A class of multivariate infinitely divisible distributions related to arcsine density

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Two transformations $\mathcal{A}_1$ and $\mathcal{A}_2$ of Lévy measures on $\mathbb{R}^d$ based on the arcsine density are studied and their relation to general Upsilon transformations is considered. The domains of definition of $\mathcal{A}_1$ and $\mathcal{A}_2$ are determined and it is shown that they have the same range. The class of infinitely divisible distributions on $\mathbb{R}^d$ with Lévy measures being in the common range is called the class A and any distribution in the class A is expressed as the law of a stochastic integral $\int_0^1 \cos(2^{-1}\pi t)dX_t$ with respect to a Lévy process $\{X_t\}$. This new class includes as a proper subclass the Jurek class of distributions. It is shown that generalized type G distributions are the image of distributions in the class A under a mapping defined by an appropriate stochastic integral. $\mathcal{A}_2$ is identified as an Upsilon transformation, while $\mathcal{A}_1$ is shown to be not.

Keywords: infinitely divisible distribution; arcsine density; Lévy measure; class A; generalized type G distribution; general Upsilon transformation

1. Introduction

Let $I(\mathbb{R}^d)$ denote the class of all infinitely divisible distributions on $\mathbb{R}^d$. For $\mu \in I(\mathbb{R}^d)$, we use the Lévy-Khintchine representation of its characteristic function $\widehat{\mu}(z)$ given by

$$\widehat{\mu}(z) = \exp\Bigg\{ -\frac{1}{2}\langle \Sigma z,z\rangle + \mathrm{i}\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(\mathrm{e}^{\mathrm{i}\langle x,z\rangle} - 1 - \frac{\mathrm{i}\langle x.z\rangle}{1+|x|^2}\right)\nu(\mathrm{d}x)\Bigg\}, \quad z \in \mathbb{R}^d,$$

where Σ is a symmetric nonnegative-definite $d\times d$ matrix, $\gamma \in \mathbb{R}^d$, and ν is a measure on $\mathbb{R}^d$ (called the Lévy measure) satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(1\wedge |x|^2)\nu(\mathrm{d}x) < \infty$. The triplet (Σ, ν, γ) is called the Lévy-Khintchine triplet of $\mu \in I(\mathbb{R}^d)$. Let $\mathfrak{M}_L(\mathbb{R}^d)$

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denote the class of Lévy measures of $\mu \in I(\mathbb{R}^d)$. The class of $\nu \in \mathfrak{M}_L(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d}(1 \wedge |x|)\nu(\mathrm{d}x) < \infty$ is denoted by $\mathfrak{M}_L^1(\mathbb{R}^d)$.

Let

$$a(x; s) = \pi^{-1}(s - x^2)^{-1/2}1_{(-s^{1/2},s^{1/2})}(x), \tag{1.1}$$

which is the density of the symmetric arcsine law with parameter $s > 0$. Here $1_{(-s^{1/2},s^{1/2})}(x)$ is the indicator of the interval $(-s^{1/2}, s^{1/2})$. In [1], a symmetric distribution such that its Lévy measure has a density ℓ of the form

$$\ell(x) = \int_{\mathbb{R}_+} a(x; s)\rho(\mathrm{d}s), \quad x \in \mathbb{R},$$

with a measure ρ on $(0,\infty)$ satisfying $\int_{(0,\infty)}(1 \wedge x)\rho(dx) < \infty$ is called a *type A distribution* on $\mathbb{R}$. Let Z be a standard normal random variable and V a positive infinitely divisible random variable independent of Z. The distribution of the one-dimensional random variable $V^{1/2}Z$ is infinitely divisible and is called of type G. It is shown in [1] that an infinitely divisible distribution $\widetilde{\mu}$ on $\mathbb{R}$ is of type G if and only if there exists a type A distribution μ on $\mathbb{R}$ which gives a stochastic integral mapping representation

$$\widetilde{\mu} = \mathcal{L}\left(\int_0^1 (-\log t)^{1/2} dX_t^{(\mu)}\right). \tag{1.2}$$

Here and in what follows, $\mathcal{L}$ means "the law of" and $\{X_t^{(\mu)}\}$ means a Lévy process on $\mathbb{R}^d$ whose distribution at time 1 is $\mu \in I(\mathbb{R}^d)$, ($d = 1$ in (1.2)).

In this paper, we define and study a class of infinitely divisible distributions on $\mathbb{R}^d$, called the class A and denoted by $A(\mathbb{R}^d)$. A distribution in $A(\mathbb{R}^d)$ is called a distribution of class A in this paper. When $d = 1$ and $\mu \in A(\mathbb{R})$ is symmetric, μ is a type A distribution in [1]. The organization of the paper is the following.

Section 2 introduces the arcsine transformation $\mathcal{A}_1$ of Lévy measures on $\mathbb{R}^d$ based on (1.1) and considers its domain and range. It is shown that the domain of $\mathcal{A}_1$ is $\mathfrak{M}_L^1(\mathbb{R}^d)$. We prove that $\mathcal{A}_1$ is a one-to-one mapping. It is shown that the range $\mathfrak{R}(\mathcal{A}_1)$ contains as a proper subclass the class of Lévy measures of distributions in the Jurek class $U(\mathbb{R}^d)$ studied in [5], [8]. The class $U(\mathbb{R}^d)$ includes several known classes of multivariate distributions characterized by the radial part of their Lévy measures, such as the Goldie-Steutel-Bondesson class $B(\mathbb{R}^d)$, the class of selfdecomposable distributions $L(\mathbb{R}^d)$ and the Thorin class $T(\mathbb{R}^d)$, see [1]. Recently, other bigger classes than the Jurek class have been discussed in the study of extension of selfdecomposability, see [6] and [13].

Section 3 deals with the class $A(\mathbb{R}^d)$ whose elements are defined as infinitely divisible distributions on $\mathbb{R}^d$with Lévy measures $\in \mathfrak{R}(\mathcal{A}_1)$. Some probabilistic interpretations of $A(\mathbb{R}^d)$ are given and the relation to the class $G(\mathbb{R}^d)$ of generalized type G distributions on $\mathbb{R}^d$ introduced in [8] is studied. It is shown that $A(\mathbb{R}^d) = \Phi_{\cos}(I(\mathbb{R}^d))$, where $\Phi_{\cos}$ is the stochastic integral mapping

$$\Phi_{\cos}(\mu) = \mathcal{L}\left(\int_0^1 \cos(2^{-1}\pi t)\mathrm{d}X_t^{(\mu)}\right), \quad \mu \in I(\mathbb{R}^d). \tag{1.3}$$

It is also shown that the class $\mathfrak{M}_L^G(\mathbb{R}^d)$ is the image of the class $\mathfrak{M}_L^B(\mathbb{R}^d) \cap \mathfrak{M}_L^1(\mathbb{R}^d)$ under $\mathcal{A}_1$, where $\mathfrak{M}_L^G(\mathbb{R}^d)$ and $\mathfrak{M}_L^B(\mathbb{R}^d)$ are the classes of Lévy measures of distributions in $G(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$, respectively. In addition, the class $G(\mathbb{R}^d)$ is described as the image of $A(\mathbb{R}^d)$ under the stochastic integral mapping (1.2), $d \geq 1$, including the multivariate and non-symmetric cases. In order to prove these facts, a modification $\mathcal{A}_2$ of the transformation $\mathcal{A}_1$ with the property $\mathfrak{R}(\mathcal{A}_2) = \mathfrak{R}(\mathcal{A}_1)$ is introduced and utilized effectively. It is shown that $\mathcal{A}_2$ is an Upsilon transformation in the sense of [3]. This is in contrast to the fact that $\mathcal{A}_1$ is not an Upsilon transformation as it is not commuting with a specific Upsilon transformation, which is different from other cases considered so far. Finally, Section 4 contains examples of $\mathcal{A}_1$ and $\mathcal{A}_2$ transformations of Lévy measures where the modified Bessel function K_0 plays an important role.

2. Arcsine transformation $\mathcal{A}_1$ on $\mathbb{R}^d$

2.1. **Definition and domain.** Besides the arcsine density (1.1), we consider the one-sided arcsine density

$$a_1(r; s) = 2\pi^{-1}(s - r^2)^{-1/2}1_{(0,s^{1/2})}(r)$$

with parameter $s > 0$. Then we consider the following arcsine transformation $\mathcal{A}_1$ of measures on $\mathbb{R}^d$.

