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The upper envelope of positive self-similar Markov processes.

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Abstract: We establish integral tests in connection with laws of the iterated logarithm at 0 and at $+\infty$, for the upper envelope of positive self-similar Markov processes. Our arguments are based on the Lamperti representation and on the study of the upper envelope of the future infimum due to the author (see [23]). These results extend laws of the iterated logarithm for Bessel processes due to Dvoretsky and Erdős [14] and stable Lévy processes with no postive jumps conditioned to stay positive due to Bertoin [1].

Key words: Self-similar Markov process, self-similar additive processes, future infimum process, Lévy process, Lamperti representation, first and last passage time, integral test, law of the iterated logarithm.

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1 Introduction.

A real Markov process $X = (X_t, t \geq 0)$ with càdlàg paths is a self-similar process if for every $k > 0$ and every initial state $x \geq 0$ it satisfies the scaling property, i.e., for some $\alpha > 0$

$$\text{the law of } (kX_{k^{-\alpha}t}, t \geq 0) \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{kx},$$

where $\mathbb{P}_x$ denotes the law of the process X starting from $x \geq 0$.
In this article, we focus on positive self-similar Markov processes and we will refer to them as *PSSMP*. We use the notation $X^{(x)}$ or $(X, \mathbb{P}_x)$ for the PSSMP starting from $x \geq 0$. These processes are to be found in many areas of probability theory. To give just a few examples, let us mention: the continuous-state branching process which is associated to a self-similar Lévy tree (see for instance Duquesne and Le Gall [13]) and

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whose genealogy may be described by the beta-coalescent (see Birkner et al. [8]), the size of a tagged individual in the self-similar branching Markov chain (see Krell [19]), the so-called tagged fragment of a self-similar fragmentation (see Bertoin [6]), ... These processes also appear as the limit of local structure of random quadrangulations (see Krikun [20]).

Self-similar processes were the object of a systematic study first done by Lamperti [21]. In later work, Lamperti [22] studied the Markovian case in detail. The main result in [22] proves that any PSSMP starting from a strictly positive state is a time-change of the exponential of a Lévy process. More precisely, let $X^{(x)}$ be a self-similar Markov process started from $x > 0$ that fulfills the scaling property for some $\alpha > 0$, then

$$X_t^{(x)} = x \exp\Big\{\xi_{\tau(tx^{-\alpha})}\Big\}, \qquad 0 \le t \le x^\alpha I(\xi), \tag{1.1}$$

where

$$\tau_t = \inf\Big\{s \ge 0 : I_s(\xi) > t\Big\}, \qquad I_s(\xi) = \int_0^s \exp\big\{\alpha\xi_u\big\}\mathrm{d}u, \qquad I(\xi) = \lim_{t\to+\infty} I_t(\xi),$$

and ξ is a real Lévy process possibly killed at an independent exponential time. This famous path construction of PSSMP is well-known as the Lamperti representation.
Several authors have studied the problem of the existence of an entrance law at 0 for $(X, \mathbb{P}_x)$, see for instance Bertoin and Caballero [3], Bertoin and Yor [4], and Caballero and Chaumont [9]. Bertoin and Caballero [3] studied for the first time this problem for the increasing case. Later Bertoin and Yor [4] generalized the results obtained in [3]. They also gave sufficient conditions for the weak convergence of $\mathbb{P}_x$ to hold when x tends to 0, in the sense of finite dimensional distributions. The entrance law was also computed in the mentioned works. Recently, Caballero and Chaumont [9] gave necessary and sufficient conditions for the weak convergence of $X^{(x)}$ on the Skorokhod's space. In [9], the authors also gave a path construction of this weak limit, that we will denote by $X^{(0)}$ or $(X, \mathbb{P}_0)$.

The aim of this work is to describe the upper envelope at 0 and at $+\infty$ for a large class of PSSMP such that $\limsup_{t\to\infty} X_t^{(x)} = \infty$ almost surely, through integral tests and laws of the iterated logarithm. We will give special attention to the case when the process has no positive jumps since we may obtain general integral tests and compare the rate of growth of $X^{(x)}$ with that of its future infimum process and the PSSMP $X^{(x)}$ reflected at its future infimum.
Several partial results on the upper envelope of $X^{(0)}$ have already been established before, the most important of which is due to Dvoretsky and Erdős [14] who studied the case of Bessel processes. More precisely, we have the well known Kolmogorov-Dvoretsky-Erdös integral test, see for instance Itô and McKean [17].

Theorem 1 (Kolmogorov-Dvoretsky-Erdős) *Let h be a nondecreasing, positive and unbounded function as t goes to $+\infty$, and $X^{(0)}$ be a Bessel process of dimension $\delta > 2$ starting from 0. Then its upper envelope at 0 is as follows,*

$$\mathbb{P}\big(X_t^{(0)} > \sqrt{t}h(t),\ \textit{i.o., as } t \to 0\big) = 0 \textit{ or } 1,$$

according as, $\int_0 h^\delta(t)\exp\{-h^2(t)/2\}t^{-1}\mathrm{d}t$ *is finite or infinite.*

Thanks to the time inversion property of Bessel processes, we also have the above integral test for large times. The integral test for transient Bessel processes will be improved in Section 4 (Theorems 2 and 3).
Recall that the future infimum of a PSSMP starting at $x \geq 0$ is defined by

$$J_s^{(x)} = \inf_{t\geq s} X_t^{(x)}.$$

Khoshnevisan et al. [18] studied the asymptotic behaviour of the future imfimum of transient Bessel processes. In that paper, the authors obtained the following law of the iterated logarithm (LIL for short) for $J^{(0)}$ and for the Bessel process reflected at its future infimum, $X^{(0)} - J^{(0)}$:

$$\limsup_{t\to+\infty} \frac{J_t^{(0)}}{\sqrt{2t\log|\log t|}} = 1 \quad \text{and} \quad \limsup_{t\to+\infty} \frac{X_t^{(0)} - J_t^{(0)}}{\sqrt{2t\log|\log t|}} = 1 \qquad \text{a.s.} \tag{1.2}$$

In Section 6, we extend these results and we also study the small time behaviour.
If $X^{(0)}$ is a stable Lévy process with no positive jumps conditioned to stay positive and with index $1 < \alpha \leq 2$, we have the following LIL due to Bertoin [1],

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{t^{1/\alpha}\big(\log|\log t|\big)^{1-1/\alpha}} = c(\alpha) \qquad \text{almost surely,} \tag{1.3}$$

where $c(\alpha)$ is a positive constant. Recently in [23], the author studied the asymptotic behaviour of the upper envelope of the future infimum of PSMMP under general hypothesis. In particular, when $X^{(0)}$ is the aforementioned class of conditioned stable processes, we have

$$\limsup_{t\to 0} \frac{J_t^{(0)}}{t^{1/\alpha}\big(\log|\log t|\big)^{1-1/\alpha}} = c(\alpha), \qquad \text{a. s.} \tag{1.4}$$

We remark that the above LIL is also satisfied for large times and any starting point. In Section 6, we will see that under the assumption that $\mathbb{P}(J_1^{(0)} > t)$ is log-regular, i.e.

$$-\log \mathbb{P}\big(J_1^{(0)} > t\big) \sim \lambda t^\beta L(t) \quad \text{as} \quad t \to +\infty,$$

where $\lambda, \beta > 0$ and L is a function which varies slowly at $+\infty$, the upper envelope of $X^{(x)}$ will be described by an explicit LIL which agrees with the LIL describing the upper envelope of its future infimum. The asymptotic results presented in Section 6 are consequences of general integral tests which are stated in Sections 3 and 4.
The rest of this paper is organized as follows. Section 2 is devoted to some preliminaries of Lévy processes and PSSMP. In Section 3, we study the asymptotic properties of the first passage time of $X^{(0)}$. In Section 4, we give the general integral tests for the upper envelope of $X^{(x)}$, $x \geq 0$. Sections 5 and 6 are devoted to applications of the results of Sections 3 and 4. Finally in Section 7, we will give some examples.

2 Preliminaries.

2.1 Weak convergence and entrance law of PSSMP.

Let $\mathcal{D}$ denote the Skorokhod's space of càdlàg paths. We consider a probability measure in $\mathcal{D}$ denoted by $\mathbb{P}$ under which ξ will always be a real Lévy process such that $\xi_0 = 0$. Let Π be the Lévy measure of ξ, that is the measure satisfying

$$\int_{(-\infty,\infty)} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty,$$

and such that the characteristic exponent Ψ, defined by $\mathbb{E}(e^{iu\xi_1}) = e^{-\Psi(u)}$, $u \in \mathbb{R}$, is given, for some $b \geq 0$ and $a \in \mathbb{R}$, by

$$\Psi(u) = iau + \frac{1}{2}b^2u^2 + \int_{(-\infty,\infty)} \Big(1 - e^{iux} + iux\mathbb{1}_{\{|x|\leq 1\}}\Big)\Pi(\mathrm{d}x), \quad u \in \mathbb{R}.$$

Define for $x \geq 0$,

$$\overline{\Pi}^+(x) = \Pi\big((x,\infty)\big), \quad \overline{\Pi}^-(x) = \Pi\big((-\infty,-x)\big), \quad M(x) = \int_0^x \mathrm{d}y \int_y^\infty \overline{\Pi}^-(z)\mathrm{d}z,$$

and

$$J = \int_{[1,\infty)} \frac{x\overline{\Pi}^+(x)}{1 + M(x)}\mathrm{d}x.$$

Let us denote by $\mathbb{P}_x$ the law, under $\mathbb{P}$, of the PSSMP $X^{(x)}$ starting from $x > 0$ associated to the Lévy process ξ via the Lamperti representation. Then according to Caballero and Chaumont [9], necessary and sufficient conditions for the weak convergence of $X^{(x)}$ in the Skorokhod's space are

$$(\mathbf{H}) \quad \xi \text{ is not arithmetic and } \quad \begin{cases} \text{either } 0 < \mathbb{E}(\xi_1) \leq \mathbb{E}(|\xi_1|) < \infty, \\ \text{or } \mathbb{E}(|\xi_1|) < \infty, \mathbb{E}(\xi_1) = 0 \text{ and } J < \infty, \end{cases}$$

and

$$\mathbb{E}\left(\log^+ \int_0^{T_1} \exp\big\{\alpha\xi_s\big\}\mathrm{d}s\right) < \infty, \tag{2.5}$$

where T_x is the first passage time above $x \geq 0$, i.e. $T_x = \inf\{t \geq 0 : \xi_t \geq x\}$.
The weak limit found in [9], denoted by $X^{(0)}$ or $(X, \mathbb{P}_0)$, is a PSSMP starting from 0 which fulfills the Feller property on $[0,\infty)$ and with the same transition probabilities as $X^{(x)}$, $x > 0$. In the sequel, we suppose that the Lévy processes considered here satisfy conditions ($\mathbf{H}$) and (2.5). We will distinguish the case $0 < \mathbb{E}(\xi_1) \leq \mathbb{E}(|\xi_1|) < \infty$ by saying that $m := \mathbb{E}(\xi_1) > 0$. Note that if $m > 0$, we have that $\mathbb{E}(T_1) < \infty$ and hence condition (2.5) is satisfied. Another example when (2.5) is satisfied is when ξ has no positive jumps (see Section 2 in [9]).
Now, we introduce the so-called first and last passage times of $X^{(0)}$ by

$$S_y = \inf\Big\{t \geq 0 : X_t^{(0)} \geq y\Big\} \quad \text{and} \quad U_y = \sup\Big\{t \geq 0 : X_t^{(0)} \leq y\Big\},$$

for $y > 0$. Note that when the process $X^{(0)}$ drifts to $+\infty$ (i.e. $m > 0$), the last passage time U_x is finite a.s. for $x \geq 0$. Moreover, if the process $X^{(0)}$ satisfies the scaling property with some index $\alpha > 0$, then we deduce that the first passage time process $S = (S_x, x \geq 0)$ and the last passage time process $U = (U_x, x \geq 0)$ are increasing self-similar processes with scaling index α^{-1}. From the path properties of $X^{(0)}$ we easily see that both processes start from 0 and go to $+\infty$ as x increases.

