E}(|\xi_1|) < +\infty\,.$$

In fact, the condition of ξ being non-lattice is not essential, whenever the process is lattice the limit exists when taken along adequately chosen subsequences. As proved by Caballero and Chaumont in [11], the latter condition is also a NASC for the weak convergence of the family $(X^{(x)})$, $x > 0$ on the Skohorod space of càdlàg trajectories. In the same article, the authors also provided a path construction of the process $X^{(0)}$.

The entrance law of $X^{(0)}$ has been described in [5] and [8] as follows: for every $t > 0$ and for every measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\mathbb{E}\left(f\left(X_t^{(0)}\right)\right) = \frac{1}{m}\mathbf{E}\left(I(-\alpha\xi)^{-1}f(tI(-\alpha\xi)^{-1})\right) . \tag{2.8}$$

Caballero and Chaumont [11] actually studied the more general case where X is only required to be such that

$$\limsup_{t\to\infty} X_t^{(x)} = \infty, \qquad \mathbb{P}_x - \text{a.s. } \forall x > 0, \tag{2.9}$$

and they were able to prove that a necessary and sufficient condition for the weak convergence to hold is that the mean of the upward ladder height process, say $h = (h_t, t \geq 0)$, associated to ξ, is finite and a further technical condition. For further details see [11]. Latter Chaumont, Kyprianou Pardo, and Rivero [16] improved the result of Chaumont and Caballero [11] by showing that the technical condition is irrelevant. Moreover, these authors obtained an expression for the entrance law which extend that obtained in [5] and [8], namely

$$\mathbb{E}_{0+}\left(f(X_t)\right) = \int_0^\infty f\left(\frac{t^{1/\alpha}}{x^{1/\alpha}}\right)\frac{1}{x}\eta(\mathrm{d}x),$$

where η is a measure defined by

$$\eta(f) = \frac{1}{\alpha\mu_+}\int_{\mathbb{R}_+^3}\mathbf{P}(\widetilde{I}\in \mathrm{d}t)\widehat{\mathcal{V}}(\mathrm{d}x)\,\mathbf{P}_x^\dagger\left(\int_0^{\varsigma_0} e^{-\alpha\xi_u}\mathrm{d}u \in \mathrm{d}s\right) f\left(e^{\alpha x}\left(t+s\right)\right),$$

and $\int_0^\infty x^{-1}\eta\left(\mathrm{d}x\right) = 1$, with $\widetilde{I}$ the exponential functional of the negative of the Lévy process ξ conditioned to stay positive, $\widehat{\mathcal{V}}$ the renewal measure of the downward ladder height process associated to ξ, and $\mathbf{P}^\dagger$ being the law of ξ conditioned to hit zero continuously.

Knowing the results of Bertoin and Caballero [5] and Bertoin and Yor [8], it may result surprising that the searched necessary and sufficient condition for the weak convergence to hold is the finiteness of the mean of the upward ladder height process. An heuristic for the necessity of this condition is as follows. Assume there is a process $X^{(0)}$ that is obtained as a weak limit of $X^{(x)}$ as $x \downarrow 0+$. This process has the scaling property and hence for any $c > 0$

$$(cX^{(0)}_{tc^{-\alpha}}, t \geq 0) \stackrel{\text{Law}}{=} (X_t^{(0)}, t \geq 0).$$

For $a > 0$, let $T_a = \inf\{t > 0 : X_t^{(0)} > a\}$, this r.v. is a finite stopping time because the original process satisfies (2.9). By the scaling property we have the equality in law

$$X^{(0)}_{T_a} \stackrel{\text{Law}}{=} cX^{(0)}_{T_{a/c}}, \qquad \text{for all } c > 0.$$

Making $c \to \infty$ and some elementary manipulations we get

$$X^{(0)}_{T_a} \stackrel{\text{Law}}{=} a\exp\left\{\lim_{c\to\infty}\left(\log(X^{(0)}_{T_{a/c}}) - \log(a/c)\right)\right\}. \tag{2.10}$$

Finally, the process $\log X^{(0)}$ should have the same hitting probabilities as ξ, because $\log X^{(\cdot)}$ is obtained by time changing ξ, and hence we should have the equality in law

$$\lim_{c\to\infty}\left(\log\left(X^{(0)}_{T_{a/c}}\right)-\log(a/c)\right\}\stackrel{\text{Law}}{=}\lim_{z\to\infty}\xi_{T^+_{\log(z)}}-\log(z),$$

with $T^+_{\log(z)}=\inf\{t>0:\xi_t>\log(z)\}$. We conclude from (2.10) that the latter limit exists and it is not degenerate. It is well known that this is equivalent to the weak convergence of the overshoots of the underlying upward ladder height subordinator. The latter condition is in turn equivalent to the finiteness of the mean of the upward ladder height subodinator, see for instance [22]. The details about the whole argument can be found in [42] where the authors obtain precise descriptions about the distribution of random variables associated to the events of first passage above a level and last passage below a level, among other results.

Besides, observe that if the mean of the upward ladder height is finite then by the results in [5] and [11] the pssMp, $H=(H_t,t\geq 0)$, associated to the upward ladder height subordinator has a non-degenerate weak limit, $H^{(0)}$, and so one may wonder whether it is possible to understand the limit process $X^{(0)}$ using the process $H^{(0)}$. We remark that we may understand the process H as the process of the past supremum of X in an adequate time scale. The main motivation of the paper [16] was to construct the process H from the process X, via a time change, establish the convergence of H, and finally construct the limit process $X^{(0)}$ from $H^{(0)}$, by *hanging* into the paths of $H^{(0)}$ the excursions from the supremum. One of the main results in [16] is the following description.

Theorem 2. *Let X be the Lamperti transform of a L.p. ξ that does not jump or drift towards $-\infty$, and define the maximum process $(M_s,s\geq 0)$, by $M_t:=\sup_{s\leq t}X_s$, $t\geq 0$. We have the following facts.*

(i) There exists a function $j(\varepsilon)$ for $\varepsilon>0$, such that

$$\lim_{\varepsilon\to 0}\frac{1}{j(\log(1+\varepsilon))}\int_0^t 1_{\left\{\frac{M_s}{X_s}\in[1,1+\varepsilon[\right\}}\mathrm{d}s=L^{\Theta}_t,$$

uniformly over bounded intervals in probability. The process L^{Θ} is a local time at the past supremum for X, i.e. and additive functional whose support is given by the closure of the random set

$$\Theta:=\{t\geq 0:X_t=\sup_{0\leq s\leq t}M_s\}.$$

(ii) Let $\{R_t,t\geq 0\}$ be the right continuous inverse of L^{Θ}, that is

$$R_t=\inf\{s>0:L^{\Theta}_s>t\},\qquad t\geq 0,$$

and put $H_t:=X_{R_t}$, $t\geq 0$. The process H is the Lamperti transform of the upward ladder height subordinator of ξ.

The term "hanging into the paths of H^{0+} the excursions from the supremum" is made precise in [16] by constructing an exit system associated to the random set Θ

defined above, and then using this exit system to prove the convergence of the resolvent of X, as the starting point tends to 0, which gives the finite dimensional convergence. Finally, the weak convergence is obtained by proving tightness. We do not provide further details. Instead we mention that the papers [16], [20] and [42] contributed to the foundation of a fluctuation theory for pssMp analogous to the well developed one of real valued Lévy processes. In the paper [20] the path of a pssMp is decomposed into the path of the pre- and post-overall infimum. A precise description of these segments of paths is provided, and the limit of the path when the overall minimum tends to zero is obtained. In particular, it is shown that the post-minimum process converges to the path of $X^{(0)}$, and hence that the pre-infimum path squeezes to the path equal to zero with 0 length when the starting point tends to 0.

2.1.2 Recurrent extensions

We now deal with the results around the question *What are the positive α–self-similar Markov processes $\widetilde{X}$ which behave like $(X,\mathbb{P})$ up to the first hitting time of* 0 *for $\widetilde{X}$ and such that* 0 *is a regular and recurrent state?* A process that has this characteristics is usually called a recurrent extension of the process $(X,\mathbb{P})$. Lamperti [44] solved this question in the special case of a Brownian motion killed at 0, using results specific to Brownian motion. After Lamperti, Vuolle-Apiala [59] used excursion theory to give a more general answer under some regularity assumptions for the resolvent of the process $(X,\mathbb{P})$. Then the question was solved in full generality by Rivero [54], [55], and Fitzsimmons [27]. In order to describe the results in those papers we introduce some further notions, but before we mention that a different approach using stochastic differential equations has been used in [24] (see Section 2.1.3).

Definition 2 (Self-similar excursion measures)**.** A measure $\mathbf{n}$ on $(\mathbb{D}^+,\mathcal{G}_\infty)$ having infinite mass is a self-similar excursion measure compatible with $(X,\mathbb{P})$ if

(i) $\mathbf{n}$ is carried by

$$\{\omega\in\mathbb{D}+|0<T_0<\infty \text{ and } X_t(\omega)=0,\ \forall t\geq T_0\}\,;$$

(ii) For every bounded $\mathcal{G}_\infty$-measurable functional H and each $t>0$ and $\Lambda\in\mathcal{G}_t$,

$$\mathbf{n}\left(H\circ\theta_t,\Lambda\cap\{t<T_0\}\right)=\mathbf{n}\left(\mathbb{E}_{X_t}\left(H\right),\Lambda\cap\{t<T_0\}\right),$$

with θ_t the shift operator;

(iii) $\mathbf{n}\left(1-e^{-T_0}\right)<\infty$;

(iv) there exists a $\gamma\in]0,\alpha[$ such that for every $c>0$ the image of $\mathbf{n}$ under the mapping $H_c:\mathbb{D}^+\to\mathbb{D}^+$, defined by $H_c(\omega)(t)=c\omega(tc^{-1/\alpha})$, for $t>0$, is

$$\mathbf{n}\circ H_c=c^{\gamma/\alpha}\,\mathbf{n}\,.$$

It is well known in the theory of Markov processes that a way to construct recurrent extensions of self–similar Markov processes is by means of the so called Itô's program or pathwise approach, which consists on *pasting together excursions.* Precise results about this topic can be found in [10] and [54]. The main results from the latter references allow us to ensure that there is a bijection between the existence of self-similar recurrent extensions and self-similar excursion measures compatible to $(X, \mathbb{P})$. Actually, the latter is the Itô excursion measure for the excursions from zero of the self-similar recurrent extension of $(X, \mathbb{P})$. We will now describe necessary and sufficient conditions for the existence of such a measure.

Vuolle-Apiala [59] proved that there are two types of excursion measures, namely those for which the recurrent extension *exits* 0 *by jumps*, which in terms of the excursion measure means $\mathbf{n}\,(X_{0+} = 0) = 0$, and those for which the recurrent extension leaves zero continuously, $\mathbf{n}\,(X_{0+} > 0) = 0$. And furthermore, a self-similar excursion measure is either of one type or the other, but not both. It has been shown in [54] that the reason for this is that they have different self-similarity index. Vuolle-Apiala proved that a consequence of the scaling property is that all the self-similar excursion measures can be written as

$$\mathbf{n}(\cdot) = c_{\alpha,\beta} \int_{x>0} \frac{\mathrm{d}x}{x^{1+\beta}} \mathbb{P}_x(\cdot), \tag{2.11}$$

for some β such that $\beta/\alpha \in]0,1[$ and $c_{\alpha,\beta} \in]0,\infty[$, is a normalizing constant. Thus, to determine the existence of a recurrent extension that leaves 0 by a jump all we are ought to do is to verify when a measure of this form bears all the condition to be a self-similar excursion measure. That is the purpose of the following theorem.

