Eigenvalues of Hermitian Lévy Processes
(Jumps of rank one)

Victor Pérez-Abreu
Department of Probability and Statistics
CIMAT, Guanajuato, Mexico

joint work with José Pérez-Garmendia, Alfonso Rocha-Arteaga

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Plan of the talk

I. Motivation: GUE.
   1. Wigner theorem.
   2. Matrix Brownian motion.
   3. Dynamics of the eigenvalue process (Dyson process).
   4. Functional asymptotics to free Brownian motion.

II. Hermitian Lévy processes.
   1. Infinitely divisible random matrix.
   2. Lévy Khintchine representation.
   3. Condition for simple spectrum.

III. Hermitian Lévy processes with jumps of rank one.
   1. Examples.
   2. Simultaneity of jumps of the eigenvalues.
   3. Simultaneity of jumps of an Hermitian Lévy process and its eigenvalues.

IV. Dynamics of the eigenvalues and noncolliding property.

V. Functional asymptotics to free Lévy processes.

I. Gaussian Unitary Ensemble (GUE)

- GUE(t), $t > 0$: $\mathbf{B}(t) = (B_n(t))_{n\geq 1}$, $B_n(t)$ is $n \times n$ Hermitian random matrix

$$B_n(t) = (b_n^{i,j}(t))_{1\leq i,j\leq n}, \quad b_n^{j,i}(t) = \overline{b}_n^{i,j}(t),$$

$$\mathrm{Re}\left(b_n^{j,i}(t)\right) \sim \mathrm{Im}\left(b_n^{j,i}(t)\right) \sim N(0, t(1+\delta_{ij})/2),$$

$$\mathrm{Re}\left(b_n^{j,i}(t)\right), \mathrm{Im}\left(b_n^{j,i}(t)\right), 1 \leq i \leq j \leq n \text{ **independent** r.v.}$$

- For each $t > 0$, the distribution of $B_n(t)$ is **invariant under unitary conjugations**: $UB_n(t)U^* \stackrel{d}{=} B_n(t), \forall U \in \mathbb{U}(n)$.
- $(B_n(t))_{n\geq 1}$ is the $n \times n$ **Hermitian Brownian motion.**
- Let $\lambda_1(t), ..., \lambda_n(t)$ be the eigenvalues of $B_n(t)$.
- Empirical Spectral Distribution (ESD) of rescaled $B_n(t)/\sqrt{n}$.

$$\mu_t^{(n)}(x) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{\{\lambda_j(t)/\sqrt{n}\leq x\}}.$$

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I. Wigner theorem and semicircle distribution

Theorem (Wigner, 1955).

Fix $t > 0$. Then $\forall\, f \in C_b(\mathbb{R})$

$$\mathbb{P}\left(\lim_{n\to\infty} \int f(x)\mathrm{d}\mu_t^{(n)}(x) = \int f(x)\mathrm{w}_t(\mathrm{d}x)\right) = 1$$

where w_t is the **Wigner** or **semicircle distribution** *on* $(-2\sqrt{t}, 2\sqrt{t})$

$$\mathrm{w}_t(\mathrm{d}x) = \frac{1}{2\pi}\sqrt{4t - x^2}\,\mathrm{d}x, \quad |x| \leq 2\sqrt{t}.$$

I. Dyson-Brownian motion

Dynamics of eigenvalue process

Theorem (Dyson, 1962)
Let $\{B(t) : t \geq 0\}$ be $n \times n$ Hermitian matrix Brownian motion and $\{(\lambda_1(t), ..., \lambda_n(t)); t \geq$
its process of eigenvalues.
Assume $\lambda_1(0) > \cdots > \lambda_n(0)$ almost surely. Then

a) The eigenvalues never meet at any time,

$$\mathbb{P}\left(\lambda_1(t) > \lambda_2(t) > ... > \lambda_n(t) \quad \forall t > 0\right) = 1.$$

b) There exist n independent one dimensional standard Brownian motions $W_1, .., W_n(t)$ such that w.p.1 for $i = 1, ..., n$

$$d\lambda_i(t) = W_i(t) + \sum_{j \neq i} \frac{dt}{\lambda_j(t) - \lambda_i(t)}, \quad \forall t > 0. \tag{1}$$

