Random Matrices: A bridge between Classical and Free Infinite Divisibility

Free Probability, Random Matrices and Infinite Divisibility

Victor Pérez-Abreu
Probability and Statistics Department
Center for Research in Mathematics CIMAT
Guanajuato, Mexico

Probability Seminar
Department of Mathematics
University of Tennessee
Knoxville, TN,
April 23, 2012

Plan of the Lecture

1. Review Lecture I and II.
   1.1 Gaussian random matrices and Wigner law.
   1.2 Free central limit theorem.
   1.3 Random matrices models for Marchenko-Pastur law.
2. Infinitely Divisible Random Matrices.
3. Free Infinite Divisibility.
   3.1 Free cumulant transform and infinite divisibility.
   3.2 Main features and characterization.
   3.3 In search of examples.
4. BP-Bijection between classical and free infinite divisibility.
5. Random Matrices Approach to the BP-Bijection.
   5.1 General results.
   5.2 Concrete realizations.

I. Wigner law for a Gaussian Unitary Ensemble (GUE)

- GUE: $\mathbf{Z} = (Z_n)_{n \geq 1}$, Z_n is $n \times n$ Hermitian random matrix

$$Z_n = (Z_n^{i,j})_{1 \leq i,j \leq n}, \quad Z_n^{j,i} = \overline{Z}_n^{i,j},$$

$$\mathrm{Re}\left(Z_n^{j,i}\right) \sim \mathrm{Im}\left(Z_n^{j,i}\right) \sim N(0, (1 + \delta_{ij})/2),$$

$$\mathrm{Re}\left(Z_n^{j,i}\right), \mathrm{Im}\left(Z_n^{j,i}\right), 1 \leq i \leq j \leq n \text{ independent r.v.}$$

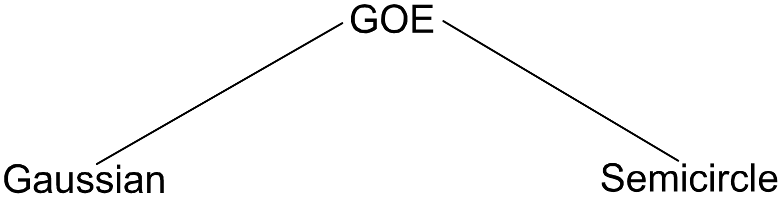
- Distribution of Z_n is invariant under unitary transformations.
- If $\lambda_{n,1}, \ldots, \lambda_{n,n}$ are eigenvalues of Z_n, ESD is

$$\widehat{F}_n(x) = \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{\{\lambda_{n,j} \leq x\}}.$$

- ASD: $\widehat{F}_n$ converges, as $n \to \infty$, to semicircle distribution

$$\mathrm{w}(x)\mathrm{d}x = \frac{1}{2\pi} \sqrt{4 - x^2}\,\mathbf{1}_{|x| \leq 2}\mathrm{d}x.$$

- Similar to GOE and universal under appropriate conditions.


GOE
Gaussian
Semicircle

I. Free Central Limit Theorem

Semicircle law as the free Gaussian

- Free independence was defined in Lecture 1 for elements of a noncommutative probability space.

- Asymptotic free independence was also defined for ensembles of random matrices with asymptotic spectral distributions.

- Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be a sequence of freely independent random variables with the same distribution with all moments, zero mean and variance one. Then the distribution of

$$\mathbf{Z}_n = \frac{1}{\sqrt{n}}(\mathbf{X}_1 + \ldots + \mathbf{X}_n)$$

  converges in distribution to the semicircle distribution.

- **Free Gaussian distribution**: the semicircle distribution plays in free probability the role Gaussian distribution does in classical probability.

I. Marchenko-Pastur law for covariance matrices

- $X_n = X_{p\times n} = (Z_{j,k} : j = 1, .., p, k = 1, ..., n)$ complex i.i.d. under second moment assumptions.
- $W_n = X_n^* X_n$ is Wishart random matrix if

$$\mathrm{Re}\,(Z_{j,k}) \sim \mathrm{Im}\,(Z_{j,k}) \sim N(0, (1+\delta_{jk})/2).$$

- Distribution of W_n is invariant under unitary conjugations.
- Covariance matrix $S_n = \frac{1}{n} X_n^* X_n$, with ESD $\widehat{F}_n$ of nonnegative eigenvalues $\lambda_{n,1}, ..., \lambda_{n,n}$ of S_n.
- If $p/n \to c > 0$, $\widehat{F}_n$ converges to MP distribution

$$\mathrm{m}_c(\mathrm{d}x) = \begin{cases} f_c(x)\mathrm{d}x, & \textit{if} \quad c \geq 1 \\ (1-c)\delta_0(\mathrm{d}x) + f_c(x)\mathrm{d}x, & \textit{if} \quad 0 < c < 1, \end{cases}$$

$$f_c(x) = \frac{c}{2\pi x}\sqrt{(x-a)(b-x)}\,\mathbf{1}_{[a,b]}(x)$$
$$a = (1-\sqrt{c})^2, \;\; b = (1+\sqrt{c})^2.$$

I. MP Law for a non covariance random matrix

Cavanal-Duvillard (2006)

- $(N_t)_{t \geq 0}$ Poisson distribution with mean p.
- $(u_j)_{j \geq 1}$ a sequence of i.i.d. random vectors with uniform distribution on the unit sphere of $\mathbb{C}^n$.
- Consider the $n \times n$ compound Poisson random matrix

$$M_n = \sum_{j=1}^{N} u_j^* u_j.$$

I. MP Law for a non covariance random matrix

Cavanal-Duvillard (2006)

- $(N_t)_{t \geq 0}$ Poisson distribution with mean p.
- $(u_j\ )_{j \geq 1}$ a sequence of i.i.d. random vectors with uniform distribution on the unit sphere of $\mathbb{C}^n$.
- Consider the $n \times n$ compound Poisson random matrix

$$M_n = \sum_{j=1}^{N} u_j^* u_j .$$

- Distributions of M_n is invariant under unitary conjugations.

I. MP Law for a non covariance random matrix

Cavanal-Duvillard (2006)

- $(N_t)_{t\geq 0}$ Poisson distribution with mean p.
- $(u_j)_{j\geq 1}$ a sequence of i.i.d. random vectors with uniform distribution on the unit sphere of $\mathbb{C}^n$.
- Consider the $n \times n$ compound Poisson random matrix

$$M_n = \sum_{j=1}^{N} u_j^* u_j .$$

- Distributions of M_n is invariant under unitary conjugations.
- ASD of $\mathbf{M} = (M_n)$, when $p/n \to c$, is MP distribution m_c.

I. MP Law for a non covariance random matrix

Cavanal-Duvillard (2006)

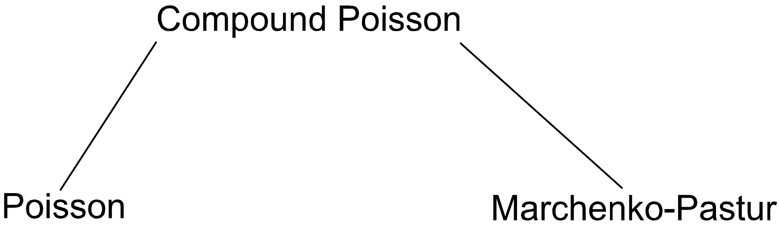
- $(N_t)_{t\geq 0}$ Poisson distribution with mean p.
- $(u_j)_{j\geq 1}$ a sequence of i.i.d. random vectors with uniform distribution on the unit sphere of $\mathbb{C}^n$.
- Consider the $n \times n$ compound Poisson random matrix

$$M_n = \sum_{j=1}^{N} u_j^* u_j.$$

- Distributions of M_n is invariant under unitary conjugations.
- ASD of $\mathbf{M} = (M_n)$, when $p/n \to c$, is MP distribution m_c.
- As random matrices, M_n is infinitely divisible, but the Wishart random matrix W_n is not.

