Sean $X, Y \in T_eG$ y $\tilde{X}, \tilde{Y} \in \mathfrak{X}(G)$

$\therefore$ $\tilde{X}_g = dL_g(X)$, $dL_g(\tilde{X}) = \tilde{X}$ $\forall g \in G$
$\tilde{Y}_g = dL_g(Y)$, $dL_g(\tilde{Y}) = \tilde{Y}$

$\tilde{X}$ está L_g-relacionado con $\tilde{X}$
$\tilde{Y}$ " " " $\tilde{Y}$

$\Rightarrow$ $[\tilde{X}, \tilde{Y}]$ está L_g-relaciondo con $[\tilde{X}, \tilde{Y}]$

$\therefore$ $dL_g([\tilde{X}, \tilde{Y}]) = [\tilde{X}, \tilde{Y}]$ $\forall g \in G$

Si $X, Y \in \mathfrak{X}(G)$:

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

pero si $X, Y \in T_eG$:

$$X(Y(f)) - Y(X(f))$$

no tiene sentido.
Sin embargo:

$$[X, Y] = [\tilde{X}, \tilde{Y}]_e$$

$$[X, Y](f) = (\tilde{X}(\tilde{Y}(f)) - \tilde{Y}(\tilde{X}(f)))_e$$

Sea $\mathcal{A}$ un álgebra asociativa.

$x, y \in \mathcal{A}$ $\quad x \cdot y = xy$

el conmutador de x, y:

$$[x, y] = xy - yx$$

Afirmación: $[\ ,\ ]$ satisface Jacobi:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

En particular

M variedad

$\mathfrak{X}(M)$ es álgebra de Lie

con $[X, Y] = XY - YX$

Para grupos de Lie G, como:

$$[X, Y] = [\tilde{X}, \tilde{Y}]_e \qquad \forall X, Y \in T_eG$$

$\therefore$ $\mathfrak{g} = (T_eG, [\ ,\ ])$ es álgebra de Lie.

$\mathfrak{a}$ álgebra de Lie
$\mathfrak{b} \trianglelefteq \mathfrak{a}$ ideal
$\Rightarrow$ $\mathfrak{a}/\mathfrak{b}$ es álgebra de Lie con

$$[X+\mathfrak{b}, Y+\mathfrak{b}] = [X,Y]+\mathfrak{b}$$

$$X_1 = X+Z_1 \, , \quad Y_1 = Y+Z_2 \, , \quad Z_1, Z_2 \in \mathfrak{b}.$$

$$[X_1, Y_1] = [X,Y] + \underbrace{[X,Z_2]+[Z_1,Y]+[Z_1,Z_2]}_{\text{en } \mathfrak{b}}$$

Tenemos

$(\mathfrak{a}, [\,,\,])$

$X \in \mathfrak{a} \qquad ad(X): \mathfrak{a} \longrightarrow \mathfrak{a}$

$$ad(X)(Y) = [X,Y]$$

$$ad([X,Y])(Z) = [[X,Y],Z] = -[Z,[X,Y]]$$

$$(ad(X)\,ad(Y) - ad(Y)\,ad(X))(Z) =$$
$$= [X,[Y,Z]] - [Y,[X,Z]]$$
$$= [X,[Y,Z]] + [Y,[Z,X]]$$

$\therefore$ Jacobi implica:

$$ad([X,Y]) = [ad(X), ad(Y)]$$

Luego:

$$ad : \mathfrak{a} \longrightarrow \mathfrak{gl}(\mathfrak{a})$$

es una representación (la rep. adjunta)

V espacio vectorial

álgebra tensorial de V es:

$$T(V) = k \oplus V \oplus (V \otimes V) \oplus \ldots$$
$$= \bigoplus_{n=0}^{+\infty} V^{\otimes n}$$

$X \in T(V)$:

$$X = \sum_i a_i v_{i1} \otimes \ldots \otimes v_{ij_i}$$

$$Y = \sum_i b_i u_{i1} \otimes \ldots \otimes u_{ik_i}$$

$$X \otimes Y = \sum_{r,s} a_r b_s v_{r1} \otimes \ldots \otimes v_{rj_r} \otimes u_{s1} \otimes \ldots \otimes u_{sk_s}$$

$T(V)$ es álgebra asociativa

$V \subseteq T(V)$

$T(V)$ es de Lie con

$$X \otimes Y - Y \otimes X \ , \ X, Y \in T(V)$$

Si $\mathfrak{a}$ es álgebra de Lie, tomamos

$T(\mathfrak{a})$ asociativa con $\otimes$ y de Lie con $X \otimes Y - Y \otimes X$

$\mathfrak{a} \subseteq T(\mathfrak{a})$

Si $X, Y \in \mathfrak{a}$, $[X, Y] \in \mathfrak{a}$, pero

$X \otimes Y - Y \otimes X \in \mathfrak{a} \otimes \mathfrak{a}$

y $\mathfrak{a} \cap (\mathfrak{a} \otimes \mathfrak{a}) = 0$.