With no loss of generality, we will suppose that $\alpha = 1$. Indeed, we see from the scaling property that if $X^{(x)}$, $x \geq 0$, is a PSSMP with index $\alpha > 0$, then $\big(X^{(x)}\big)^{\alpha}$ is a PSSMP with index equal to 1. Therefore, the integral tests and LIL established in the sequel can easily be interpreted for any $\alpha > 0$.

2.2 PSSMP with no positive jumps and some path transformations.

Here, we suppose that ξ has no positive jumps. It is known that under this assumption, the process ξ has finite exponential moments of arbitrary positive order (see [2] for background). In particular, we have that $\mathbb{E}\big(\exp\{u\xi_t\}\big) = \exp\{t\psi(u)\}$, for $u \geq 0$ where $\psi(u) = -\Psi(-iu)$. Note that if $m > 0$, then for all $x \geq 0$, S_x and U_x are finite and $X^{(0)}_{S_x} = X^{(0)}_{U_x} = x$, almost surely.
From the Caballero-Chaumont construction and the path decomposition in Corollary 1 in [11], we deduce that the first and last passage time processes are increasing self-similar processes with independent increments. We also remark that when $m = 0$, the process S is still an increasing self-similar process with independent increments.
We are interested in describing the law of the process $(X^{(0)}_{(S_x - t)^-}, 0 \leq t \leq S_x)$ and in obtaining the law of the first passage time in terms of the underlying Lévy process. To this end we briefly recall the definition of the Lévy process conditioned to stay positive, denoted by $\xi^{\uparrow}$, and refer to [10] for a complete account on this subject. The process $\xi^{\uparrow}$ is an h-process of ξ killed when it first exists $(0, \infty)$, i.e. at time $R = \inf\{t : \xi_t \leq 0\}$. The law of this strong Markov process $\xi^{\uparrow}$ is defined by its semi-group:

$$\mathbb{P}(\xi^{\uparrow}_{t+s} \in dy \,|\, \xi^{\uparrow}_s = x) = \frac{h(y)}{h(x)} \mathbb{P}(\xi_t + x \in dy \,, t < R) \,, \quad s, t \geq 0, \quad x, y > 0$$

and its entrance law:

$$\mathbb{P}(\xi^{\uparrow}_t \in dx) = h(x) \hat{N}(\xi_t \in dx, \, t < \zeta) \,,$$

where $\hat{N}$ is the excursion measure of the reflected process $\xi - I = (\xi_t - \inf_{s \leq t} \xi_s, t \geq 0)$, ζ is the lifetime of the generic excursion and h is the positive harmonic function (for ξ killed at time R) defined by:

$$h(x) = \mathbb{E}\left(\int_0^{\infty} \mathbb{1}_{\{I_t \geq -x\}} \, dL_t \right) \,, \quad x \geq 0 \,,$$

where $I_t = \inf_{s \leq t} \xi_s$ and L is the local time of the reflected process $\xi - I$.
Note that $\xi^{\uparrow}$ has no positive jumps and that almost surely

$$\lim_{t \downarrow 0} \xi^{\uparrow}_t = 0, \qquad \lim_{t \uparrow +\infty} \xi^{\uparrow}_t = +\infty, \qquad \text{and} \qquad \xi^{\uparrow}_t > 0 \quad \text{for all } t > 0.$$

The following time reversal property of $\xi^{\uparrow}$ is an important tool for our next result. Its proof can be found in Theorem VII.18 in Bertoin [2].

Lemma 1 *The law of* $(x - \xi_t, 0 \le t \le T_x)$ *is the same as that of the time-reversed process* $(\xi^{\uparrow}_{(\gamma^{\uparrow}(x)-t)-}, 0 \le t \le \gamma^{\uparrow}(x))$*, where* $\gamma^{\uparrow}(x) = \sup\{t \ge 0, \xi^{\uparrow}_t \le x\}$*. Moreover, for every* $x > 0$*, the process* $\big(\xi^{\uparrow}_t, 0 \le t \le \gamma^{\uparrow}(x)\big)$ *is independent of* $\big(\xi^{\uparrow}_{\gamma^{\uparrow}(x)+t} - x, t \ge 0\big)$*, and the latter has the same law as that of the process* $\xi^{\uparrow}$*.*

For every $y > 0$ let us define

$$\tilde{X}^{(y)}_t = y \exp\Big\{ - \xi^{\uparrow}_{\tau^{\uparrow}(t/y)}\Big\} \qquad t \ge 0,$$

where

$$\tau^{\uparrow}_t = \inf\Big\{s \ge 0 : I_s\big(-\xi^{\uparrow}\big) > t\Big\}, \quad \text{ and } \quad I_s\big(-\xi^{\uparrow}\big) = \int_0^s \exp\big\{-\xi^{\uparrow}_u\big\}\mathrm{d}u.$$

We denote by $\tilde{\mathbb{P}}_y$, the law of $\tilde{X}^{(y)}$. Since $\xi^{\uparrow}$ drifts towards $+\infty$, we deduce that $\tilde{X}^{(y)}$, reaches 0 at an almost surely finite random time, denoted by $\tilde{\rho}_y = \inf\{t \ge 0, \tilde{X}^{(y)}_t = 0\}$.

Proposition 1 *Suppose* $m \ge 0$*. The law of the process time-reversed at its first passage time above* x*,* $\big(X^{(0)}_{(S_x - t)-}, 0 \le t \le S_x\big)$ *is the same as that of the process* $(\tilde{X}^{(x)}_t, 0 \le t \le \tilde{\rho}_x)$*.*

Proof: Let us take any decreasing sequence (x_n) of positive real numbers which converges to 0 and such that $x_1 = x$. By Lemma 1, we can split $(\tilde{X}^{(x)}_t, 0 \le t \le \tilde{\rho}_x)$ into the sequence

$$\Big(x_1 \exp\big\{-\xi^{\uparrow}_{\tau^{\uparrow}(t/x_1)}\big\}, x_1 I_{\gamma^{(n)}}\big(-\xi^{\uparrow}\big) \le t \le x_1 I_{\gamma^{(n+1)}}\big(-\xi^{\uparrow}\big)\Big), \quad n \ge 1,$$

where $\gamma^{(n)} = \sup\{t \ge 0 : \xi^{\uparrow}_t \le \log(x_n/x_1)\}$.
To complete the proof, it is enough to show that, for each $n \ge 1$

$$\Big(X^{(0)}_{(S_{x_n}-t)-}, 0 \le t \le S^{(n)}\Big) \overset{(d)}{=} \Big(x_1 \exp\big\{-\xi^{\uparrow}_{\tau^{\uparrow}(t/x_1)}\big\}, x_1 I_{\gamma^{(n)}}\big(-\xi^{\uparrow}\big) \le t \le x_1 I_{\gamma^{(n+1)}}\big(-\xi^{\uparrow}\big)\Big)$$

where $S^{(n)} = S_{x_n} - S_{x_{n+1}}$.
Fix $n \ge 1$. From Theorem 1 in [9], we know that the left-hand side of the above identity has the same law as

$$\bigg(x_{n+1} \exp\Big\{\xi^{(n+1)}_{\tau^{(n+1)}\left(I_{T^{(n+1)}}\left(\xi^{(n+1)}\right) - t/x_{n+1}\right)}\Big\}, 0 \le t \le x_{n+1} I_{T^{(n+1)}}\big(\xi^{(n+1)}\big)\bigg), \qquad (2.6)$$

where $T^{(n+1)} = \inf\{t \ge 0 : \xi^{(n+1)} \ge \log(x_n/x_{n+1})\}$. On the other hand by Lemma 1, we know that $(-\xi^{\uparrow}_t, 0 \le t \le \gamma^{(n)})$ is independent of $\xi^{\uparrow(n)} = (\log(x/x_n) - \xi^{\uparrow}_{\gamma^{(n)}+t}, t \ge 0)$ and that the latter has the same law as $-\xi^{\uparrow}$. Since

$$\tau^{\uparrow}\Big(I_{\gamma^{(n)}}\big(-\xi^{\uparrow}\big) + t/x\Big) = \gamma^{(n)} + \inf\bigg\{s \ge 0 : \int_0^s \exp\Big\{-\xi^{\uparrow(n)}_u\Big\}\mathrm{d}u \ge t/x_n\bigg\},$$

it is clear that the right-hand side of the above identity the same law as

$$\Big(x_n \exp\big\{-\xi^{\uparrow}_{\tau^{\uparrow}(t/x_n)}\big\}, 0 \le t \le x_n I_{\gamma^{\uparrow}(\log(x_n/x_{n+1}))}\big(-\xi^{\uparrow}\big)\Big). \tag{2.7}$$

Therefore, it is enough to show that (2.6) and (2.7) have the same distribution.
Now, let us define the exponential functional of $\big(\xi^{(n+1)}_{(T^{(n+1)}-t)^-}, 0 \le t \le T^{(n+1)}\big)$ as follows,

$$B^{(n+1)}_s = \int_0^s \exp\Big\{\xi^{(n+1)}_{T^{(n+1)}-u}\Big\}\mathrm{d}u \qquad \text{for } s \in [0, T^{(n+1)}],$$

and set $\mathcal{H}(t) = \inf\big\{0 \le s \le T^{(n+1)}, B^{(n+1)}_s > t\big\}$, the right continuous inverse of the exponential functional $B^{(n+1)}$.
By a change of variable, it is clear that $B^{(n+1)}_s = I_{T^{(n+1)}}\big(\xi^{(n+1)}\big) - I_{T^{(n+1)}-s}\big(\xi^{(n+1)}\big)$. If we set $t = x_{n+1}B^{(n+1)}_s$, then $s = \mathcal{H}(t/x_{n+1})$ and hence

$$\begin{aligned}\tau^{(n+1)}\Big(I_{T^{(n+1)}}\big(\xi^{(n+1)}\big) - t/x_{n+1}\Big) &= \tau^{(n+1)}\Big(I_{T^{(n+1)}-s}\big(\xi^{(n+1)}\big)\Big)\\ &= T^{(n+1)} - \mathcal{H}(t/x_{n+1}).\end{aligned}$$

Therefore, we can rewrite (2.6) as

$$\bigg(x_{n+1}\exp\Big\{\xi^{(n+1)}_{T^{(n+1)}-\mathcal{H}(t/x_{n+1})}\Big\}, 0 \le t \le x_{n+1}B^{(n+1)}_{T^{(n+1)}}\bigg). \tag{2.8}$$

Applying Lemma 1, we get that (2.8) has the same law as that of the process defined in (2.7). ■

An important consequence of this Proposition is the following time-reversed identity. For any $y < x$,

$$\Big(X^{(0)}_{(S_x-t)-}, S_y \le t \le S_x\Big) \overset{(d)}{=} \big(\tilde{X}^{(x)}_t, 0 \le t \le \tilde{U}_y\big), \tag{2.9}$$

where $\tilde{U}_y = \sup\big\{t \ge 0, \tilde{X}^{(x)}_t \le y\big\}$.
Recall that a random variable X is self-decomposable if for every constant $0 < c < 1$ there exists a variable Y_c which is independent of X and such that $Y_c + cX$ has the same law as X.