- Brownian part + repulsion part (at any time t).
- (1) is an $\mathbb{R}^n$-valued SDE with **non smooth** drift coefficient.
  - Anderson, Guionnet, Zeitouni 2010 (clear proof).
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I. Associated measure valued processes

- Empirical measure-valued process associated to $B_n(t)/\sqrt{n}$

$$\mu_t^{(n)} = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_{,j}(t)/\sqrt{n}}.$$

- w_t *semicircle or Wigner distribution on* $(-2\sqrt{t}, 2\sqrt{t})$

$$\mathrm{w}_t(\mathrm{d}x) = \frac{1}{2\pi}\sqrt{4t - x^2}\mathrm{d}x, \quad |x| \leq 2\sqrt{t}.$$

- For each $t > 0$, w_t is free infinitely divisible.
- What about functional convergence?
  - Measure valued empirical process

$$\left\{ (\mu_t^{(n)})_{n\geq 1}; t \geq 0 \right\}$$

  Measure valued function

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I. Notation

- $\mathcal{P}(\mathbb{R})$ be the set of probability measures on $\mathbb{R}$.

- Let $C(\mathbb{R}_+, \mathcal{P}(\mathbb{R}))$ be the spaces of continuous functions from $\mathbb{R}_+ \to \mathcal{P}(\mathbb{R})$, with the topology of uniform convergence on compact intervals of $\mathbb{R}_+$.

- For $\mu \in \mathcal{P}(\mathbb{R})$ and a function $f : \mathbb{R} \to \mathbb{R}$ that is μ-integrable we write

$$\langle \mu, f \rangle = \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x).$$

I. Functional limit theorems

Biane (1997), Cabanal-Duvillard and Guionnet (2001)

Theorem.

a) **(Functional Wiener theorem)** $\forall\, f \in C_b(\mathbb{R})$ and $T > 0$

$$\mathbb{P}\left(\lim_{n\to\infty} \sup_{0\le t\le T} \left|\left\langle \mu_t^{(n)}, f\right\rangle - \langle \mathrm{w}_t, f\rangle\right| = 0\right) = 1.$$

b) **(Measure-valued equation for the limit)**. If $\mu_0^{(n)} \to \delta_0$, the family $\left(\mu_t^{(n)}\right)_{t\ge 0}$ of measure valued-processes converges weakly in $C\left(\mathbb{R}_+, \mathcal{P}(\mathbb{R})\right)$ to a unique continuous probability-measure valued function such that $\forall\, f \in C_b^2(\mathbb{R})$

$$\langle \mu_t, f\rangle = f(0) + \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x-y}\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y).$$

Moreover, $\mu_t = \mathrm{w}_t$, $t \ge 0$.

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II. Hermitian Lévy process

$\mathbb{H}_n$ is the linear space of $n \times n$ Hermitian matrices with inner product.

$$\langle A, B \rangle = \operatorname{tr}(AB).$$

Definition
A $n \times n$ matrix process $\{M(t)\}_{t \geq 0}$ with values in $\mathbb{H}_n$ is a Hermitian Levy process iff:

1. $M(0) = 0$ w.p.1.

2. It has independent increments: $\forall\ 0 < t_1 < \cdots < t_n, \quad n \geq 1,$
$M(t_n) - M(t_{n-1}), ..., M(t_2) - M(t_1)$ are independent random matrices.

3. It has stationary increments: $\forall\ 0 < s < t$, $M(t) - M(s)$, and $M(t-s)$ have the same matrix distribution.

4. With probability 1 the path $t \to M(t)$ is right continuous with left limits in the topology of $\mathbb{H}_n$.

II. Background: Infinitely divisible random matrix

Definition
A random matrix M in $\mathbb{H}_n$ is infinitely divisible (ID) iff $\forall\ n \geq 1$ there exist n independent identically distributed random matrices $M_1, ..., M_n$ in $\mathbb{H}_n$ such that

$$M \overset{d}{=} M_1 + ... + M_n.$$

Fact
Given an ID random matrix M in $\mathbb{H}_n$, there exists an Hermitian Lévy process $\{M(t)\}_{t\geq 0}$ such that

$$M \overset{d}{=} M(1).$$

Definition
A random matrix M is called self-decomposable if for any $c \in (0,1)$ there exists M_c independent of M such that