Definition 2.1. Let ν be a measure on $\mathbb{R}^d$. Define the *arcsine transformation* $\mathcal{A}_1$ of ν by

$$\mathcal{A}_1(\nu)(B) = \int_{\mathbb{R}^d\setminus\{0\}} \nu(\mathrm{d}x) \int_0^\infty a_1(r; |x|)1_B\left(r\frac{x}{|x|}\right) \mathrm{d}r, \quad B \in \mathcal{B}(\mathbb{R}^d), \tag{2.1}$$

for $\nu \in \mathfrak{D}(\mathcal{A}_1)$, where the domain $\mathfrak{D}(\mathcal{A}_1)$ is the class of measures ν on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and the right-hand side of (2.1) belongs to $\mathfrak{M}_L(\mathbb{R}^d)$ as B runs in $\mathcal{B}(\mathbb{R}^d)$. The range is $\mathfrak{R}(\mathcal{A}_1) = \{\mathcal{A}_1(\nu) \colon \nu \in \mathfrak{D}(\mathcal{A}_1)\}$.

Proposition 2.2. $\mathfrak{D}(\mathcal{A}_1) = \mathfrak{M}_L^1(\mathbb{R}^d)$.

Proof. Suppose that $\nu \in \mathfrak{D}(\mathcal{A}_1)$. Write $\widetilde{\nu} = \mathcal{A}_1(\nu)$. Then

$$\int_{\mathbb{R}^d} (1 \wedge |x|^2)\widetilde{\nu}(\mathrm{d}x) = c \int_{\mathbb{R}^d} \nu(\mathrm{d}x) \int_0^{|x|^{1/2}} (|x| - r^2)^{-1/2}(1 \wedge r^2)\mathrm{d}r \tag{2.2}$$

with $c = 2\pi^{1/2}$. Changing variables $r = |x|^{1/2}u$ and using $\int (1 \wedge |x|^2)\widetilde{\nu}(\mathrm{d}x) < \infty$, we see that

$$c \int_{\mathbb{R}^d} (1 \wedge |x|)\nu(\mathrm{d}x) \int_0^1 (1 - u^2)^{-1/2} u^2 \mathrm{d}u < \infty.$$

Hence $\nu \in \mathfrak{M}_L^1(\mathbb{R}^d)$. This shows $\mathfrak{D}(\mathcal{A}_1) \subset \mathfrak{M}_L^1(\mathbb{R}^d)$.

Next suppose that $\nu \in \mathfrak{M}_L^1(\mathbb{R}^d)$. Let $\widetilde{\nu}(B)$ denote the right-hand side of (2.1). Then $\widetilde{\nu}$ is a measure on $\mathbb{R}^d$ with $\widetilde{\nu}(\{0\}) = 0$ and (2.2) holds. Hence

$$\int_{\mathbb{R}^d} (1 \wedge |x|^2)\widetilde{\nu}(\mathrm{d}x) \le \text{const} \int_{\mathbb{R}^d} (1 \wedge |x|)\nu(\mathrm{d}x) < \infty.$$

This shows that $\nu \in \mathfrak{D}(\mathcal{A}_1)$. Thus $\mathfrak{M}_L^1(\mathbb{R}^d) \subset \mathfrak{D}(\mathcal{A}_1)$. $\square$

2.2. **One-to-one property.** We next show that the arcsine transformation $\mathcal{A}_1$ is one-to-one. Our proof is different from usual proofs of the one-to-one property by the use of Laplace transform.

Let us prepare a lemma. For a measure ρ on $(0, \infty)$, define a measure $\sigma_{(\rho)}$ on $(0, \infty)$ by

$$\sigma_{(\rho)}(\mathrm{d}u) = \left(\int_{(u,\infty)} \pi^{-1/2}(s - u)^{-1/2}\rho(\mathrm{d}s) \right) \mathrm{d}u. \tag{2.3}$$

This is fractional integral of order 1/2.

Lemma 2.3. *Let ρ be a measure on $(0, \infty)$ satisfying*

$$\rho((b, \infty)) < \infty \quad \textit{for all } b > 0. \tag{2.4}$$

Then

$$\sigma_{(\sigma_{(\rho)})}(\mathrm{d}u) = \rho((u, \infty))\,\mathrm{d}u, \tag{2.5}$$

which implies that ρ is determined by $\sigma_{(\rho)}$ under the condition (2.4).

Proof. Using Fubini's theorem, notice that

$$\begin{aligned} &\int_u^\infty \pi^{-1/2}(s - u)^{-1/2}\,\sigma_{(\rho)}(\mathrm{d}s) \\ &\qquad = \pi^{-1} \int_u^\infty (s - u)^{-1/2}\mathrm{d}s \int_{(s,\infty)} (v - s)^{-1/2}\rho(\mathrm{d}v) \end{aligned}$$

$$= \pi^{-1} \int_{(u,\infty)} \rho(\mathrm{d}v) \int_u^v (s-u)^{-1/2}(v-s)^{-1/2}\mathrm{d}s = \rho((u,\infty)),$$

because

$$\int_u^v (s-u)^{-1/2}(v-s)^{-1/2}\mathrm{d}s = \int_0^1 s^{-1/2}(1-s)^{-1/2}\mathrm{d}s = B(1/2,1/2) = \pi,$$

where $B(\cdot,\cdot)$ is the Beta function. Hence (2.5) is true. □

Theorem 2.4. *The transformation $\mathcal{A}_1$ is one-to-one.*

Proof. Suppose that $\nu, \nu' \in \mathfrak{M}_L^1(\mathbb{R}^d)$ and $\mathcal{A}_1(\nu) = \mathcal{A}_1(\nu')$. Let (λ, ν_ξ) and (λ', ν'_ξ) be polar decompositions (radial decompositions) of ν and ν', respectively, as in Proposition 3.1 of [13]. That is, λ is a measure on the unit sphere $\mathbb{S} = \{\xi \in \mathbb{R}^d\colon\ |\xi| = 1\}$ with $0 \le \lambda(\mathbb{S}) \le \infty$ and ν_ξ, $\xi \in \mathbb{S}$, are measures on $(0,\infty)$ such that $\nu_\xi(E)$ is measurable in ξ for each $E \in \mathcal{B}((0,\infty))$, $0 < \nu_\xi((0,\infty)) \le \infty$, and

$$\nu(B) = \int_{\mathbb{S}} \lambda(\mathrm{d}\xi) \int_{(0,\infty)} 1_B(u\xi)\nu_\xi(\mathrm{d}u), \quad B \in \mathcal{B}(\mathbb{R}^d);$$

λ' and ν'_ξ have similar properties with respect to ν'. It follows from Definition 2.1 that

$$\begin{aligned}
\mathcal{A}_1(\nu)(B) &= \int_{\mathbb{R}^d\setminus\{0\}} \nu(\mathrm{d}x) \int_0^{|x|^{1/2}} 2\pi^{-1}(|x|-r^2)^{-1/2} 1_B\left(r\frac{x}{|x|}\right)\mathrm{d}r \\
&= \int_{\mathbb{S}} \lambda(\mathrm{d}\xi) \int_{(0,\infty)} \nu_\xi(\mathrm{d}u) \int_0^{u^{1/2}} 2\pi^{-1}(u-r^2)^{-1/2}1_B(r\xi)\mathrm{d}r \\
&= \int_{\mathbb{S}} \lambda(\mathrm{d}\xi) \int_0^\infty 1_B(r\xi)\mathrm{d}r \int_{(r^2,\infty)} 2\pi^{-1}(u-r^2)^{-1/2}\nu_\xi(\mathrm{d}u)
\end{aligned}$$

and

$$\mathcal{A}_1(\nu')(B) = \int_{\mathbb{S}} \lambda'(\mathrm{d}\xi) \int_0^\infty 1_B(r\xi)\mathrm{d}r \int_{(r^2,\infty)} 2\pi^{-1}(u-r^2)^{-1/2}\nu'_\xi(\mathrm{d}u).$$

These give polar decompositions of $\mathcal{A}_1(\nu) = \mathcal{A}_1(\nu')$. Hence, by Proposition 3.1 of [13], there is a measurable function $c(\xi)$ satisfying $0 < c(\xi) < \infty$ such that $\lambda'(\mathrm{d}\xi) = c(\xi)\lambda(\mathrm{d}\xi)$ and, for λ-a. e. ξ,

$$\left(\int_{(r^2,\infty)} (u-r^2)^{-1/2}\nu'_\xi(\mathrm{d}u)\right)\mathrm{d}r = \left(c(\xi)^{-1}\int_{(r^2,\infty)} (u-r^2)^{-1/2}\nu_\xi(\mathrm{d}u)\right)\mathrm{d}r.$$

Using a new variable $v = r^2$, we see that

$$\left(\int_{(v,\infty)} (u-v)^{-1/2}\nu'_\xi(\mathrm{d}u)\right)\mathrm{d}v = \left(c(\xi)^{-1}\int_{(v,\infty)} (u-v)^{-1/2}\nu_\xi(\mathrm{d}u)\right)\mathrm{d}v.$$

Since ν_ξ and ν'_ξ satisfy (2.4) for λ-a. e. ξ, we obtain $\nu_\xi = c(\xi)^{-1}\nu'_\xi$ for λ-a. e. ξ from Lemma 2.3. It follows that $\nu = \nu'$. $\square$

2.3. **Range.** Let us show some necessary conditions for $\widetilde{\nu}$ to belong to the range of $\mathcal{A}_1$.