Corollary 1 *Let $m \ge 0$. For every $x > 0$, S_x has the same law as $x\displaystyle\int_0^\infty \exp\{-\xi^{\uparrow}_u\}\mathrm{d}u$. Moreover, S_1 is self-decomposable.*

Proof: Let $I(-\xi^{\uparrow}) := \displaystyle\int_0^\infty \exp\{-\xi^{\uparrow}_u\}\mathrm{d}u$. From Proposition 1, we see that S_x and $\tilde{\rho}_x$ have the same law. By the Lamperti representation of $\tilde{X}^{(x)}$, we deduce that $\tilde{\rho}_x = xI\big(-\xi^{\uparrow}\big)$ and then the identity in law follows.
Now, let $0 < c < 1$. Since $\xi^{\uparrow}$ does not have positive jumps, it is clear that

$$I\big(-\xi^{\uparrow}\big) = \int_0^{\gamma^{\uparrow}(\log(1/c))} \exp\{-\xi^{\uparrow}_u\}\mathrm{d}u + c\int_0^{+\infty} \exp\big\{-\xi^{\uparrow}_{\gamma^{\uparrow}(\log(1/c))+u} - \log c\big\}\mathrm{d}u.$$

Hence, the self-decomposability follows from Lemma 1. ■

3 The lower envelope of the first passage time.

In this section, we are interested in describing the lower envelope at 0 and at $+\infty$ of the first passage time of $X^{(0)}$ with the help of integral tests. We first deal with the case when $X^{(0)}$ has no positive jumps since we may obtain explicit integral tests in terms of the tail probability of $I(-\xi^{\uparrow})$ and also we may consider the case when $X^{(0)}$ oscillates (i.e. that $m = 0$).

3.1 The case with no positive jumps.

Let us define $F^{\uparrow}(t) \stackrel{\text{(def)}}{=} \mathbb{P}\big(I(-\xi^{\uparrow}) < t\big)$ and denote by $\mathcal{H}_0^{-1}$, the totality of positive increasing functions $h(x)$ on $(0, \infty)$ that satisfy

i) $h(0) = 0$ and

ii) there exists $\beta \in (0, 1)$ such that $\sup_{t<\beta} h(t)/t < \infty$.

The lower envelope at 0 of the first passage process S is given as follows.

Proposition 2 *Let $m \geq 0$ and $h \in \mathcal{H}_0^{-1}$.*

i) If $\displaystyle\int_{0+} F^{\uparrow}\left(h(x)/x\right) x^{-1}\mathrm{d}x < \infty$, *then for all* $\epsilon > 0$

$$\mathbb{P}\Big(S_x < (1-\epsilon)h(x), \text{ i.o., as } x \to 0\Big) = 0.$$

ii) If $\displaystyle\int_{0+} F^{\uparrow}\left(h(x)/x\right) x^{-1}\mathrm{d}x = \infty$, *then for all* $\epsilon > 0$

$$\mathbb{P}\Big(S_x < (1+\epsilon)h(x), \text{ i.o., as } x \to 0\Big) = 1.$$

The above result is a consequence of Lemma 3.1 of Watanabe [25], Corollary 1 and the fact that S is an increasing self-similar process with independent increments. It is important to note that Proposition 2 may be proved using similar arguments to those of Theorems 3 in [23]. In this respects, it is enough to exchange $I(\hat{\xi})$ by $I(-\xi^{\uparrow})$ and note that $\Gamma = 1$.

The integral test for large times is very similar. In this case, we need to define $\mathcal{H}_{\infty}^{-1}$, the totality of positive increasing functions $h(t)$ on $(0, \infty)$ that satisfy

i) $\lim_{t\to\infty} h(t) = +\infty$ and

ii) there exists $\beta \in (1, +\infty)$ such that $\displaystyle\sup_{t>\beta} \frac{h(t)}{t} < \infty$,

and replace $\displaystyle\int_{0+} F^{\uparrow}\left(h(x)/x\right) x^{-1}\mathrm{d}x$ by $\displaystyle\int^{\infty} F^{\uparrow}\left(h(x)/x\right) x^{-1}\mathrm{d}x$.

3.2 The general case.

Let us define

$$G(t) \stackrel{\text{(def)}}{=} \mathbb{P}\big(S_1 < t\big) \quad \text{and} \quad \overline{F}(t) \stackrel{\text{(def)}}{=} \mathbb{P}\big(I(\hat{\xi}) < t\big).$$

The lower envelope at 0 of the first passage process S is given as follows.

Proposition 3 *Let $m > 0$ and $h \in \mathcal{H}_0^{-1}$.*

i) If $\int_{0+} G\left(h(x)/x\right) x^{-1}\mathrm{d}x < \infty$, then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_x < (1-\epsilon)h(x),\ i.o.,\ as\ x \to 0\Big) = 0.$$

ii) If $\int_{0+} \overline{F}\left(h(x)/x\right) x^{-1}\mathrm{d}x = \infty$, then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_x < (1+\epsilon)h(x),\ i.o.,\ as\ x \to 0\Big) = 1.$$

Proof: We first prove part (i). Let (x_n) be a decreasing sequence of positive numbers which converges to 0 and let us define the events $A_n = \{S_{x_{n+1}} < h(x_n)\}$.
Now, we choose $x_n = r^n$, for $r < 1$. From the first Borel Cantelli's Lemma, if $\sum_n \mathbb{P}(A_n) < \infty$, it follows

$$S_{r^{n+1}} \geq h\big(r^n\big) \qquad \mathbb{P}\text{-a.s.,}$$

for all sufficiently large n. Since the function h and the process S are increasing, we have

$$S_x \geq h(x) \qquad \text{for} \quad r^{n+1} \leq x \leq r^n.$$

On the other hand, from the scaling property, we get that

$$\begin{aligned}\sum_n \mathbb{P}\Big(S_{r^n} < h\big(r^{n+1}\big)\Big) &\leq \int_1^\infty \mathbb{P}\Big(r^t S_1 < h\big(r^t\big)\Big)\mathrm{d}t \\ &= -\frac{1}{\log r}\int_0^r G\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x}.\end{aligned}$$

From our hypothesis, this last integral is finite. Then from the above discussion, there exist x_0 such that for every $x \geq x_0$

$$S_x \geq r^2 h(x), \qquad \text{for all} \quad r < 1.$$

Clearly, this implies that

$$\mathbb{P}_0\Big(S_x < r^2 h(x),\ \text{i.o., as } x \to 0\Big) = 0,$$

which proves part (i).
Part (ii) follows from part (ii) of Theorem 3 in [23]. ■

As in the case with no positive jumps, there is an integral test for large times which holds for $h \in \mathcal{H}_\infty^{-1}$.

Note now that the integral tests of Proposition 3 no longer depends on $F^\uparrow$ as in the case with no positive jumps and that we are assuming that $m > 0$. We also remark that $\overline{F}(t) \leq G(t)$, for $t \geq 0$, but for our purpose Proposition 3 will be very useful for our applications (see Sections 5 and 6 below).

4 The upper envelope of PSSMP.

Here, we are interested in describing the upper envelope at 0 and at $+\infty$ of PSSMP through integral tests. For the same reasons as those mentioned in the preceding section, we will first study the case with no positive jumps.

4.1 The case with no positive jumps.

The following theorem shows in particular that the upper envelope at 0 of $X^{(0)}$ only depends on the tail behaviour of the law of $I(-\xi^\uparrow)$ and on the additional hypothesis

$$\mathbb{E}\Big(\log^+ I\big(-\xi^\uparrow\big)^{-1}\Big) < \infty. \tag{4.10}$$

Let us recall that $F^\uparrow(t) = \mathbb{P}(I(-\xi^\uparrow) < t)$, and denote by $\mathcal{H}_0$ the totality of positive increasing functions $h(t)$ on $(0, \infty)$ that satisfy

i) $h(0) = 0$ and

ii) there exists $\beta \in (0,1)$ such that $\sup_{t<\beta} \dfrac{t}{h(t)} < \infty$.

Theorem 2 *Let $m \geq 0$ and $h \in \mathcal{H}_0$.*

i) If $\int_{0+} F^\uparrow\left(t/h(t)\right) t^{-1}\mathrm{d}t < \infty$, then for all $\epsilon > 0$

$$\mathbb{P}_0\Big(X_t > (1+\epsilon)h(t),\ \textit{i.o., as } t \to 0\Big) = 0.$$

ii) Assume that (4.10) is satisfied. If $\int_{0+} F^\uparrow\left(t/h(t)\right) t^{-1}\mathrm{d}t = \infty$, then for all $\epsilon > 0$

$$\mathbb{P}_0\Big(X_t > (1-\epsilon)h(t),\ \textit{i.o., as } t \to 0\Big) = 1.$$

Proof: Let (x_n) be a decreasing sequence which converges to 0. We define the events $A_n = \big\{\text{There exists } t \in [S_{x_{n+1}}, S_{x_n}] \text{ such that } X_t^{(0)} > h(t)\big\}$. From the fact that S_{x_n} tends to 0, a.s. when n goes to $+\infty$, we see

$$\Big\{X_t^{(0)} > h(t),\ \text{i.o., as } t \to 0\Big\} = \limsup_{n\to+\infty} A_n.$$

Since h is an increasing function the following inclusions hold

$$\Big\{x_n > h\big(S_{x_n}\big)\Big\} \subset A_n \subset \Big\{x_n > h\big(S_{x_{n+1}}\big)\Big\}. \tag{4.11}$$

Now, we prove part (i). We choose $x_n = r^n$, for $r < 1$ and $h_r(t) = r^{-2}h(t)$. Since h is increasing, we deduce that

$$\sum_n \mathbb{P}\Big(r^n > h_r\big(S_{r^{n+1}}\big)\Big) \le \int_1^{+\infty} \mathbb{P}\Big(r^t > h\big(S_{t^r}\big)\Big)\mathrm{d}t \le -\frac{1}{\log r}\int_0^r \mathbb{P}\Big(t > h\big(S_t\big)\Big)\frac{\mathrm{d}t}{t}.$$

Replacing h by h_r in (4.11), we see that we can obtain our result if

$$\int_0^r \mathbb{P}\Big(t > h\big(S_t\big)\Big)\frac{\mathrm{d}t}{t} < \infty.$$

From elementary calculations and Corollary 1, we deduce that

$$\int_0^r \mathbb{P}\Big(t > h\big(S_t\big)\Big)\frac{\mathrm{d}t}{t} = \mathbb{E}\left(\int_0^{h^{-1}(r)} \mathbb{I}_{\big\{t/r < I(-\xi^{\uparrow}) < t/h(t)\big\}}\frac{\mathrm{d}t}{t}\right),$$

where $h^{-1}(s) = \inf\{t > 0, h(t) > s\}$, the right inverse function of h. Then, this integral converges if $\displaystyle\int_0^{h^{-1}(r)} \mathbb{P}\left(I(-\xi^{\uparrow}) < t/h(t)\right) t^{-1}\mathrm{d}t < \infty$. This proves part (i).