Theorem 3. *Let $(X, \mathbb{P})$ be an α-self-similar Markov process that hits the cemetery point 0 in a finite time a.s. and $(\xi, \mathbf{P})$ the Lévy process associated to it via Lamperti's transformation. For $0 < \beta < \alpha$, the following are equivalent*

(i) $\mathbf{E}(e^{\beta\xi_1}, 1 < \zeta) < 1$,

(ii) $\mathbf{E}\left(\left(\int_0^\infty \exp\{\alpha\xi_s\}\mathrm{d}s\right)^{\beta/\alpha}\right) < \infty$,

(iii) There exists a recurrent extension of $(X, \mathbb{P})$, say $X^{(\beta)}$, that leaves 0 by a jump and its associated excursion measure $\mathbf{n}^\beta$ is such that

$$\mathbf{n}^\beta(X_{0+} \in \mathrm{d}x) = c_{\alpha,\beta}\beta x^{-1-\beta}\mathrm{d}x, \qquad x > 0,$$

where $c_{\alpha,\beta}$ is a constant.

In this case, the process $X^{(\beta)}$ is the unique recurrent extension of $(X, \mathbb{P})$ that leaves 0 by a jump distributed as above.

The proof of this theorem resides in the fact that a measure of the form in (2.11) satisfies the conditions (i), (ii) and (iv) from the definition 2, as it can be easily verified, but the condition (iii) is only satisfied when the condition (ii) of Theorem 3 is satisfied, and hence what is left to prove is the equivalence between (i) and (ii), because the equivalence between (ii) and (iii) is obtained from the previous discussion.

It is actually more difficult to establish the existence of entrance laws that are carried by the paths that leave 0 continuously. The definitive answer is given in the following result obtained by Rivero [54, 55] and Fitzsimmons [27].

Theorem 4. *Let $(X, \mathbb{P})$ be an α-self-similar Markov process that hits its cemetery state 0 in a finite time $\mathbb{P}$-a.s. and $(\xi, \mathbf{P})$ be the Lévy process associated to $(X, \mathbb{P})$ via Lamperti's transformation. The following are equivalent:*

(i) $\exists\ \theta \in]0,\alpha[$, *s. t.* $\mathbf{E}(e^{\theta\xi_1}, 1 < \zeta) = 1$, ***Cramér's condition.***

(ii) There exists a recurrent extension of $(X, \mathbb{P})$ *that leaves* 0 *continuously and such that its associated excursion measure from 0, say* $\mathbf{N}$, *satisfies* $\mathbf{N}(1 - e^{-T_0}) = 1$.

In this case, the recurrent extension in (ii) is unique and the entrance law associated to the excursion measure $\mathbf{N}$ *is given by, for any* f *positive and measurable*

$$\mathbf{N}(f(X_t), t < T_0) = \frac{1}{t^{\theta/\alpha}\Gamma(1-(\theta/\alpha))\,\mathbf{E}^{\natural}(J^{(\theta/\alpha)-1})}\,\mathbf{E}^{\natural}\left(f\left(\frac{t^{1/\alpha}}{J^{1/\alpha}}\right)J^{(\theta/\alpha)-1}\right),$$

for $t > 0$, *where* $\mathbf{P}^{\natural} = e^{\theta\xi_t}\,\mathbf{P}$ *on* $\sigma(\xi_s, s \le t)$ *and* $J = \int_0^\infty \exp\{-\alpha\xi_s\}\mathrm{d}s$.

The above description of the measure $\mathbf{N}$ is reminiscent of Imhof's construction of the excursion measure of the Brownian process out from 0 which relates the law of a brownian motion conditioned to stay positive and started from zero, that is a Bessel process issued from 0, and the excursion measure. Further results in this direction and a description of the excursion measure conditionally on the length as well as its image under time reversal are provided in [54]. A description of the excursion measure in terms of the height of the excursion is provided in [3].

2.1.3 A stochastic differential equation approach

Motivated by the problem of existing zero and the description of the recurrent extension of positive self-similar Markov process, Barczy and Döring [24] studied a stochastic differential equation (SDE for short) approach. More precisely, recall that the Lévy-Itô representation of a Lévy process $\xi = (\xi_t, t \ge 0)$ is given as follows

$$\xi_t = bt + \sigma B_t + \int_0^t \int_{\{|u|\le 1\}} u\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_{\{|u|\ge 1\}} u\mathcal{N}(\mathrm{d}s, \mathrm{d}u),$$

where $b \in \mathbb{R}$, $\sigma \ge 0$, B is a Brownian motion and $\mathcal{N}$ is an independent Poisson random measure on $(0,\infty) \times \mathbb{R}$ with intensity $\mathrm{d}s \otimes \Pi(\mathrm{d}u)$ and $\widetilde{\mathcal{N}}$ represents its compensated version. Assuming that $\mathbf{E}[e^{\xi_1}] < \infty$, hence the proposed SDE can be written as follows

$$\begin{aligned} X_t = x + \big(\log \mathbf{E}[e^{\xi_1}; \zeta > 1]\big)\, t + \sigma \int_0^t \sqrt{X_t}\mathrm{d}B_s - \int_0^t \int_0^\infty \mathbf{1}_{\{rX_{s-}\le 1\}} X_{s-}\widetilde{\mathcal{M}}(\mathrm{d}s, \mathrm{d}r) \\ + \int_0^t \int_0^\infty \int_{-\infty}^\infty \mathbf{1}_{\{rX_{s-}\le 1\}} X_{s-}(e^u - 1)\widetilde{\mathcal{N}_1}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u), \end{aligned}$$

for $t \le T_0$. Here B is a Brownian motion, $\mathcal{N}_1$ is an independent Poisson random measure on $(0,\infty)\times(0,\infty)\times\mathbb{R}$ with intensity $\mathrm{d}s\otimes\mathrm{d}r\otimes\Pi(\mathrm{d}u)$ and $\mathcal{M}$ is an independent Poisson random measure on $(0,\infty)\times(0,\infty)$ with intensity $q\mathrm{d}s\otimes\mathrm{d}r$. The random measures $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$ represents the compensated version of $\mathcal{M}$ and $\mathcal{N}$, respectively. The intuition of the above SDE follows from applying Itô's formula to e^{ξ_t} and afterwards including a correction which is given by the random time change.

It is important to note that whenever $0 < \log \mathbf{E}[e^{\xi_1}; \zeta > 1] < \infty$, then this new representation is not restricted to $t \le T_0$. We also note that the SDE defined above posses weak solutions up to a first hitting time. If ξ posses only negative jumps and satisfies that $0 < \log \mathbf{E}[e^{\xi_1}; \zeta > 1]$, then for any initial condition $x > 0$ there is a pathwise unique non-negative strong solution $(X_t, t \geq 0)$ which is self-similar with index $\alpha = 1$ and such that $(X_t, 0 \leq t \leq T_0)$ its underlying Lévy process in the Lamperti transform has the same law as ξ killed at rate q. If the Lévy process ξ satisfies the Cramér condition then $(X_t, t \geq 0)$ is the unique recurrent self-similar extension of $(X_t, 0 \leq t \leq T_0)$. Moreover if ξ does not drift to $-\infty$ or ξ drifts to $-\infty$ and it satisfies the Cramér condition, then the process $(X_t, t \geq 0)$ is the unique strong solution of the above SDE with initial condition $X_0 = 0$.

2.2 Exponential functionals

As we have seen in previous sections a recurrent object in the theory of pssMp is the so-called *exponential functional* of a Lévy process ξ with lifetime ζ, i.e.

$$I_\zeta(\xi) := \int_0^\zeta \exp\left\{\xi_s\right\} \mathrm{d}s.$$

For instance, a consequence of Lamperti's transformation is that the first hitting time of a positive valued ssMp has the same law as the exponential functional of a Lévy process; we have also seen that it describes the entrance law of a pssMp that never hits zero, or the entrance law under the excursion measure of the recurrent extension associated to a pssMp that hits zero in a finite time. Latter in this note, Section 2.3, we will see that the density of an exponential functional plays a crucial role in establishing integral test for describing the upper and lower envelopes of a pssMp; also we will see in Section 2.4 that the Yaglom limit for a pssMp is determined by the asymptotic behaviour of the tail distribution of the first hitting time of zero. Hence a good understanding of the law of an exponential functional is necessary in order to obtain precise information about pssMp's. Furthermore, one could say that exponential functionals of Lévy processes and pssMp leave in symbiosis because facts about pssMp have been used to obtain properties of exponential functionals.

Moreover, this is not the only fact that has motivated many research works on the topic over the last two decades. The law of $I_\zeta(\xi)$ plays an important role in many other areas of probability theory, for instance in fragmentation, coalescence and branching processes, financial and insurance mathematics, Brownian motion in hyperbolic spaces, random processes in random environment, etc. For more details about these topics and other aspects not covered in this section we refer to the survey paper [9]. In this section we intend to provide a collection of results that partially complements the latter paper.

Because in our setting ζ is taken as the life time of the Lévy process ξ, we will hence focus in the case where $\zeta = \mathbf{e}_q$, an exponential random variable with parameter $q \geq 0$ which is independent of the process ξ. Many authors have been interested in the existence (and determine explicitly, as well) of the density associated to $I_{\mathbf{e}_q}(\xi)$. When $q = 0$, then $\mathbf{e}_q$ is understood as ∞. In this case, we assume that the process ξ drifts towards $-\infty$ since it is a necessary and sufficient condition for the almost sure finiteness of $I(\xi) := I_\infty(\xi)$, see for instance Theorem 1 in Bertoin and Yor [9]. Carmona, Petit and Yor [15] were the first in studying the existence of the density of $I_\infty(\xi)$. More precisely, they proved the existence of such density in the case when the jump structure of the Lévy process is of finite variation and also provided an integral equation that the density must hold, we will recall the integral equation below. Recently Bertoin et al. [6] (see Theorem 3.9) proved the existence of the density in the general case.

The first result that we present in this section is about the existence of the density of $I_{\mathbf{e}_q}(\xi)$ in the case $q > 0$, in terms of its associated positive self-similar process $(X, \mathbb{P}_1)$, it has been obtained in [50].

Theorem 5. *Let $q > 0$, then the function*

$$h(t) := q\mathbb{E}_1\left[\frac{1}{X_t}\mathbf{1}_{\{t<T_0\}}\right], \qquad t \geq 0,$$

is a density for the law of $I_{\mathbf{e}_q}(\xi)$.

A consequence of this result in the case where ξ is a subordinator gives the following important property of h.

Corollary 1. *Assume $q > 0$ and that ξ is a subordinator. Then the law of the r.v. $I_{\mathbf{e}_q}(\xi)$ is a mixture of exponentials, that is its law has a density h on $(0, \infty)$ which is completely monotone. Furthermore, $\lim_{t\downarrow 0} h(t) = q$.*

Carmona, et al. [15] integral equation give some information of the density and allow us to compute it explicitly in some particular cases. A generalizaton and extension of this integral equation in the case of the negative of a subordinator has been obtained in [50]

Theorem 6. *Assume that $\xi = -\sigma$, with σ a subordinator with drift $c \geq 0$, killing term q and Lévy measure Π. Let $q \geq 0$. The random variable $I_{\mathbf{e}_q}$ has a density that we denote by k, and it solves the equations*

$$\int_y^\infty k(x)\mathrm{d}x = \int_0^\infty k(ye^x)U_q(\mathrm{d}x), \qquad \textit{almost everywhere,} \tag{2.12}$$

and

$$(1 - cx)k(x) = \int_x^\infty \overline{\Pi}(\log(y/x))k(y)\mathrm{d}y + q\int_x^\infty k(y)\mathrm{d}y, \qquad x \in (0, 1/c). \tag{2.13}$$

with $\mathbf{E}\left[\int_0^{\mathbf{e}_q} 1_{\{\sigma_t \in dx\}}\mathrm{d}t\right] = U_q(\mathrm{d}x)$, on $x \geq 0$, and $\overline{\Pi}(y) := \Pi(y, \infty)$, for $y > 0$. Conversely, if a density on $(0, 1/c)$ satisfies any of the equations (2.12) or (2.13) then it is the density of $I_{\mathbf{e}_q}$.