$$M \overset{d}{=} cM + M_c$$

II. Lévy-Khintchine representation

Theorem
Let $\{X(t) : t \geq 0\}$ be a $n \times n$ Hermitian Lévy process. Then, for each $t \geq 0$ and $\Theta \in \mathbb{H}_n$,

$$\log \mathbb{E} e^{\mathrm{itr}(\Theta X(t))} = t \left\{ \mathrm{itr}(\Theta \Psi) - \frac{1}{2} \mathrm{tr}\left(\Theta \mathcal{A} \Theta\right) + \int_{\mathbb{H}_n} \left(e^{\mathrm{itr}(\Theta \xi)} - 1 - \mathrm{i} \frac{\mathrm{tr}(\Theta \xi)}{1 + \|\xi\|^2} \right) \nu(\mathrm{d}\xi) \right\},$$

$\Psi \in \mathbb{H}_n$,

$\mathcal{A} : \mathbb{H}_n \to \mathbb{H}_n$ is a positive symmetric linear operator,

ν is the Lévy measure on $\mathbb{H}_n$ such that $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{H}_n} (1 \wedge \|x\|^2) \nu(dx) < \infty.$$

The triplet $(\mathcal{A}, \nu, \Psi)$ uniquely determines the distribution of X.

II. Hermitian Lévy process with simple spectrum

A Hermitian random matrix M has simple spectrum $w.p.1$ if it has an absolutely continuous distribution, since almost every Hermitian matrix has simple spectrum.

Facts:

a) If $\{X(t)\}_{t\geq 0}$ is a nondegenerate Lévy process in $\mathbb{H}_n$ with Gaussian component then $X(t)$ has absolutely continuous distribution for each $t > 0$.

b) If $\{X(t)\}_{t\geq 0}$ is a self-decomposable process in $\mathbb{H}_n$ without Gaussian component, then $X(t)$ has absolutely continuous distribution for each $t > 0$.

- Let $X = \{X(t) : t \geq 0\}$ be a $n \times n$ Hermitian Lévy process.

  The process X has jumps of rank one if $\Delta X(s) \neq 0$ for $s \geq 0$,

  $$\Delta X(s) = ruv^{\top},$$

  where $r = r(s) \in \mathbb{C} \setminus \{0\}, \quad u = u(s), v = v(s) \in \mathbb{C}^n \setminus \{0\}$.

- **Example 1**
  Let $Y = \{Y(t) : t \geq 0\}$ be a $\mathbb{C}^n$-valued Lévy process. Define

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  Let $Y_1(t), ..., Y_n(t)$ be independent one dimensional Lévy processes.
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$$X(t) := U\text{diag}\left(Y_1(t), ..., Y_n(t)\right) U^*, \qquad t \geq 0.$$

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III. Hermitian Lévy processes with jumps of rank one

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A $n \times n$ Hermitian Lévy process M_n with jumps of rank one has invariant distribution under unitary conjugations iff its Lévy triplet $(\mathcal{A}, \Gamma, \nu)$ is of the form

a) $\Gamma = \gamma \mathrm{I}_n$
b) $\mathcal{A}\Theta = \sigma^2(\Theta + \mathrm{tr}(\Theta)\mathrm{I}_n)$, $\Theta \in \mathbb{H}_n$.
c) The Lévy measure is concentrated on matrices of rank one

$$\nu(E) = \int_{\mathbb{S}(\mathbb{H}_n^1)} \int_0^\infty 1_E(rV)\,\nu_V(\mathrm{d}r)\,\Pi(\mathrm{d}V), \quad E \in \mathcal{B}(\mathbb{H}_n \backslash \{0\}), \tag{2}$$

$\nu_V = \nu|_{(0,\infty)}$ or $\nu|_{(-\infty,0)}$ according to $V \geq 0$ or $V \leq 0$ and $\Pi(\mathrm{d}V)$ is a measure on $\mathbb{S}(\mathbb{H}_n^1)$ such that

$$\int_{\mathbb{S}(\mathbb{H}_n^1)} 1_{\mathrm{D}}(V)\,\Pi(\mathrm{d}V) = \int_{\mathbb{S}(\mathbb{H}_n^1)\cap\overline{\mathbb{H}}_n^+} \int_{\{-1,1\}} 1_D(tV)\,\lambda(\mathrm{d}t)\,\omega_n(\mathrm{d}V), \quad D \in \mathcal{B}\left(\mathbb{S}(\mathbb{H}_n^1)\right), \tag{3}$$