I. Covariance vs. Covariation process

- Covariance matrix
$$S_n = X_n^* X_n.$$
- Compound Poisson $n \times n$ random matrix
$$M_n = \sum_{j=1}^{N} u_j^* u_j.$$
- Distribution of M_n and Wishart W_n are invariant under unitary conjugations and have m_c as their same ASD.
- M_n comes from a quadratic variation process
$$M_n(t) = [X^*, X](t) = \sum_{s<t} (\Delta X(s))^* \Delta X(s) = \sum_{j=1}^{N_t} u_j^* u_j$$
$$X(t) = \sum_{j=1}^{N_t} u_j, \quad M_n = [X^*, X](1).$$
- The Wishart process $W_n(t)$ is a covariance process.
- $M_n(t)$ is an infinitely divisible process, but $W_n(t)$ is not.


Compound Poisson
Poisson
Marchenko-Pastur

II. Infinitely divisible random matrices

- $\mathbb{M}_d$ space of $d \times d$ matrices (real or complex entries).

II. Infinitely divisible random matrices

- $\mathbb{M}_d$ space of $d \times d$ matrices (real or complex entries).
- A random matrix M in $\mathbb{M}_d$ is **Infinitely Divisible** (ID) iff $\forall$ $n \geq 1$ $\exists_n$ i.i.d. random matrices $M_1, \ldots, M_n$ in $\mathbb{M}_d$ such that

$$M_1 + \ldots + M_n \stackrel{\mathcal{L}}{=} M.$$

II. Infinitely divisible random matrices

- $\mathbb{M}_d$ space of $d \times d$ matrices (real or complex entries).
- A random matrix M in $\mathbb{M}_d$ is **Infinitely Divisible** (ID) iff $\forall$ $n \geq 1$ $\exists_n$ i.i.d. random matrices $M_1, ..., M_n$ in $\mathbb{M}_d$ such that

$$M_1 + ... + M_n \stackrel{\mathcal{L}}{=} M.$$

- Gaussian random matrices in GOE and GUE are ID.

II. Infinitely divisible random matrices

- $\mathbb{M}_d$ space of $d \times d$ matrices (real or complex entries).
- A random matrix M in $\mathbb{M}_d$ is **Infinitely Divisible** (ID) iff $\forall$ $n \geq 1$ $\exists_n$ i.i.d. random matrices $M_1, ..., M_n$ in $\mathbb{M}_d$ such that

$$M_1 + ... + M_n \stackrel{\mathcal{L}}{=} M.$$

- Gaussian random matrices in GOE and GUE are ID.
- Wishart random matrix is not ID.

II. Infinitely divisible random matrices

- $\mathbb{M}_d$ space of $d \times d$ matrices (real or complex entries).
- A random matrix M in $\mathbb{M}_d$ is **Infinitely Divisible** (ID) iff $\forall$ $n \geq 1$ $\exists_n$ i.i.d. random matrices $M_1, \ldots, M_n$ in $\mathbb{M}_d$ such that

$$M_1 + \ldots + M_n \stackrel{\mathcal{L}}{=} M.$$

- Gaussian random matrices in GOE and GUE are ID.
- Wishart random matrix is not ID.
- Compound Poisson matrix process $M_n = \sum_{j=1}^{N} u_j^* u_j$ is ID.

II. Infinitely divisible random matrices

- $\mathbb{M}_d$ space of $d \times d$ matrices (real or complex entries).
- A random matrix M in $\mathbb{M}_d$ is **Infinitely Divisible** (ID) iff $\forall$ $n \geq 1$ $\exists_n$ i.i.d. random matrices $M_1, ..., M_n$ in $\mathbb{M}_d$ such that

$$M_1 + ... + M_n \stackrel{\mathcal{L}}{=} M.$$

- Gaussian random matrices in GOE and GUE are ID.
- Wishart random matrix is not ID.
- Compound Poisson matrix process $M_n = \sum_{j=1}^{N} u_j^* u_j$ is ID.
- **Open problem**: ASD for ensembles of Hermitian unitary invariant infinitely divisible random matrices.

II. Infinitely divisible random matrices

- $\mathbb{M}_d$ space of $d \times d$ matrices (real or complex entries).
- A random matrix M in $\mathbb{M}_d$ is **Infinitely Divisible** (ID) iff $\forall$ $n \geq 1$ $\exists_n$ i.i.d. random matrices $M_1, ..., M_n$ in $\mathbb{M}_d$ such that

$$M_1 + ... + M_n \stackrel{\mathcal{L}}{=} M.$$

- Gaussian random matrices in GOE and GUE are ID.
- Wishart random matrix is not ID.
- Compound Poisson matrix process $M_n = \sum_{j=1}^{N} u_j^* u_j$ is ID.
- **Open problem**: ASD for ensembles of Hermitian unitary invariant infinitely divisible random matrices.
- Partial answer today (due to Benaych-Georges (05) and Cavanal-Duvillard (05)) and more.

II. Why infinitely divisible random matrices?

Applied and theoretical reasons

1. Stochastic modelling (fixed dimension):

- There exists a matrix Lévy process $(M_t)_{t\geq 0}$ such that

$$M_1 \stackrel{\mathcal{L}}{=} M.$$

- Multivariate financial modelling via Lévy and non Gaussian Ornstein-Uhlenbeck matrix processes: Barndorff-Nielsen & Stelzer (09, 11), Pigorsch & Stelzer (09), Stelzer (10).
- ID random matrix models alternative to Wishart random matrix: Barndorff-Nielsen & PA (08), PA & Stelzer (12).

2. **Today**: (asymptotic spectral distribution)

- Random matrices approach to the relation between classical and free infinite divisibility.
- Benaych-Georges (05), Cabanal-Duvillard (05), PA & Sakuma (08), Molina & Rocha-Arteaga (12), joint work in progress with Molina & Rocha-Arteaga.

III. But before: Free infinite divisibility

Analytic tools similar to classical probability

- Fourier transform of probability measure μ on $\mathbb{R}$

$$\widehat{\mu}(s) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sx} \mu(\mathrm{d}x), \quad s \in \mathbb{R},$$

- Cauchy transform of μ

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x} \mu(\mathrm{d}x), \quad z \in \mathbb{C}/\mathbb{R}.$$

- Classical cumulant transform

$$c_\mu(s) = \log \widehat{\mu}(s), \quad s \in \mathbb{R}.$$

- Free cumulant transform

$$C_\mu(z) = zG_\mu^{-1}(z) - 1, \quad z \in \Gamma_\mu$$

III. Classical and free convolutions

- Classical convolution $\mu_1 * \mu_2$ is defined by

$$c_{\mu_1 * \mu_2}(s) = c_{\mu_1}(s) + c_{\mu_2}(s).$$

- X_1 & X_2 classical independent r.v. $\mu_i = \mathcal{L}(X_i)$,

$$\mu_1 * \mu_2 = \mathcal{L}\left(X_1 + X_2\right)$$

- Free convolution $\mu_1 \boxplus \mu_2$ is defined by

$$C_{\mu_1 \boxplus \mu_2}(z) = C_{\mu_1}(z) + C_{\mu_2}(z), \quad z \in \Gamma_{\mu_1} \cap \Gamma_{\mu_2}.$$

- $\mathbf{X}_1$ & $\mathbf{X}_2$ free independent, $\mu_i = \mathcal{L}(\mathbf{X}_i)$,

$$\mu_1 \boxplus \mu_2 = \mathcal{L}\left(\mathbf{X}_1 + \mathbf{X}_2\right)$$

- Also in Lecture 1 free *multiplicative* convolution $\mu_1 \boxtimes \mu_2$.

III. More on the free cumulant transform

- Reciprocal of Cauchy transform $\underline{G}_\mu(z) = 1/G_\mu(z)$.