Proposition 2.5. *If $\widetilde{\nu}$ is in $\mathfrak{R}(\mathcal{A}_1)$ and not zero measure, then $\widetilde{\nu}$ has a radial decomposition $(\lambda, \ell_\xi(r)\mathrm{d}r)$ having the following properties:*
(1) *$\ell_\xi(r)$ is measurable in (ξ, r) and lower semi-continuous in $r \in (0,\infty)$;*
(2) *there is $b_\xi \in (0,\infty]$ such that $\ell_\xi(r) > 0$ for $r < b_\xi$ and, if $b_\xi < \infty$, then $\ell_\xi(r) = 0$ for $r \geq b_\xi$;*
(3) $\liminf_{r\downarrow 0} \ell_\xi(r) > 0$.

Proof. Let $\widetilde{\nu} = \mathcal{A}_1(\nu)$ with $\nu \in \mathfrak{M}^1_L(\mathbb{R}^d)$ and (λ, ν_ξ) a polar decomposition of ν. Then, the proof of Theorem 2.4 shows that $\widetilde{\nu} \in \mathfrak{M}_L(\mathbb{R}^d)$ with radial decomposition $(\lambda, \ell(r)\mathrm{d}r)$ where

$$\ell(r) = 2\pi^{-1} \int_{(r^2,\infty)} (u - r^2)^{-1/2} \nu_\xi(\mathrm{d}u).$$

Then our assertion is proved in the same way as Proposition 2.13 of [13]. $\square$

2.4. **How big is $\mathfrak{R}(\mathcal{A}_1)$?** Several well-known and well studied classes of multivariate infinitely divisible distributions are the following. The Jurek class, the class of selfdecomposable distributions, the Goldie-Steutel-Bondesson class, the Thorin class and the class of generalized type G distributions. They are characterized only by the radial component of their Lévy measures and Σ and γ in the Lévy-Khintchine triplet are irrelevant. Among them, the Jurek class is the biggest. Recently, some classes bigger than the Jurek class have been discussed in the study of extension of selfdecomposability (see e.g. [6] and [13]). Then a natural question is how big $\mathfrak{R}(\mathcal{A}_1)$ is. Let $\mathfrak{M}^U_L(\mathbb{R}^d)$ be the class of Lévy measures of distributions in the Jurek class. The radial component ν_ξ of $\nu \in \mathfrak{M}^U_L(\mathbb{R}^d)$ is characterized as $\nu_\xi(\mathrm{d}r) = \ell_\xi(r)\mathrm{d}r$ with $\ell_\xi(r)$ being measurable in (ξ, r) and decreasing and right-continuous in $r > 0$. We will show below that $\mathfrak{R}(\mathcal{A}_1)$ is strictly bigger than $\mathfrak{M}^U_L(\mathbb{R}^d)$.

Theorem 2.6. $\mathfrak{M}^U_L(\mathbb{R}^d) \subsetneqq \mathfrak{R}(\mathcal{A}_1)$.

Lemma 2.7. *Let ρ be a σ-finite measure on $(0,\infty)$. Then, for $\alpha > -1$ and $b > 0$, the measure $\sigma_{(\rho)}$ in (2.3) satisfies*

$$\int_{(b,\infty)} u^{\alpha}\sigma_{(\rho)}(\mathrm{d}u) \le C_1 \int_{(b,\infty)} s^{\alpha+1/2}\rho(\mathrm{d}s) \tag{2.6}$$

and

$$\int_{(0,b]} u^{\alpha}\sigma_{(\rho)}(\mathrm{d}u) \le C_2 \left(\int_{(0,b]} s^{\alpha+1/2}\rho(\mathrm{d}s) + \int_{(b,\infty)} s^{-1/2}\rho(\mathrm{d}s)\right), \tag{2.7}$$

where C_1 and C_2 are constants independent of ρ.

Proof. Let $c = \pi^{-1/2}$. We have

$$\int_{(b,\infty)} u^{\alpha}\sigma_{(\rho)}(\mathrm{d}u) = c\int_{(b,\infty)} \rho(\mathrm{d}s)\int_b^s u^{\alpha}(s-u)^{-1/2}\mathrm{d}u$$

by Fubini's theorem, and

$$\int_b^s u^{\alpha}(s-u)^{-1/2}\mathrm{d}u = s^{\alpha+1/2}\int_{b/s}^1 v^{\alpha}(1-v)^{-1/2}\mathrm{d}v \sim s^{\alpha+1/2}B(\alpha+1,1/2), \quad s\to\infty.$$

Hence (2.6) holds. We have

$$\int_{(0,b]} u^{\alpha}\sigma_{(\rho)}(\mathrm{d}u) = c\int_{(0,b]} \rho(\mathrm{d}s)\int_0^s u^{\alpha}(s-u)^{-1/2}\mathrm{d}u + c\int_{(b,\infty)} \rho(\mathrm{d}s)\int_0^b u^{\alpha}(s-u)^{-1/2}\mathrm{d}u.$$

Notice that

$$\int_0^s u^{\alpha}(s-u)^{-1/2}\mathrm{d}u = s^{\alpha+1/2}B(\alpha+1,1/2)$$

and

$$\begin{aligned}\int_0^b u^{\alpha}(s-u)^{-1/2}\mathrm{d}u &= s^{-1/2}\int_0^b u^{\alpha}(1-u/s)^{-1/2}\mathrm{d}u \\ &\le s^{-1/2}\int_0^b u^{\alpha}(1-u/b)^{-1/2}\mathrm{d}u = s^{-1/2}b^{\alpha+1}B(\alpha+1,1/2), \quad s>b.\end{aligned}$$

Thus (2.7) holds. □

Proof of Theorem 2.6. Let $\widetilde{\nu} \in \mathfrak{M}_L^U(\mathbb{R}^d)$ and $c = \int_{\mathbb{R}^d}(1\wedge|x|^2)\widetilde{\nu}(\mathrm{d}x)$. Then $\widetilde{\nu}$ has a polar decomposition $(\lambda, \ell_\xi(r)\mathrm{d}r)$ mentioned above. Further, we may and do assume that λ is a probability measure and $\int_0^\infty(1\wedge r^2)\ell_\xi(r)\mathrm{d}r = c$ for all ξ (see the proof of Proposition 3.1 of [13], letting $f(x) = 1\wedge|x|^2$). Let ρ_ξ be a measure on $(0,\infty)$ such that $\rho_\xi((r^2,\infty)) = \ell_\xi(r)$ for $r>0$ and let $\eta_\xi = \sigma_{(\rho_\xi)}$. The proof of Lemma 2.3 shows that

$$\rho_\xi((u,\infty)) = \int_{(u,\infty)} \pi^{-1/2}(s-u)^{-1/2}\eta_\xi(\mathrm{d}s).$$

Hence we have, for $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\widetilde{\nu}(B) = \int_{\mathbb{S}} \lambda(\mathrm{d}\xi) \int_0^\infty 1_B(r\xi)\mathrm{d}r \int_{(r^2,\infty)} \pi^{-1/2}(s - r^2)^{-1/2}\eta_\xi(\mathrm{d}s).$$

We claim that

$$\int_{\mathbb{S}} \lambda(\mathrm{d}\xi) \int_0^\infty (1 \wedge u)\eta_\xi(\mathrm{d}u) < \infty. \tag{2.8}$$

This will ensure that $\widetilde{\nu} = \mathcal{A}_1(\nu)$ for $\nu \in \mathfrak{D}(\mathcal{A}_1)$ that has polar decomposition $(\lambda, 2^{-1}\pi^{1/2}\eta_\xi(\mathrm{d}r))$. First, notice that

$$\begin{aligned} c &= \frac{1}{2}\int_0^1 u^{1/2}\rho_\xi((u,\infty))\mathrm{d}u + \frac{1}{2}\int_1^\infty u^{-1/2}\rho_\xi((u,\infty))\mathrm{d}u \\ &\geq \frac{1}{3}\rho_\xi((1,\infty)) + \frac{1}{2}\int_1^\infty u^{-1/2}\rho_\xi((u,\infty))\mathrm{d}u. \end{aligned}$$

Then, use (2.6) with $\alpha = 0$ to obtain

$$\begin{aligned} \int_{(1,\infty)} \eta_\xi(\mathrm{d}u) &\leq C_1 \int_{(1,\infty)} s^{1/2}\rho_\xi(\mathrm{d}s) \\ &= C_1\rho_\xi((1,\infty)) + \frac{C_1}{2}\int_1^\infty s^{-1/2}\rho_\xi((s,\infty))\mathrm{d}s \leq 3cC_1. \end{aligned}$$

Similarly, using (2.7) with $\alpha = 1$,

$$\begin{aligned} \int_{(0,1]} u\,\eta_\xi(\mathrm{d}u) &\leq C_2\left(\int_{(0,1]} s^{3/2}\rho_\xi(\mathrm{d}s) + \int_{(1,\infty)} s^{-1/2}\rho_\xi(\mathrm{d}s)\right) \\ &\leq C_2\left(\frac{3}{2}\int_0^1 s^{1/2}\rho_\xi((s,1])\mathrm{d}s + \int_{(1,\infty)} s^{1/2}\rho_\xi(\mathrm{d}s)\right) \leq 5cC_2. \end{aligned}$$

Hence (2.8) is true. It follows that $\mathfrak{M}_L^U(\mathbb{R}^d) \subset \mathfrak{R}(\mathcal{A}_1)$.