Next, we prove the divergent case. We suppose that h satisfies $\displaystyle\int_{0+} F^{\uparrow}\left(t/h(t)\right) t^{-1}\mathrm{d}t$ diverges. Take, again, $x_n = r^n$, for $r < 1$ and define

$$B_n \overset{\text{(def)}}{=} \bigcup_{m=n}^{\infty} A_m = \Big\{\text{There exists } t \in (0, S_{r^n}] \text{ such that } X_t^{(0)} > h_r(t)\Big\}.$$

Note that the family (B_n) is decreasing and

$$B \overset{\text{(def)}}{=} \bigcap_{n\ge 1} B_n = \Big\{X_t^{(0)} > h_r(t),\ \text{i.o., as } t \to 0\Big\}.$$

Hence it is enough to prove that $\lim \mathbb{P}(B_n) = 1$ to obtain our result.
Again replacing h by h_r in inclusion (4.11), we see

$$\mathbb{P}(B_n) \ge 1 - \mathbb{P}\Big(r^j \le h_r\big(S_{r^j}\big),\ \text{for all } n \le j \le m-1\Big), \tag{4.12}$$

where m is chosen arbitrarily $m \ge n+1$.
Now, we define the events $C_n \overset{\text{(def)}}{=} \Big\{r^n > rh\Big(S_{r^n}\Big)\Big\}$. We will prove that $\sum \mathbb{P}(C_n)$ diverges. Since the function h is increasing, from the identity in law of Corollary 1 it is straightforward to show that

$$\sum_{n\ge 1}\mathbb{P}(C_n) \ge \int_0^{+\infty} \mathbb{P}\left(r^t > h\Big(S_{r^t}\Big)\right)\mathrm{d}t = -\frac{1}{\log r}\int_0^1 \mathbb{P}\Big(t > h\big(tI\big(-\xi^{\uparrow}\big)\big)\Big)\frac{\mathrm{d}t}{t}.$$

Hence, if this last integral is infinite, we get that $\sum \mathbb{P}(C_n) = \infty$. In this direction, we have

$$\int_0^r \mathbb{P}\Big(t > h\big(tI\big(-\xi^{\uparrow}\big)\big)\Big)\frac{\mathrm{d}t}{t} = \mathbb{E}\left(\int_0^{h^{-1}(r)} \mathbb{I}_{\left\{t/r<I(-\xi^{\uparrow})<t/h(t)\right\}}\frac{\mathrm{d}t}{t}\right).$$

On the other hand, we see that

$$\int_0^{h^{-1}(r)} \mathbb{P}\left(I\big(-\xi^{\uparrow}\big) < \frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} = \int_0^{h^{-1}(r)} \mathbb{P}\left(\frac{t}{r} < I\big(-\xi^{\uparrow}\big) < \frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} + \int_0^{h^{-1}(r)} \mathbb{P}\left(I\big(-\xi^{\uparrow}\big) < \frac{t}{r}\right)\frac{\mathrm{d}t}{t},$$

and

$$\int_0^{h^{-1}(r)} \mathbb{P}\left(I\big(-\xi^{\uparrow}\big) < \frac{t}{r}\right)\frac{\mathrm{d}t}{t} \le \mathbb{E}\left(\log^+\frac{h^{-1}(r)}{r} I\big(-\xi^{\uparrow}\big)^{-1}\right)$$

which is clearly finite from our assumptions. We thus deduce that

$$\mathbb{E}\left(\int_0^{h^{-1}(r)} \mathbb{I}_{\left\{t/r<I(-\xi^{\uparrow})<t/h(t)\right\}}\frac{\mathrm{d}t}{t}\right) = \infty,$$

and hence $\sum \mathbb{P}(C_n) = \infty$.
Next, for $n \le m-1$, we define

$$H(n,m) \overset{\text{(def)}}{=} \mathbb{P}\Big(r^j \le rh\big(S_{r^j} - S_{r^m}\big),\ \text{for all } n \le j \le m-1\Big).$$

We will prove that there exist two increasing sequences to ∞, (n_l) and (m_l), such that $0 \le n_l \le m_l - 1$ and $H(n_l, m_l)$ tends to 0 as l goes to infinity.
Suppose the contrary, i.e., there exist $\delta > 0$ such that $H(n,m) \ge \delta$ for all sufficiently large integers m and n. From the independence of the increments of S,

$$\begin{aligned} 1 \ge \mathbb{P}\left(\bigcup_{m=n+1}^{\infty} C_m\right) &\ge \sum_{m=n+1}^{\infty} \mathbb{P}\left(C_m \bigcap \left(\bigcap_{j=n}^{m-1} C_j^c\right)\right) \\ &\ge \sum_{m=n+1}^{\infty} \mathbb{P}\Big(r^m > rh\Big(S_{r^m}\Big)\Big)H(n,m) \ge \delta \sum_{m=n+1}^{\infty} \mathbb{P}\big(C_m\big). \end{aligned}$$

Since $\sum \mathbb{P}(C_n)$ diverges there is a contradiction and we see that our assertion is true.
Now, we define

$$\rho_{n_l,m_l}(x) \overset{\text{(def)}}{=} \mathbb{P}\Big(r^j \le rh\big(S_{r^j} - S_{r^{m_l-1}} + x\big) \text{ for, } n_l \le j \le m_l - 2\Big), \qquad x \ge 0,$$

and

$$G(n_l, m_l) \overset{\text{(def)}}{=} \mathbb{P}\Big(r^j \le rh\big(S_{r^j}\big) \text{ for, } n_l \le j \le m_l - 1\Big).$$

Since h is increasing, we see that $\rho_{n_l,m_l}(x)$ is increasing in x.
If we denote by μ and $\bar{\mu}$ the laws of S_1 and $S_1 - S_r$ respectively, by the scaling property we may express $H(n_l, m_l)$ and $G(n_l, m_l)$ as follows

$$H(n_l, m_l) = \int_0^{+\infty} \bar{\mu}(\mathrm{d}x)\mathbb{1}_{\left\{h(r^{m_l-1}x)\geq r^{m_l}\right\}}\rho_{n_l,m_l}(r^{m_l-1}x) \quad \text{and,}$$
$$G(n_l, m_l) = \int_0^{+\infty} \mu(\mathrm{d}x)\mathbb{1}_{\left\{h(r^{m_l-1}x)\geq r^{m_l}\right\}}\rho_{n_l,m_l}(r^{m_l-1}x).$$

In particular, we get that for l sufficiently large

$$H(n_l, m_l) \geq \rho_{n_l,m_l}(N)\int_N^{+\infty} \bar{\mu}(\mathrm{d}x) \qquad for \quad N \geq rC,$$

where $C = \sup_{x\leq\beta} x/h(x)$.
Since $H(n_l, m_l)$ converges to 0, as l goes to $+\infty$ and $\bar{\mu}$ does not depend on l, then $\rho_{n_l,m_l}(N)$ also converges to 0 when l goes to $+\infty$, for every $N \geq rC$.
On the other hand, we have

$$G(n_l, m_l) \leq \rho_{n_l,m_l}(N)\int_0^{N} \mu(\mathrm{d}x) + \int_N^{\infty} \mu(\mathrm{d}x).$$

Letting l and N go to infinity, we get that $G(n_l, m_l)$ tends to 0. Then, by (4.12) we get that $\lim \mathbb{P}(B_n) = 1$ and with this we finish the proof. ∎

For the integral tests at $+\infty$, we define $\mathcal{H}_\infty$, the totality of positive increasing functions $h(t)$ on $(0,\infty)$ that satisfy

i) $\lim_{t\to\infty} h(t) = \infty$ and

ii) there exists $\beta > 1$ such that $\displaystyle\sup_{t>\beta} \frac{t}{h(t)} < \infty$.

The upper envelope of $X^{(x)}$ at $+\infty$ is given by the following result.

Theorem 3 *Let $m \geq 0$ and $h \in \mathcal{H}_\infty$.*

i) If $\displaystyle\int^{+\infty} F^{\uparrow}\left(t/h(t)\right)t^{-1}\mathrm{d}t < \infty$, then for all $\epsilon > 0$ and for all $x \geq 0$,

$$\mathbb{P}_x\Big(X_t > (1+\epsilon)h(t), \ \ i.o., \ as \ t \to +\infty\Big) = 0.$$

ii) Assume that (4.10) is satisfied. If $\displaystyle\int^{+\infty} F^{\uparrow}\left(t/h(t)\right)t^{-1}\mathrm{d}t = \infty$, then for all $\epsilon > 0$ and for all $x \geq 0$

$$\mathbb{P}_x\Big(X_t > (1-\epsilon)h(t), \ \ i.o., \ as \ t \to +\infty\Big) = 1.$$

Proof: We first consider the case where $x = 0$. In this case the proof of the tests at $+\infty$ is almost the same as that of the tests at 0. It is enough to apply the same arguments to the sequence $x_n = r^n$, for $r > 1$.

Now, we prove (i) for any $x > 0$. Let $h \in \mathcal{H}_\infty$ such that $\int^{+\infty} F^{\uparrow}\left(t/h(t)\right) t^{-1}\mathrm{d}t$ is finite.