III. More on the free cumulant transform

- Reciprocal of Cauchy transform $\underline{G}_\mu(z) = 1/G_\mu(z)$.
- Bercovici & Voiculescu (93): Right inverse $\underline{G}_\mu^{-1}$ of $\underline{G}_\mu$ exists in $\Gamma = \cup_{\alpha>0}\Gamma_{\alpha,\beta_\alpha}$, where

$$\Gamma_{\alpha,\beta} = \{z = x + iy : y > \beta,\, x < \alpha y\},\ \alpha > 0, \beta > 0.$$

III. More on the free cumulant transform

- Reciprocal of Cauchy transform $\underline{G}_\mu(z) = 1/G_\mu(z)$.
- Bercovici & Voiculescu (93): Right inverse $\underline{G}_\mu^{-1}$ of $\underline{G}_\mu$ exists in $\Gamma = \cup_{\alpha>0}\Gamma_{\alpha,\beta_\alpha}$, where

$$\Gamma_{\alpha,\beta} = \{z = x + iy : y > \beta,\, x < \alpha y\}, \, \alpha > 0, \beta > 0.$$

- *Voiculescu transform*

$$\phi_\mu(z) = \underline{G}_\mu^{-1}(z) - z, \quad z \in \Gamma^\mu_{\alpha,\beta}.$$

III. More on the free cumulant transform

- Reciprocal of Cauchy transform $\underline{G}_\mu(z) = 1/G_\mu(z)$.
- Bercovici & Voiculescu (93): Right inverse $\underline{G}_\mu^{-1}$ of $\underline{G}_\mu$ exists in $\Gamma = \cup_{\alpha>0}\Gamma_{\alpha,\beta_\alpha}$, where

$$\Gamma_{\alpha,\beta} = \{z = x + iy : y > \beta,\, x < \alpha y\}, \, \alpha > 0, \beta > 0.$$

- *Voiculescu transform*

$$\phi_\mu(z) = \underline{G}_\mu^{-1}(z) - z, \quad z \in \Gamma^\mu_{\alpha,\beta}.$$

- Barndorff-Nielsen & Thorbjørnsen (06): Free cumulant

$$C_\mu(z) = z\phi_\mu(\frac{1}{z}) = z\underline{G}_\mu^{-1}(\frac{1}{z}) - 1.$$

III. More on the free cumulant transform

- Reciprocal of Cauchy transform $\underline{G}_\mu(z) = 1/G_\mu(z)$.
- Bercovici & Voiculescu (93): Right inverse $\underline{G}_\mu^{-1}$ of $\underline{G}_\mu$ exists in $\Gamma = \cup_{\alpha>0}\Gamma_{\alpha,\beta_\alpha}$, where

$$\Gamma_{\alpha,\beta} = \{z = x + iy : y > \beta,\ x < \alpha y\},\ \alpha > 0, \beta > 0.$$

- *Voiculescu transform*

$$\phi_\mu(z) = \underline{G}_\mu^{-1}(z) - z, \quad z \in \Gamma^\mu_{\alpha,\beta}.$$

- Barndorff-Nielsen & Thorbjørnsen (06): Free cumulant

$$C_\mu(z) = z\phi_\mu(\frac{1}{z}) = z\underline{G}_\mu^{-1}(\frac{1}{z}) - 1.$$

- ϕ_μ & C_μ linearize free additive convolution:

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z), \quad z \in \Gamma^{\mu_1}_{\alpha_1,\beta_1} \cap \Gamma^{\mu_2}_{\alpha_2,\beta_2}$$

$$C_{\mu_1 \boxplus \mu_2}(z) = C_{\mu_1}(z) + C_{\mu_2}(z), \quad \frac{1}{z} \in \Gamma^{\mu_1}_{\alpha_1,\beta_1} \cap \Gamma^{\mu_2}_{\alpha_2,\beta_2}.$$

III. Free infinite divisibility

- Let μ be a probability distribution on $\mathbb{R}$ ($\mu \in \mathcal{P}(\mathbb{R})$).
- μ is **infinitely divisible** w.r.t. $\star$ iff $\forall n \geq 1$, $\exists\ \mu_{1/n} \in \mathcal{P}(\mathbb{R})$,

$$\mu = \mu_{1/n} \star \mu_{1/n} \star \cdots \star \mu_{1/n}.$$

- μ is **infinitely divisible** w.r.t. $\boxplus$ iff $\forall n \geq 1$, $\exists\ \mu_{1/n} \in \mathcal{P}(\mathbb{R})$,

$$\mu = \mu_{1/n} \boxplus \mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n}.$$

- Notation: $I^{\boxplus}$ (I^*) class of all free (classical) ID distributions.
- Problems**:**
  1. Characterization of $I^{\boxplus}$, criteria, examples.
  2. In particular, characterize the class $I^{\boxplus}$ similar to I^*.
  3. Search for examples.
  4. Relations between $I^{\boxplus}$ and I^*.
- Two approaches: Combinatorial and analytic.

III. Free infinite divisibility: Combinatorial approach

Not today: Nica and Speicher (2006)

- Only for distributions μ with compact support,

$$m_n(\mu) = \int x^n \mu(\mathrm{d}x), \quad n \geq 1.$$

- Classical cumulants $(k_n(\mu))_{n\geq 1}$

$$c_\mu(s) = \sum_{n=1}^{\infty} k_n(\mu) s^n = \log \widehat{\mu(s)} = \log \left(\sum_{n=0}^{\infty} \frac{m_n(\mu)}{n!} s^n \right),$$

$$m_n(\mu) = \sum_{\pi \in P(n)} k_\pi(\mu).$$

- Free cumulants $(\kappa_n(\mu))_{n\geq 1}$

$$C_\mu(z) = \sum_{n=1}^{\infty} \kappa_n(\mu) z^n,$$

$$m_n(\mu) = \sum_{\pi \in NC(n)} k_\pi(\mu).$$

III. Examples of free ID distributions

- Semicircle distribution w_{m,σ^2} on $(m-2\sigma, m2\sigma)$

$$\mathrm{w}_{m,\sigma^2}(\mathrm{d}x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-(x-m)^2}1_{[m-2\sigma,m+2\sigma]}(x)\mathrm{d}x.$$

$$C_{\mathrm{w}_{m,\sigma^2}}(z) = mz + \sigma^2 z.$$

$$\mathrm{w}_{m_1+m_2,\sigma_1^2+\sigma_2^2} = \mathrm{w}_{m_1,\sigma_1^2} \boxplus \mathrm{w}_{m_2,\sigma_2^2}.$$

$$\kappa_1 = m,\ \kappa_2 = \sigma^2, \kappa_n = 0, n \geq 3$$

III. Examples of free ID distributions

- Semicircle distribution w_{m,σ^2} on $(m-2\sigma, m2\sigma)$

$$\mathrm{w}_{m,\sigma^2}(\mathrm{d}x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-(x-m)^2}1_{[m-2\sigma,m+2\sigma]}(x)\mathrm{d}x.$$

$$C_{\mathrm{w}_{m,\sigma^2}}(z) = mz + \sigma^2 z.$$

$$\mathrm{w}_{m_1+m_2,\sigma_1^2+\sigma_2^2} = \mathrm{w}_{m_1,\sigma_1^2} \boxplus \mathrm{w}_{m_2,\sigma_2^2}.$$

$$\kappa_1 = m,\ \kappa_2 = \sigma^2,\ \kappa_n = 0,\ n \geq 3$$

- Marchenko-Pastur distribution m_c of parameter $c > 0$

$$C_{\mathrm{m}_c}(z) = \frac{cz}{1-z},$$

$$\mathrm{m}_{c_1+c_2} = \mathrm{m}_{c_1} \boxplus \mathrm{m}_{c_2},$$

$$\kappa_n = c,\ n \geq 1.$$

III. Examples of free ID distributions

Example

Cauchy distribution of parameter $\theta > 0$

$$\mathrm{c}_\theta(\mathrm{d}x) = \frac{1}{\pi}\frac{\theta}{\theta^2 + x^2}\mathbf{1}_{x\in\mathbb{R}}\mathrm{d}x$$

Cauchy transform

$$G_{\mathrm{c}_\theta}(z) = \frac{1}{z + \theta i}$$

Free cumulant transform

$$C_{\mathrm{c}_\theta}(z) = -i\theta z$$

$\boxplus$*-convolution of Cauchy distributions is a Cauchy distribution*

$$\mathrm{c}_{\theta_1} \boxplus \mathrm{c}_{\theta_2} = \mathrm{c}_{\theta_1+\theta_2}.$$

III. Free infinite divisibility: Analytic approach

Bercovici & Voiculescu (93)

- The following three statements are equivalent:

III. Free infinite divisibility: Analytic approach

Bercovici & Voiculescu (93)

- The following three statements are equivalent:
  1. $\mu \in I^{\boxplus}$.