There are two main approaches which have been developed and used to extract more information about the law of the exponential functional. The first one uses the fact that the Mellin transform of $I_{\mathbf{e}_q}(\xi)$ is solution of the functional equation,

$$\mathbf{E}\left[I_{\mathbf{e}_q}(\xi)^{s-1}\right] = \frac{\psi_q(s)}{s}\mathbf{E}\left[I_{\mathbf{e}_q}(\xi)^s\right], \tag{2.14}$$

where $\psi_q(\lambda) = -\ln \mathbf{E}[e^{\lambda\xi_1}, 1 < \mathbf{e}_q]$. The above equation, when $q = 0$, appears for the first time in Carmona et al. [15] and was extended by Maulik and Zwart [45]. When $q = 0$, the equation (2.14) can be solved explicitly in the case when ξ is the negative of a subordinator or a spectrally positive Lévy process, which, in both cases, determine the law of $I_{\mathbf{e}_q}(\xi)$. More precisely, let $-\xi$ be a subordinator and $\Phi(\lambda) = -\ln \mathbf{E}[e^{\lambda\xi_1}]$. Carmona et al. [15] noted that the law of the exponential functional of a subordinator is determined by its entire moments which are given by the identity

$$\mathbf{E}\left[I(\xi)^k\right] = \frac{k}{\Phi(k)}\mathbf{E}\left[I(\xi)^{k-1}\right] = \frac{k!}{\Phi(1)\cdots\Phi(k)}, \qquad \text{for} \quad k = 1, 2, \ldots$$

We note that this equation can be solved explicitly in many situations, see for instance Bertoin and Yor [9]. Similarly, if ξ is a spectrally positive Lévy process, Bertoin and Yor [9] proved that the law of $I(\xi)$ is determined by its negative entire moments and can be expressed in the form

$$\mathbf{E}\left[I(\xi)^{-(k+1)}\right] = \frac{\Psi(k)}{k}\mathbf{E}\left[I(\xi)^{-k}\right] = m\frac{\Psi(1)\cdots\Psi(k-1)}{(k-1)!}, \qquad \text{for} \quad k = 1, 2, \ldots$$

where $\Psi(\lambda) = \ln \mathbf{E}[e^{-\lambda\xi_1}]$ and with the convention that the right-hand side equals m for $k = 1$.

One can prove that if Cramér's condition is satisfied then the Mellin transform of $I_{\mathbf{e}_q}(\xi)$ satisfies the functional identity (2.14), however it is clear that there are infinitely many functions which satisfy the same functional identity. The next result obtained by Kuznetsov and Pardo [39] tells us that if we have found a function $f(s)$ which satisfies (2.14), and if we can verify two conditions about the zeros of this function and its asymptotic behaviour, then we can in fact uniquely identify the Mellin transform of $I_{\mathbf{e}_q}(\xi)$.

Proposition 1. *Assume that there exists $z_0 > 0$ such that $\psi_q(z)$ is finite for all $z \in (0, z_0)$ and $\psi_q(\theta) = 0$ for some $\theta \in (0, z_0)$. If $f(s)$ satisfies the following three properties*

(i) $f(s)$ is analytic and zero-free in the strip $\mathrm{Re}(s) \in (0, 1+\theta)$,

(ii) $f(1) = 1$ and $f(s+1) = sf(s)/\psi_q(s)$ for all $s \in (0, \theta)$,

(iii) $|f(s)|^{-1} = o(\exp(2\pi|\mathrm{Im}(s)|))$ as $\mathrm{Im}(s) \to \infty$, uniformly in $\mathrm{Re}(s) \in (0, 1+\theta)$,

then $\mathbf{E}[I_{\mathbf{e}_q}(\xi)^{s-1}] \equiv f(s)$ for $\mathrm{Re}(s) \in (0, 1+\theta)$.

In particular, this proposition can be used to provide a very simple and short proof of the well-known result on exponential functional of Brownian motion with drift and of the recent results on exponential functionals of processes with double-sided hyper-exponential jumps (see [14]).

Recently in [39], the authors found a particular class of Lévy processes, called hypergeometric Lévy processes, for which the solution of the functional equation can directly be guessed from (2.14) and verified using Proposition 1, and derived the law of $I_{\mathbf{e}_q}(\alpha\xi)$ for an specific value of q.

Hypergeometric Lévy processes were first introduced in [42] and constructed using Vigon's theory of philanthropy (see [58]). The class of processes that we present next should be considered as a subclass of the hypergeometric processes studied in [42] and as a generalization of Lamperti-stable processes, which were introduced by Caballero and Chaumont [11].

We start by defining its Laplace exponent $\psi_q(z)$ as

$$\psi_q(z) = \frac{\Gamma(1-\beta+\gamma-z)}{\Gamma(1-\beta-z)}\frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(\hat{\beta}+z)}, \tag{2.15}$$

where $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ belong to the admissible set of parameters

$$\mathcal{A} = \{\beta \leq 1,\ \gamma \in (0,1),\ \hat{\beta} \geq 0,\ \hat{\gamma} \in (0,1)\}.$$

Let

$$\eta = 1-\beta+\gamma+\hat{\beta}+\hat{\gamma}.$$

The Levy density of hypergeometric Lévy processes can be computed explicitly, see Proposition 1 in [39]. Moreover, if $\beta < 1$ and $\hat{\beta} > 0$ the process ξ is killed at rate

$$q = \psi_q(0) = \frac{\Gamma(1-\beta+\gamma)}{\Gamma(1-\beta)}\frac{\Gamma(\hat{\beta}+\hat{\gamma})}{\Gamma(\hat{\beta})}.$$

The process ξ drifts to $+\infty$, $-\infty$ or oscillates whenever $\beta = 1$ and $\hat{\beta} > 0$, $\beta < 1$ and $\hat{\beta} = 0$ or $\beta = 1$ and $\hat{\beta} = 0$. The process ξ has no Gaussian component. When $\gamma + \hat{\gamma} < 1$ $(1 \leq \gamma + \hat{\gamma} < 2)$ the process has paths of bounded variation and no linear drift (paths of unbounded variation).

Three Lamperti-stable processes $\xi^*, \xi^{\uparrow}$ and $\xi^{\downarrow}$ were introduced by Caballero and Chaumont [11] by applying the Lamperti transformation to the positive self-similar Markov processes construced from a stable process. In particular, the process ξ^* is obtained from a stable process started at $x > 0$ and killed upon exit from the positive half-line, while the process $\xi^{\uparrow}$ $\{\xi^{\downarrow}\}$ is obtained from a stable process conditioned to stay positive {conditioned to hit zero continuously}. We refer to [11, 12, 17] for all the details on these processes.

The Lamperti-stable processes ξ^*, $\xi^{\uparrow}$, $\xi^{\downarrow}$ can be identified as hypergeometric processes with the following sets of parameters

	β	γ	$\hat{\beta}$	$\hat{\gamma}$
ξ^*	$1-\alpha(1-\rho)$	$\alpha\rho$	$1-\alpha(1-\rho)$	$\alpha(1-\rho)$
$\xi^{\uparrow}$	1	$\alpha\rho$	1	$\alpha(1-\rho)$
$\xi^{\downarrow}$	0	$\alpha\rho$	0	$\alpha(1-\rho)$

Let $\delta = 1/\alpha$. From the definition of the Laplace exponent (2.15) we find that ξ satisfies Cramér's condition, that is to say $\mathbf{E}[\exp(\hat{\beta}\xi_1)] = 1$, therefore applying Lemma 2 from Rivero [55] we conclude that the Mellin transform of $I_{\mathbf{e}_q}(\alpha\xi)$ exists for $s \in (0, 1+\hat{\beta}\delta)$. In order to describe the results about the law of $I_{\mathbf{e}_q}(\alpha\xi)$, we need to define the double gamma function, $G(z;\tau)$. The double gamma function is defined by an infinite product in Weierstrass's form

$$G(z;\tau) = \frac{z}{\tau}e^{a\frac{z}{\tau}+b\frac{z^2}{2\tau}}\prod_{m\geq 0}\prod_{n\geq 0}{}' \left(1+\frac{z}{m\tau+n}\right)e^{-\frac{z}{m\tau+n}+\frac{z^2}{2(m\tau+n)^2}}, \quad |\arg(\tau)| < \pi, \quad z \in \mathbb{C}.$$

Here the prime in the second product means that the term corresponding to $m = n = 0$ is omitted. Note that by definition $G(z;\tau)$ is an entire function in z and if $\tau \notin \mathbb{Q}$ it has simple zeros on the lattice $m\tau + n$, $m \leq 0$, $n \leq 0$. We refer to Kuznetsov [36] or Kuznetsov and Pardo [39] for more properties of this function. The following result, lifted from [39], characterize the Mellin transform of the exponential functional of hypergeometric Lévy processes.

Theorem 7. *Assume that $\alpha > 0$, $(\beta, \gamma, \hat{\beta}, \hat{\gamma}) \in \mathcal{A}$ and $\hat{\beta} > 0$. Then for $s \in \mathbb{C}$ we have*

$$\mathbf{E}[I_{\mathbf{e}_q}(\alpha\xi)^{s-1}] \equiv C\Gamma(s)\frac{G((1-\beta)\delta + s;\delta)}{G((1-\beta+\gamma)\delta + s;\delta)}\frac{G((\hat{\beta}+\hat{\gamma})\delta + 1 - s;\delta)}{G(\hat{\beta}\delta + 1 - s;\delta)}, \qquad (2.16)$$

where the constant C is such that the above identity equals 1 when $s = 1$.

We now want to study the density of the exponential functional, which is defined by

$$p(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{P}(I_{\mathbf{e}_q}(\alpha\xi) \leq x), \quad x \geq 0.$$

In order to do so, we have to compute the inverse Mellin transform of (2.16) which is not a simple inversion exercise. From the paper by Kuznetsov & Pardo [39] in the next result it is provided an asymptotic expansion of the density $p(x)$ in the case when $\alpha \notin \mathbb{Q}$.

Theorem 8. *Assume that $\alpha \notin \mathbb{Q}$. Then*

$$\begin{aligned} p(x) &\sim \sum_{n\geq 0} a_n x^n + \sum_{m\geq 0}\sum_{n\geq 0} b_{m,n} x^{(m+1-\beta+\gamma)\delta + n}, \quad && x \to 0^+, \\ p(x) &\sim \sum_{m\geq 0}\sum_{n\geq 0} c_{m,n} x^{-(m+\hat{\beta})\delta - n - 1}, && x \to +\infty. \end{aligned}$$

The series $(a_n)_{n\geq 0}$, $(b_{m,n})_{m,n\geq 0}$ and $(c_{m,n})_{m,n\geq 0}$ can be computed explicitly. We refer to Kuznetsov and Pardo [39] for more details about this series.

It turns out that for almost all parameters α the asymptotic series from above converge to $p(x)$ for all $x > 0$. In order to state this result, we need to define the following set of real numbers.

Definition 3. Let $\mathcal{L}$ be the set of real irrational numbers x, for which there exists a constant $b > 1$ such that the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{b^q}$$

is satisfied for infinitely many integers p and q.

For more details about this set of irrational numbers see Kuznetsov [36] and Kuznetsov and Hubalek [32]. The following result was obtained in [39].

Theorem 9. *Assume that $\alpha \notin \mathcal{L} \cup \mathbb{Q}$. Then for all $x > 0$*

$$p(x) = \begin{cases} \sum\limits_{n\geq 0} a_n x^n + \sum\limits_{m\geq 0}\sum\limits_{n\geq 0} b_{m,n} x^{(m+1-\beta+\gamma)\delta + n}, & \text{if } \gamma + \hat{\gamma} < 1, \\ \sum\limits_{m\geq 0}\sum\limits_{n\geq 0} c_{m,n} x^{-(m+\hat{\beta})\delta - n - 1}, & \text{if } \gamma + \hat{\gamma} > 1. \end{cases}$$

It is worth recalling that, in general, it is not an easy exercise to invert the Mellin (or moments) transform of $I_{\mathbf{e}_q}(\xi)$ since a fine analysis of its asymptotic behavior is required.