- λ is the spherical measure of ν,
- ω_n is the probability measure on $\mathbb{S}(\mathbb{H}_n^1) \cap \overline{\mathbb{H}}_n^+$ induced by the transformation $u \to V = uu^*$, with u is uniformly distributed in the unit sphere of $\mathbb{C}_n$,
- $\mathbb{H}_n^0 = \mathbb{H}_n \backslash \{0\}$ and $\mathbb{H}_n^1$ is the set of rank one matrices in $\mathbb{H}_n$,
- $\overline{\mathbb{H}}_n^+$ the cone of nonnegative definite matrices,
- $\mathbb{S}(\mathbb{H}_n^1)$ the unit sphere of $\mathbb{H}_n^1$.

III. Jumps of the Hermitian Lévy process and its eigenvalues

Let $\{X(t) : t \geq 0\}$ be a $n \times n$ Hermitian Lévy process with jumps of rank one.

1) If $X(s)$ and $X(s-)$ commute

$$\Longrightarrow X(s) \text{ and } X(s-) \text{ simultaneously diagonalizable}$$

$$\Longrightarrow \Delta X(s) = X(s) - X(s-) = U_s diag\left(0, ..., \Delta\lambda_i\left(s\right), ..., 0\right) U_s^*$$

hence only one eigenvalue jumps at time s

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Completely different situation!.

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Completely different situation!.

- A rank one perturbation of a $n \times n$ complex matrix M is a matrix

$$M + ruv^{\top}$$

where $u, v \in \mathbb{C}^n$ and $r \in \mathbb{C} \setminus \{0\}$.

- A generic set S is a non empty subset of $\mathbb{C}^n$ such that $\mathbb{C}^n \setminus S$ is contained in a subset of the form

$$\{u \in \mathbb{C}^n : f(u) = 0 \text{ for all } f \in F\} \subsetneq \mathbb{C}^n,$$

where F is a finite subset of polynomials in $\mathbb{C}[x_1, ..., x_n]$.

- **Proposition.**
Let u, v be generic. Let m be the degree of the minimal polynomial of M. There are exactly m eigenvalues of

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If a jump $\Delta X(s)$ is of rank one,

$$X_{s-} = X_s + ruv^\top, \quad r \in \mathbb{C} \setminus \{0\}, \quad u, v \in \mathbb{C}^n \setminus \{0\}.$$

Let $\lambda(t) = (\lambda_1(t), ..., \lambda_n(t))$ be the eigenvalues of $X(t)$ for each $t \geq 0$.

Lemma
Let $\{X(t) : t \geq 0\}$ be an absolutely continuous $n \times n$ Hermitian Lévy process.

If X has a rank one jump at time $s \geq 0$, then

$$\mathbb{P}\left(\{\lambda_1(X_s), \cdots, \lambda_n(X_s)\} \cap \{\lambda_1(X_{s-}), \cdots, \lambda_n(X_{s-})\} = \varnothing\right) = 1.$$

Remark
If $X_s - X_{s-} \neq 0 \quad \Longrightarrow \quad \lambda_j(X_s) - \lambda_j(X_{s-}) \neq 0$ for all j

$$\Longrightarrow \quad \lambda(X_s) - \lambda(X_{s-}) \neq 0.$$

III. Simultaneity of jumps

Theorem (PA, Rocha-Arteaga, 2015).

a) Let $\{X(t) : t \geq 0\}$ be a $n \times n$ Hermitian Lévy process with jumps of rank one.
b) Assume that $X(t)$ has absolutely continuous distribution for each $t \geq 0$.

Then

i) If $X(s)$ and $X(s-)$ do not commute then $\Delta X(s) \neq 0$ if and only if $\Delta\lambda(s) \neq 0$.

ii) $\Delta\lambda_i(s) \neq 0$ for some i if and only if $\Delta\lambda_j(s) \neq 0$ for all $j \neq i$.