To see the inclusion is strict, consider $\eta \in \mathfrak{R}(\mathcal{A}_1)$ defined by

$$\eta(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_0^1 1_B(r\xi)2\pi^{-1}(1 - r^2)^{-1/2}\mathrm{d}r.$$

Then $\eta \notin \mathfrak{M}_L^U(\mathbb{R}^d)$, since $(1 - r^2)^{-1/2}$ is strictly increasing on $(0, 1)$. □

3. Distributions of class A

3.1. Definition and representation by another arcsine transformation $\mathcal{A}_2$.

Definition 3.1. A probability distribution in $I(\mathbb{R}^d)$ is said to be a *distribution of class A* on $\mathbb{R}^d$ if its Lévy measure ν belongs to $\mathfrak{R}(\mathcal{A}_1)$. There is no restriction on

Σ and γ in its Lévy-Khintchine triplet. We denote by $A(\mathbb{R}^d)$ the totality of such distributions on $\mathbb{R}^d$.

Let, for $s > 0$,

$$a_2(r; s) = a_1(r; s^2) = 2\pi^{-1}(s^2 - r^2)^{-1/2} 1_{(0,s)}(r).$$

Definition 3.2. Let ν be a measure on $\mathbb{R}^d$. Define the *arcsine transformation* $\mathcal{A}_2$ of ν by

$$\mathcal{A}_2(\nu)(B) = \int_{\mathbb{R}^d \setminus \{0\}} \nu(\mathrm{d}x) \int_0^\infty a_2(r; |x|) 1_B\left(r \frac{x}{|x|}\right) \mathrm{d}r, \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{3.1}$$

The domain $\mathfrak{D}(\mathcal{A}_2)$ is defined to be the class of ν such that $\nu(\{0\}) = 0$ and the right-hand side of (3.1) belongs to $\mathfrak{M}_L(\mathbb{R}^d)$.

For any measure ν on $\mathbb{R}^d$ with $\nu(\{0\}) = 0$, define a measure $\nu^{(2)}$ on $\mathbb{R}^d$ by

$$\nu^{(2)}(B) = \int_{\mathbb{R}^d \setminus \{0\}} 1_B\left(|x|^2 \frac{x}{|x|}\right) \nu(\mathrm{d}x), \quad B \in \mathcal{B}(\mathbb{R}^d).$$

The transformation $\nu \mapsto \nu^{(2)}$ is one-to-one, since we have $\nu = (\nu^{(2)})^{(1/2)}$, defining $\rho^{(1/2)}$ for ρ with $\rho(\{0\}) = 0$ as

$$\rho^{(1/2)}(B) = \int_{\mathbb{R}^d \setminus \{0\}} 1_B\left(|x|^{1/2} \frac{x}{|x|}\right) \rho(\mathrm{d}x).$$

The following propositions give the connections between $\mathcal{A}_1$ and $\mathcal{A}_2$.

Proposition 3.3. $\mathfrak{D}(\mathcal{A}_2) = \mathfrak{M}_L(\mathbb{R}^d)$.

Proposition 3.4. $\nu \in \mathfrak{M}_L(\mathbb{R}^d)$ *if and only if* $\nu^{(2)} \in \mathfrak{M}_L^1(\mathbb{R}^d)$, *and in this case*

$$\mathcal{A}_2(\nu) = \mathcal{A}_1(\nu^{(2)}).$$

Proposition 3.5. $\mathfrak{R}(\mathcal{A}_1) = \mathfrak{R}(\mathcal{A}_2)$.

Proof of Propositions 3.3–3.5. Since $\int_{\mathbb{R}^d} f(x)\nu^{(2)}(\mathrm{d}x) = \int_{\mathbb{R}^d} f(|x|x)\nu(\mathrm{d}x)$ for any nonnegative measurable function f, we have $\int_{\mathbb{R}^d}(1 \wedge |x|)\nu^{(2)}(\mathrm{d}x) = \int_{\mathbb{R}^d}(1 \wedge |x|^2)\nu(\mathrm{d}x)$ and the first two propositions follow. Proposition 3.5 is a direct consequence of them. □

We will also use the following fact.

Proposition 3.6. $\mathcal{A}_2$ *is a one-to-one transformation.*

Proof. This follows from Theorem 2.4 and Proposition 3.4. □

The usefulness of introduction of $\mathcal{A}_2$ is based on the next property.

3.2. **Transformation $\mathcal{A}_2$ as an Upsilon transformation.** Barndorff-Nielsen, Rosiński and Thorbjørnsen [3] considered general Upsilon transformations (see also [2] and [12]). Given a measure τ on $(0,\infty)$, a transformation Υ_τ from measures on $\mathbb{R}^d$ into $\mathfrak{M}_L(\mathbb{R}^d)$ is called an Upsilon transformation associated to τ (or with dilation measure τ) when

$$\Upsilon_\tau(\nu)(B) = \int_0^\infty \nu(u^{-1}B)\tau(\mathrm{d}u), \qquad B \in \mathcal{B}(\mathbb{R}^d). \tag{3.2}$$

The domain of Υ_τ is the class of σ-finite measures ν such that the right-hand side of (3.2) is a measure in $\mathfrak{M}_L(\mathbb{R}^d)$.

Theorem 3.7. *$\mathcal{A}_2$ is an Upsilon transformation given by*

$$\mathcal{A}_2(\nu)(B) = \int_0^1 \nu(u^{-1}B)2\pi^{-1}(1-u^2)^{-1/2}\mathrm{d}u, \qquad B \in \mathcal{B}(\mathbb{R}^d) \tag{3.3}$$

for $\nu \in \mathfrak{M}_L(\mathbb{R}^d)$.

Proof. Let (λ, ν_ξ) be a polar decomposition of $\nu \in \mathfrak{M}_L(\mathbb{R}^d)$. Then, with $c = 2\pi^{-1}$,

$$\begin{aligned}
\mathcal{A}_2(\nu)(B) &= c\int_{\mathbb{S}} \lambda(\mathrm{d}\xi) \int_0^\infty 1_B(r\xi)\mathrm{d}r \int_{(r,\infty)} (s^2-r^2)^{-1/2}\nu_\xi(\mathrm{d}s)\\
&= c\int_{\mathbb{S}} \lambda(\mathrm{d}\xi) \int_0^\infty \nu_\xi(\mathrm{d}s) \int_0^s 1_B(r\xi)(s^2-r^2)^{-1/2}\mathrm{d}r\\
&= c\int_{\mathbb{S}} \lambda(\mathrm{d}\xi) \int_0^\infty \nu_\xi(\mathrm{d}s) \int_0^1 1_B(us\xi)(1-u^2)^{-1/2}\mathrm{d}u\\
&= c\int_0^1 (1-u^2)^{-1/2}\mathrm{d}u \int_{\mathbb{S}} \lambda(\mathrm{d}\xi) \int_0^\infty 1_B(us\xi)\nu_\xi(\mathrm{d}s)\\
&= c\int_0^1 (1-u^2)^{-1/2}\mathrm{d}u \int_{\mathbb{R}^d} 1_B(ux)\nu(\mathrm{d}x),
\end{aligned}$$

which shows (3.3).

Conversely, let ν be a σ-finite measure on $\mathbb{R}^d$, define $\widetilde{\nu}(B)$ by the right-hand side of (3.3), and suppose that $\widetilde{\nu} \in \mathfrak{M}_L(\mathbb{R}^d)$. Then $0 = \widetilde{\nu}(\{0\}) = \nu(\{0\})$ and the calculation above shows that $\nu \in \mathfrak{D}(\mathcal{A}_2)$ and $\mathcal{A}_2(\nu) = \widetilde{\nu}$. Hence $\mathcal{A}_2 = \Upsilon_\tau$ of the form (3.3). $\square$

The mapping $\mathcal{A}_1$ is not an Upsilon transformation for any dilation measure τ. This remarkable result will be proved in Section 3.8, as a byproduct of Theorem 3.13 shown in Section 3.7.