Let $x > 0$ and $S_x = \inf\{t \geq 0 : X_t^{(0)} \geq x\}$. Since clearly $\int^{+\infty} F^{\uparrow}\left(t/h(t - S_x)\right) t^{-1}\mathrm{d}t$ is also finite, from the Markov property at time S_x, we have for all $\epsilon > 0$

$$\mathbb{P}_0\Big(X_t > (1+\epsilon)h(t-S_x), \text{ i. o., as } t \to \infty\Big) = \mathbb{P}_x\Big(X_t > (1+\epsilon)h(t), \text{ i. o., as } t \to \infty\Big) = 0,$$

which proves part (i). Part (ii) can be proved in the same way. ■

4.2 The general case.

Proposition 4 *Let $m > 0$ and $h \in \mathcal{H}_0$.*

i) If $\int_{0+} G\left(t/h(t)\right) t^{-1}\mathrm{d}t < \infty$, then for all $\epsilon > 0$

$$\mathbb{P}\Big(X_t^{(0)} > (1+\epsilon)h(t), \text{ i.o., as } t \to 0\Big) = 0.$$

ii) If $\int_{0+} \overline{F}\left(t/h(t)\right) t^{-1}\mathrm{d}t = \infty$, then for all $\epsilon > 0$

$$\mathbb{P}\Big(X_t^{(0)} < (1-\epsilon)h(t), \text{ i.o., as } t \to 0\Big) = 1.$$

Proof: Let (x_n) be a decreasing sequence which converges to 0. We define the events $A_n = \big\{\text{There exists } t \in [S_{x_{n+1}}, S_{x_n}) \text{ such that } X_t^{(0)} > h(t)\big\}$. From the fact that S_{x_n} tends to 0, a.s. when n goes to $+\infty$, we see

$$\Big\{X_t^{(0)} > h(t), \text{ i.o., as } t \to 0\Big\} = \limsup_{n\to+\infty} A_n.$$

Since h is an increasing function the following inclusion holds

$$A_n \subset \Big\{x_n > h\big(S_{x_{n+1}}\big)\Big\}. \tag{4.13}$$

Next, we prove the convergent part. We choose $x_n = r^n$, for $r < 1$ and $h_r(t) = r^{-2}h(t)$. Since h is increasing, we deduce

$$\sum_n \mathbb{P}\Big(r^n > h_r\big(S_{r^{n+1}}\big)\Big) \leq -\frac{1}{\log r}\int_0^r \mathbb{P}\Big(t > h\big(S_t\big)\Big)\frac{\mathrm{d}t}{t}.$$

Replacing h by h_r in (4.13), we see that we can obtain our result if $\int_0^r \mathbb{P}\Big(t > h\big(S_t\big)\Big)t^{-1}\mathrm{d}t$ is finite. From elementary calculations, we get

$$\int_0^r \mathbb{P}\Big(t > h\big(S_t\big)\Big)\frac{\mathrm{d}t}{t} = \mathbb{E}\left(\int_0^{h^{-1}(r)} \mathbb{I}_{\left\{t/r < S_1 < t/h(t)\right\}}\frac{\mathrm{d}t}{t}\right),$$

where $h^{-1}(s) = \inf\{t > 0, h(t) > s\}$, the right inverse function of h. Then, this integral converges if $\displaystyle\int_0^{h^{-1}(r)} \mathbb{P}\left(S_1 < t/h(t)\right) t^{-1}\mathrm{d}t$ is finite. This proves part (i).
The divergent part follows from part (ii) of Theorem 1 in [23]. ■

Proposition 5 *Let $m > 0$ and $h \in \mathcal{H}_\infty$.*

i) If $\displaystyle\int^{+\infty} G\left(t/h(t)\right) t^{-1}\mathrm{d}t < \infty$, then for all $\epsilon > 0$ and for all $x \geq 0$

$$\mathbb{P}\Big(X_t^{(x)} > (1+\epsilon)h(t),\ i.o.,\ as\ t \to +\infty\Big) = 0.$$

ii) If $\displaystyle\int^{+\infty} \overline{F}\left(t/h(t)\right) t^{-1}\mathrm{d}t = \infty$, then for all $\epsilon > 0$ and for all $x \geq 0$

$$\mathbb{P}\Big(X_t^{(x)} < (1-\epsilon)h(t),\ i.o.,\ as\ t \to +\infty\Big) = 1.$$

Proof: We first consider the case where $x = 0$. In this case the proof of the tests at $+\infty$ is almost the same as that of the tests at 0. It is enough to apply the same arguments to the sequence $x_n = r^n$, for $r > 1$.
Now, we prove (i) for any $x > 0$. Let $h \in \mathcal{H}_\infty$ such that $\displaystyle\int^{+\infty} G\left(t/h(t)\right) t^{-1}\mathrm{d}t$ is finite. Let $x > 0$ and S_x and note by μ_x the law of $X^{(0)}_{S_x}$. Since clearly $\displaystyle\int^{+\infty} G\left(t/h(t - S_x)\right) t^{-1}\mathrm{d}t$ is also finite, from the Markov property at time S_x, we have for all $\epsilon > 0$

$$\begin{aligned}&\mathbb{P}_0\Big(X_t > (1+\epsilon)h(t - S_x),\ \text{i. o., as } t \to \infty\Big)\\ &= \int_{[x,+\infty)} \mathbb{P}_y\Big(X_t > (1+\epsilon)h(t),\ \text{i. o., as } t \to \infty\Big)\mu_x(\mathrm{d}y) = 0.\end{aligned} \tag{4.14}$$

If x is an atom of μ_x, then equality (4.14) shows that

$$\mathbb{P}\Big(X_t^{(x)} > (1+\epsilon)h(t),\ \text{i. o., as } t \to \infty\Big) = 0$$

and the result is proved. Suppose that x is not an atom of μ_x. From Theorem 1 in [9], we know that $X^{(0)}_{S_x} \overset{(d)}{=} xe^\theta$, where θ is a positive r.v. such that

$$\xi_{T_z} - z \xrightarrow[z \to +\infty]{(w)} \theta.$$

Then from Section 2 in [9], the law of θ is given by

$$\mathbb{P}(\theta > t) = \mathbb{E}(\sigma_1) \int_{(t,\infty)} s\nu(\mathrm{d}s), \quad t \geq 0,$$

where σ is the upward ladder height process associated with ξ and ν its Lévy measure. Hence, $\mathbb{P}(e^{\theta} > z) > 0$ for $z > 1$, and for any $\alpha > 0$, $\mu_x(x, x+\alpha) > 0$. Now (4.14) shows that there exists $y > x$ such that

$$\mathbb{P}\Big(X_t^{(y)} > (1+\epsilon)h(t), \text{ i. o., as } t \to \infty\Big) = 0,$$

for all $\epsilon > 0$. The previous arguments allow us to conclude part (i). Part (ii) can be proved in the same way. ■

5 The regular case.

In this section, we will assume that $m > 0$. According to Chaumont and Pardo [11] the law of U_1, the last passage time below level 1, is the same as $\nu I(\hat{\xi})$ where ν is a positive random variable bounded above by 1 and independent of the exponential functional $I(\hat{\xi})$. Hence, we have the following inequality $\nu I(\hat{\xi}) \leq I(\hat{\xi})$ a.s.
Now, let us define $\overline{F}_\nu(t) := \mathbb{P}\big(\nu I(\hat{\xi}) < t\big)$, and suppose that

$$ct^\beta L(t) \leq \overline{F}(t) \leq \overline{F}_\nu(t) \leq Ct^\beta L(t) \qquad \text{as} \quad t \to 0, \tag{5.15}$$

where $\beta > 0$, c and C are two positive constants such that $c \leq C$ and L is a slowly varying function at 0. An important example included in this case is when $\overline{F}$ and $\overline{F}_\nu$ are regularly varying functions at 0.

Proposition 6 *Under condition (5.15), we have that*

$$ct^\beta L(t) \leq G(t) \leq C_\epsilon t^\beta L(t) \qquad \textit{as} \quad t \to 0,$$

where C_ϵ is a positive constant bigger than C.

Proof: The lower bound is clear since $\overline{F}(t) \leq G(t)$, for all $t \geq 0$ and our assumption. Now, let us define $M_t^{(0)} = \sup_{0\leq s\leq t} X_s^{(0)}$ and fix $\epsilon > 0$. Then, by the Markov property and the fact that $J^{(x)}$ is an increasing process, we have

$$\begin{aligned}
\mathbb{P}_0\left(J_1 > \frac{1-\epsilon}{t}\right) &\geq \mathbb{P}_0\left(J_1 > \frac{1-\epsilon}{t}, M_1 \geq \frac{1}{t}\right)\\
&= \mathbb{E}\left(S_{1/t} \leq 1, \mathbb{P}_{X^{(0)}_{S_{1/t}}}\left(J_{1-S_{1/t}} > \frac{1-\epsilon}{t}\right)\right)\\
&\geq \mathbb{E}\left(S_{1/t} \leq 1, \mathbb{P}_{X^{(0)}_{S_{1/t}}}\left(J_0 > \frac{1-\epsilon}{t}\right)\right).
\end{aligned}$$

Since $X^{(0)}_{S_{1/t}} \geq 1/t$ a.s., using the Lamperti representation (1.1), we deduce that

$$\mathbb{E}\left(S_{1/t} \leq 1, \mathbb{P}_{X^{(0)}_{S_{1/t}}}\left(J_0 > \frac{1-\epsilon}{t}\right)\right) \geq \mathbb{P}\big(S_{1/t} < 1\big)\mathbb{P}\Big(\inf_{s\geq 0} \xi_s > \log(1-\epsilon)\Big).$$

On the other hand, under the assumption that ξ drifts towards $+\infty$, we know from Section 2 of Chaumont and Doney [10] (see also Proposition VI.17 in [2]) that for all $\epsilon > 0$

$$K_\epsilon := \mathbb{P}\Big(\inf_{s\geq 0} \xi_s > \log(1-\epsilon)\Big) > 0.$$

Hence

$$K_\epsilon^{-1}\mathbb{P}_0\left(J_1 > \frac{1-\epsilon}{t}\right) \geq \mathbb{P}\big(S_1 < t\big)$$

which implies that

$$CK_\epsilon^{-1}\left(\frac{t}{1-\epsilon}\right)^\beta L(t) \geq K_\epsilon^{-1}\mathbb{P}\left(U_1 < \frac{t}{1-\epsilon}\right) \geq \mathbb{P}\big(S_1 < t\big), \qquad \text{as} \quad t \to 0.$$

The proposition is proved. ■

Theorem 4 *Under condition (5.15), the lower envelope of S at 0 and at $+\infty$ is as follows.*

i) Let $h \in \mathcal{H}_0^{-1}$, such that either $\lim_{x\to 0} h(x)/x = 0$ or $\liminf_{x\to 0} h(x)/x > 0$, then

$$\mathbb{P}\Big(S_x < h(x),\ \textit{i.o., as } x \to 0\Big) = 0 \textit{ or } 1,$$

accordingly as $\int_{0+} \overline{F}\,(h(x)/x)\, x^{-1}\mathrm{d}x$ is finite or infinite.

ii) Let $h \in \mathcal{H}_\infty^{-1}$, such that either $\lim_{x\to+\infty} h(x)/x = 0$ or $\liminf_{x\to+\infty} h(x)/x > 0$, then

$$\mathbb{P}\Big(S_x < h(x),\ \textit{i.o., as } x \to \infty\Big) = 0 \textit{ or } 1,$$

accordingly as $\int^{+\infty} \overline{F}\,(h(x)x)\, x^{-1}\mathrm{d}x$ is finite or infinite.