The second methodology is based on the well-known relation between the law of $I_{\mathbf{e}_q}(\xi)$ and the distribution of the absorption time of positive self-similar Markov processes. Indeed, in Carmona et al. [15] it is shown that the law of $I_{\mathbf{e}_q}(\xi)$ can be expressed as an invariant function of a transient Ornstein-Uhlenbeck process associated to self-similar Markov process.

In Pardo et al. [49], starting from a large class of Lévy processes and assuming that $q = 0$, it is shown that the law of $I(\xi)$ can be factorized into the product of independent exponential functionals associated with two companion Lévy processes, namely the descending ladder height process of ξ and a spectrally positive Lévy process constructed from its ascending ladder height process. It is known that these two subordinators appear in the Wiener-Hopf factorization of Lévy processes. The laws of these exponential functionals are uniquely determined either by their positive or negative integer moments. Moreover, whenever the law of any of these can be expanded in series we can in general develop the law of $I(\xi)$ in series. Thus, for example, the requirements put on the Lévy measure of ξ in Kuznetsov and Pardo [39] can be relaxed to conditions only on the positive jumps (the Lévy measure on the positive half-line) of ξ thus enlarging considerably the class of Lévy processes ξ, for which we can obtain a series expansion of the law of $I(\xi)$.

Before stating the next results let us introduce some notation. First, since in our setting ξ drifts to $-\infty$, it is well-known that the ascending (resp. descending) ladder height process $H^+ = (H^+(t))_{t\geq 0}$ (resp. $H^- = (-H^-(t))_{t\geq 0}$) is a killed (resp. proper) subordinator. Then, we write, for any $z \in i\mathbb{R}$,

$$\phi_+(z) = \log \mathbf{E}\left[\exp(zH^+(1))\right] = \delta_+ z + \int_{(0,\infty)} (e^{zy} - 1)\mu_+(\underset{\cdot}{\mathrm{y}}) - k_+\,, \tag{2.17}$$

where $\delta_+ \geq 0$ is the drift and $k_+ > 0$ is the killing rate. Similarly, with $\delta_- \geq 0$, we have

$$\phi_-(z) = \log \mathbf{E}\left[\exp(zH^-(1))\right] = -\delta_- z - \int_{(0,\infty)} (1 - e^{-zy})\mu_-(\underset{\cdot}{\mathrm{y}})\,. \tag{2.18}$$

We recall that the integrability condition $\int_0^\infty (1 \wedge y)\mu_\pm(dy) < \infty$ holds. The Wiener-Hopf factorization then reads off as follows

$$\Psi(z) = -\phi_+(z)\phi_-(z), \text{ for any } z \in i\mathbb{R}. \tag{2.19}$$

Definition 4. We denote by $\mathcal{P}$ the set of positive measures on $\mathbb{R}_+$ which admit a non-increasing density.

Before we formulate the next result we introduce the two main hypothesis:

($\mathcal{H}_1$) Assume further that $-\infty < \mathbf{E}[\xi_1]$ and that one of the following conditions holds:

- **i)** $\mu_+ \in \mathcal{P}$ and there exists $z_+ > 0$ such that for all z with, $\Re(z) \in (0, z_+)$, we have $|\Psi(z)| < \infty$.
- **ii)** $\Pi_+ \in \mathcal{P}$.

($\mathcal{H}_2$) Assume that

i) $\mu_+ \in \mathcal{P},\ k_+ > 0$ and $\mu_- \in \mathcal{P}$.

Then the following result has been proved by Pardo, Patie & Savov [49].

Theorem 10. *Assume that ξ is a Lévy process that drifts to $-\infty$ with characteristics of the ladder height processes as in* (2.17) *and* (2.18)*. Let either ($\mathcal{H}_1$) or ($\mathcal{H}_2$) holds. Then, in both cases, there exists a spectrally positive Lévy process Y with a negative mean whose Laplace exponent ψ_+ takes the form*

$$\psi_+(-s) = -s\phi_+(-s) = \delta_+ s^2 + k_+ s + s^2 \int_0^\infty e^{-sy} \mu_+(y,\infty)dy,\ s \geq 0, \tag{2.20}$$

and the following factorization holds

$$\mathrm{I}(\xi) \stackrel{d}{=} \mathrm{I}(-H^-) \times \mathrm{I}(Y) \tag{2.21}$$

where $\stackrel{d}{=}$ stands for the identity in law and $\times$ for the product of independent random variables.

The above result has been recently improved by Patie and Savov [52]. The obtained identity can be looked from another perspective. Let us have two subordinators with Lévy measures $\mu_\pm$ such that $\mu_+ \in \mathcal{P},\ k_+ > 0$ and $\mu_- \in \mathcal{P}$. Then according to Vigon's theory of philanthropy, see [58], we can construct a process ξ such that its ladder height processes have exponents as in (2.17) and (2.18) and hence ξ satisfies the conditions of the previous Theorem. Therefore this method can be used to synthesize examples starting from the building blocks, i.e. the ladder height processes. This was observed in [49].

Corollary 2. *Let $\mu_\pm$ be the Lévy measures of two subordinators and $\mu_+ \in \mathcal{P},\ k_+ > 0$ and $\mu_- \in \mathcal{P}$. Then there exists a Lévy process which drifts to $-\infty$ whose ascending and descending ladder height processes have the Laplace exponents respectively given by* (2.17) *and* (2.18)*. Then all the claims of the Theorem 10 hold and in particular we have the factorization* (2.21)*.*

Another interesting problem is determining the behaviour of the density of the exponential functional $I(\xi)$ at 0 and at ∞. This problem has been recently studied by Kuznetzov [37] for Lévy processes with rational Laplace exponent (at 0 and at ∞) and by Patie [51] for spectrally negative Lévy processes (at ∞). In most of the applications, it is enough to have estimates of the tail behaviour $\mathbf{P}(I(\xi) \leq t)$ when t goes to 0 and/or $\mathbf{P}(I(\xi) \geq t)$ when t goes to ∞. The tail behaviour $\mathbf{P}(I(\xi) \leq t)$ was studied by Pardo [48] in the case where $-\xi$ is spectrally positive Lévy process and its Laplace exponent is regularly varying at infinity with index $\gamma \in (1,2)$. That is the content of the following result.

Proposition 2. *Let $I(\xi)$ be the exponential functional associated to a spectrally negative Lévy process ξ. Suppose that ψ, the Laplace exponent of $-\xi$, varies regularly at $+\infty$ with index $\beta \in (1,2)$. Then*

$$-\log \mathbf{P}\left(I(\xi) < 1/x\right) \sim (\beta - 1)\overleftarrow{H}(x) \quad \text{as} \quad x \to +\infty, \tag{2.22}$$

where

$$\overleftarrow{H}(x) = \inf\left\{s > 0\ ,\ \psi(s)/s > x\right\}.$$

In the case where the Lévy process is the negative of a subordinator several results are available to describe the left and right tail distribution of the exponential functional. Haas and Rivero [30] studied the case when ξ is the negative of a subordinator under several different frameworks, obtaining very precise estimates of the right tail behavior of the law of I, and described the maximum domain of attraction of I. One of the main results in [30] is the following description of the hazard rate function of an exponential functional of the negative of a subordinator.

Theorem 11. *Let $-\xi$ be a subordinator, $q \geq 0$, and $\boldsymbol{\varphi_{\Pi,q}}$ the function defined in (2.34) below. $I_{\mathbf{e}_q}(\xi)$ has a density k such that*

1. *if $a = 0$ and $\liminf_{x\to 0+} \frac{x\overline{\Pi}(x)}{\int_0^x \overline{\Pi}(u)du} > 0$,*

$$\frac{k(t)}{\mathbf{P}(I_{\mathbf{e}_q}(\xi) > t)} \sim_{t\to\infty} \frac{\boldsymbol{\varphi_{\Pi,q}}(t)}{t}, \qquad -\log \mathbf{P}(I_{\mathbf{e}_q}(\xi) > t) \sim_{t\to\infty} \int_c^t \frac{\boldsymbol{\varphi_{\Pi,q}}(u)}{u}du.$$

2. *if $a > 0$ and $0 < \liminf_{x\to 0+} \frac{x\overline{\Pi}(x)}{\int_0^x \overline{\Pi}(u)du} \leq \limsup_{x\to 0+} \frac{x\overline{\Pi}(x)}{\int_0^x \overline{\Pi}(u)du} < 1$,*

$$\frac{k(t)}{\mathbf{P}(I_{\mathbf{e}_q}(\xi) > t)} \sim_{t\to 1/a} a\boldsymbol{\varphi_{\Pi,q}}\left(\frac{t}{1-at}\right).$$

3. *if $a > 0$ and $\Pi(0,\infty) < \infty$, $\left(\frac{1}{a} - t\right)\frac{k(t)}{\mathbf{P}(I_{\mathbf{e}_q}(\xi) > t)} \sim_{t\to 1/a} \frac{\Pi(0,\infty) + q}{a}$.*

In the first two cases the Von-mises condition is satisfied

$$k(t)\int_t^\infty \mathbf{P}(I_{\mathbf{e}_q}(\xi) > s)ds/(\mathbf{P}(I_{\mathbf{e}_q}(\xi) > t))^2 \xrightarrow[t\to t_F]{} 1.$$

To describe the behaviour of the distribution at 0 of the negative of a subordinator we introduce the following assumption.
(A) *The Lévy measure Π belongs to the class $\mathcal{L}_\alpha$ for some $\alpha \geq 0$, that is to say that the tail Lévy measure $\overline{\Pi}$ satisfies*

$$\lim_{x\to\infty} \frac{\overline{\Pi}(x+y)}{\overline{\Pi}(x)} = e^{-\alpha y}, \quad \textit{for all } y \in \mathbb{R}. \tag{2.23}$$

Observe that regularly varying and subexponential tail Lévy measures satisfy this assumption with $\alpha = 0$ and that convolution equivalent Lévy measures are examples of Lévy measures satisfying (2.23) for some index $\alpha > 0$.

Theorem 12. *Let $q \geq 0$ and $\xi = -\sigma$, where σ is a subordinator such that when $q = 0$ the Lévy measure Π satisfies assumption* **(A)**. *The following asymptotic behaviour holds for the density function k of the exponential functional $I_{\mathbf{e}_q}$.*

i) If $q > 0$, then

$$k(x) \to q \qquad as \;\; x \downarrow 0.$$

ii) If $q = 0$, then $\mathbf{E}[I^{-\alpha}] < \infty$ and

$$k(x) \sim \mathbf{E}\big[I^{-\alpha}\big]\overline{\Pi}(\log 1/x) \quad as \;\; x \downarrow 0.$$

From this result it is possible to derive estimates for the exponential functional of the negative of a type of spectrally negative Lévy process, see [50].

Furthermore, the tail behaviour $\mathbb{P}(I \geq t)$ has been studied in a general setting, see for instance [19, 45, 54, 56], which is far from being an exhaustive list of references. We quote the following result from [54, 56].

Theorem 13. (i) *Assume that ξ_1 is no-lattice and it satisfies the conditions*

$$\mathbf{E}\left[\exp(\gamma\xi_1)\right] = 1 \qquad and \qquad \mathbf{E}\left[\xi_1^+ \exp(\gamma\xi_1)\right] < \infty.$$

In this case we have that $t^\gamma\, \mathbf{P}(I > t) \xrightarrow[t\to\infty]{} C_\gamma \in (0,\infty)$.