3.3. **Stochastic integral representation of $A(\mathbb{R}^d)$.** We study a probabilistic interpretation of distributions class A, representing them by stochastic integrals with respect to Lévy processes.

Let $T \in (0,\infty)$ and let $f(t)$ be a square integrable function on $[0,T]$. Then the stochastic integral $\int_0^T f(t)\mathrm{d}X_t^{(\mu)}$ is defined for any $\mu \in I(\mathbb{R}^d)$ and is infinitely divisible. Define the stochastic integral mapping Φ_f based on f as $\Phi_f(\mu) = \mathcal{L}\left(\int_0^T f(t)\mathrm{d}X_t^{(\mu)}\right)$ for $\mu \in I(\mathbb{R}^d)$. If $\mu \in I(\mathbb{R}^d)$ has the Lévy-Khintchine triplet (Σ,ν,γ), then, as in [10, 11, 13], $\widetilde{\mu} = \Phi_f(\mu)$ has the triplet $(\widetilde{\Sigma},\widetilde{\nu},\widetilde{\gamma})$ expressed as

$$\widetilde{\Sigma} = \int_0^T f(t)^2\, \Sigma\, \mathrm{d}t, \tag{3.4}$$

$$\widetilde{\nu}(B) = \int_0^T \mathrm{d}t \int_{\mathbb{R}^d} 1_B(f(t)x)\,\nu(\mathrm{d}x), \qquad B \in \mathcal{B}(\mathbb{R}^d), \tag{3.5}$$

$$\widetilde{\gamma} = \int_0^T f(t)\mathrm{d}t\left(\gamma + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(t)x|^2} - \frac{1}{1+|x|^2}\right)\nu(\mathrm{d}x)\right). \tag{3.6}$$

Let us characterize the class $A(\mathbb{R}^d)$ as the range of a stochastic integral mapping.

Theorem 3.8. *Define $\Phi_{\cos}$ by (1.3). Then $\Phi_{\cos}$ is a one-to-one mapping and*

$$A(\mathbb{R}^d) = \Phi_{\cos}(I(\mathbb{R}^d)). \tag{3.7}$$

Proof. Let $\widetilde{\mu} \in A(\mathbb{R}^d)$ with triplet $(\widetilde{\Sigma},\widetilde{\nu},\widetilde{\gamma})$. Then $\widetilde{\nu} \in \mathfrak{R}(\mathcal{A}_2)$ by Proposition 3.5 and hence we have (3.3) with some $\nu \in \mathfrak{M}_L(\mathbb{R}^d)$ by Theorem 3.7. Let $t = g(u) = \int_u^1 2\pi^{-1}(1-v^2)^{-1/2}\mathrm{d}v = 2\pi^{-1}\arccos(u)$ for $0 < u < 1$. Then $u = \cos(2^{-1}\pi t)$ for $0 < t < 1$. Thus

$$\widetilde{\nu}(B) = -\int_0^1 \mathrm{d}g(u) \int_{\mathbb{R}^d} 1_B(ux)\nu(\mathrm{d}x) = \int_0^1 \mathrm{d}t \int_{\mathbb{R}^d} 1_B(x\cos(2^{-1}\pi t))\nu(\mathrm{d}x).$$

That is, (3.5) is satisfied with $T = 1$ and $f(t) = \cos(2^{-1}\pi t)$. Using ν, $\widetilde{\Sigma}$, and $\widetilde{\gamma}$, we can find Σ and γ satisfying (3.4) and (3.6). Let μ be the distribution in $I(\mathbb{R}^d)$ with triplet (Σ,ν,γ). Then $\widetilde{\mu} = \Phi_{\cos}(\mu)$. Hence $A(\mathbb{R}^d) \subset \Phi_{\cos}(I(\mathbb{R}^d))$.

Conversely, suppose that $\widetilde{\mu} \in \Phi_{\cos}(I(\mathbb{R}^d))$. Then $\widetilde{\mu} = \Phi_{\cos}(\mu)$ for some $\mu \in I(\mathbb{R}^d)$ by a similar argument, showing that $\Phi_{\cos}(I(\mathbb{R}^d)) \subset A(\mathbb{R}^d)$.

The mapping $\Phi_{\cos}$ is one-to-one, since ν is determined by $\widetilde{\nu}$ (Proposition 3.6) and Σ and γ are determined by $\widetilde{\Sigma}$, $\widetilde{\gamma}$, and ν. □

3.4. **Υ^0-transformation.** We use Υ and Υ^0 defined by

$$\Upsilon(\mu) = \mathcal{L}\left(\int_0^1 (-\log t)\mathrm{d}X_t^{(\mu)}\right), \quad \mu \in I(\mathbb{R}^d),$$

$$\Upsilon^0(\nu)(B) = \int_0^\infty \nu(u^{-1}B)\mathrm{e}^{-u}\mathrm{d}u, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Let $\mathfrak{M}_L^B(\mathbb{R}^d)$ be the class of Lévy measures of the Goldie-Steutel-Bondesson class $B(\mathbb{R}^d)$. In [1], it is shown that $\Upsilon(I(\mathbb{R}^d)) = B(\mathbb{R}^d)$ and $\Upsilon^0(\mathfrak{M}_L(\mathbb{R}^d)) = \mathfrak{M}_L^B(\mathbb{R}^d)$. Both Υ^0 and Υ are one-to-one.

Proposition 3.9. *Let $\nu \in \mathfrak{M}_L(\mathbb{R}^d)$. Then $\Upsilon^0(\nu) \in \mathfrak{M}_L^1(\mathbb{R}^d)$ if and only if $\nu \in \mathfrak{M}_L^1(\mathbb{R}^d)$.*

Proof. Notice that

$$\begin{aligned}\int_{|x|\le 1} |x|\Upsilon^0(\nu)(\mathrm{d}x) &= \int_{\mathbb{R}^d} |x|\nu(\mathrm{d}x)\int_0^{1/|x|} u\mathrm{e}^{-u}\mathrm{d}u,\\ \int_{|x|>1} \Upsilon^0(\nu)(\mathrm{d}x) &= \int_{\mathbb{R}^d} \mathrm{e}^{-1/|x|}\nu(\mathrm{d}x),\end{aligned}$$

to see the equivalence. □

3.5. **A representation of completely monotone functions.** In [8], the class of generalized type G distributions on $\mathbb{R}^d$, denoted by $G(\mathbb{R}^d)$, is defined as follows: $\mu \in G(\mathbb{R}^d)$ if and only if the radial component ν_ξ of the Lévy measure of μ satisfies $\nu_\xi(\mathrm{d}r) = g_\xi(r^2)\mathrm{d}r$, where $g_\xi(u)$ is measurable in (ξ, u) and completely monotone in $u > 0$. $\mathfrak{M}_L^G(\mathbb{R}^d)$ denotes the class of all Lévy measures of $\mu \in G(\mathbb{R}^d)$. We use the following result when dealing with $G(\mathbb{R}^d)$. It is a result on the arcsine transformation representation of a function $g(r^2)$ when g is completely monotone on $(0,\infty)$.

Proposition 3.10. *Let $g(u)$ be a function on $(0,\infty)$. Then the following three conditions are equivalent.*

(a) The function $g(u)$ is completely monotone on $(0,\infty)$ and satisfies

$$\int_0^\infty (1\wedge r^2)g(r^2)\mathrm{d}r < \infty. \tag{3.8}$$

(b) There exists a completely monotone function $h(s)$ on $(0,\infty)$ satisfying

$$\int_0^\infty (1\wedge s)h(s)\mathrm{d}s < \infty \tag{3.9}$$

such that

$$g(r^2) = \int_0^\infty a_1(r;s)h(s)\mathrm{d}s, \quad r > 0. \tag{3.10}$$

(c) There exists a measure ρ on $(0,\infty)$ satisfying

$$\int_0^\infty (1\wedge s)\rho(\mathrm{d}s) < \infty$$

such that

$$g(r^2) = \int_{(0,\infty)} a_1(r;s)\Upsilon^0(\rho))(r)(\mathrm{d}s), \quad r > 0. \tag{3.11}$$

Proof. (a) $\Rightarrow$ (b): From Bernstein's theorem, there exists a measure Q on $[0,\infty)$ such that

$$g(u) = \int_{[0,\infty)} \mathrm{e}^{-uv} Q(\mathrm{d}v), \quad u > 0. \tag{3.12}$$

It follows from (3.8) that $Q(\{0\}) = 0$, since $Q(\{0\}) = \lim_{u\to\infty} g(u)$. We need the fact that the one-dimensional Gaussian density $\varphi(x;t)$ of mean 0 and variance t is the arcsine transform of the exponential distribution with mean $t > 0$. More precisely,

$$\varphi(x;t) = (2\pi t)^{-1/2}\mathrm{e}^{-x^2/(2t)} = t^{-1}\int_0^\infty \mathrm{e}^{-s/t} a(x;2s)\mathrm{d}s,\ t > 0,\ x \in \mathbb{R}. \tag{3.13}$$

This is the well-known Box-Muller method to generate normal random variables. Its proof can be given by change of variables $s = tu + x^2/2$. Using (3.13), we have

$$\begin{aligned} g(r^2) &= \int_{(0,\infty)} \mathrm{e}^{-r^2 v} Q(\mathrm{d}v) = \int_{(0,\infty)} v^{1/2} Q(\mathrm{d}v) \int_{r^2/2}^\infty \mathrm{e}^{-2sv} 2\pi^{-1/2}(2s - r^2)^{-1/2}\mathrm{d}s \\ &= \int_{r^2}^\infty \pi^{-1/2}(s - r^2)^{-1/2}\mathrm{d}s \int_{(0,\infty)} \mathrm{e}^{-sv} v^{1/2} Q(\mathrm{d}v) = \int_0^\infty a_1(r;s)h(s)\mathrm{d}s, \end{aligned}$$

where

$$h(s) = 2^{-1}\pi^{1/2} \int_{(0,\infty)} \mathrm{e}^{-sv} v^{1/2} Q(\mathrm{d}v). \tag{3.14}$$

Applying Proposition 2.2 for $d = 1$, we see (3.9) from (3.8).