III. Free infinite divisibility: Analytic approach

Bercovici & Voiculescu (93)

- The following three statements are equivalent:
  1. $\mu \in I^{\boxplus}$.
  2. ϕ_μ has an analytic extension $\mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$.

III. Free infinite divisibility: Analytic approach

Bercovici & Voiculescu (93)

- The following three statements are equivalent:
  1. $\mu \in I^{\boxplus}$.
  2. ϕ_μ has an analytic extension $\mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$.
  3. There exists $a \in \mathbb{R}$ & finite measure σ on $\mathbb{R}$ such that

$$\phi_\mu(z) = a + \int_{\mathbb{R}} \frac{1 + tz}{z - t} \sigma(\mathrm{d}x), \; z \in \mathbb{C}^+.$$

III. Free infinite divisibility: Analytic approach

Bercovici & Voiculescu (93)

- The following three statements are equivalent:
  1. $\mu \in I^{\boxplus}$.
  2. ϕ_μ has an analytic extension $\mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$.
  3. There exists $a \in \mathbb{R}$ & finite measure σ on $\mathbb{R}$ such that

$$\phi_\mu(z) = a + \int_{\mathbb{R}} \frac{1 + tz}{z - t} \sigma(\mathrm{d}x), \; z \in \mathbb{C}^+.$$

- Facts:

III. Free infinite divisibility: Analytic approach

Bercovici & Voiculescu (93)

- The following three statements are equivalent:
  1. $\mu \in I^{\boxplus}$.
  2. ϕ_μ has an analytic extension $\mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$.
  3. There exists $a \in \mathbb{R}$ & finite measure σ on $\mathbb{R}$ such that

$$\phi_\mu(z) = a + \int_{\mathbb{R}} \frac{1+tz}{z-t}\sigma(\mathrm{d}x), \ z \in \mathbb{C}^+.$$

- Facts:
  - If $\mu_n \in I^{\boxplus}$, $n \geq 1$, and $\mu_n \Rightarrow \mu$, then $\mu \in I^{\boxplus}$.

III. Free infinite divisibility: Analytic approach

Bercovici & Voiculescu (93)

- The following three statements are equivalent:
  1. $\mu \in I^{\boxplus}$.
  2. ϕ_μ has an analytic extension $\mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$.
  3. There exists $a \in \mathbb{R}$ & finite measure σ on $\mathbb{R}$ such that

$$\phi_\mu(z) = a + \int_{\mathbb{R}} \frac{1+tz}{z-t}\sigma(\mathrm{d}x), \; z \in \mathbb{C}^+.$$

- Facts:
  - If $\mu_n \in I^{\boxplus}$, $n \geq 1$, and $\mu_n \Rightarrow \mu$, then $\mu \in I^{\boxplus}$.
  - If $\mu \in I^{\boxplus}$, μ has at most one atom.

III. Free infinite divisibility: Analytic approach

Bercovici & Voiculescu (93)

- The following three statements are equivalent:
  1. $\mu \in I^{\boxplus}$.
  2. ϕ_μ has an analytic extension $\mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$.
  3. There exists $a \in \mathbb{R}$ & finite measure σ on $\mathbb{R}$ such that

$$\phi_\mu(z) = a + \int_{\mathbb{R}} \frac{1+tz}{z-t}\sigma(\mathrm{d}x), \; z \in \mathbb{C}^+.$$

- Facts:
  - If $\mu_n \in I^{\boxplus}$, $n \geq 1$, and $\mu_n \Rightarrow \mu$, then $\mu \in I^{\boxplus}$.
  - If $\mu \in I^{\boxplus}$, μ has at most one atom.
  - A non trivial discrete distribution is not in $I^{\boxplus}$.

III. Free infinite divisibility: Analytic approach

Bercovici & Voiculescu (93)

- The following three statements are equivalent:
  1. $\mu \in I^{\boxplus}$.
  2. ϕ_μ has an analytic extension $\mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$.
  3. There exists $a \in \mathbb{R}$ & finite measure σ on $\mathbb{R}$ such that

$$\phi_\mu(z) = a + \int_{\mathbb{R}} \frac{1+tz}{z-t}\sigma(\mathrm{d}x), \; z \in \mathbb{C}^+.$$

- Facts:
  - If $\mu_n \in I^{\boxplus}$, $n \geq 1$, and $\mu_n \Rightarrow \mu$, then $\mu \in I^{\boxplus}$.
  - If $\mu \in I^{\boxplus}$, μ has at most one atom.
  - A non trivial discrete distribution is not in $I^{\boxplus}$.
  - If $I^{\boxplus} \ni \mu \neq \delta_x$, then for n sufficiently large $\mu^{\boxplus n}$ has no atoms.

III. Free infinite divisibility: Analytic approach

Bercovici & Voiculescu (93)

- The following three statements are equivalent:
  1. $\mu \in I^{\boxplus}$.
  2. ϕ_μ has an analytic extension $\mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$.
  3. There exists $a \in \mathbb{R}$ & finite measure σ on $\mathbb{R}$ such that

$$\phi_\mu(z) = a + \int_{\mathbb{R}} \frac{1+tz}{z-t}\sigma(\mathrm{d}x), \ z \in \mathbb{C}^+.$$

- Facts:
  - If $\mu_n \in I^{\boxplus}$, $n \geq 1$, and $\mu_n \Rightarrow \mu$, then $\mu \in I^{\boxplus}$.
  - If $\mu \in I^{\boxplus}$, μ has at most one atom.
  - A non trivial discrete distribution is not in $I^{\boxplus}$.
  - If $I^{\boxplus} \ni \mu \neq \delta_x$, then for n sufficiently large $\mu^{\boxplus n}$ has no atoms.
- Proofs based on Pick-Nevanlinna theory of analytic functions.

III. Not free infinitely divisible distribution

Examples

Arcsine distribution

$$\mathrm{a}(\mathrm{d}x) = \frac{1}{\pi\sqrt{1-x^2}} 1_{(-1,1)}(x)\mathrm{d}x$$

is not free infinitely divisible:

(i) Its Voiculescu transform is not analytic:

$$\phi_{\mathrm{a}}(z) = \sqrt{z^2+4} - z$$

(ii) But also, from Lecture 1, $\mathrm{a} = \mathrm{b} \boxplus \mathrm{b}$ with

$$\mathrm{b}(\mathrm{d}x) = \frac{1}{2}\left\{\delta_{\{-1\}}(\mathrm{d}x) + \delta_{\{1\}}(\mathrm{d}x)\right\}.$$

and b is not free infinitely divisible.

III. Classical and free infinite divisibility

Lévy-Khintchine representations

- Classical L-K: $\mu \in I^*$

$$c_\mu(s) = \eta s - \frac{1}{2} a s^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}sx} - 1 - sx 1_{[-1,1]}(x) \right) \rho(\mathrm{d}x), \; s \in \mathbb{R}.$$

- Free L-K: $\nu \in I^{\boxplus}$

$$C_\nu(z) = \eta z + a z^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 - xz 1_{[-1,1]}(x) \right) \rho(\mathrm{d}x), \; z \in \mathbb{C}^-.$$

- In both cases (η, a, ρ) is a unique *Lévy triplet*: $\eta \in \mathbb{R}$, $a \geq 0$, $\rho(\{0\}) = 0$ and

$$\int_{\mathbb{R}} \min(1, x^2) \rho(\mathrm{d}x) < \infty.$$

IV. Relation between classical and free infinite divisibility

Bercovici, Pata (Biane), Ann. Math. (1999)

- Classical Lévy-Khintchine representation for $\mu \in I^*$

$$c_\mu(s) = \eta s - \frac{1}{2} a s^2 + \int_{\mathbb{R}} \left(e^{isx} - 1 - sx 1_{[-1,1]}(x) \right) \rho(\mathrm{d}x).$$

- Free Lévy-Khintchine representation for $\nu \in I^{\boxplus}$

$$C_\nu(z) = \eta z + a z^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 - xz 1_{[-1,1]}(x) \right) \rho(\mathrm{d}x).$$