(ii) *Assume $q = 0$, that there exists a $\gamma > 0$ s.t. ξ is convolution equivalent with index γ,*

$$\lim_{t\to\infty} \frac{\mathbf{P}(\xi_1 > t+s)}{\mathbf{P}(\xi_1 > t)} = e^{-\gamma s}, \;\; s \in \mathbb{R}, \quad \lim_{t\to\infty} \frac{\mathbf{P}(\xi_2 > t)}{\mathbf{P}(\xi_1 > t)} = 2\,\mathbf{E}[e^{\gamma\xi_1}],$$

and $\mathbf{E}[e^{\gamma\xi_1}] < 1$. If $0 < \gamma \leq 1$, we assume furthermore that $\mathbf{E}[\xi_1] \in (-\infty, 0)$. Under these hypotheses

$$\mathbf{P}(I > t) \sim c_\gamma\, \mathbf{E}(I^{|1-\gamma|})\Pi_\xi(\log(t), \infty) = t^{-\gamma}\ell(t), \qquad t \to \infty,$$

with ℓ an slowly varying function.

More results about exponential functionals and their relations with other areas of probability theory can be found in the thorough review by Bertoin and Yor [9].

2.3 Asymptotic behaviour

The asymptotic behaviour of positive self-similar Markov processes $X^{(x)}$ with initial state $x > 0$ was studied by Lamperti (Theorem 7.1 in [44]). This property is inherited by the asymptotic behaviour of its associated Lévy process ξ and the fact that,

$$\lim_{t\to 0} \frac{\tau(t)}{t} = 1 \qquad \mathbb{P}_x - a.s.$$

Particularly, we have the following result due to Lamperti [44].

Theorem 14. *Let ξ a Lévy process that admits a law of the iterated logarithm, i.e. for some function $g : [0,+\infty) \to [0,+\infty)$ and some constant $c \in \mathbb{R}$*

$$\liminf_{t\to 0} \frac{\xi_t}{g(t)} = c \quad or \quad \limsup_{t\to 0} \frac{\xi_t}{g(t)} = c, \quad almost\ surely.$$

Then for $x > 0$, $X^{(x)}$, its associated self-similar Markov process, satisfies

$$\liminf_{t\to 0} \frac{X_t^{(x)} - x}{g(t)} = C(x,c) \quad or \quad \limsup_{t\to 0} \frac{X_t^{(x)} - x}{g(t)} = C(x,c), \quad almost\ surely,$$

where $C(x,c)$ is a constant that only depends on x and c.

Of course, we would like to know if we can use the Lamperti representation to the study of the asymptotic behaviour of $X^{(x)}$ at $+\infty$. Also, we would like to know if we can study the lower and upper functions of positive self-similar Markov processes starting from 0 at small times.

Several partial results on the lower envelope of $X^{(0)}$ have already been established before, the oldest of which are due to Dvoretsky and Erdös [25] and Motoo [46] who studied the special case of Bessel processes. More precisely, when $X^{(0)}$ is a Bessel process with dimension $\delta > 2$, we have the following integral test at 0: if f is an increasing function then

$$\mathbb{P}(X_t^{(0)} < f(t), \text{i.o., as } t \to 0) = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according as} \quad \int_{0+} \left(\frac{f(t)}{t}\right)^{\frac{\delta-2}{4}} \frac{dt}{t} \begin{cases} < \infty \\ = \infty \end{cases} .$$

The time inversion property of Bessel processes, induces the same integral test for the behaviour at $+\infty$ of $X^{(x)}$, $x \geq 0$.

Rivero [53] studied the lower functions of increasing self-similar Markov processes via the Lamperti representation. Following the method of Motoo [46] applied to $(e^{-t}X^{(x)}_{e^t-1}, t \geq 0)$, the Ornstein-Uhlenbeck process associated to $X^{(x)}$ (see Carmona et al. [15] for a proper definition), and under the assumption that the density ρ is decreasing in a neighborhood of $+\infty$ and bounded, Rivero [53] gave the following integral test for the lower envelope at $+\infty$.

Theorem 15. *Let $x > 0$ and $X^{(x)}$ an increasing self-similar Markov processes starting from x. If h is a decreasing function then*

$$\mathbb{P}\Big(X_s^{(x)} < s^{1/\alpha}h(s), \text{ i.o., as } s \to +\infty\Big) = 0 \text{ or } 1$$

according as

$$\int^{\infty} \rho\big(1/h(s)\big)\frac{\mathrm{d}s}{s} \qquad \text{is finite or infinite.}$$

A similar integral test for the lower envelope at 0 is obtained by Rivero via some reversal properties of $X^{(x)}$. From estimates of ρ and from the above result, Rivero [53] deduced the following laws of the iterated logarithm.

Theorem 16. *Let $\alpha > 0$, and ξ be a subordinator whose Laplace exponent ϕ is regularly varying at $+\infty$ with index $\beta \in (0,1)$. Suppose that the density ρ, of the Lévy exponential functional $I(-\alpha\xi)$ of ξ satisfies that is decreasing in a neighborhood of $+\infty$, and bounded. For $x \geq 0$, let $X^{(x)}$ be the increasing positive self-similar Markov process associated to ξ with scaling index $1/\alpha$. Define*

$$f(t) = \frac{\phi(\log|\log t|)}{\log|\log t|}, \qquad t \neq e, \quad t > 0,$$

then for any $x \geq 0$

$$\liminf_{t \to +\infty} \frac{X^{(x)}}{(tf(t))^{1/\alpha}} = \alpha^{\beta/\alpha}(1-\beta)^{(1-\beta)/\alpha} \qquad \text{almost surely,}$$

and

$$\liminf_{t \to 0} \frac{X^{(0)}}{(tf(t))^{1/\alpha}} = \alpha^{\beta/\alpha}(1-\beta)^{(1-\beta)/\alpha} \qquad \text{almost surely.}$$

This result extends the laws of the iterated logarithm of Friested [28] and Xiao [61].

We now present some general results on the lower envelope of $X^{(0)}$ at 0 and at ∞. The next result obtained by Chaumont and Pardo [19] means in particular that the asymptotic behaviour of $X^{(0)}$ only depends on the tail behaviour of the law of $I(-\xi)$, and on this of the law of

$$I_q(-\xi) \stackrel{\text{(def)}}{=} \int_0^{\hat{T}_{-q}} \exp\Big\{-\xi_s\Big\} \mathrm{d}s,$$

with $\hat{T}_x = \inf\{t : -\xi_t \le x\}$, for $x \le 0$. So also we set

$$F(t) \stackrel{\text{(def)}}{=} \mathbf{P}(I(-\xi) > t) \qquad \text{and} \qquad F_q(t) \stackrel{\text{(def)}}{=} \mathbf{P}(I_q(-\xi) > t).$$

Theorem 17. *The lower envelope of $X^{(0)}$ at 0 is described as follows. Let f be an increasing function.*

(i) If

$$\int_{0+} F\left(\frac{t}{f(t)}\right) \frac{dt}{t} < \infty \,,$$

then for all $\varepsilon > 0$,

$$\mathbb{P}(X_t^{(0)} < (1-\varepsilon) f(t), \textit{i.o., as } t \to 0) = 0 \,.$$

(ii) If for all $q > 0$,

$$\int_{0+} F_q\left(\frac{t}{f(t)}\right) \frac{dt}{t} = \infty \,,$$

then for all $\varepsilon > 0$,

$$\mathbb{P}(X_t^{(0)} < (1+\varepsilon) f(t), \textit{i.o., as } t \to 0) = 1 \,.$$

(iii) Suppose that $t \mapsto f(t)/t$ is increasing. If there exists $\gamma > 1$ such that,

$$\limsup_{t\to+\infty} \mathbf{P}(I > \gamma t)/\,\mathbf{P}(I > t) < 1 \textit{ and if } \int_{0+} F\left(\frac{t}{f(t)}\right) \frac{dt}{t} = \infty \,,$$

then for all $\varepsilon > 0$,

$$\mathbb{P}(X_t^{(0)} < (1+\varepsilon) f(t), \textit{i.o., as } t \to 0) = 1 \,.$$

As can be expected, there is a version of the last result for large times but Chaumont and Pardo proved that it could be extended also for $X^{(x)}$, for $x > 0$.

We now consider two types of behaviour of $F(t)$. The first type of tail behaviour that we consider is the case where F is regularly varying at infinity, i.e.

$$F(t) \sim \lambda t^{-\gamma} L(t) \,, \quad t \to +\infty \,, \tag{2.24}$$

where $\gamma > 0$ and L is a slowly varying function at $+\infty$. It is not difficult to see that, under this assumption, for any $q > 0$ the functions F_q and F are equivalent, i.e. $F_q \asymp F$. More precisely, if (2.24) holds then for all $q > 0$,

$$(1 - e^{-\gamma q}) F(t) \le F_q(t) \le F(t) \,, \tag{2.25}$$

for all t large enough. This last inequality is consequence from (2.24) and the independence of the processes $(\xi_s, 0 \le s \le \hat{T}_{-q})$ and $(\xi_{s+\hat{T}_{-q}} - \xi_{\hat{T}_{-q}}, s \ge 0)$. It is important to note that the Dvoretzky and Erdös integral test is consequence of this result when $X^{(0)}$ is a transient Bessel process. Another example of such behaviour is when the process ξ satisfies the hypotheses of Theorem 13.

The second type of behaviour that we shall consider is when $\log F$ is regularly varying at $+\infty$, i.e.

$$-\log F(t) \sim \lambda t^{\beta} L(t)\,, \quad \text{as } t \to \infty, \tag{2.26}$$

where $\lambda > 0$, $\beta > 0$ and L is a function which varies slowly at $+\infty$. Under this assumption, the conditions of part (iii) of the general integral tests due to Chaumont and Pardo are satisfied.

Define the function ψ by

$$\psi(t) \stackrel{\text{(def)}}{=} \frac{t}{\inf\{s : 1/F(s) > |\log t|\}}\,, \quad t > 0\,,\ t \neq 1\,. \tag{2.27}$$

Chaumont and Pardo found that the lower envelope of $X^{(0)}$ satisfies the following law of the iterated logarithm:

(i)

$$\liminf_{t\to 0} \frac{X_t^{(0)}}{\psi(t)} = 1\,, \quad \text{almost surely.} \tag{2.28}$$

(ii) For all $x \ge 0$,

$$\liminf_{t\to +\infty} \frac{X_t^{(x)}}{\psi(t)} = 1\,, \quad \text{almost surely.} \tag{2.29}$$

Note that this result extends the laws of the iterated logarithm found by Xiao [61] and Rivero [53] in the increasing case, but not their integral tests. Also, note that the assumption that the density of $I(-\xi)$ is decreasing in a neigbourhood of $+\infty$ in the law of the iterated logarithm due to Rivero [53] is not necessary but it is really important for his integral tests. In what follows we will describe some integral tests obtained in [47].

Let us define

$$\bar{G}(t) \stackrel{\text{(def)}}{=} \mathbf{P}\Big(S_1 < t\Big) \quad \text{and} \quad \bar{F}(t) \stackrel{\text{(def)}}{=} \mathbf{P}\Big(I(-\xi) < t\Big),$$

where S_1 denotes the first passage time of $X^{(0)}$ above the level 1. We also denote by $\mathcal{H}_0$ the totality of positive increasing functions $h(t)$ on $(0, \infty)$ that satisfy

i) $h(0) = 0$, and

ii) there exists $\beta \in (0, 1)$ such that $\sup_{t<\beta} \dfrac{t}{h(t)} < \infty$.

The following result is extracted from [47].