Proof: First let us check that under condition (5.15) we have

$$\int_0^\lambda \overline{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty \quad \text{if and only if} \quad \int_0^\lambda G\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty. \tag{5.16}$$

Since $\overline{F}(t) \leq G(t)$ for all $t \geq 0$, it is clear that we only need to prove that

$$\int_0^\lambda \overline{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty \quad \text{implies that} \quad \int_0^\lambda G\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty.$$

From the hypothesis, either $\lim_{x\to 0} h(x)/x = 0$ or $\liminf_{x\to 0} h(x)/x > 0$. In the first case, from condition (5.15) there exists $\lambda > 0$ such that, for every $x < \lambda$

$$c\left(\frac{h(x)}{x}\right)^\beta L\left(\frac{h(x)}{x}\right) \leq \overline{F}\left(\frac{h(x)}{x}\right) \leq C\left(\frac{h(x)}{x}\right)^\beta L\left(\frac{h(x)}{x}\right).$$

Since, we suppose that $\int_0^\lambda \overline{F}\left(h(x)/x\right)x^{-1}\mathrm{d}x$ is finite, then

$$\int_0^\lambda \left(\frac{h(x)}{x}\right)^\beta L\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty.$$

Hence from Proposition 6, we get that $\int_0^\lambda G\left(h(x)/x\right)x^{-1}\mathrm{d}x$ is also finite. In the second case, since, for any $0 < \delta < \infty$ $\mathbb{P}\big(I < \delta\big) > 0$, and $\liminf_{x\to 0} h(x)/x > 0$, we have for any $y \in \mathbb{R}$

$$0 < \mathbb{P}\left(I < \liminf_{x\to 0}\frac{h(x)}{x}\right) < \mathbb{P}\left(I < \frac{h(y)}{y}\right). \tag{5.17}$$

As, $\overline{F}(t) \le G(t)$ for all $t \ge 0$, we deduce that

$$\int_0^\lambda \overline{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} = \int_0^\lambda G\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} = \infty.$$

Now, let us check that for any constant $\beta > 0$,

$$\int_0^\lambda \overline{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty \quad \text{if and only if} \quad \int_0^\lambda \overline{F}\left(\frac{\beta h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty. \tag{5.18}$$

Again, from the hypothesis, either $\lim_{x\to 0} h(x)/x = 0$ or $\liminf_{x\to 0} h(x)/x > 0$. In the first case, we deduce (5.18) from (5.15). In the second case, from (5.17) both of the integrals in (5.18) are infinite.

Next, it follows from Proposition 3 part (i) and (5.16) that if $\int_{0+} \overline{F}\left(h(x)/x\right)x^{-1}\mathrm{d}x$ is finite, then for all $\epsilon > 0$,

$$\mathbb{P}\big(S_x < (1-\epsilon)h(x), \text{ i.o., as } x \to 0\big) = 0.$$

If $\int_{0+} \overline{F}\left(h(x)/x\right)x^{-1}\mathrm{d}x$ diverges, then from Proposition 3 part (ii) we have that for all $\epsilon > 0$,

$$\mathbb{P}\big(S_x < (1+\epsilon)h(x), \text{ i.o., as } x \to 0\big) = 1.$$

Equation (5.18) allows us to drop ϵ in these implications. ■

From the previous Theorem, we deduce that under condition (5.15) the first and the last passage time processes have the same lower functions (see Theorem 5 in [23]). In fact, we may deduce that under the same condition a PSSSMP and its future infimum have the same upper functions (see Theorem 6 in [23]). We state the following result without proof since it follows from the same arguments as those presented for the lower envelope of the first passage time.

Theorem 5 *Under condition (5.15), the upper envelope of the PSSMP at* 0 *and at* $+\infty$ *is as follows:*

i) Let $h \in \mathcal{H}_0$, such that either $\lim_{t\to 0} t/h(t) = 0$ or $\liminf_{t\to 0} t/h(t) > 0$, then

$$\mathbb{P}\Big(X_t^{(0)} > h(t),\ \textit{i.o., as } t \to 0\Big) = 0 \ \textit{or}\ 1,$$

accordingly as $\displaystyle\int_{0+} \overline{F}\,(t/h(t))\, t^{-1}\mathrm{d}t$ is finite or infinite.

ii) Let $h \in \mathcal{H}_\infty$, such that either $\lim_{t\to +\infty} t/h(t) = 0$ or $\liminf_{t\to +\infty} t/h(t) > 0$, then for all $x \geq 0$

$$\mathbb{P}\Big(X_t^{(x)} > h(t),\ \textit{i.o., as } t \to \infty\Big) = 0 \ \textit{or}\ 1,$$

accordingly as $\displaystyle\int^{+\infty} \overline{F}\,(t/h(t))\, t^{-1}\mathrm{d}t$ is finite or infinite.

6 The log-regular case.

In this section, we also assume that $m > 0$ and study two types of behaviour of $\overline{F}$ and $\overline{F}_\nu$, which allow us to obtain laws of the iterated logarithm for the upper envelope of $X^{(0)}$. The first type of behaviour that we will consider is when $\log \overline{F}$ and $\log \overline{F}_\nu$ are regularly varying at 0, i.e.

$$-\log \overline{F}_\nu(1/t) \sim -\log \overline{F}(1/t) \sim \lambda t^\delta L(t), \qquad \text{as } t \to +\infty, \tag{6.19}$$

where $\lambda > 0$, $\delta > 0$ and L is a slowly varying function at $+\infty$.
The second type of behaviour that we will consider is when $\log \overline{F}$ and $\log \overline{F}_\nu$ satisfy that

$$-\log \overline{F}_\nu(1/t) \sim -\log \overline{F}(1/t) \sim K(\log t)^\gamma, \qquad \text{as } t \to +\infty, \tag{6.20}$$

where K and γ are strictly positive constants.

6.1 Laws of the iterated logarithm for PSSMP.

Proposition 7 *Under condition (6.19), the tail probability of S_1 satisfies*

$$-\log G(1/t) \sim \lambda t^\delta L(t) \qquad \textit{as} \quad t \to +\infty. \tag{6.21}$$

Similarly, under condition (6.20), the tail probability of S_1 satisfies

$$-\log G(1/t) \sim K(\log t)^\gamma \qquad \textit{as} \quad t \to +\infty. \tag{6.22}$$

Proof: First, we prove the upper bound of (6.21). With the same notation as in the proof of Proposition 6, we see

$$-\log \mathbb{P}\Big(\nu I(\hat{\xi}) < 1/t\Big) = -\log \mathbb{P}_0\Big(J_1 > t\Big) \geq -\log \mathbb{P}_0\Big(M_1 > t\Big),$$

which implies

$$1 \geq \limsup_{t\to\infty} \frac{-\log \mathbb{P}_0\Big(M_1 > t\Big)}{\lambda t^\delta L(t)}.$$

Since $\mathbb{P}_0\big(M_1 > t\big) = \mathbb{P}\big(S_1 < 1/t\big)$, we get the upper bound.
Now, fix $\epsilon > 0$. Again from the proof of Proposition 6, we have

$$\mathbb{P}_0\big(J_1 > (1-\epsilon)t\big) \geq \mathbb{P}\big(S_t < 1\big)\mathbb{P}\Big(\inf_{s\geq 0}\xi_s > \log(1-\epsilon)\Big).$$

On the other hand, we know

$$K_\epsilon := \mathbb{P}\Big(\inf_{s\geq 0}\xi_s > \log(1-\epsilon)\Big) > 0,$$

Hence,

$$-\log \mathbb{P}_0\big(J_1 > (1-\epsilon)t\big) \leq -\log \mathbb{P}\Big(S_1 < 1/t\Big) - \log K_\epsilon,$$

which implies the following lower bound

$$(1-\epsilon)^\delta \leq \liminf_{t\to\infty} \frac{-\log \mathbb{P}\Big(S_1 < 1/t\Big)}{\lambda t^\delta L(t)}.$$

Since ϵ can be chosen arbitrarily small, (6.21) is proved.
The upper bound of tail behaviour (6.22) is proven in the same way. For the lower bound, we follow the same arguments as above and we get that

$$-\log \mathbb{P}_0\big(J_1 > (1-\epsilon)t\big) \leq -\log \mathbb{P}\Big(S_1 < 1/t\Big) - \log K_\epsilon,$$

which implies that

$$1 = \liminf_{t\to\infty}\left(\frac{\log(1-\epsilon)t}{\log t}\right)^\gamma \leq \liminf_{t\to\infty} \frac{-\log \mathbb{P}\Big(S_1 < 1/t\Big)}{K(\log t)^\gamma},$$

and the proposition is proved. ■

The following result gives us laws of the iterated logarithm for the first passage time process when condition (6.19) is satisfied. Define the functions

$$\varphi(x) := \frac{x}{\inf\big\{s : 1/\overline{F}(1/s) > |\log x|\big\}}, \quad x > 0, \quad x \neq 1,$$

and

$$\vartheta(t) := \frac{t^2}{\varphi(t)}, \quad t > 0, \quad t \neq 1.$$

Theorem 6 *Under condition (6.19), we have the following law of the iterated logarithm for S.*

$$\limsup_{x\to 0} \frac{S_x}{\varphi(x)} = 1 \quad \textit{and} \quad \limsup_{x\to\infty} \frac{S_x}{\varphi(x)} = 1 \quad \textit{almost surely.}$$

The upper envelope of PSSMP, under condition (6.19), are described by the following law of the iterated logarithm. For all $x \geq 0$

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{\vartheta(t)} = 1, \qquad \textit{and} \qquad \limsup_{t\to +\infty} \frac{X_t^{(x)}}{\vartheta(t)} = 1, \qquad \textit{a.s.}$$

Proof: This Theorem is a consequence of Propositions 3,4, 5, and 7, and it is proven in the same way as Theorem 4 in [11], we only need to emphasize that we can replace $\log G$ by $\log \overline{F}$, since they are asymptotically equivalent. ■

Note that under condition (6.19) a PSSMP and its future infimum satisfy the same law of the iterated logarithm (see Theorem 8 in [23]) but they do not necessarily have the same upper functions.
Similarly, under condition (6.20) we may establish laws of the iterated logarithm for the upper envelope of PSSMP and their future infimum. In this direction, let us define

$$\phi(x) := x \exp\Big\{ - \big(K^{-1} \log|\log x|\big)^{1/\gamma}\Big\}, \quad x > 0, \quad x \neq 1,$$

and

$$\Phi(t) := \frac{t^2}{\phi(t)}, \quad t > 0, \quad t \neq 1.$$

We recall that $J^{(x)} = (J^{(x)}_t, t \geq 0)$ is the future infimum process of $X^{(x)}$, for $x \geq 0$, where $J^{(x)}_t = \inf_{s \geq t} X^{(x)}$.

