# PCD Simulation with Uniform Distribution on a Klein Bottle

## *Part 1: Hausdorff Measure and Simulation*


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**Abstract**

Diaconis et al. [2] proposed a method to simulate point cloud data (PCD) with uniform distribution over a manifold, illustrating the case of the 2-torus. We use this method and a parametrization of the Klein bottle due to Franzoni [4] in 2012 to simulate PCD on this non-orientable surface. We also simulate uniformly on the parameters and compare the corresponding persistence summaries. Furthermore we discuss the convergence of the Hausdorff distance between these and other point clouds, leading to get some insight about the concentration inequality in Fasy et al. [3].


## Introduction

The Klein bottle emerges in many contexts. In Carlsson et al. [1] it is seen that a high-density submanifold in the space of natural images has topological features of a Klein bottle. They also find an embedding of the Klein bottle in this space with the purpose of developping a compression algorithm. In Martin et al. [5] it is shown that the energy landscape of the cyclo-octane has the structure of a reducible algebraic variety that is the union of a sphere and a Klein bottle.

The simulation of point clouds on the Klein bottle and other manifolds is useful itself as is pointed in Diaconis et al. [2]. These samples are used to calibrate and study the existing topological algorithms. An example of this use is seen in Otter et al. [6].

An extensive review on the Klein bottle can be found in Franzoni [4]. He compiles many representations of it in $\mathbb{R}^3$.

In the first part we present an application of the method by Diaconis et al. [2], where the only example related to this context is the 2-torus. In the second part we simulate point clouds from other distributions.

## Background

Given a parametrized manifold we would like to simulate a point cloud on it such that it is, in some sense, uniformly distributed. We would like to have a notion of uniformity such that any two regions with the same "volume" have equal probability measure. This notion is given by the Hausdorff measure: for every $A \subset \mathbb{R}^d$ the $m$-dimensional Hausdorff measure of $A$ is defined as

$$\mathcal{H}^m(A) = \lim_{\delta \longrightarrow 0} \inf_{A \subseteq \cup B_i,\, \mathrm{diam}(B_i) \leq \delta} \sum \omega_m \left(\frac{\mathrm{diam}(B_i)}{2}\right)^m,$$

where the infimum is taken over all countable coverings $\{B_i\}$ of $A$ and $\omega_m$ is the volume of the unit ball in $\mathbb{R}^m$.

The area formula gives a distribution on the parameters such that a sample from it maps to a uniformly (in the previous sense) distributed sample on the manifold:

$$\int_A g(f(x)) J_m f(x) \lambda^m(dx) = \int_{\mathbb{R}^n} g(y) N(f|_A, y) \mathcal{H}^n(dy),$$

where $f : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz, $m \leq n$, $A \subset \mathbb{R}^m$ is $\lambda^m$-measurable and $N(f|A, y) = \#\{x \in A : f(x) = y\}$.

For a manifold $\mathcal{M}$ parametrized by $f$, we take $g = 1/\mathrm{vol}(\mathcal{M})$ so by the area formula, the desired density on the parameters is given by $J_m f(x)/\mathrm{vol}(\mathcal{M})$.

## Simulations

### Klein Bottle

We use the following parametrization shown in Franzoni [4] for the Klein bottle:

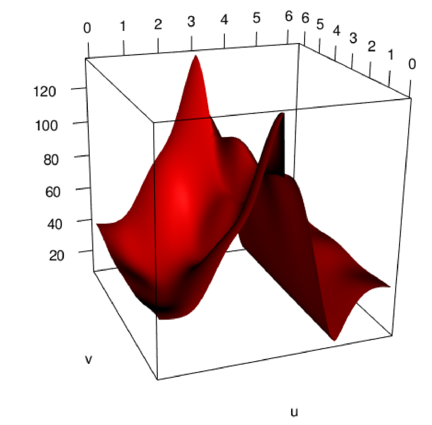
$$x = \begin{cases} 6\cos(u)(1+\sin(u)) + 4(1-\frac{1}{2}\cos(u))\cos(u)\cos(v), & 0 \leq u \leq \pi, \\ 6\cos(u)(1+\sin(u)) + 4(1-\frac{1}{2}\cos(u))\cos(v+\pi), & \pi < u \leq 2\pi, \end{cases}$$

$$y = \begin{cases} 16\sin(u) + 4(1-\frac{1}{2}\cos(u))\sin(u)\cos(v), & 0 \leq u \leq \pi, \\ 16\sin(u), & \pi < u \leq 2\pi, \end{cases}$$