Theorem 18. *Let $h \in \mathcal{H}_0$.*

i) If

$$\int_{0+} \bar{G}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}_0\Big(X_t^{(0)} > (1+\epsilon)h(t),\ \textit{i.o., as } t \to 0\Big) = 0.$$

ii) If

$$\int_{0+} \bar{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}_0\Big(X_t > (1-\epsilon)h(t),\ \textit{i.o., as } t \to 0\Big) = 1.$$

A similar integral test holds for large times. As in the case for the lower envelope of $X^{(0)}$, these results can be applied to two type of estimates of the tail behaviour of I and νI. First, we discuss the case when $\bar{F}$ and $\bar{G}$ satisfy

$$ct^{\alpha}L(t) \le \bar{F}(t) \le \bar{G}(t) \le Ct^{\alpha}L(t) \qquad \text{as} \qquad t \to 0, \tag{2.30}$$

where $\alpha > 0$, c and C are two positive constants such that $c \le C$ and L is a slowly varying function at 0. An important example included in this case is when $\bar{F}$ and $\bar{G}$ are regularly varying functions at 0. The "regularity" of the behaviour of $\bar{F}$ and $\bar{G}$ gives the following integral tests obtained in [47].

Theorem 19. (Regular case) *Under condition (2.30), the upper envelope of* $X^{(0)}$ *at* 0 *and at* $+\infty$ *is as follows. Let* $h \in \mathcal{H}_0$*, such that either* $\lim_{t\to 0} t/h(t) = 0$ *or* $\liminf_{t\to 0} t/h(t) > 0$*, then*

$$\mathbb{P}\Big(X_t^{(0)} > h(t),\ \textit{i.o., as } t \to 0\Big) = 0 \textit{ or } 1,$$

according as

$$\int_{0+} \bar{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty \qquad \textit{is finite or infinite.}$$

Note that under condition (2.30), we may drop the factor $(1+\epsilon)$ and that the previous integral test depends only of $\bar{F}$. This result extends the integral test of Khintchine [34] for a stable subordinator. This is consequence of the following estimate of $\bar{F}$,

$$\bar{F}(t) \sim kt^{\beta+1} \qquad \text{as} \qquad t \to 0,$$

and since in the increasing case this integral test determines the upper envelope of increasing pssMp.

The second type of behaviour that we shall consider is when $\log \bar{F}$ and $\log \bar{G}$ are regularly varying at 0, i.e

$$-\log \bar{G}(1/t) \sim -\log \bar{F}(1/t) \sim \lambda t^{\beta} L(t), \qquad \text{as } t \to +\infty, \tag{2.31}$$

where $\lambda > 0$, $\beta > 0$ and L is a slowly varying function at $+\infty$. Under this assumption, the upper envelope of $X^{(0)}$ may be described as follows. Define the function

$$\phi(t) \stackrel{\text{(def)}}{=} t \inf\big\{s : 1/\bar{F}(1/s) > |\log t|\big\}, \quad t > 0, \quad t \ne 1.$$

The following result has been obtained in [47].

Theorem 20. (Log-regular case) *Under condition (2.31), the future infimum process satisfies the following law of the iterated logarithm:*

i)

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{\phi(t)} = 1, \qquad \textit{almost surely.}$$

ii) For all $x \geq 0$,

$$\limsup_{t\to +\infty} \frac{X_t^{(x)}}{\phi(t)} = 1, \qquad \textit{almost surely.}$$

It can be seen using the results in Theorem 2 that there is a large family that satisfies the condition (2.31). From this estimate Pardo obtained the following law of iterated logarithm for the future infimum process in terms of the following function. Let us define the function

$$f(t) = \frac{\psi(\log|\log t|)}{\log|\log t|} \qquad \text{for} \qquad t > 1, \quad t \neq e,$$

with ψ the Laplace exponent of $-\xi$, with ξ a spectrally negative Lévy process. By integration by parts, we can see that the function $\psi(\lambda)/\lambda$ is increasing, hence it is straightforward that the function $tf(t)$ is also increasing in a neighbourhood of ∞. Using this in [47] it has been proved that if ψ is regularly varying at $+\infty$ with index $\beta \in (1, 2)$, then

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{\big(tf(t)\big)^{1/\alpha}} = \alpha^{\beta/\alpha}(\beta-1)^{-(\beta-1)/\alpha} \qquad \text{almost surely,} \tag{2.32}$$

and,

$$\limsup_{t\to +\infty} \frac{X_t^{(x)}}{\big(tf(t)\big)^{1/\alpha}} = \alpha^{\beta/\alpha}(\beta-1)^{-(\beta-1)/\alpha} \qquad \text{almost surely.} \tag{2.33}$$

We finish this section with the following interpretation of the result of existence of a limit measure $\mathbb{P}_0$. If $(\xi, \mathbf{P})$ is a subordinator, associated to a pssMp $(X, \mathbb{P})$ via Lamperti's transformation, and has finite mean $m := \mathbf{E}(\xi_1) < \infty$, we know that there exists a measure $\mathbb{P}_{0+}$ such that

$$\mathbb{P}_x(X_1 \in dy) \xrightarrow[x\to 0+]{\text{weakly}} \mathbb{P}_{0+}(X_1 \in dx) = m^{-1} y^{\alpha}\, \mathbf{P}\left(I^{-1/\alpha} \in dy\right),$$

where $I = \int_0^\infty e^{-\alpha\xi_s} ds$. It is furthermore known that if $\mathbf{E}(\xi_1) = \infty$, then

$$\mathbb{P}_x(X_1 \in dy) \xrightarrow[x\to 0+]{\text{weakly}} \delta_\infty(dy).$$

Due to the self-similarity

$$\mathbb{P}_x(X_1 \in dy) = \mathbb{P}_1\left(xX_{1/x^\alpha} \in dy\right),$$

hence the latter is equivalent to

$$\frac{X_t}{t^{1/\alpha}} \xrightarrow[t\to\infty]{\text{Law}} \begin{cases} Z, & \text{if } \mathbf{E}(\xi_1) < \infty, \ Z \text{ has the same law as } X_1 \text{ under } \mathbb{P}_{0+}; \\ \infty, & \text{if } \mathbf{E}(\xi_1) = \infty. \end{cases}$$

A further problem that has been addressed in [13] describes the rate at which $X_t/t^{1/\alpha}$ tends towards ∞, when the mean of the underlying subordinator ξ is not finite. The main result by Caballero and Rivero [13] is the following.

Theorem 21. *Let $\{X(t), t \geq 0\}$ be a positive $1/\alpha$-self-similar Markov process with increasing paths. The following assertions are equivalent:*

(i) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, *is regularly varying at* 0 *with an index* $\beta \in [0, 1]$.

(ii) Under $\mathbb{P}_1$ the random variables $\left\{\log(X(t)/t^{1/\alpha})/\log(t), t > 1\right\}$ converge weakly as $t \to \infty$ towards a r.v. V.

(iii) For any $x > 0$, under $\mathbb{P}_x$ the random variables $\left\{\log(X(t)/t^{1/\alpha})/\log(t), t > 1\right\}$ converge weakly as $t \to \infty$ towards a r.v. V.

In this case, the law of V is given by: $V = 0$ a.s. if $\beta = 1$; $V = \infty$, a.s. if $\beta = 0$, and if $\beta \in]0, 1[$, its law has a density given by

$$\frac{\alpha^{1-\beta}2^{\beta}\sin(\beta\pi)}{\pi} v^{-\beta}(2 + \alpha v)^{-1}\mathrm{d}v, \qquad v > 0.$$

2.4 Quasi-stationary distributions

Another topic that has been studied related with the asymptotic behaviour of pssMp is the existence of Yaglom limits and quasi-stationary distributions. The main difference with the asymptotic behaviour described before relies on the idea that pssMp that hit zero at a finite time may be at an equilibrium state before being absorbed at 0. The main questions that were addressed in the paper [30] are the following. Assuming that the self-similar Markov process hits zero in a finite time with probability one, determine

QS-I whether there exists a probability measure μ on $\mathbb{R}^+$ such that for any $t > 0$,

$$\int_{\mathbb{R}^+} \mu(dx)\mathbb{P}_x(X_t \in dy | t < T_0) = \mu(dy), \qquad y \geq 0,$$

i.e. μ is a *quasi-stationary measure* for the pssMp $(X, \mathbb{P})$;

QS-II whether there exists a function $g : \mathbb{R}^+ \to \mathbb{R}^+ \setminus \{0\}$ and a non-degenerate probability measure ν on $\mathbb{R}^+$ such that

$$\mathbb{P}_1\left(\frac{X_t}{g(t)} \in dy | t < T_0\right) \xrightarrow[t\to t_F]{\text{weakly}} \nu(dy),$$

we will say that ν *is the limit in the Yaglom sense of X normalized by g*; $t_F = \sup\{t > 0 : \mathbb{P}_1(T_0 > t) > 0\}$.

QS-III what is the relation between μ and ν.

These questions were studied by Haas in [29] under the assumption that X has non-increasing paths and that the Lévy measure of the associated Lévy process is regularly varying at 0. In [30] the general case has been studied. We will assume wlog that the self-similarity index is equal to 1. In other case, the process

$$Y_t = X_t^{\alpha}, t \geq 0,$$

is a 1-pssMp. A QS-law for X exists iff a QS-law for Y exists. Analogously for Yaglom limits.

The problem of existence of quasi-stationary distributions was tangentially studied by Bertoin and Yor [7] in the case where the process has non-increasing paths. A slight modification of their main results read as follows.

Theorem 22. *If X has non-increasing paths, that is $-\xi$ is a subordinator, then*

- *there exists a QS-law for X,*

$$\int_{\mathbb{R}^+} \mu(dx)\mathbb{P}_x(X_t \in dy | t < T_0) = \mu(dy), \qquad y \geq 0.$$

- *μ is characterized by its entire moments; there is a $\beta > 0$*

$$\int_{\mathbb{R}^+} x^n \mu(dx) = \beta^{-n} \prod_{i=1}^{n} \phi(i), \qquad n \geq 1,$$

  with $\phi(\lambda) = -\log \mathbf{E}(e^{\lambda \xi_1})$, $\lambda > 0$,

- *let R follow the law μ and assume it is independent of ξ, then*

$$R \times \int_0^{\infty} e^{\xi_s} ds \sim \mathit{Exponential}(\beta).$$

So, in order to answer the question (QS-I) one should deal with the case where the paths are allowed to increase. In Theorem 1.1 in [30] it is proved that a necessary and sufficient condition for the existence of a QS-law for a pssMp is that the process has non-increasing paths. The main argument in proving the necessity of this condition is the following. Assume μ is a QS-law for X, the simple Markov property implies that there exists an index $\theta > 0$ such that

$$\int_{(0,\infty)} \mu(\mathrm{d}x)\mathbb{P}_x(t < T_0) = e^{-\theta t}, \qquad t \geq 0.$$

The self-similarity of X implies

$$e^{-\theta t} = \int_{(0,\infty)} \mu(\mathrm{d}x)\mathbb{P}_x(t < T_0) = \int_{[0,\infty)} \mu(\mathrm{d}x)\mathbb{P}_1(t < xT_0).$$

Recall then that $(T_0, \mathbb{P}_1) \stackrel{\text{Law}}{=} (I, \mathbf{P})$, where $I = \int_0^{\zeta} e^{\xi_s} \mathrm{d}s$. Then if R $\sim \mu$ and R is independent of I, we have that

$$\mathrm{R}I \stackrel{\text{Law}}{=} \mathbf{e}/\theta.$$

This identity and the independence imply that I has moments of all positive orders. An easy argument allows to prove that the totality of Lévy processes for which the exponential functional I has moments of all positive orders are exactly non-increasing Lévy processes. This concludes the argument because Lamperti's transform preserves the path of the process.

A by product of the above discussion is that the existence of QS-laws is closely related to factorizations of he exponential r.v. as the product of two independent r.v., one of which is an exponential functional. As we will see below this extends to Yaglom limits and give also rise to factorizations of Pareto and Beta r.v. Before giving a precise result we state a key observation that allows to transform the problem of existence of Yaglom limits into a problem of maximum domain of attraction for exponential functionals.