Theorem 7 *Under condition (6.20), we have the following laws of the iterated logarithm.*

i) Almost surely, $\displaystyle \limsup_{x\to 0} \frac{S_x}{\phi(x)} = 1$ *and* $\displaystyle \limsup_{x\to \infty} \frac{S_x}{\phi(x)} = 1.$

ii) Almost surely, $\displaystyle \limsup_{x\to 0} \frac{U_x}{\phi(x)} = 1$ *and* $\displaystyle \limsup_{x\to +\infty} \frac{U_x}{\phi(x)} = 1.$

The upper envelope of a PSSMP and its future infimum processes under condition (6.20) are described by the following laws of the iterated logarithm:

iii) Almost surely, $\displaystyle \limsup_{t\to 0} \frac{X^{(0)}_t}{\Phi(t)} = 1$ *and* $\displaystyle \limsup_{t\to 0} \frac{J^{(0)}_t}{\Phi(t)} = 1.$

iv) For all $x \geq 0$, almost surely $\displaystyle \limsup_{t\to +\infty} \frac{X^{(x)}_t}{\Phi(t)} = 1$ *and* $\displaystyle \limsup_{t\to 0} \frac{J^{(x)}_t}{\Phi(t)} = 1.$

Proof: We first prove part (i) for small times. Note that it is easy to check that both $\phi(x)$ and $\phi(x)/x$ are increasing in a neighbourhood of 0, moreover the function $\phi(x)/x$ is bounded by 1, for $x \in [0, 1)$.
From condition (6.20) and Proposition 7, we have for all $k_1 < 1$ and $k_2 > 1$ and for all x sufficiently small,

$$k_1 K(\log(1/x))^\gamma \leq -\log G(x) \leq k_2 K(\log(1/x))^\gamma,$$

so that for ϕ defined above,

$$k_1 \log|\log x| \leq -\log G\left(\phi(x)/x\right) \leq k_2 \log|\log x|.$$

Hence $G(\phi(x)/x) \geq (|\log x|)^{-k_2}$. Since $k_2 > 1$, we obtain the convergence of the integral $\int_{0+} G(\phi(x)x)\, x^{-1}\mathrm{d}x$. This allows us to deduce with the help of Proposition 3 (i) that for all $\varepsilon > 0$,

$$\mathbb{P}(S_x < (1-\varepsilon)\phi(x), \text{i.o., as } x \to 0) = 0.$$

The divergent part is proven in the same way using that from Proposition 4 part (ii), one has for all $\varepsilon > 0$,

$$\mathbb{P}(S_x < (1+\varepsilon)\phi(x), \text{i.o., as } x \to 0) = 1.$$

Condition (6.20) implies that $\phi(x)$ is increasing in a neighbourhood of $+\infty$ whereas $\phi(x)/x$ is decreasing in a neighbourhood of $+\infty$. Hence, the proof of the result at $+\infty$ is done in the same way as at 0.
The parts (ii), (iii) and (iv) can be proved following the same arguments. It is enough to note that the LIL of the last passage process and the future infimum process will use the integral tests in [23], (see Theorems 1,2,3 and 4 in [23]) ■

6.2 The case with no positive jumps.

Here, we suppose that the PSSMP $X^{(x)}$ has no positive jumps. Our next result means in particular that if there exists a positive function that describes the upper envelope of $X^{(x)}$ by a LIL then the same function describes the upper envelope of the future infimum of $X^{(x)}$ and the PSSMP $X^{(x)}$ reflected at its future infimum.

Theorem 8 *Let us suppose that* $\limsup_{t\to 0} \dfrac{X_t^{(0)}}{\Lambda(t)} = 1$ *a.s., where* Λ *is a positive function such that* $\Lambda(0) = 0$*, then*

$$\limsup_{t\to 0} \frac{J_t^{(0)}}{\Lambda(t)} = 1 \quad \textit{and} \quad \limsup_{t\to 0} \frac{X_t^{(0)} - J_t^{(0)}}{\Lambda(t)} = 1 \quad \textit{almost surely.}$$

Moreover, if for all $x \geq 0$, $\limsup_{t\to+\infty} \dfrac{X_t^{(x)}}{\Lambda(t)} =$ *a.s., where* Λ *is a positive function such that* $\lim_{t\to+\infty} \Lambda(t) = +\infty$*, then*

$$\limsup_{t\to+\infty} \frac{J_t^{(x)}}{\Lambda(t)} = 1 \quad \textit{and} \quad \limsup_{t\to+\infty} \frac{X_t^{(0)} - J_t^{(0)}}{\Lambda(t)} = 1 \quad \textit{almost surely.}$$

Proof: First, we prove the result for large times. Let $x \geq 0$. Since $J_t^{(x)} \leq X_t^{(x)}$ for every $t \geq 0$, in light of our hypothesis tha is clear

$$\limsup_{t\to+\infty} \frac{J_t^{(x)}}{\Lambda(t)} \leq 1 \qquad \text{almost surely.}$$

Now, fix $\epsilon \in (0, 1/2)$ and define

$$R_n = \inf\left\{ s \geq n : \frac{X_s^{(x)}}{\Lambda(s)} \geq (1-\epsilon) \right\}.$$

From the above definition, it is clear that $R_n \geq n$ and that R_n diverges a.s. as n goes to $+\infty$. From our hypothesis, we deduce that R_n is finite, a.s.
Since $X^{(x)}$ has no positive jumps, applying the strong Markov property and the Lamperti representation (1.1), we have

$$\begin{aligned}
\mathbb{P}\left(\frac{J_{R_n}^{(x)}}{\Lambda(R_n)} \geq (1-2\epsilon) \right) &= \mathbb{P}\left(J_{R_n}^{(x)} \geq \frac{(1-2\epsilon)X_{R_n}^{(x)}}{(1-\epsilon)} \right) \\
&= \mathbb{E}\left(\mathbb{P}\left(J_{R_n}^{(x)} \geq \frac{(1-2\epsilon)X_{R_n}^{(x)}}{(1-\epsilon)} \middle| X_{R_n}^{(x)} \right) \right) \\
&= \mathbb{P}\left(\inf_{t\geq 0} \xi \geq \log \frac{(1-2\epsilon)}{(1-\epsilon)} \right) = cW\left(\log \frac{1-\epsilon}{1-2\epsilon} \right) > 0,
\end{aligned}$$

where $W : [0, +\infty) \to [0, +\infty)$ is the unique absolutely continuous increasing function with Laplace exponent

$$\int_0^{+\infty} e^{-\lambda x} W(x) \mathrm{d}x = \frac{1}{\psi(\lambda)} \qquad \text{for } \lambda > 0,$$

and $c = 1/W(+\infty)$, (see Bertoin [2] Theorem VII.8).
Since $R_n \geq n$,

$$\mathbb{P}\left(\frac{J_t^{(x)}}{\Lambda(t)} \geq (1-2\epsilon), \text{ for some } t \geq n \right) \geq \mathbb{P}\left(\frac{J_{R_n}^{(x)}}{\Lambda(R_n)} \geq (1-2\epsilon) \right).$$

Therefore, for all $\epsilon \in (0, 1/2)$,

$$\mathbb{P}\left(\frac{J_t^{(x)}}{\Lambda(t)} \geq (1-2\epsilon), \text{i.o., as } t \to +\infty \right) \geq \lim_{n\to+\infty} \mathbb{P}\left(\frac{J_{R_n}^{(x)}}{\Lambda(R_n)} \geq (1-2\epsilon) \right) > 0.$$

The event of the left hand side is in the upper-tail sigma-field $\cap_t \sigma\{X_s^{(x)} : s \geq t\}$ which is trivial. Therefore

$$\limsup_{t\to+\infty} \frac{J_t^{(x)}}{\Lambda(t)} \geq 1 - 2\epsilon \qquad \text{almost surely.}$$

The proof of part (ii) is very similar, in fact

$$\begin{aligned}
\mathbb{P}\left(\frac{X_{R_n}^{(x)} - J_{R_n}^{(x)}}{\Lambda(R_n)} \geq (1-2\epsilon) \right) &= \mathbb{P}\left(J_{R_n}^{(x)} \leq \frac{\epsilon X_{R_n}^{(x)}}{1-\epsilon} \right) = \mathbb{E}\left(\mathbb{P}\left(J_{R_n}^{(x)} \leq \frac{\epsilon X_{R_n}^{(x)}}{1-\epsilon} \middle| X_{R_n}^{(x)} \right) \right) \\
&= \mathbb{P}\left(\inf_{t\geq 0} \xi \leq \log \frac{\epsilon}{1-\epsilon} \right) = 1 - cW\left(\log \frac{1-\epsilon}{\epsilon} \right) > 0.
\end{aligned}$$

Since $R_n \geq n$,

$$\mathbb{P}\left(\frac{X_t^{(x)} - J_t^{(x)}}{\Lambda(t)} \geq (1-2\epsilon), \text{ for some } t \geq n\right) \geq \mathbb{P}\left(\frac{X_{R_n}^{(x)} - J_{R_n}^{(x)}}{\Lambda(R_n)} \geq (1-2\epsilon)\right).$$

Hence, for all $\epsilon \in (0, 1/2)$,

$$\mathbb{P}\left(\frac{X_t^{(x)} - J_t^{(x)}}{\Lambda(t)} \geq (1-2\epsilon), \text{i.o., as } t \to \infty\right) \geq \lim_{n\to+\infty} \mathbb{P}\left(\frac{X_{R_n}^{(x)} - J_{R_n}^{(x)}}{\Lambda(R_n)} \geq (1-2\epsilon)\right) > 0.$$

The event of the left hand side of the above inequality is in the upper-tail sigma-field $\cap_t \sigma\{X_s^{(x)} : s \geq t\}$ which is trivial and this establishes part (ii) for large times.
In order to prove the LIL for small times, we now define the following stopping time

$$R_n = \inf\left\{\frac{1}{n} \leq s : \frac{X_s^{(0)}}{\Lambda(s)} \geq (1-\epsilon)\right\}.$$

Following same arguments as above, we get that for a fixed $\epsilon \in (0, 1/2)$ and n sufficiently large,

$$\mathbb{P}\left(\frac{J_{R_n}^{(0)}}{\Lambda(R_n)} \geq (1-2\epsilon)\right) > 0 \quad \text{and} \quad \mathbb{P}\left(\frac{X_{R_n}^{(0)} - J_{R_n}^{(0)}}{\Lambda(R_n)} \geq (1-2\epsilon)\right) > 0.$$

Next, we note

$$\mathbb{P}\left(\frac{J_{R_p}^{(0)}}{\Lambda(R_p)} \geq (1-2\epsilon), \text{ for some } p \geq n\right) \geq \mathbb{P}\left(\frac{J_{R_n}^{(0)}}{\Lambda(R_n)} \geq (1-2\epsilon)\right),$$

and

$$\mathbb{P}\left(\frac{X_{R_p}^{(0)} - J_{R_p}^{(0)}}{\Lambda(R_p)} \geq (1-2\epsilon), \text{ for some } p \geq n\right) \geq \mathbb{P}\left(\frac{X_{R_n}^{(0)} - J_{R_n}^{(0)}}{\Lambda(R_n)} \geq (1-2\epsilon)\right).$$

Since R_n converges a.s. to 0 as n goes to ∞, the conclusion follows taking the limit when n tends to $+\infty$. ■

Hence, when $F^{\uparrow}$ satisfies condition (6.19) we have the following laws of the iterated logarithm for the future infimum of $X^{(x)}$ and the PSSMP $X^{(x)}$ reflected at its future infimum.