- *Bercovici-Pata bijection*: $\Lambda : I^* \to I^{\boxplus}$, $\Lambda(\mu) = \nu$

$$I^* \ni \mu \sim (\eta, a, \rho) \leftrightarrow \Lambda(\mu) \sim (\eta, a, \rho)$$

- Λ preserves convolutions (and weak convergence)

$$\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$$

IV. Image of classical ID distributions under BP bijection

- *Free Gaussian:* For classical Gaussian distribution γ_{m,σ^2},

$$\mathrm{w}_{m,\sigma^2} = \Lambda(\gamma_{m,\sigma^2})$$

is Wigner distribution on $(m - 2\sigma, m + 2\sigma)$ with

$$C_{\mathrm{w}_{\eta,\sigma^2}}(z) = mz + \sigma^2 z^2.$$

IV. Image of classical ID distributions under BP bijection

- *Free Gaussian:* For classical Gaussian distribution γ_{m,σ^2},

$$\mathrm{w}_{m,\sigma^2} = \Lambda(\gamma_{m,\sigma^2})$$

is Wigner distribution on $(m - 2\sigma, m + 2\sigma)$ with

$$C_{\mathrm{w}_{\eta,\sigma^2}}(z) = mz + \sigma^2 z^2.$$

- *Free Poisson:* For classical Poisson distribution p_c, $c > 0$,

$$\mathrm{m}_c = \Lambda(\mathrm{p}_c)$$

is the M-P distribution with

$$C_{\mathrm{m}_c}(z) = \frac{cz}{1-z} = \int_{\mathbb{R}} \left(\frac{1}{1-xz} - 1 \right) c\delta_1(\mathrm{d}x).$$

IV. Image of classical ID distributions under BP bijection

- *Free Gaussian:* For classical Gaussian distribution γ_{m,σ^2},

$$\mathrm{w}_{m,\sigma^2} = \Lambda(\gamma_{m,\sigma^2})$$

is Wigner distribution on $(m - 2\sigma, m + 2\sigma)$ with

$$C_{\mathrm{w}_{\eta,\sigma^2}}(z) = mz + \sigma^2 z^2.$$

- *Free Poisson:* For classical Poisson distribution p_c, $c > 0$,

$$\mathrm{m}_c = \Lambda(\mathrm{p}_c)$$

is the M-P distribution with

$$C_{\mathrm{m}_c}(z) = \frac{cz}{1-z} = \int_{\mathbb{R}} \left(\frac{1}{1-xz} - 1 \right) c\delta_1(\mathrm{d}x).$$

- Belinschi, Bozejko, Lehner & Speicher (11): γ_{m,σ^2} is free ID.

IV. Image of classical ID distributions under BP bijection

- *Free Gaussian:* For classical Gaussian distribution γ_{m,σ^2},

$$\mathrm{w}_{m,\sigma^2} = \Lambda(\gamma_{m,\sigma^2})$$

  is Wigner distribution on $(m - 2\sigma, m + 2\sigma)$ with

$$C_{\mathrm{w}_{\eta,\sigma^2}}(z) = mz + \sigma^2 z^2.$$

- *Free Poisson:* For classical Poisson distribution p_c, $c > 0$,

$$\mathrm{m}_c = \Lambda(\mathrm{p}_c)$$

  is the M-P distribution with

$$C_{\mathrm{m}_c}(z) = \frac{cz}{1 - z} = \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 \right) c\delta_1(\mathrm{d}x).$$

- Belinschi, Bozejko, Lehner & Speicher (11): γ_{m,σ^2} is free ID.
- Open problems: $\gamma_{m,\sigma^2} = \Lambda(?)$ and what is its Lévy measure?.

IV. Image of classical ID distributions under BP bijection

- *Free compound Poisson distributions* $\{\sigma \in \mathcal{P}(\mathbb{R}), \lambda > 0\}$

$$CP^{\boxplus} = \{\Lambda(\mu); \mu \text{ is classical } CP\}\text{, i.e.}$$

$$c_\mu(t) = \lambda \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}t}x - 1\right) \sigma(\mathrm{d}x),$$

$$C_{\Lambda(\mu)}(z) = \lambda \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1\right) \sigma(\mathrm{d}x).$$

- *Free Cauchy:* $\Lambda(\mathrm{c}_\lambda) = \mathrm{c}_\lambda$ for the Cauchy distribution

$$\mathrm{c}_\lambda(\mathrm{d}x) = \frac{1}{\pi} \frac{\lambda}{\lambda^2 + x^2} \mathrm{d}x$$

with free cumulant transform $C_\lambda(z) = -i\lambda z$.

- *Free stable* (Bercovici, Pata, Biane, (99))

$$S^{\boxplus} = \{\Lambda(\mu); \mu \text{ is classical stable}\}.$$

IV. Image of classical ID distributions under BP bijection

- *Free GGC* (PA-Sakuma (08))

$$GGC\,(\boxplus) = \left\{ \Lambda(\mu); \mu \text{ is } GGC(*) \right\}.$$

- *Free subordinators* (Arizmendi, Hasebe, Sakuma (11))

$$I_+^{\boxplus} = \left\{ \Lambda(\mu); \mu \text{ is } I_+^* \right\},$$

I_+^* class of classical ID distributions with support on $[0, \infty)$

$$c_\mu(t) = it\eta_0 + \int_{\mathbb{R}_+} \left(e^{it}x - 1 \right) \rho(dx),$$

$$C_{\Lambda(\mu)}(z) = iz\eta_0 + \int_{\mathbb{R}_+} \left(\frac{1}{1 - xz} - 1 \right) \rho(dx),$$

$$\int_{\mathbb{R}_+} \min(1, x)\rho(dx) < \infty,\ \eta_0 \geq 0,\ \rho(-\infty, 0] = 0.$$

IV. Search for new examples of free ID distributions

Arizmendi, Barndorff-Nielsen & PA (2009)

- Special symmetric Beta distribution

$$\beta_s(\mathrm{d}x) = \frac{1}{2\pi} |x|^{-1/2} (2 - |x|)^{1/2} \mathrm{d}x, \quad |x| < 2$$

- Cauchy transform

$$G_{\beta_s}(z) = -\frac{1}{2}\sqrt{1 - \sqrt{z^{-2}(z^2 - 4)}}$$

- Free additive cumulant transform is $C_{\beta_s}(z) = \sqrt{z^2 + 1} - 1$.
- β_s is free ID with triplet $(0, 0, \mathrm{a})$, a is arcsine on $(-1, 1)$
- For $A_1, A_2, \ldots$, i.i.d. with distribution a & independent of standard Poisson r.v. N

$$\beta_s = \Lambda(\sum_{j=1}^{N} A_j).$$

- Interpretation as multiplicative convolution $\beta_s = \mathrm{m}_1 \boxtimes \mathrm{a}$.

IV. Search for new examples of free ID distributions

Motivated by the symmetric Beta distribution

- Important facts from the last example:
  - β_s has Cauchy transform

$$G_{\beta_s}(z) = -\frac{1}{2}\sqrt{1-\sqrt{z^{-2}(z^2-4)}}.$$

  - Free infinite divisibility of $\beta_s = \mathrm{m}_1 \boxtimes \mathrm{a}$
- Arizmendi & Hasebe (11):

$$G^{\alpha}_{s,r}(z) = -r^{1/\alpha}\left(\frac{1-(1-s(-\frac{1}{z})^{\alpha})^{1/r}}{s}\right)^{1/\alpha}$$

$r > 0, 0 < \alpha \leq 2, s \in \mathbb{C}\backslash\{0\}$.

$$\mu^{\alpha}_{s,2} = \mathrm{m}_1 \boxtimes \mathrm{a}^{\alpha}_{s/4} \text{ is free ID,}$$

- $\mathrm{a}^{\alpha}_{s/4}$ is stable with respect to monotone convolution, where the arcsine law $\mathrm{a}^{1}_{4/4} = \mathrm{a}$ plays the role of Gaussian distribution.