Theorem (PA, Rocha-Arteaga, 2015).
Let $\{X(t) : t \geq 0\}$ be an absolutely continuous $n \times n$ Hermitian Lévy process with distribution invariant under unitary transformations $((\sigma^2 \mathrm{I}_n, \gamma \mathrm{I}_n, \nu))$ and let $\lambda(t) = (\lambda_1(t), ..., \lambda_n(t))$ its eigenvalue process where $\lambda_1(t) \geq \lambda_2(t) \geq \cdots \geq \lambda_n(t)$ for each $t \geq 0$. Then, for $m = 1, ..., n$,

$$\lambda_m(X_t) = \lambda_m(X_0) + \gamma \sum_{i=1}^{n} \int_0^t (D\lambda_m(X_{s-}))_{ii} \mathrm{d}s + 4\sigma^2 \int_0^t \sum_{j \neq m} \frac{1}{\lambda_m(s) - \lambda_j(s)} \mathrm{d}s + M_t^m$$

$$+ \int_{(0,t] \times \mathbb{H}_n^0} (\lambda_m(X_{s-} + y) - \lambda_m(X_{s-}) - \mathrm{tr}(D\lambda_m(X_{s-})y)) \nu(\mathrm{d}y)\mathrm{d}s,$$

with

$$M_t^m = \sigma \sum_{r=1}^{n} \sum_{\text{ł}=1}^{n} \int_0^t (D\lambda_m(X_{s-}))_{ij} \mathrm{d}B_s^{ij} + \int_0^t \int_{(0,t] \times \mathbb{H}_n^0} (\lambda_m(X_{s-} + y) - \lambda_m(X_{s-})) \tilde{J}_X(\mathrm{d}s, \mathrm{d}y)$$

$J_X(\cdot, \cdot)$ is Poisson random measure of jumps of X on $[0, \infty) \times \mathbb{H}_n^0$ with intensity measure $Leb \otimes \nu$, independent of the independent Brownian motions $B_s^{ij}, i, j = 1, ..., n$ and $\widetilde{J}_X(dt,dy) = J_X(dt,dy) - dt\nu(dy)$.

IV. Noncolliding property

Theorem (PA, Rocha-Arteaga, 2015) If in addition X has jumps of rank one, then

$$\mathbb{P}\left(\lambda_1(t) > \lambda_2(t) > ... > \lambda_n(t) \quad \forall t > 0\right) = 1.$$

- The repulsion term between each pair of eigenvalues $1/(\lambda_i(s) - \lambda_j(s))$, $i \neq j$, appears only when there is a Gaussian component.
- The last two theorems give the analogous of the Dyson–Brownian motion for a Hermitian Lévy process.

V. Towards free Lévy processes

Pérez-Garmendia, PA and Rocha-Arteaga (2015)

- Let $(X(t))_{t\geq 0}$ be a $n\times n$ Hermitian Lévy process invariant under unitary conjugations and absolutely continuos law.
- For each $t\geq 0$ let $\lambda_1(t)<\cdots<\lambda_n(t)$ be the eigenvalues of X and the ESD

$$\mu_t^n(\mathrm{d}x)=\frac{1}{n}\sum_{m=1}^{n}\delta_{\lambda_m(t)}(\mathrm{d}x).$$

- For a family of measures $(\mu_t)_{t\geq 0}$ consider the Cauchy transforms

$$\psi_\mu(t,z):=\int_{\mathbb{R}}(z-x)^{-1}\mu_t(\mathrm{d}x).$$

- *Assume that $\mu_0^{(n)}$ converges weakly to δ_0.* **Then**, *$\{(\mu_t^{(n)})_{t\geq 0}:n\geq 1\}$ converges weakly in $C(\mathbb{R}_+,\Pr(\mathbb{R}))$ to the unique continuous probability-measure valued function $(\mu_t)_{t\geq 0}$ satisfying, for each $t\geq 0$,*

$$\begin{aligned}\psi_\mu(t,z)=&-2\int_0^t\psi_\mu(s,z)\frac{\partial}{\partial z}\psi_\mu(s,z)\mathrm{d}s\\&-2\int_0^t\gamma\frac{\partial}{\partial z}\psi_\mu(s,z)\mathrm{d}s-\int_0^t\int_{\mathbb{R}}\frac{\psi_\mu(s,z)}{1-t\psi_\mu(s,z)}\nu(\mathrm{d}t)\mathrm{d}s.\end{aligned}$$

- *The family $(\mu_t)_{t\geq 0}$ corresponds to the law of a* **free Lévy process.**