(b) $\Rightarrow$ (c): There is $\rho \in \mathfrak{M}_L(\mathbb{R})$ such that $h(s)\mathrm{d}s = \Upsilon^0(\rho)$ (see Theorem A of [1]). Since $\Upsilon^0(\rho)$ is concentrated on $(0,\infty)$, ρ is concentrated on $(0,\infty)$. Using Proposition 3.9, we see that $\int_{(0,1]} s\,\rho(\mathrm{d}s) < \infty$.

(c) $\Rightarrow$ (a): It follows from Proposition 3.9 that $\int_{(0,\infty)}(1\wedge s)\Upsilon^0(\rho)(\mathrm{d}s) < \infty$. Hence it follows from (3.11) that $g(r^2)$ satisfies (3.8) (use Proposition 2.2 for $d = 1$). Finally let us prove that $g(u)$ is completely monotone. There is a completely monotone function $h(s)$ such that $\Upsilon^0(\rho)(\mathrm{d}s) = h(s)\mathrm{d}s$ (see Theorem A of [1] again). Hence we can find a measure R on $[0,\infty)$ such that $h(s) = \int_{[0,\infty)} \mathrm{e}^{-sv} R(\mathrm{d}v)$, $s > 0$. We have $R(\{0\}) = 0$. Thus

$$g(r^2) = \int_{r^2}^\infty 2\pi^{-1}(s - r^2)^{-1/2}\mathrm{d}s \int_{(0,\infty)} \mathrm{e}^{-sv} R(\mathrm{d}v) = \int_{(0,\infty)} \mathrm{e}^{-r^2 v} 2\pi^{-1/2} v^{-1/2} R(\mathrm{d}v),$$

where the last equality is from the same calculus as in the proof that (a) $\Rightarrow$ (b). Now we see that $g(u)$ is completely monotone. $\square$

3.6. **A representation of $G(\mathbb{R}^d)$ in terms of $\mathcal{A}_1$.** We now give an alternative representation for Lévy measures of distributions in $G(\mathbb{R}^d)$.

Theorem 3.11. *Let $\widetilde{\mu}$ be an infinitely divisible distribution on $\mathbb{R}^d$ with the Lévy-Khintchine triplet $(\widetilde{\Sigma}, \widetilde{\nu}, \widetilde{\gamma})$. Then the following three conditions are equivalent.*
(a) $\widetilde{\mu} \in G(\mathbb{R}^d)$.
(b) $\widetilde{\nu} = \mathcal{A}_1(\nu)$ *with some* $\nu \in \mathfrak{M}_L^B(\mathbb{R}^d) \cap \mathfrak{M}_L^1(\mathbb{R}^d)$.
(c) $\widetilde{\nu} = \mathcal{A}_1(\Upsilon^0(\rho))$ *with some* $\rho \in \mathfrak{M}_L^1(\mathbb{R}^d)$.
In condition (b) or (c), the representation of $\widetilde{\nu}$ by ν or ρ is unique.

Proof. (a) $\Rightarrow$ (b): By definition of $G(\mathbb{R}^d)$ and Proposition 3.10, the Lévy measure $\widetilde{\nu}$ of $\widetilde{\mu}$ has polar decomposition $(\lambda, g_\xi(r^2)\mathrm{d}r)$ where $g_\xi(u)$ is measurable in (ξ, u) and satisfies (3.9) and (3.10) with $g_\xi(r^2)$ and $h_\xi(s)$ in place of $g(r^2)$ and $h(s)$. The measure Q_ξ in the representation (3.12) of $g_\xi(u)$ has the property that $Q_\xi(E)$ is measurable in ξ for every Borel set E in $[0, \infty)$ (see Remark 3.2 of [1]). Hence the function $h_\xi(s)$ given as in (3.14) is measurable in (ξ, s). Thus we have $\widetilde{\nu} = \mathcal{A}_1(\nu)$, letting ν denote the Lévy measure with polar decomposition $(\lambda,\ h_\xi(s)\mathrm{d}s)$. Notice that $\nu \in \mathfrak{M}_L^B(\mathbb{R}^d)$ and show that $\nu \in \mathfrak{M}_L^1(\mathbb{R}^d)$ by an argument similar to the proof of Proposition 2.2.

(b) $\Rightarrow$ (c): Use Proposition 3.9 and the representation of $\mathfrak{M}_L^B(\mathbb{R}^d)$ by Υ^0.

(c) $\Rightarrow$ (a): Use Proposition 3.9 again together with Proposition 3.10.

Uniqueness of the representations comes from Theorem 2.4 and that of Υ^0. $\square$

3.7. **$G(\mathbb{R}^d)$ as image of $A(\mathbb{R}^d)$ under a stochastic integral mapping.** Following [7], we define the transformation $\Upsilon_{\alpha,\beta}(\nu)$ for $-\infty < \alpha < 2$ and $0 < \beta \le 2$. For a measure ν on $\mathbb{R}^d$ with $\nu(\{0\}) = 0$ define

$$\Upsilon_{\alpha,\beta}(\nu)(B) = \int_0^\infty \nu(s^{-1}B)\beta s^{-\alpha-1}\mathrm{e}^{-s^\beta}\mathrm{d}s, \quad B \in \mathcal{B}(\mathbb{R}^d),$$

whenever the right-hand side gives a measure in $\mathfrak{M}_L(\mathbb{R}^d)$. This definition is different from that of [7] in the constant factor β. A special case with $\beta = 1$ coincides with the transformation of Lévy measures in the stochastic integral mapping Ψ_α studied by Sato [11]. Of particular interest in this work is the mapping $\Upsilon_{-2,2}$. Notice that $\Upsilon_{-1,1} = \Upsilon^0$.

Proposition 3.12. $\mathfrak{D}(\Upsilon_{-2,2}) = \mathfrak{M}_L(\mathbb{R}^d)$. *The mapping $\Upsilon_{-2,2}$ is one-to-one.*

Proof. Let $\widetilde{\nu}(B) = \int_0^\infty \nu(s^{-1}B)2s\mathrm{e}^{-s^2}\mathrm{d}s$. Then

$$\int_{\mathbb{R}^d}(1 \wedge |x|^2)\widetilde{\nu}(\mathrm{d}x) = \int_0^\infty 2s\mathrm{e}^{-s^2}\mathrm{d}s \int_{\mathbb{R}^d}(1 \wedge |sx|^2)\nu(\mathrm{d}x)$$

$$= \int_{\mathbb{R}^d} |x|^2 \nu(\mathrm{d}x) \int_0^{1/|x|} 2s^3 \mathrm{e}^{-s^2} \mathrm{d}s + \int_{\mathbb{R}^d} \nu(\mathrm{d}x) \int_{1/|x|}^{\infty} 2s\mathrm{e}^{-s^2} \mathrm{d}s.$$

Observe that $\int_0^{1/|x|} 2s^3 \mathrm{e}^{-s^2} \mathrm{d}s$ is convergent as $|x| \downarrow 0$ and $\sim 2^{-1}|x|^{-4}$ as $|x| \to \infty$ and $\int_{1/|x|}^{\infty} 2s\mathrm{e}^{-s^2} \mathrm{d}s$ is $\sim \mathrm{e}^{-1/|x|^2}$ as $|x| \downarrow 0$ and convergent as $|x| \to \infty$. Then we see that $\int_{\mathbb{R}^d} (1 \wedge |x|^2) \widetilde{\nu}(\mathrm{d}x)$ is finite if and only if $\int_{\mathbb{R}^d} (1 \wedge |x|^2) \nu(\mathrm{d}x)$ is finite. To prove that $\Upsilon_{-2,2}$ is one-to-one, make a similar argument to the proof of Proposition 4.1 of [11]. $\square$

The following result is needed in giving the characterization of $G(\mathbb{R}^d)$ in terms of distributions of class A. It shows that $\mathcal{A}_1$ and Υ^0 are not commutative. However, $\mathcal{A}_2$ and Υ^0 are commutative, both being Upsilon transformations with domain equal to $\mathfrak{M}_L(\mathbb{R}^d)$.