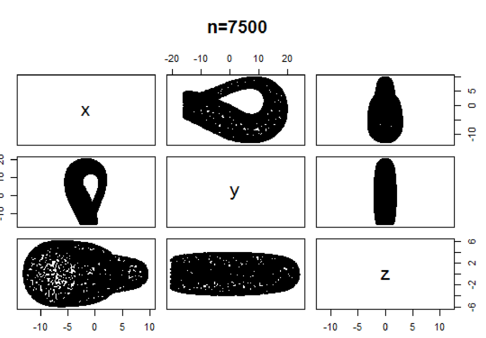
$$z = 4\left(1 - \frac{1}{2}\cos(u)\right)\sin(v),$$

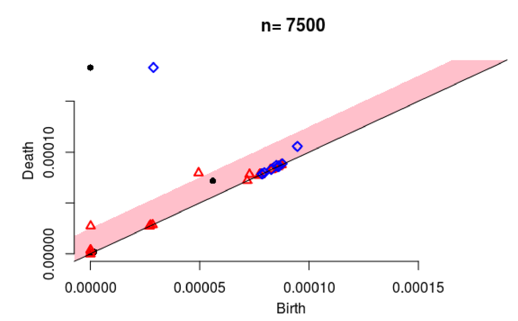
with $(u, v) \in [0, 2\pi] \times [0, 2\pi]$.

The density from which we simulate to obtain the desired sample looks as follows:

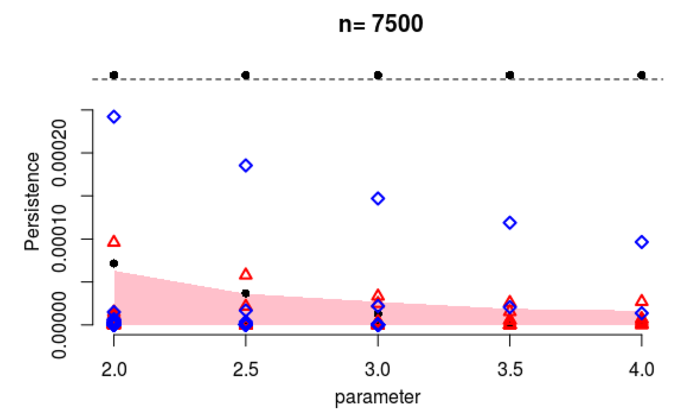


We show an example of one simulation of a 7500 points cloud uniformly distributed over the Klein bottle. We also show a persistence diagram obtained from the Morse-Smale filtration for the same point cloud, with a 95% confidence band and a smoothing parameter of 3.0 for the kernel density estimator:

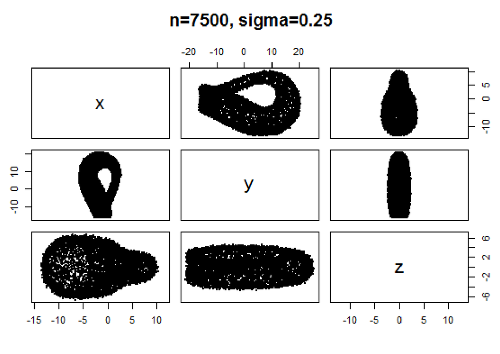


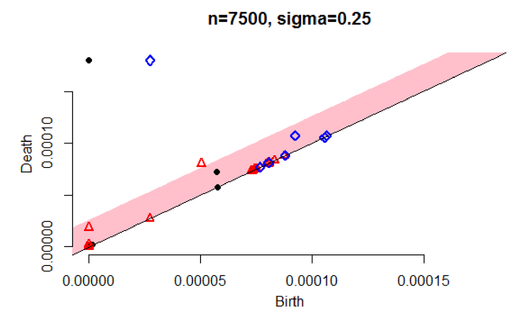



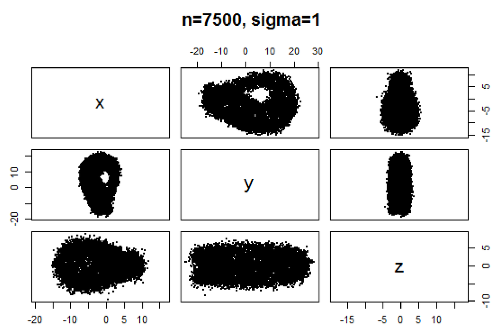

The following graph is printed by the function `maxPersistence` from the `R` library `TDA`.

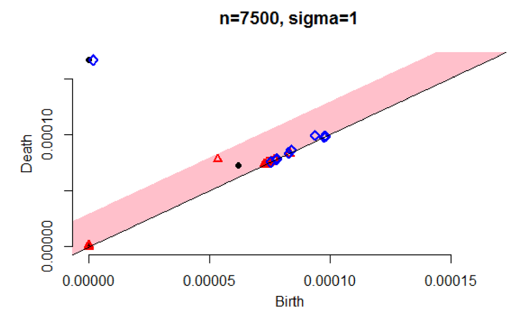



The samples obtained above were noiseless. Now we show examples of persistence diagrams from normal noise. We can see that one of the two cycles is sensitive to noise.