IV. Search for new examples of free ID distributions

Type W distributions

- PA & Sakuma (12): Multiplicative convolutions with the Wigner, $\sigma \in \mathcal{P}(\mathbb{R}_+)$
$$\mu = \sigma \boxtimes \mathrm{w}$$
- Is free infinitely divisible iff
$$\sigma \boxtimes \sigma \in \Lambda(I_+^*).$$
- For any $\sigma \in \mathcal{P}(\mathbb{R}_+)$
$$\mu^2 = \sigma \boxtimes \sigma \boxtimes \mathrm{m}_1 \in \Lambda(I_+^*).$$
- Arizmendi, Hasebe & Sakuma (11):
$$\sigma \in \Lambda(I_+^*) \Rightarrow \sigma \boxtimes \sigma \in \Lambda(I_+^*),$$
$$\sigma \in \Lambda(I_+^*) \Rightarrow \sigma^{\boxtimes t} \in \Lambda(I_+^*), t \geq 1.$$

IV. A remarkable semigroup

Belinschi & Nica (08)

- $$\mathbb{B}_t(\mu) = \left(\mu^{\boxplus(1+t)}\right)^{\uplus \frac{1}{1+t}}, \ t \geq 0,$$

  $\uplus$ is Boolean convolution.

- $$\mathbb{B}_t(\mu_1 \boxtimes \mu_2) = \mathbb{B}_t(\mu_1) \boxtimes \mathbb{B}_t(\mu_2).$$

- Free divisibility indicator

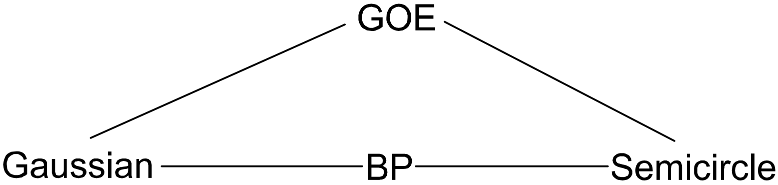
$$\varphi(\mu) = \sup\left\{t \geq 0 : \mu \in \mathbb{B}_t(\mathcal{P}(\mathbb{R}))\right\}.$$

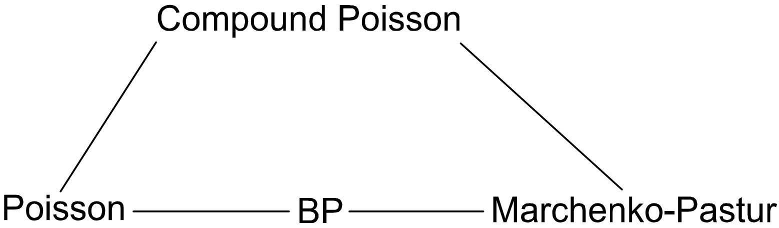
- There exists $\nu \in \mathcal{P}(\mathbb{R})$ such that

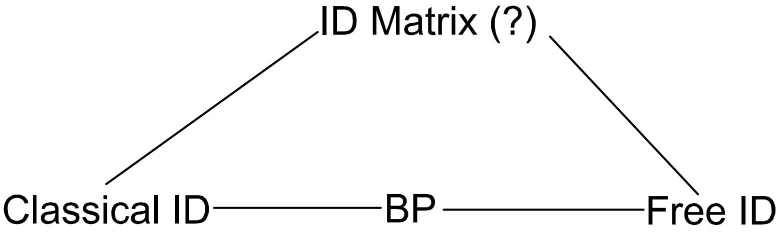
$$\varphi_{\mathbb{B}_t(\mu)}(\nu) = \mu.$$

- μ is free infinitely divisible distribution iff $\varphi(\mu) \geq 1$.
- Divisibility indicator for free multiplicative convolution (Arizmendi & Hasebe (12)).

Classical ID ——— BP ——— Free ID


GOE
Gaussian
BP
Semicircle


Compound Poisson
Poisson
BP
Marchenko-Pastur


ID Matrix (?)
Classical ID
BP
Free ID

V. Random matrix approach to BP bijection

- Benachy-Georges (05, AP), Cavanal-Duvillard (05, EJP): For $\mu \in I^*$ there is an ensemble of unitary invariant random matrices $(M_d)_{d\geq 1}$, such that with probability one its ESD converges in distribution to $\Lambda(\mu) \in I^{\boxplus}$.

V. Random matrix approach to BP bijection

- Benachy-Georges (05, AP), Cavanal-Duvillard (05, EJP): For $\mu \in I^*$ there is an ensemble of unitary invariant random matrices $(M_d)_{d \geq 1}$, such that with probability one its ESD converges in distribution to $\Lambda(\mu) \in I^{\boxplus}$.

- Some properties and questions:

V. Random matrix approach to BP bijection

- Benachy-Georges (05, AP), Cavanal-Duvillard (05, EJP): For $\mu \in I^*$ there is an ensemble of unitary invariant random matrices $(M_d)_{d\geq 1}$, such that with probability one its ESD converges in distribution to $\Lambda(\mu) \in I^{\boxplus}$.

- Some properties and questions:
  - M_d is infinitely divisible in the space of matrices $\mathbb{M}_d$.

V. Random matrix approach to BP bijection

- Benachy-Georges (05, AP), Cavanal-Duvillard (05, EJP): For $\mu \in I^*$ there is an ensemble of unitary invariant random matrices $(M_d)_{d\geq 1}$, such that with probability one its ESD converges in distribution to $\Lambda(\mu) \in I^{\boxplus}$.
- Some properties and questions:
  - M_d is infinitely divisible in the space of matrices $\mathbb{M}_d$.
  - The existence of $(M_d)_{d\geq 1}$ is not constructive.

V. Random matrix approach to BP bijection

- Benachy-Georges (05, AP), Cavanal-Duvillard (05, EJP): For $\mu \in I^*$ there is an ensemble of unitary invariant random matrices $(M_d)_{d\geq 1}$, such that with probability one its ESD converges in distribution to $\Lambda(\mu) \in I^{\boxplus}$.
- Some properties and questions:
  - M_d is infinitely divisible in the space of matrices $\mathbb{M}_d$.
  - The existence of $(M_d)_{d\geq 1}$ is not constructive.
  - How are the random matrix $(M_d)_{d\geq 1}$ realized?

V. Random matrix approach to BP bijection

- Benachy-Georges (05, AP), Cavanal-Duvillard (05, EJP): For $\mu \in I^*$ there is an ensemble of unitary invariant random matrices $(M_d)_{d \geq 1}$, such that with probability one its ESD converges in distribution to $\Lambda(\mu) \in I^{\boxplus}$.
- Some properties and questions:
  - M_d is infinitely divisible in the space of matrices $\mathbb{M}_d$.
  - The existence of $(M_d)_{d \geq 1}$ is not constructive.
  - How are the random matrix $(M_d)_{d \geq 1}$ realized?
  - How are the corresponding matrix Lévy processes $\{M_d(t)\}_{t \geq 0}$ realized?

V. Random matrix approach to BP bijection

- Benachy-Georges (05, AP), Cavanal-Duvillard (05, EJP): For $\mu \in I^*$ there is an ensemble of unitary invariant random matrices $(M_d)_{d\geq 1}$, such that with probability one its ESD converges in distribution to $\Lambda(\mu) \in I^{\boxplus}$.
- Some properties and questions:
  - M_d is infinitely divisible in the space of matrices $\mathbb{M}_d$.
  - The existence of $(M_d)_{d\geq 1}$ is not constructive.
  - How are the random matrix $(M_d)_{d\geq 1}$ realized?
  - How are the corresponding matrix Lévy processes $\{M_d(t)\}_{t\geq 0}$ realized?
  - The jump $\Delta M_d(t) = M_d(t) - M_d(t^-)$ has rank one!

V. Random matrix approach to BP bijection

- Benachy-Georges (05, AP), Cavanal-Duvillard (05, EJP): For $\mu \in I^*$ there is an ensemble of unitary invariant random matrices $(M_d)_{d\geq 1}$, such that with probability one its ESD converges in distribution to $\Lambda(\mu) \in I^{\boxplus}$.
- Some properties and questions:
  - M_d is infinitely divisible in the space of matrices $\mathbb{M}_d$.
  - The existence of $(M_d)_{d\geq 1}$ is not constructive.
  - How are the random matrix $(M_d)_{d\geq 1}$ realized?
  - How are the corresponding matrix Lévy processes $\{M_d(t)\}_{t\geq 0}$ realized?
  - The jump $\Delta M_d(t) = M_d(t) - M_d(t^-)$ has rank one!
  - *Open problem:* $\Delta M_d(t)$ has rank $k \geq 2$.