Theorem 3.13. *It holds that*

$$\Upsilon_{-2,2}(\mathcal{A}_1(\rho)) = \mathcal{A}_1(\Upsilon^0(\rho)) \quad \text{for } \rho \in \mathfrak{M}_L^1(\mathbb{R}^d).$$

Proof. Let $\rho \in \mathfrak{M}_L^1(\mathbb{R}^d)$, $\nu = \mathcal{A}_1(\rho)$, and $\widetilde{\nu} = \Upsilon_{-2,2}(\nu)$ with polar decompositions (λ, ρ_ξ), (λ, ν_ξ), and $(\lambda, \widetilde{\nu}_\xi)$, respectively. From Lemma 2.5 of [7] we have $\widetilde{\nu}_\xi(\mathrm{d}r) = rg_\xi(r^2)\mathrm{d}r$ with $rg_\xi(r^2) = 2r \int_0^\infty s^{-2} \mathrm{e}^{-r^2/s^2} \nu_\xi(\mathrm{d}s)$. Hence

$$\begin{aligned} rg_\xi(r^2) &= 2r \int_0^\infty \mathrm{e}^{-r^2/s^2} s^{-2} \mathrm{d}s \int_{(0,\infty)} a_1(s;u) \rho_\xi(\mathrm{d}u) \\ &= \int_0^\infty \mathrm{e}^{-t} t^{-1/2} \mathrm{d}t \int_{(0,\infty)} a_1(t^{-1/2}r;u) \rho_\xi(\mathrm{d}u) \\ &= \int_0^\infty \mathrm{e}^{-t} \mathrm{d}t \int_0^\infty a_1(r;tu) \rho_\xi(\mathrm{d}u) \\ &= \int_0^\infty a_1(r;u) \Upsilon^0(\rho_\xi)(\mathrm{d}u). \end{aligned}$$

It follows that $\widetilde{\nu} = \mathcal{A}_1(\Upsilon^0(\rho))$. $\square$

The following result shows that $G(\mathbb{R}^d)$ is the class of distributions of stochastic integrals with respect Lévy processes with distribution of class A at time 1. This is a multivariate and not necessarily symmetric generalization of (1.2).

Theorem 3.14. *Let*

$$\Psi_{-2,2}(\mu) = \mathcal{L}\left(\int_0^1 (-\log t)^{1/2} \mathrm{d}X_t^{(\mu)} \right), \quad \mu \in I(\mathbb{R}^d).$$

Then $\Psi_{-2,2}$ is one-to-one and

$$G(\mathbb{R}^d) = \Psi_{-2,2}(A(\mathbb{R}^d)) = \Psi_{-2,2}(\Phi_{\cos}(I(\mathbb{R}^d))). \tag{3.15}$$

Proof. Suppose that $\widetilde{\mu} \in G(\mathbb{R}^d)$ with triplet $(\widetilde{\Sigma}, \widetilde{\nu}, \widetilde{\gamma})$. Then it follows from Theorems 3.11 and 3.13 that

$$\widetilde{\nu} = \mathcal{A}_1(\Upsilon^0(\rho)) = \Upsilon_{-2.2}(\mathcal{A}_1(\rho))$$

for some $\rho \in \mathfrak{M}_L^1(\mathbb{R}^d)$. Let $\nu = \mathcal{A}_1(\rho)$. Since $\widetilde{\nu} = \Upsilon_{-2,2}(\nu)$ and since $u = f(t) = (-\log t)^{1/2}$ is the inverse function of $t = \int_u^\infty 2v\mathrm{e}^{-v^2}\mathrm{d}v = \mathrm{e}^{-u^2}$, we have (3.5) for this $f(t)$ and $T = 1$. Choose Σ and γ satisfying (3.4) and (3.6). Let $\mu \in I(\mathbb{R}^d)$ having triplet (Σ, ν, γ). Then $\mu \in A(\mathbb{R}^d)$ and $\widetilde{\mu} = \Psi_{-2,2}(\mu)$. Notice that $\mathfrak{D}(\Psi_{-2,2}) = I(\mathbb{R}^d)$, as $f(t)$ is square-integrable on $(0,1)$. Conversely, we can see that if $\mu \in A(\mathbb{R}^d)$, then $\Psi_{-2,2}(\mu) \in G(\mathbb{R}^d)$. Thus the first equality in (3.15) is proved. The second equality follows from (3.7) of Theorem 3.8. The one-to-one property of $\Psi_{-2,2}$ follows from that of $\Upsilon_{-2,2}$ in Proposition 3.12. □

Remark 3.15. (a) The two representations of $\widetilde{\mu} \in G(\mathbb{R}^d)$ in Theorems 3.11 and 3.14 are related in the following way. Theorem 3.14 shows that $\widetilde{\mu} \in G(\mathbb{R}^d)$ if and only if $\widetilde{\mu} = \Upsilon_{-2.2}(\Phi_{\cos}(\mu))$ for some $\mu \in I(\mathbb{R}^d)$. This μ has Lévy measure $\rho^{(1/2)}$ if ρ is the Lévy measure in the representation of $\widetilde{\mu}$ in Theorem 3.11 (c). For the proof, use Proposition 3.4, Theorems 3.8 and 3.13.

(b) We have another representation of the class $G(\mathbb{R}^d)$. Let $h(u) = \int_u^\infty \mathrm{e}^{-v^2}\mathrm{d}v$, $u > 0$, and denote its inverse function by $h^*(t)$. For $\mu \in I(\mathbb{R}^d)$, we define

$$\mathcal{G}(\mu) = \mathcal{L}\left(\int_0^{\sqrt{\pi}/2} h^*(t)\mathrm{d}X_t^{(\mu)}\right).$$

It is known that $G(\mathbb{R}^d) = \mathcal{G}\left(I(\mathbb{R}^d)\right)$, see Theorem 2.4 (5) in [8]. This suggests us that $\mathcal{G}$ is decomposed into

$$\mathcal{G} = \Psi_{-2.2} \circ \Phi_{\cos} = \Phi_{\cos} \circ \Psi_{-2.2}$$

with the same domain $I(\mathbb{R}^d)$, where $\circ$ means composition of mappings. The proof is easy to obtain.

3.8. $\mathcal{A}_1$ **is not an Upsilon transformation.** From Theorem 3.13 we obtain the following result.

Theorem 3.16. *The transformation $\mathcal{A}_1$ is not an Upsilon transformation Υ_τ for any dilation measure τ.*

Proof. Suppose that there is a measure τ on $(0,\infty)$ such that $\mathcal{A}_1(\rho)(B) = \Upsilon_\tau(\rho)(B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$. Then, we can show that $\mathcal{A}_1(\Upsilon^0(\rho)) = \Upsilon^0(\mathcal{A}_1(\rho))$ for $\rho \in \mathfrak{M}_L^1(\mathbb{R}^d)$, using the Fubini theorem. Then, it follows from Theorem 3.13 that $\Upsilon_{-2,2}(\mathcal{A}_1(\rho)) = \Upsilon^0(\mathcal{A}_1(\rho))$ for $\rho \in \mathfrak{M}_L^1(\mathbb{R}^d)$. Let $\widetilde{\rho} = \mathcal{A}_1(\rho)$. If $\int_{\mathbb{R}^d} |x|\,\rho(\mathrm{d}x) < \infty$, then

$$\int_{\mathbb{R}^d} x\Upsilon^0(\widetilde{\rho})(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-u}\mathrm{d}u \int_{\mathbb{R}^d} ux\widetilde{\rho}(\mathrm{d}x) = \int_{\mathbb{R}^d} x\widetilde{\rho}(\mathrm{d}x),$$

while

$$\int_{\mathbb{R}^d} x\Upsilon_{-2,2}(\widetilde{\rho})(\mathrm{d}x) = \int_0^\infty 2u\mathrm{e}^{-u^2}\mathrm{d}u \int_{\mathbb{R}^d} ux\widetilde{\rho}(\mathrm{d}x) = 2^{-1}\pi^{1/2}\int_{\mathbb{R}^d} x\widetilde{\rho}(\mathrm{d}x).$$

Hence $\Upsilon_{-2,2}(\widetilde{\rho}) \neq \Upsilon^0(\widetilde{\rho})$ whenever $\int_{\mathbb{R}^d} x\widetilde{\rho}(\mathrm{d}x) \neq 0$ (for example, choose $\rho = \delta_{e_1}, e_1 = (1, 0, ..., 0)$). This is a contradiction. Hence the measure τ does not exist. $\square$

4. Examples

We conclude this paper with examples for Theorems 3.11 and 3.13, where the modified Bessel function K_0 plays a role. We only consider the one-dimensional case of Lévy measures concentrated on $(0, \infty)$. Multivariate extensions are possible by using polar decomposition.