### Moebius Strip

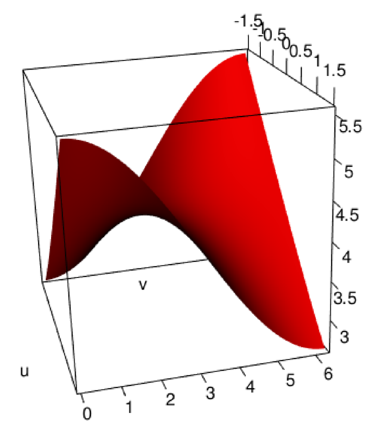
We use the folloing parametrization for the Moebius strip:

$$x = (R + u\cos(\frac{v}{2}))\cos(v),$$

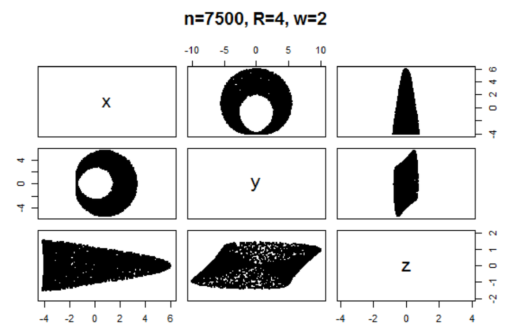
$$y = (R + u\cos(\frac{v}{2}))\sin(v),$$

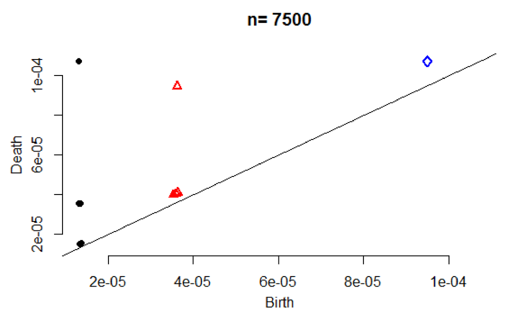
$$z = u\sin(\frac{v}{2}),$$

with $(u, v) \in [-w, w] \times [0, 2\pi]$, where $w > 0$ is the half of the bandwidth and $R$ is the radius. The density from which we simulate to obtain the desired sample looks as follows:

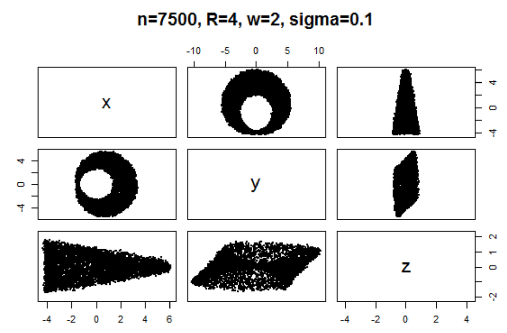


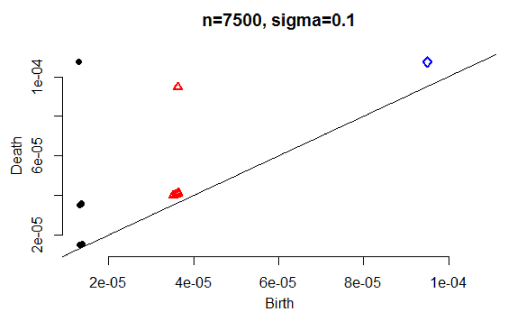
We show an example of a simulation of a point cloud uniformly distributed and a persistence diagram:

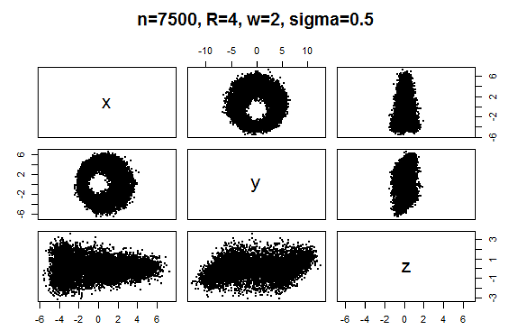


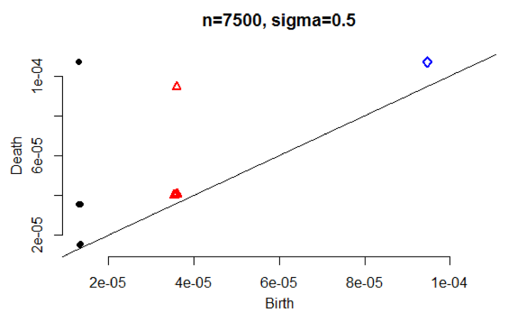




Compared to the Klein bottle case, the change in the persistence diagrams is smaller when we add noise to the Moebius strip.




## References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