V. Concrete realization for RM models to BP bijection

- Cavanal-Duvillard (05):

V. Concrete realization for RM models to BP bijection

- Cavanal-Duvillard (05):
  - If μ is Gaussian, Z_d *GUE* independent of $g \stackrel{\mathcal{L}}{=} N(0,1)$

$$M_d = \frac{1}{\sqrt{d+1}}(Z_d + dg\mathrm{I}_d)$$

V. Concrete realization for RM models to BP bijection

- Cavanal-Duvillard (05):
  - If μ is Gaussian, Z_d *GUE* independent of $g \stackrel{\mathcal{L}}{=} N(0,1)$

$$M_d = \frac{1}{\sqrt{d+1}}(Z_d + dg\mathrm{I}_d)$$

  - If μ is Poisson with parameter $c > 0$

$$M_d = \sum_{k=1}^{N} u_k^* u_k$$

V. Concrete realization for RM models to BP bijection

- Cavanal-Duvillard (05):
  - If μ is Gaussian, Z_d *GUE* independent of $g \stackrel{\mathcal{L}}{=} N(0,1)$
    $$M_d = \frac{1}{\sqrt{d+1}}(Z_d + dg\mathrm{I}_d)$$
  - If μ is Poisson with parameter $c > 0$
    $$M_d = \sum_{k=1}^{N} u_k^* u_k$$
- Molina & Rocha-Arteaga (12): If for some 1-dim Lévy process $\{X_t\}_{t\geq 0}$ and for a non random function $h : \mathbb{R}_+ \to \mathbb{R}$
  $$\mu = \mathcal{L}\left(\int_0^\infty h(t)\mathrm{d}X_t\right),$$
  then, there exists a $d \times d$ matrix Lévy process $\mathbf{X}_t$ such that
  $$M_d \stackrel{\mathcal{L}}{=} \int_0^\infty h(t)\mathrm{d}\mathbf{X}_t.$$

V. Concrete realization for RM models to BP bijection

- Cavanal-Duvillard (05):
  - If μ is Gaussian, Z_d *GUE* independent of $g \stackrel{\mathcal{L}}{=} N(0,1)$
  $$M_d = \frac{1}{\sqrt{d+1}}(Z_d + dg\mathrm{I}_d)$$
  - If μ is Poisson with parameter $c > 0$
  $$M_d = \sum_{k=1}^{N} u_k^* u_k$$
- Molina & Rocha-Arteaga (12): If for some 1-dim Lévy process $\{X_t\}_{t\geq 0}$ and for a non random function $h : \mathbb{R}_+ \to \mathbb{R}$
  $$\mu = \mathcal{L}\left(\int_0^\infty h(t)\mathrm{d}X_t\right),$$
  then, there exists a $d \times d$ matrix Lévy process $\mathbf{X}_t$ such that
  $$M_d \stackrel{\mathcal{L}}{=} \int_0^\infty h(t)\mathrm{d}\mathbf{X}_t.$$
- PA-Sakuma (08): $X_t, \mathbf{X}_t$ 1-dim and matrix Gamma processes.

V. Matrix Lévy processes for BP bijection

Molina, PA, Rocha-Arteaga:

- How is the matrix Lévy process $M_d(t)$ realized?

V. Matrix Lévy processes for BP bijection

Molina, PA, Rocha-Arteaga:

- How is the matrix Lévy process $M_d(t)$ realized?
- Simple case: μ $CP(\nu, \psi)$, ν p.m. on $\mathbb{R}$, $\psi \in \mathbb{R}$

$$M_1(t) = t\psi + \sum_{j=1}^{N_t} R_j$$

N_t PP independent of $(R_j)_{j \geq 1}$, i.i.d, $\mathcal{L}(R_j) = \nu$.

V. Matrix Lévy processes for BP bijection

Molina, PA, Rocha-Arteaga:

- How is the matrix Lévy process $M_d(t)$ realized?
- Simple case: μ $CP(\nu, \psi)$, ν p.m. on $\mathbb{R}$, $\psi \in \mathbb{R}$

$$M_1(t) = t\psi + \sum_{j=1}^{N_t} R_j$$

  N_t PP independent of $(R_j)_{j\geq 1}$, i.i.d, $\mathcal{L}(R_j) = \nu$.
- $\Lambda(\mu) = \nu \boxtimes \mathrm{m}_1$, free multiplicative convolution, m_1 is MP.

V. Matrix Lévy processes for BP bijection

Molina, PA, Rocha-Arteaga:

- How is the matrix Lévy process $M_d(t)$ realized?
- Simple case: μ $CP(\nu, \psi)$, ν p.m. on $\mathbb{R}$, $\psi \in \mathbb{R}$

$$M_1(t) = t\psi + \sum_{j=1}^{N_t} R_j$$

N_t PP independent of $(R_j)_{j \geq 1}$, i.i.d, $\mathcal{L}(R_j) = \nu$.

- $\Lambda(\mu) = \nu \boxtimes \mathrm{m}_1$, free multiplicative convolution, m_1 is MP.
- For each $d \geq 2$

$$M_d(t) = \psi t \mathrm{I}_d + \sum_{j=1}^{N_t} R_j u_j^* u_j$$

$(u_j)_{j \geq 1}$ independent d-vectors uniform on unit sphere of $\mathbb{C}^d$, independent of (N_t) and $(R_j)_{j \geq 1}$.

V. Matrix Lévy processes for BP bijection

Molina, PA, Rocha-Arteaga:

- $$M_d(t) = \psi t \mathrm{I}_d + \sum_{j=1}^{N_t} R_j u_j^* u_j$$

V. Matrix Lévy processes for BP bijection

Molina, PA, Rocha-Arteaga:

- $$M_d(t) = \psi t \mathrm{I}_d + \sum_{j=1}^{N_t} R_j u_j^* u_j$$
- Realization as quadratic covariation $M_d(t) = [X_d, Y_d]_t$:

V. Matrix Lévy processes for BP bijection

Molina, PA, Rocha-Arteaga:

- $$M_d(t) = \psi t \mathrm{I}_d + \sum_{j=1}^{N_t} R_j u_j^* u_j$$
- Realization as quadratic covariation $M_d(t) = [X_d, Y_d]_t$:
- $\{X_d(t)\}_{t\geq 0}$, $\{Y_d(t)\}_{t\geq 0}$ are $\mathbb{C}_d$-Lévy processes

$$X_d(t) = \sqrt{|\psi|}B_t + \sum_{j=1}^{N_t} \sqrt{|R_j|}u_j, \quad t \geq 0,$$

$$Y_d(t) = \mathrm{sign}(\psi)\sqrt{|\psi|}B_t + \sum_{j=1}^{N_t} \mathrm{sign}\,(R_j)\,\sqrt{|R_j|}u_j, \quad t \geq 0,$$

$\{B_t\}$ is $\mathbb{C}_d$-Brownian motion independent of (R_j), (u_j), $\{N_t\}$.

V. Open problems

- Lecture 2: Matrix Brownian motion $B_n(t) = (b_{ij}(t)), t \geq 0$
  - $(\lambda_1(t), \cdots, \lambda_n(t))$ eigenvalues process of $B_n(t)$.
  - Dyson-Brownian motion: $\exists_n$ n independent 1-dim Brownian motions $b_1^{(n)}, \ldots, b_n^{(n)}$ such that if $\lambda_{n,1}(0) < \cdots < \lambda_{n,n}(0)$

$$\lambda_{n,i}(t) = \lambda_{n,i}(0) + b_i^{(n)}(t) + \sum_{j \neq i} \int_0^t \frac{1}{\lambda_{n,j}(s) - \lambda_{n,i}(s)} \mathrm{d}s.$$

  - Corresponding measure valued process

$$\mu_t^{(n)} = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_{n,j}(t)},$$

converges weakly in $C(\mathbb{R}_+ \mathcal{P}(\mathbb{R}))$ to $\{\mathrm{w}_t, t \geq 0\}$.