Lemma 1. *Let X be a pssMp that hits 0 in a finite time and ξ the Lévy process associated to it via Lamperti's transformation, which necessarily drift towards $-\infty$ or has a finite lifetime. We denote $I := \int_0^\infty e^{\xi_s} ds$. For $t > 0$, we have the equality of measures*

$$\mathbf{P}(I - t \in dy | t < I) = \mathbb{P}_1(X_t \widetilde{I} \in dy | t < T_0),$$

where $\widetilde{I}$ has the same law as I and is independent of $(X_s, s \leq t)$.

This result is an straith forward consequence of Lamperti's transformation. With this result at hand we have the following answer to question (QS-II) about Yaglom limits.

Theorem 23. *Let X be a pssMp that hits 0 in a finite time. The following assertions are equivalent.*

1. *The process $(X_t, t \geq 0)$ admits a Yaglom limit.*
2. *The process $(X\widetilde{I}, t \geq 0)$ admits a Yaglom limit, with $\widetilde{I} \stackrel{Law}{=} I$ and independent of X.*
3. *There exists a function $g : \mathbb{R}^+ \to \mathbb{R}^+ \setminus \{0\}$ and a non-degenerate probability measure $\widetilde{\Lambda}$ on $\mathbb{R}^+$ s.t.*
$$\mathbf{P}\left(\frac{I - t}{g(t)} \in dy | t < I\right) \xrightarrow[t\to\infty]{weakly} \widetilde{\Lambda}(dy).$$
4. *I is in the maximum domain of attraction of a Gumbel, Weibul or Fréchet distribution.*

In this case,

- *if $I \in MDA(Gumbel)$, $\widetilde{\Lambda}(dy) = e^{-y} dy$, $y \geq 0$;*
- *if $I \in MDA(Weibul)$, $\exists \gamma > 0$ s.t. $\widetilde{\Lambda}(dy) = \gamma(1-y)^{\gamma-1} dy$, $y \in (0, 1)$;*
- *if $I \in MDA(Frechet)$, $\exists \gamma > 0$ s.t. $\widetilde{\Lambda}(dy) = \gamma(1+y)^{-\gamma-1} dy$, $y \geq 0$.*

As we mentioned before the problem of existence of Yaglom limits is also related to factorisations of r.v. The Lemma 1 and the latter theorem have the following consequence: if X admits a Yaglom limit, then there exists a non trivial independent random variable

$$R \stackrel{\text{Law}}{=} \text{weak-}\lim_{t\to\infty} \frac{X_t}{g(t)}|t < T_0,$$

s.t.

$$R \times I \sim \begin{cases} \text{Exponential}(\beta), & \text{for some } \beta > 0, \\ \text{Beta}(1,\gamma), & \text{for some } \gamma > 0, \\ \text{Pareto}(\gamma), & \text{for some } \gamma > 0. \end{cases}$$

A systematic study of this type of factorisations and some consequences is carried out in [31].

We will next quote two of the main results from [30] providing some necessary and sufficient conditions for I to be in a maximum domain of attraction of a Gumbel or Frechet distribution. Further cases are treated in [30], together with precise results about descriptions of the Weibull case.

Assume X has non-increasing paths and let ξ the underlying Lévy process. It is known that

$$-\log \mathbf{E}(e^{\lambda \xi_1}) = q + a\lambda + \int_0^\infty 1 - e^{-\lambda x}\Pi(dx), \qquad \lambda > 0,$$

with Π a measure on $(0,\infty)$ such that $\int_{(0,\infty)} 1 \wedge x\Pi(dx) < \infty$, $q, a \geq 0$. We denote by $\boldsymbol{\varphi_{\Pi,q}}$ the inverse function of the mapping

$$t \mapsto \frac{t}{\int_0^\infty (1-e^{-tx})\Pi(\mathrm{d}x) + q}; \tag{2.34}$$

the inverse is well defined on $[0,\infty)$ if $q > 0$ and on $\left[(\int_0^\infty x\Pi(\mathrm{d}x))^{-1}, \infty\right)$ in other case. We denote $\overline{\Pi}(x) := \Pi(x,\infty)$.

Theorem 24. *Let X be a pssMp with non-increasing paths, and ξ the underlying Lévy process. Assume that $-\xi$ is a subordinator with killing term $q \geq 0$, drift $a = 0$ and Lévy measure Π such that*

$$\liminf_{x\to 0} \frac{x\overline{\Pi}(x)}{\int_0^x \overline{\Pi}(u)\mathrm{d}u} > 0. \tag{2.35}$$

In this case $I \in \mathrm{MDA}_{\mathrm{Gumbel}}$, $t_F = \infty$ and

$$\mathbb{P}_1\left(\frac{\boldsymbol{\varphi_{\Pi,q}}(t)X_t}{t} \in \cdot \ \mid \ t < T_0\right) \xrightarrow[t\to\infty]{} \mu_I^{(\mathbf{e})}.$$

$\mu_I^{(\mathbf{e})}$ is the unique probability measure such that if $R \sim \mu_I^{(\mathbf{e})}$, and $R \perp I$ then $RI \sim Exp(1)$.

Reciprocally, if $I \in \mathrm{MDA}_{\mathrm{Gumbel}}$ and $t_F = \infty$, then $-\xi$ is a subordinator with drift zero and $g(t) \sim \frac{\int_t^\infty \mathbf{P}(I>s)ds}{\mathbf{P}(I>t)} = \mathbb{E}_1(X_t|t < T_0)\,\mathbf{E}(I)$.

The proof of this result relay on precise estimates for the tail distribution of an exponential functional of a non-increasing Lévy processes, some of which are described in Theorem 11 here, and this allow to verify that the so-called Von-Mises condition is satisfied which is well known to be a necessary and sufficient condition for a r.v. to be in the maximum domain of attraction of a Gumbel distribution.

To deal with the non-monotone case, which happens to be the one corresponding to the domain of attraction of a Frechet distribution, the following result has been obtained in [30].

Theorem 25. *The following are equivalent:*

- *X is a pssMp with non-monotone paths and that admits a Yaglom limit.*
- *$I \in \mathrm{MDA}_{\text{Fréchet}}$.*
- *$t \mapsto \mathbb{P}(I > t)$ is regularly varying at infinity with some index $-\gamma$, $\gamma > 0$.*

In this case

$$\mathbb{P}_1\left(\frac{X_t}{t} \in \cdot \;\mid\; t < T_0\right) \xrightarrow[t\to\infty]{} \mu_I^{(\mathbf{P}_\gamma)}. \tag{2.36}$$

The probability measure $\mu_I^{(\mathbf{P}_\gamma)}$ is the unique p.m. such that if $R \sim \mu_I^{(\mathbf{P}_\gamma)}$, and $R \perp I$ then

$$\mathbf{P}(RI \in dy) = \gamma(1+y)^{-\gamma}, y > 0.$$

A necessary condition for $I \in \mathrm{MDA}_{\text{Fréchet}}$ is

$$\mathbb{E}[e^{\theta\xi_1}] \leq 1, \;\forall 0 \leq \theta \leq \gamma, \quad \textit{and } \mathbb{E}[e^{\delta\xi_1}] > 1, \forall \delta > \gamma,$$

for some $\gamma > 0$.

Sufficient conditions for the above theorem to hold were given in Theorem 13.

3 Real valued self-similar Markov processes

In previous sections, we studied positive self-similar Markov processes and their relationship with Lévy processes via the Lamperti representation. In this section we survey some recent results on real valued self-similar Markov processes which turn out to be associated to Markov additive processes via a Lamperti-type representation.

The structure of real valued self-similar Markov processes has been investigated by Chybiryakov [21] in the symmetric case, and by Kiu [35] and Chaumont et al. [18] in general. Inspired from [38], here we give an interpretation of those authors' results in terms of a two-state Markov additive process. We begin with some relevant definitions.

We focus on real-valued self-similar Markov processes (rssMp) $X = (X_t, t \geq 0)$ with self- similarity index $\alpha > 0$ and starting from $x \in \mathbb{R} \setminus \{0\}$. Let $(\mathbb{P}_x)_{x\in\mathbb{R}\setminus\{0\}}$ denote its probability laws starting from x.

In [18], the authors confine their attention to processes in "class **C.4**". A real valued self-similar Markov process X is in this class if,

$$\mathbb{P}_x(\exists t > 0 : X_t X_{t-} < 0) = 1, \qquad \forall x \neq 0.$$

By the strong Markov property this implies that, with probability one, the process X changes sign infinitely often. This assumption will be in force in the sequel.

The other cases introduced in [18], namely **C.1-3**, are those where either the process changes once, and only once, of sign, and those where the process never changes of sign. In the former cases the construction of real valued ssMp can be easily deduced using the ideas to deal with the cases where the process changes of sign infinitely many times. In the latter case, the real-ssMp is constructed by applying Lamperti's transformation to the process when started at a positive (negative) position using two Lévy processes, one to describe the positive values of the path and the other for the negative values. We do not provide further details about these cases.

3.1 Markov additive processes

Let E be a finite state space and $(\mathcal{G}_t)_{t\geq 0}$ a standard filtration. A càdlàg process (ξ, J) in $\mathbb{R}\times E$ with law $\mathbf{P}$ is called a Markov additive process (MAP) with respect to $(\mathcal{G}_t)_{t\geq 0}$ if $(J(t))_{t\geq 0}$ is a continuous-time Markov chain in E, and the following property is satisfied, for any $i\in E$, $s,t\geq 0$:

$$\begin{aligned}&\text{given } \{J(t)=i\}, \text{ the pair } (\xi(t+s)-\xi(t), J(t+s)) \text{ is independent of } \mathcal{G}_t,\\ &\qquad \text{and has the same distribution as } (\xi(s)-\xi(0), J(s)) \text{ given } \{J(0)=i\}.\end{aligned} \tag{3.37}$$

Aspects of the theory of Markov additive processes are covered in a number of texts, among them [1] and [2].

Let us introduce some notation. We write $\mathbf{P}_i=\mathbf{P}(\cdot|\xi(0)=0, J(0)=i)$; and if μ is a probability distribution on E, we write $\mathbf{P}_\mu=\mathbb{P}(\cdot|\xi(0)=0, J(0)\sim\mu)=\sum_{i\in E}\mu(i)\mathbf{P}_i$. We adopt a similar convention for expectations.

It is well-known that a Markov additive process (ξ, J) also satisfies (3.37) with t replaced by a finite stopping time. Furthermore, it has the structure given by the following proposition; see [2, §XI.2a] and [33, Proposition 2.5].

Proposition 3. *The pair (ξ, J) is a Markov additive process if and only if, for each $i,j\in E$, there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geq 0}$ and a sequence of iid random variables $(U_{ij}^n)_{n\geq 0}$, independent of the chain J, such that if $T_0=0$ and $(T_n)_{n\geq 1}$ are the jump times of J, the process ξ has the representation*

$$\xi(t)=\mathbf{1}_{\{n>0\}}(\xi(T_n-)+U^n_{J(T_n-),J(T_n)})+\xi^n_{J(T_n)}(t-T_n), \textit{for } t\in[T_n,T_{n+1}),\ n\geq 0.$$

For each $i\in E$, it will be convenient to define, on the same probability space, ξ_i as a Lévy process whose distribution is the common law of the ξ_i^n processes in the above representation; and similarly, for each $i,j\in E$, define U_{ij} to be a random variable having the common law of the U_{ij}^n variables.

Let us now fix the following setup. Firstly, we confine ourselves to irreducible Markov chains J. Let the state space E be the finite set $\{1,\ldots,N\}$, for some $N\in\mathbb{N}$. Denote the transition rate matrix of the chain J by $Q=(q_{ij})_{i,j\in E}$. For each $i\in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i, in the sense that $e^{\psi_i(z)}=\mathbf{E}[e^{z\xi_i(1)}]$, for all $z\in\mathbb{C}$ for which the right-hand side exists. For each pair of $i,j\in E$, define the Laplace transform $G_{ij}(z)=\mathbf{E}[e^{zU_{ij}}]$ of the jump distribution U_{ij}, where this exists; write $G(z)$ for the $N\times N$ matrix whose (i,j)–th element is $G_{ij}(z)$.