Corollary 2 *Under condition (6.19), we have the following laws of the iterated logarithm.*

i) For all $x \geq 0$, $\displaystyle\limsup_{t\to 0} \frac{J_t^{(0)}}{\vartheta(t)} = 1$ *and* $\displaystyle\limsup_{t\to+\infty} \frac{J_t^{(x)}}{\vartheta(t)} = 1$ *a.s.*

ii) For all $x \geq 0$, $\displaystyle\limsup_{t\to 0} \frac{X_t^{(0)} - J_t^{(0)}}{\vartheta(t)} =$ *and* $\displaystyle\limsup_{t\to+\infty} \frac{X_t^{(x)} - J_t^{(x)}}{\vartheta(t)} = 1$ *a.s.*

7 Examples.

Example 1. The first examples that we will consider are the only continuous PSSMP up to a power rate: Bessel processes. Recall that a Bessel process of dimension $\delta \geq 0$ with starting point $x \geq 0$ is the diffusion R whose square satisfies the stochastic differential equation

$$R_t^2 = x^2 + 2\int_0^t R_s \mathrm{d}\beta_s + \delta t, \qquad t \geq 0, \tag{7.23}$$

where β is a standard Brownian Motion.
From Itô's formula, we get that the underlying Lévy process of a Bessel process of dimension δ is $\xi = (2(B_t + at), t \geq 0)$, where B is a standard Brownian motion and $a = (\delta - 2)/2$. The process ξ satisfies the conditions under which we can define $X^{(x)}$ when $x = 0$. When $a > 0$, $X^{(x)}$ is transient and when $a = 0$, the process $X^{(x)}$ oscillates. Gruet and Shi [16] proved that there exist a finite constant $K > 1$, such that for any $0 < s \leq 2$,

$$K^{-1}s^{1-\delta/2}\exp\left\{-\frac{1}{2s}\right\} \leq \mathbb{P}(S_1 < s) = F^{\uparrow}(s) \leq Ks^{1-\delta/2}\exp\left\{-\frac{1}{2s}\right\}. \tag{7.24}$$

Hence, from (7.24) and Proposition 2, we may obtain integral tests for the lower envelope of the first passage time of the squared Bessel process $X^{(0)}$. Similarly, but now using Theorems 3 and 4 , we may obtain a variant of the Kolmogorov-Dvoretsky-Erdős integral tests for the upper envelope of Bessel processes. Finally, we also note that $F^{\uparrow}$ satisfies condition (6.19).
Example 2. Let $X^{(0)}$ be a stable Lévy process conditioned to stay positive and index $0 < \alpha \leq 2$ (see [10] for a proper definition). From Chaumont et al. [12], we know that when $X^{(0)}$ has positive jumps then there exists a constant C such that

$$\overline{F}(t) \sim Ct^{\alpha\rho - 1}, \qquad \text{as} \quad t \to 0,$$

where ρ is the positivity parameter of the stable Lévy processes.
On the other hand, from Fourati [15], we have that

$$\overline{F}_\nu(t) \sim Ct^{\alpha\rho}, \qquad \text{as} \quad t \to 0.$$

Hence under the assumption that $X^{(0)}$ has positive jumps, we get the following corollary. Here, we only the state the results at 0 but we recall that the results for large times also holds.

Corollary 3 *Let S be the first passage time of the stable Lévy process conditioned to stay positive $X^{(0)}$ with index $\alpha \in (0, 2]$. The lower envelope at* 0 *is described as follows. Let $h \in \mathcal{H}_0^{-1}$ be such that either* $\lim_{x\to 0} h(x)/x = 0$ *or* $\liminf_{x\to 0} h(x)/x > 0$. *Then*

$$\mathbb{P}\Big(S_x < h(t), \ \textit{i.o., as } x \to 0\Big) = 0 \ \textit{or } 1,$$

accordingly as

$$\int_{0+} \left(\frac{h(x)}{x}\right)^{\alpha\rho-1} \frac{\mathrm{d}x}{x} \quad \textit{is finite} \quad \textit{or} \quad \int_{0+} \left(\frac{h(x)}{x}\right)^{\alpha\rho} \frac{\mathrm{d}x}{x} \qquad \textit{is infinite.}$$

The upper envelope of $X^{(0)}$ at 0 is described as follows. Let $h \in \mathcal{H}_0$, such that either

$$\lim_{t\to 0} t/h(t) = 0 \qquad \textit{or} \qquad \liminf_{t\to 0} t/h(t) > 0.$$

Then

$$\mathbb{P}\Big(X_t^{(0)} > h(t),\ \textit{i.o., as } t \to 0\Big) = 0 \ \textit{or}\ 1,$$

accordingly as

$$\int_{0+} \left(\frac{h(x)}{x}\right)^{\alpha\rho-1} \frac{\mathrm{d}x}{x} \quad \textit{is finite} \quad \textit{or} \quad \int_{0+} \left(\frac{h(x)}{x}\right)^{\alpha\rho} \frac{\mathrm{d}x}{x} \qquad \textit{is infinite.}$$

Now, suppose that $X^{(0)}$ has no positive jumps. In this case, it is known that $1 < \alpha \le 2$. From [23], we know that $X^{(0)}$ drifts towards $+\infty$ and that

$$-\log \overline{F}(1/t) \sim \frac{\alpha-1}{\alpha}\left(\frac{1}{\alpha}\right)^{1/(\alpha-1)} t^{1/\alpha-1} \quad \text{as} \quad t \to +\infty.$$

Then applying Proposition 6, Theorem 6 and Corollary 2, we get the following LIL.

Corollary 4 *Let $X^{(0)}$ be a stable Lévy process with no positive jumps conditioned to stay positive and $\alpha > 1$. Then, the first passage time process satisfies*

$$\liminf_{t\to 0} \frac{S_t\big(\log|\log t|\big)^{\alpha-1}}{t^\alpha} = \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1}, \qquad \textit{almost surely.}$$

The processes $X^{(0)}$, $J^{(0)}$ and $X^{(0)} - J^{(0)}$ satisfy the following laws of the iterated logarithm

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{t^{1/\alpha}\big(\log|\log t|\big)^{1-1/\alpha}} = \alpha\,(\alpha-1)^{-\frac{\alpha-1}{\alpha}}, \qquad \textit{almost surely,}$$

$$\limsup_{t\to 0} \frac{J_t^{(0)}}{t^{1/\alpha}\big(\log|\log t|\big)^{1-1/\alpha}} = \alpha\,(\alpha-1)^{-\frac{\alpha-1}{\alpha}}, \qquad \textit{almost surely,}$$

$$\limsup_{t\to 0} \frac{X_t^{(0)} - J_t^{(0)}}{t^{1/\alpha}\big(\log|\log t|\big)^{1-1/\alpha}} = \alpha\,(\alpha-1)^{-\frac{\alpha-1}{\alpha}}, \qquad \textit{almost surely.}$$

Example 3. Let ξ be a Lévy process which drifts towards $+\infty$, with finite exponential moments of arbitrary positive order. Note that this condition is satisfied, for example, when the jumps of ξ are bounded from above by some fixed number and in particular when ξ is a Lévy process with no positive jumps. In that case, we have

$$\mathbb{E}\big(e^{\lambda\xi_t}\big) = \exp\big\{t\psi(\lambda)\big\} < +\infty, \qquad t, \lambda \ge 0.$$

From Theorem 25.3 in Sato [24], we know that this hypothesis is equivalent to assuming that the Lévy measure Π of ξ satisfies

$$\int_{[1,\infty)} e^{\lambda x}\Pi(\mathrm{d}x) < +\infty \qquad \text{for every } \lambda > 0.$$

Under this condition and with the hypothesis that ψ varies regularly at $+\infty$ with index $\beta \in (1,2)$, the author in [23] gave the following estimates of the tail probabilities of $I(\hat{\xi})$ and $\nu I(\hat{\xi})$,

$$-\log \mathbb{P}\Big(\nu I(\hat{\xi}) < 1/x\Big) \sim -\log \mathbb{P}\left(I(\hat{\xi}) < 1/x\right) \sim (\beta-1)\overleftarrow{H}(x) \quad \text{as} \quad x \to +\infty,$$

where

$$\overleftarrow{H}(x) = \inf\Big\{s > 0\ ,\ \psi(s)/s > x\Big\},$$

is a regularly varying function with index $(\beta-1)^{-1}$.
Hence, the PSSMP associated to ξ satisify condition (6.19). This allows us to obtain an LIL for the first passage time process and for the PSSMP in terms of the function

$$f(t) := \frac{\log|\log t|}{\psi(\log|\log t|)} \qquad \text{for} \qquad t > 1, \quad t \neq e.$$

Using integration by parts, we can see that the function $\psi(\lambda)/\lambda$ is increasing. Hence it is straightforward that the function $tf(t)$ is also increasing in a neighbourhood of ∞. Similarly to Example 2, we only state the LIL for small times.

Corollary 5 *If ψ is regularly varying at $+\infty$ with index $\beta \in (1,2)$, then*

$$\liminf_{x\to 0} \frac{S_x}{xf(x)} = (\beta-1)^{\beta-1}, \qquad \textit{almost surely.}$$

Let us define

$$g(t) := \frac{\psi(\log|\log t|)}{\log|\log t|} \qquad \text{for} \quad t > 1, \quad t \neq e.$$

Corollary 6 *If ψ is regularly varying at $+\infty$ with index $\beta \in (1,2)$, then*

$$\limsup_{t\to 0} \frac{X_t}{tg(t)} = (\beta-1)^{-(\beta-1)}, \qquad \textit{almost surely.}$$

Moreover, if the processes $X^{(0)}$ has no positive jumps, $J^{(0)}$ and $X^{(0)} - J^{(0)}$ satisfy the following laws of the iterated logarithm:

$$\limsup_{t\to 0} \frac{J^{(0)}_t}{tg(t)} = (\beta-1)^{-(\beta-1)} \quad \textit{and} \quad \limsup_{t\to 0} \frac{X^{(0)}_t - J^{(0)}_t}{tg(t)} = (\beta-1)^{-(\beta-1)} \quad \textit{a.s.}$$

Example 4. Let $\xi = N$ be a standard Poisson process. We remark that in this case the limit process $X^{(0)}$ may not be defined with Caballero and Chaumont's construction. This is a consequence of the fact that N is arithmetic.
According to Bertoin and Caballero [3], we can define a limit process for the PSSMP $X^{(x)}$ associated to N in the sense of finite dimensional distributions. With an abuse of notation, we will denote such limit process by $X^{(0)}$. From Proposition 3 in Bertoin and Yor [5] and Example 1 in [23], we know that

$$-\log \mathbb{P}\big(I(\hat{\xi}) < t\big) \sim -\log \mathbb{P}\big(\nu I(\hat{\xi}) < t\big) \sim \frac{1}{2}(\log 1/t)^2, \qquad \text{as } t \to 0.$$

Hence, we obtain the following LIL. Let us define

$$m(t) := t \exp\Big\{ - \sqrt{2 \log |\log t|}\Big\}.$$

Corollary 7 *Let N be a Poisson process, then the PSSMP $X^{(x)}$ associated to N by the Lamperti representation satisfies the following law of the iterated logarithm,*

$$\limsup_{t \to 0} \frac{X_t^{(0)} m(t)}{t^2} = 1 \qquad \textit{almost surely.}$$

For $x \geq 0$

$$\limsup_{t \to +\infty} \frac{X_t^{(x)} m(t)}{t^2} = 1 \qquad \textit{almost surely.}$$

The first passage time process S associated to $X^{(0)}$ satisfies the following law of the iterated logarithm,

$$\liminf_{x \to 0} \frac{S_x}{m(x)} = 1, \qquad \textit{and} \qquad \liminf_{x \to +\infty} \frac{S_x}{m(x)} = 1, \qquad \textit{almost surely.}$$

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