Sea $J \subseteq T(\mathfrak{a})$ el ideal (bilateral) generado por:

$$X \otimes Y - Y \otimes X - [X, Y] \quad \text{con } X, Y \in \mathfrak{a}.$$

El álgebra universal envolvente de $\mathfrak{a}$ es:

$$U(\mathfrak{a}) = T(\mathfrak{a}) / J$$

con el producto

$$X, Y \in U(\mathfrak{a})$$

$$X = X_1 + J, \quad Y = Y_1 + J, \quad X_1, Y_1 \in T(\mathfrak{a})$$

$$XY = (X_1 + J)(Y_1 + J) = X_1 \otimes Y_1 + J$$

En primer lugar:

$$\mathfrak{a} \hookrightarrow T(\mathfrak{a}) \xrightarrow{\pi} T(\mathfrak{a})/J = U(\mathfrak{a})$$

$$X \longmapsto X^*$$

y además: $\forall X, Y \in \mathfrak{a}$

$$[X^*, Y^*] = [X+J, Y+J]$$

$$= (X+J)(Y+J) - (Y+J)(X+J)$$

$$= (X \otimes Y - Y \otimes X) + J$$

$$= [X, Y] + J = [X, Y]^*$$

Es decir:

$$\mathfrak{a} \longrightarrow U(\mathfrak{a})$$

$$X \longmapsto X^*$$

asociativa y por tanto de Lie

es homomorfismo de álgebras de Lie.

Veremos que si $\mathfrak{a} = \mathfrak{g}$ el álgebra de Lie de un grupo de Lie G, entonces $U(\mathfrak{g})$ es el álgebra asociativa de operadores diferenciales en G invariantes por la izquierda

La proposición 1.1 se lee como sigue:

$$\begin{aligned} \mathfrak{a} &\xrightarrow{\varphi} U(\mathfrak{a}) \\ X &\longmapsto X^* \end{aligned}$$

Dada $\rho : \mathfrak{a} \longrightarrow \mathfrak{gl}(V)$ rep. de Lie $\exists !\ \rho^* : U(\mathfrak{a}) \longrightarrow \mathfrak{gl}(V)$ rep. asociativa $\ni$:

$$\begin{tikzcd} \mathfrak{a} \arrow[r, "\rho"] \arrow[dr, "\varphi"'] & \mathfrak{gl}(V) \\ & U(\mathfrak{a}) \arrow[u, "\rho^*"'] \end{tikzcd}$$

i.e.: $\rho^*(X^*) = \rho^*(\varphi(X)) = \rho(X)$

$\forall X \in \mathfrak{a}$.

Dem.

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\rho"] \arrow[d, hook] & \mathfrak{gl}(V) \\
T(\mathfrak{g}) \arrow[r, "\pi"'] \arrow[ur, dashed, "\tilde{\rho}"] & T(\mathfrak{g})/J = U(\mathfrak{g}) \arrow[u, dashed, "\rho^*"']
\end{tikzcd}$$

ρ lineal $\Rightarrow \exists\ \tilde{\rho} : T(\mathfrak{g}) \longrightarrow \mathfrak{gl}(V)$
rep. asociativa

ρ homomorfismo de Lie $\Rightarrow \tilde{\rho}(J) = 0$

$\Rightarrow \exists\ \rho^* : U(\mathfrak{g}) \longrightarrow \mathfrak{gl}(V)$
rep. asociativa.

$n = 2$, $\quad X^*(t) = t_1 X_1^* + t_2 X_2^*$

$M : \ (1, 0)$, $|M| = 1$

$X^*(t)^1$, $t^M = t_1^1 t_2^0 = t_1$

$X^*(1, 0) = X_1^*$

$M = (1,1)$

$|M| = 2 \;, \quad t^M = t_1^1 t_2^1 = t_1 t_2$

$$X^*(t)^2 = (t_1 X_1^* + t_2 X_2^*)^2$$

$$= t_1^2 X_1^{*2} + t_2^2 X_2^{*2} +$$

$$+ t_1 t_2 (X_1^* X_2^* + X_2^* X_1^*)$$

$$X^*(1,1) = \frac{1}{2}(X_1^* X_2^* + X_2^* X_1^*)$$

$$X^*(2,0) = \frac{1}{2} X_1^{*2}$$