- **Open problems:**

V. Open problems

- Lecture 2: Matrix Brownian motion $B_n(t) = (b_{ij}(t)), t \geq 0$
  - $(\lambda_1(t), \cdots, \lambda_n(t))$ eigenvalues process of $B_n(t)$.
  - Dyson-Brownian motion: $\exists_n$ n independent 1-dim Brownian motions $b_1^{(n)}, \ldots, b_n^{(n)}$ such that if $\lambda_{n,1}(0) < \cdots < \lambda_{n,n}(0)$

$$\lambda_{n,i}(t) = \lambda_{n,i}(0) + b_i^{(n)}(t) + \sum_{j \neq i} \int_0^t \frac{1}{\lambda_{n,j}(s) - \lambda_{n,i}(s)} \mathrm{d}s.$$

  - Corresponding measure valued process

$$\mu_t^{(n)} = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_{n,j}(t)},$$

converges weakly in $C(\mathbb{R}_+ \mathcal{P}(\mathbb{R}))$ to $\{\mathrm{w}_t, t \geq 0\}$.

- **Open problems:**
  - Dyson process associated to the matrix Lévy process $M_d(t)$?

V. Open problems

- Lecture 2: Matrix Brownian motion $B_n(t) = (b_{ij}(t)), t \geq 0$
  - $(\lambda_1(t), \cdots, \lambda_n(t))$ eigenvalues process of $B_n(t)$.
  - Dyson-Brownian motion: $\exists_n$ n independent 1-dim Brownian motions $b_1^{(n)}, \ldots, b_n^{(n)}$ such that if $\lambda_{n,1}(0) < \cdots < \lambda_{n,n}(0)$

$$\lambda_{n,i}(t) = \lambda_{n,i}(0) + b_i^{(n)}(t) + \sum_{j \neq i} \int_0^t \frac{1}{\lambda_{n,j}(s) - \lambda_{n,i}(s)} \mathrm{d}s.$$

  - Corresponding measure valued process

$$\mu_t^{(n)} = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_{n,j}(t)},$$

converges weakly in $C(\mathbb{R}_+ \mathcal{P}(\mathbb{R}))$ to $\{\mathrm{w}_t, t \geq 0\}$.

- **Open problems:**
  - Dyson process associated to the matrix Lévy process $M_d(t)$?
  - Asymptotics for corresponding measure valued process?

References for free ID: analytic approach

- H. Bercovici & D. Voiculescu (1993). Free convolution of measures with unbounded supports. *Indiana Univ. Math. J.*
- O. Arizmendi, O.E. Barndorff-Nielsen & VPA (2009). On free and classical type G distributions. *Rev. Braz. Probab. Statist.*
- Belinschi S. & A. Nica (2008). A remarkable semigroup with respect to multiplicative convolution. *Adv. Math.*
- VPA & Sakuma Noriyoshi (2008). Free generalized gamma convolutions. *Elect. Comm. Probab.*
- O. Arizmendi and VPA (2010). On the non-classical infinite divisibility of power semicircle distributions. *COSA.*.
- VPA & Sakuma Noriyoshi (2012). Free multiplicative convolutions of free multiplicative mixtures of the Wigner distribution. *J. Theoretical Probab.*
- O. Arizmendi, T. Hasebe & N. Sakuma (2011). On free regular infinitely divisible distributions. arXiv:1201.0311.
- O. Arizmendi & Hasebe (2011). On a class of explicit Cauchy-Stieltjes transforms related to monotone stable and free Poisson laws. *Bernoulli.*

References for Bercovici-Pata bijection

- H. Bercovici & V. Pata with an appendix by P. Biane (1999). Stable laws and domains of attraction in free probability theory. *Ann. Math.*
- O. E. Barndorff-Nielsen & S. Thorbjørnsen (2004). A connection between free and classical infinite divisibility. *Inf. Dim. Anal. Quantum Probab.*
- O. E. Barndorff-Nielsen and S. Thorbjørnsen (2006). Classical and free infinite divisibility and Lévy processes. *LNM* 1866.
- F. Benaych-Georges, F. (2005). Classical and free i.d. distributions and random matrices. *Ann. Probab.*
- T. Cabanal-Duvillard (2005): A matrix representation of the Bercovici-Pata bijection. *Electron. J. Probab.*
- A. Dominguez & A. Rocha Arteaga (2012). Random matrix models of stochastic integral type for free infinitely divisible distributions. *Period. Math. Hung.*

References for free multiplicative convolutions

- D. Voiculescu (1987). Multiplication of certain non-commuting random variables. *J. Operator Theory.*
- H. Bercovici & D. Voiculescu (1993). Free convolution of measures with unbounded supports. *Indiana Univ. Math. J.*
- H. Bercovici & J.C. Wang (2008). Limit theorems for free multiplicative convolutions. *Trans. Amer. Math. Soc.*
- N. Raj Rao & R. Speicher (2007). Multiplication of free random variables and the S-transform: The case of vanishing mean. *Elect. Comm. Probab.*
- O. Arizmendi & VPA (2009). *The S-transform of symmetric probability measures with unbounded supports. Proc. Amer. Math. Soc.*

References for Lévy matrix modelling

- O.E. Barndorff-Nielsen & VPA (2008). Matrix subordinators and related Upsilon transformations. *Theory Probab. Appl.*
- O.E. Barndorff-Nielsen & R. Stelzer (2011). The multivariate supOU stochastic volatility model. *Math. Finance.*
- O.E. Barndorff-Nielsen & R. Stelzer (2011): Multivariate supOU processes. *Ann. Appl. Probab.*
- VPA & R. Stelzer (2012). A class of ID multivariate and matrix Gamma distributions and cone-valued GGC.
- C. Pigorsch & R. Stelzer (2009). A multivariate Ornstein-Uhlenbeck type stochastic volatility model.
- R. Stelzer (2010). Multivariate COGARCH(1, 1) processes. *Bernoulli.*

Matrix covariation

- If X, Y are $\mathbb{M}_{p\times r}$-semimartingales
$$[X,Y] := ([X,Y]_t)_{t\geq 0}$$
$$[X,Y]^{ij}_t = \sum_{k=1}^{q} [x_{ik}, y_{kj}]_t \, .$$
- In general,
$$[X,Y]_t = X_0 Y_0 + [X^c, Y^c]_t + \sum_{s\leq t} (\Delta X_s)(\Delta Y_s) \, ,$$
$$[X^c, Y^c]^{ij}_t := \textstyle\sum_{k=1}^{q} [x_{ik}, y_{kj}]^c_t \, .$$
- If continuous part is zero
$$[X,Y]_t = X_0 Y_0 + \sum_{s\leq t} (\Delta X_s)(\Delta Y_s) \, .$$
- It holds
$$X_t Y_t = \int_0^t X_{s^-} \mathrm{d} Y_s + \int_0^t \mathrm{d} X_s Y_{s_-} + [X,Y]_t .$$

Infinitely divisible random matrices

Lévy-Khintchine representation

- Random matrix M is ID iff its Fourier transform $\mathbb{E}e^{\mathrm{itr}(\Theta^* M)} = \exp(\psi(\Theta))$ has Laplace exponent

$$\psi(\Theta) = \mathrm{itr}(\Theta^* \Psi\ ) - \frac{1}{2}\mathrm{tr}\left(\Theta^* \mathcal{A}\Theta^*\right) + \int_{\mathbb{M}_d} \left(e^{\mathrm{itr}(\Theta^* \xi)} - 1 - \mathrm{i}\frac{\mathrm{tr}(\Theta^* \xi)}{1 + \|\xi\|^2} \right) \nu(\mathrm{d}\xi),$$

  - $\Psi \in \mathbb{M}_d$
  - $\mathcal{A} : \mathbb{M}_d \to \mathbb{M}_d$ positive symmetric operator
  - ν Lévy measure on $\mathbb{M}_d$, $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{M}_d} (\|x\|^2 \wedge 1)\nu(\mathrm{d}x) < \infty.$$

  - The triplet $(\mathcal{A}, \nu, \Psi)$ is unique.
- Scalar product $\mathrm{tr}\,(AB^*)$, norm $\|A\| = \left[\mathrm{tr}\,(AA^*)\right]^{1/2}$.