By the well-known formula for the modified Bessel functions we have

$$K_0(x) = \frac{1}{2}\int_0^\infty \mathrm{e}^{-t-x^2/(4t)}t^{-1}\mathrm{d}t,\ x > 0.$$

An alternative expression is

$$K_0(x) = \int_1^\infty (t^2-1)^{-1/2}\mathrm{e}^{-xt}\mathrm{d}t,\ x > 0, \tag{4.1}$$

see (3.387.3) in [4, p.350]. It follows that $K_0(x)$ is completely monotone on $(0, \infty)$ and that $\int_0^\infty K_0(x)\mathrm{d}x = \pi/2$. The Laplace transform of K_0 in $x > 0$ is

$$\varphi_{K_0}(s) := \int_0^\infty \mathrm{e}^{-sx}K_0(x)\mathrm{d}x = \begin{cases} (1-s^2)^{-1/2}\arccos(s), & 0 < s < 1 \\ 1, & s = 1 \\ (1-s^2)^{-1/2}\log(s + (s^2-1)^{1/2}), & s > 1, \end{cases} \tag{4.2}$$

see (6.611.9) in [4, p.695].

The following is an example of ν and $\widetilde{\nu}$ in Theorem 3.11 (b).

Example 4.1. Let

$$\widetilde{\nu}(\mathrm{d}x) = K_0(x)1_{(0,\infty)}(x)\mathrm{d}x$$

and

$$\nu(\mathrm{d}x) = 4^{-1}\pi x^{-1/2}\mathrm{e}^{-x^{1/2}}1_{(0,\infty)}(x)\mathrm{d}x. \tag{4.3}$$

Then $\nu \in \mathfrak{M}_L^B(\mathbb{R}) \cap \mathfrak{M}_L^1(\mathbb{R})$, and $\widetilde{\nu} = \mathcal{A}_1(\nu) \in \mathfrak{M}_L^G(\mathbb{R})$.

The proof is as follows. Since the function $x^{-1/2}\mathrm{e}^{-x^{1/2}}$ is completely monotone on $(0,\infty)$ and $\int_0^1 x\nu(\mathrm{d}x) < \infty$, we have $\nu \in \mathfrak{M}_L^B(\mathbb{R}) \cap \mathfrak{M}_L^1(\mathbb{R})$. We can show

$$\mathcal{A}_1(\nu)(B) = \int_0^1 \nu^{(1/2)}(u^{-1}B)2\pi^{-1}(1-u^2)^{-1/2}\mathrm{d}u, \quad B \in \mathcal{B}(\mathbb{R}), \tag{4.4}$$

like (3.3). Hence

$$\begin{aligned}
\mathcal{A}_1(\nu)(B) &= \int_0^1 2^{-1}(1-u^2)^{-1/2}\mathrm{d}u \int_0^\infty 1_B(us^{1/2})s^{-1/2}\mathrm{e}^{-s^{1/2}}\mathrm{d}s \\
&= \int_0^1 (1-u^2)^{-1/2}\mathrm{d}u \int_0^\infty 1_B(r)\mathrm{e}^{-r/u}\mathrm{d}r \\
&= \int_0^\infty 1_B(r)\mathrm{d}r \int_1^\infty (y^2-1)^{-1/2}\mathrm{e}^{-ry}\mathrm{d}y \\
&= \int_0^\infty 1_B(r)K_0(r)\mathrm{d}r = \widetilde{\nu}(B).
\end{aligned}$$

The fact that $\widetilde{\nu} \in \mathfrak{M}_L^G(\mathbb{R})$ can also be shown directly, since $K_0(x^{1/2})$ is again completely monotone in $x \in (0,\infty)$.

It follows from $\widetilde{\nu} \in \mathfrak{M}_L^G(\mathbb{R})$ that $\widetilde{\nu}$ is the Lévy measure of a generalized type G distribution $\widetilde{\mu}$ on $\mathbb{R}$. Using (4.2), we find that this $\widetilde{\mu}$ is supported on $[0,\infty)$ if and only if it has Laplace transform

$$\int_{[0,\infty)} \mathrm{e}^{-sx}\widetilde{\mu}(\mathrm{d}x) = \exp\left\{-\gamma_0 s + \varphi_{K_0}(s) - 2^{-1}\pi\right\}$$

for some $\gamma_0 \geq 0$.

Remark 4.2. $\mathcal{A}_1(\nu)$ in Example 4.1 actually belongs to a smaller class $\mathfrak{M}_L^B(\mathbb{R})$. Therefore, in connection to Theorem 3.11, it might be interesting to find a necessary and sufficient condition on ν in order that $\widetilde{\mu} \in B(\mathbb{R}^d)$. The ν in Example 4.1 also belongs to a smaller class than $\mathfrak{M}_L^B(\mathbb{R}) \cap \mathfrak{M}_L^1(\mathbb{R})$. It belongs to the class of Lévy measures of distributions in $\mathfrak{R}(\Psi_{-1/2})$ studied in Theorem 4.2 of [11].

We now give an example of ρ in Theorem 3.11 (c).

Example 4.3. Consider the following Lévy measure in $\mathfrak{M}_L^B(\mathbb{R})$:

$$\rho(\mathrm{d}x) = 4^{-1}\pi^{1/2}x^{-1/2}\mathrm{e}^{-x/4}1_{(0,\infty)}(x)\mathrm{d}x. \tag{4.5}$$

Then ν in (4.3) satisfies $\nu = \Upsilon^0(\rho)$.

To prove this, we write $\Upsilon^0(\rho)$ as

$$\Upsilon^0(\rho)(\mathrm{d}x) = \int_0^\infty \rho(u^{-1}\mathrm{d}x)\mathrm{e}^{-u}\mathrm{d}u = 4^{-1}\pi^{1/2}x^{-1/2}\mathrm{d}x\int_0^\infty u^{-1/2}\mathrm{e}^{-u-x/(4u)}\mathrm{d}u$$

on $(0,\infty)$. By formula (3.475.15) in [4, pp 369], we have

$$\int_0^\infty u^{-1/2}\mathrm{e}^{-u-x/(4u)}\mathrm{d}u = \pi^{1/2}\mathrm{e}^{-x^{1/2}}.$$

Hence, $\Upsilon^0(\rho) = \nu$ from (4.3).

Since $\mathcal{A}_1(\nu) = \mathcal{A}_1(\Upsilon^0(\rho)) = \Upsilon_{-2,2}(\mathcal{A}_1(\rho))$ by Theorem 3.13, $\mathcal{A}_1(\rho)$ is also of interest.

Example 4.4. Let ρ be as in (4.5). Then

$$\mathcal{A}_1(\rho)(\mathrm{d}x) = 2^{-1}\pi^{-1/2}\mathrm{e}^{-x^2/8}K_0(x^2/8)1_{(0,\infty)}(x)\mathrm{d}x. \tag{4.6}$$

The proof is as follows. We have, from (4.4),

$$\begin{aligned}\mathcal{A}_1(\rho)(B) &= 2^{-1}\pi^{-1/2}\int_0^1(1-u^2)^{-1/2}\mathrm{d}u\int_0^\infty 1_{u^{-1}B}(s^{1/2})s^{-1/2}\mathrm{e}^{-s/4}\mathrm{d}s\\ &= \pi^{-1/2}\int_0^\infty 1_B(r)\mathrm{d}r\int_0^1 u^{-1}(1-u^2)^{-1/2}\mathrm{e}^{-r^2/(4u^2)}\mathrm{d}u\\ &= 2^{-1}\pi^{-1/2}\int_0^\infty 1_B(r)\mathrm{d}r\int_1^\infty y^{-1/2}(y-1)^{-1/2}\mathrm{e}^{-r^2y/4}\mathrm{d}y.\end{aligned}$$

Use (3.383.3) in [4, pp 347] to obtain

$$\int_1^\infty y^{-1/2}(y-1)^{-1/2}\mathrm{e}^{-r^2y/4}\mathrm{d}y = \mathrm{e}^{-r^2/8}K_0(r^2/8).$$

Thus we obtain (4.6).

Remark 4.5. The ρ in (4.5) also belongs to $\mathfrak{M}_L^B(\mathbb{R})\cap\mathfrak{M}_L^1(\mathbb{R})$. Therefore $\mathcal{A}_1(\rho)$ itself is another example of a measure in $\mathfrak{M}_L^G(\mathbb{R})$.

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