We will adopt the convention that $U_{ij} = 0$ if $q_{ij} = 0$, $i \neq j$, and also set $U_{ii} = 0$ for each $i \in E$.

A multidimensional analogue of the Laplace exponent of a Lévy process is provided by the matrix-valued function

$$F(z) = \operatorname{diag}(\psi_1(z), \dots, \psi_N(z)) + Q \circ G(z), \tag{3.38}$$

for all $z \in \mathbb{C}$ where the elements on the right are defined, where $\circ$ indicates elementwise multiplication, also called Hadamard multiplication. It is then known that

$$\mathbf{E}_i[e^{z\xi(t)}; J(t) = j] = \left(e^{F(z)t}\right)_{ij}, \text{ for } i, j \in E,$$

for all $z \in \mathbb{C}$ where one side of the equality is defined. For this reason, F is called the matrix exponent of the MAP ξ.

3.2 Lamperti type representation of real valued self similar Markov processes

Let X be a real valued self-similar Markov process. In [18] it has been proved that X may be identified up to the first hitting time of 0,

$$T_0 = \inf\{t \geq 0 : X_{t-} = 0 \text{ or } X_t = 0\},$$

as the time-changed exponential of a certain complex-valued process $\mathcal{E}$, which from the terminology used in [18] it will be called the Lamperti–Kiu representation of X. The main result in [18] is summarised in the following theorem.

Theorem 26. *Let X be a rssMp in class **C.4**, and let $x \neq 0$. It is possible to define independent sequences $(\xi^{\pm,k})_{k\geq 0}$, $(\zeta^{\pm,k})_{k\geq 0}$, $(U^{\pm,k})_{k\geq 0}$ of iid random objects with the following proprties:*

1. *The elements of these sequences are distributed such that: the $\xi^{\pm}$ are real-valued Lévy processes; $\zeta^{\pm}$ are exponential random variables with parameters $q^{\pm}$; and $U^{\pm}$ are real-valued random variables.*

2. *For each $x \neq 0$, define the following objects:*

$$(\xi^{(x,k)}, \zeta^{(x,k)}, U^{(x,k)}) = \begin{cases} (\xi^{+,k}, \zeta^{+,k}, U^{+,k}), & \textit{if } \operatorname{sgn}(x)(-1)^k = 1 \\ (\xi^{-,k}, \zeta^{-,k}, U^{-,k}), & \textit{if } \operatorname{sgn}(x)(-1)^k = -1, \end{cases}$$

$$\mathcal{T}_0^{(x)} = 0, \quad \mathcal{T}_n^{(x)} = \sum_{k=0}^{n-1} \zeta^{(x,k)},$$

$$N_t^{(x)} = \max\{n \geq 0 : \mathcal{T}_n^{(x)} \leq t\},$$

$$\sigma_t^{(x)} = t - \mathcal{T}_{N_t^{(x)}}^{(x)},$$

$$\mathcal{E}_t^{(x)} = \xi_{\sigma_t^{(x)}}^{(N_t^{(x)})} + \sum_{k=0}^{N_t - 1} (\xi_{\zeta^{(x,k)}}^{(x,k)} + U^{(x,k)}) + i\pi N_t^{(x)}, \qquad t \geq 0,$$

$$\tau(t) = \inf\left\{ s > 0 : \int_0^s |\exp(\alpha \mathcal{E}_u^{(x)})| \,\mathrm{d}u > t|x|^{-\alpha} \right\}, \quad t < T_0.$$

Then, the process X under the measure $\mathbb{P}_x$ has the representation

$$X_t = x \exp(\mathcal{E}_{\tau(t)}^{(x)}), \qquad 0 \le t < T_0.$$

3. *Reciprocally, any process constructed in this form is a real-valued ssMp.*

The case where X is a stable process killed at its first hitting time of 0 or conditioned to avoid zero is studied in detail in [18].

The abundance of notation necessary to be precise in this context may obscure the fundamental idea, which is as follows. At any given time, the process $\mathcal{E}$ evolves as a Lévy process $\xi^{\pm}$, moving along a line $\Im(z) = \pi N$, up until an exponential 'clock' $\zeta^{\pm}$ (corresponding to the process X changing sign) rings. At this point the imaginary part of $\mathcal{E}$ is incremented by π, the real part jumps by $U^{\pm}$, and the process begins to evolve as the other Lévy process, $\xi^{\mp}$.

Particularly in light of the discussion in the previous section, the latter result can be formulated using Markov additive processes. This is the purpose of the following result proved in [38].

Proposition 4. *Let X be an rssMp, with Lamperti–Kiu representation $\mathcal{E}$. Define furthermore*

$$[n] = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

Then, for each $x \neq 0$, the process

$$(\xi(t), J(t)) = \left(\Re(\mathcal{E}_t^{(x)}), \left[\Im(\mathcal{E}^{(x)})_t/\pi + 1_{\{x>0\}}\right]\right)$$

defined in Proposition 3 is a Markov additive process with state space $E = \{1, 2\}$, and X under $\mathbb{P}_x$ has the representation

$$X_t = x \exp\big(\xi(\tau(t)) + i\pi(J(\tau(t)) + 1)\big), \text{for } \ 0 \le t < T_0,$$

where we note that $(\xi(0), J(0))$ is equal to $(0, 1)$ if $x > 0$, or $(0, 2)$ if $x < 0$. Furthermore, the time-change τ has the representation

$$\tau(t) = \inf\left\{ s > 0 : \int_0^s \exp(\alpha \xi(u)) \,\mathrm{d}\,u > t|x|^{-\alpha} \right\}, \qquad \text{for } t < T_0, \tag{3.39}$$

in terms of the real-valued process ξ.

Note that the MAP (ξ, J) under $\mathbf{P}_1$ corresponds to the rssMp X started at a point $x > 0$, and the MAP under $\mathbf{P}_2$ corresponds to the rssMp started at a point $x < 0$.

Furthermore, we observe from the form (3.39) of the time-change τ that under $\mathbb{P}_x$, for any $x \neq 0$, the following identity holds for T_0, the hitting time of zero:

$$|x|^{-\alpha} T_0 \overset{\text{Law}}{=} \int_0^\infty e^{\alpha \xi(u)} \,\mathrm{d}\,u.$$

Implicit in this statement is that the MAP on the right-hand side has law $\mathbf{P}_1$ if $x > 0$, and law $\mathbf{P}_2$ if $x < 0$.

3.3 Exponential functionals of MAPs

We start by describing the existence of the leading eigenvalue of the matrix F, which will play a key role in our analysis of MAPs. This is sometimes also called the Perron-Frobenius eigenvalue; see [2, §XI.2c] and [33, Proposition 2.12].

Proposition 5. *Suppose that $z \in \mathbb{C}$ is such that $F(z)$ is defined. Then, the matrix $F(z)$ has a real simple eigenvalue $\kappa(z)$, which is larger than the real part of all its other eigenvalues. Furthermore, the corresponding right-eigenvector $v(z)$ may be chosen so that $v_i(z) > 0$ for every $i \in E$, and normalised such that $\pi v(z) = 1$ where π is the equilibrium distribution of the chain J.*

This leading eigenvalue features in the following probabilistic result, which identifies a martingale (also known as the Wald martingale) and change of measure analogous to the exponential martingale and Esscher transformation of a Lévy process; cf. [2, Proposition XI.2.4, Theorem XIII.8.1].

Proposition 6. *Let*

$$M(t, \gamma) = e^{\gamma\xi(t) - \kappa(\gamma)t} \frac{v_{J(t)}(\gamma)}{v_{J(0)}(\gamma)}, \text{ for } t \geq 0,$$

for some γ such that the right-hand side is defined. Then,

i) $M(\cdot, \gamma)$ is a unit-mean martingale with respect to $(\mathcal{G}_t)_{t\geq 0}$ under any initial distribution of $(\xi(0), J(0))$.

ii) Define the change of measure

$$\left.\frac{\mathrm{d}\mathbf{P}^{(\gamma)}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{G}_t} = M(t, \gamma).$$

Under $\mathbf{P}^{(\gamma)}$, the process ξ is still a Markov additive process, and it has the following characteristics, for each $i, j \in E$:

- $\mathbf{P}^{(\gamma)}(U_{ij} \in \mathrm{d}\,x) = \dfrac{e^{\gamma x}}{G_{ij}(\gamma)} \mathbf{P}(U_{ij} \in \mathrm{d}\,x)$, *and hence* $G^{(\gamma)}_{ij}(z) = \dfrac{G_{ij}(z+\gamma)}{G_{ij}(\gamma)}$,
- $q^{(\gamma)}_{ij} = \dfrac{v_j(\gamma)}{v_i(\gamma)} q_{ij} G_{ij}(\gamma)$ *and*
- $\psi^{(\gamma)}_i(z) = \psi_i(z+\gamma) - \psi_i(\gamma)$.

Furthermore,

$$F^{(\gamma)}(z) = (\mathrm{diag}(v_i(\gamma),\, i \in E))^{-1}[F(z+\gamma) - \kappa(\gamma)\,\mathrm{Id}]\mathrm{diag}(v_i(\gamma),\, i \in E),$$

and hence,

$$\kappa^{(\gamma)}(z) = \kappa(z+\gamma) - \kappa(\gamma).$$

Making use of this, the following proposition with properties of κ are often used in the literature.

Proposition 7. *Suppose that F is defined in some open interval D of $\mathbb{R}$. Then, the leading eigenvalue κ of F is smooth and convex on D.*

In Section 2.2 , we studied the exponential functional of Lévy processes, now we are interested in obtaining some results which will assist us in computing the law of an integrated exponential functional associated to Markov additive processes.

For a MAP ξ, let

$$I(-\xi) = \int_0^\infty \exp(-\xi(t)) \, \mathrm{d}\, t.$$

One way to characterise the law of $I(-\xi)$ is via its Mellin transform, which we write as $\mathcal{M}(s)$. This is the vector in $\mathbb{R}^N$ whose ith element is given by

$$\mathcal{M}_i(s) = \mathbf{E}_i[I(-\xi)^{s-1}], \text{for} i \in E.$$

We will shortly express a functional equation for $\mathcal{M}$, analogous to the functional equation for the Mellin transform for the exponential functional of Lévy processes which we saw in Section 2.2. For Lévy processes, proofs of the result can be found in [15], [45] and [55].

We make the following assumption, which is analogous to the Cramér condition for a Lévy process; recall that κ is the leading eigenvalue of the matrix F, as discussed in section 3.1.

Definition 5 (Cramér condition for a Markov additive process)**.** There exists $z_0 < 0$ such that $F(s)$ exists on $(z_0, 0)$, and some $\theta \in (0, -z_0)$, called the Cramér number, such that $\kappa(-\theta) = 0$.

Since the leading eigenvalue κ is smooth and convex where it is defined, it follows also that $\kappa(-s) < 0$ for $s \in (0, \theta)$. In particular, this renders the matrix $F(-s)$ negative definite, and hence invertible. Furthermore, it follows that $\kappa'(0-) > 0$, and hence (see [2, Corollary XI.2.7] and [33, Lemma 2.14]) that ξ drifts to $+\infty$ independently of its initial state. This implies that $I(-\xi)$ is an a.s. finite random variable.

Proposition 8. *Suppose that ξ satisfies the Cramér condition (Assumption 5) with Cramér number $\theta \in (0,1)$. Then, $\mathcal{M}(s)$ is finite and analytic when $\Re(s) \in (0, 1+\theta)$, and we have the following vector-valued functional equation:*

$$\mathcal{M}(s+1) = -s(F(-s))^{-1}\mathcal{M}(s), \qquad \textit{for } s \in (0, \theta).$$

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