La representación adjunta de $\mathfrak{a}$ es:

$$\begin{aligned} \mathrm{ad}: \mathfrak{a} &\longrightarrow \mathfrak{gl}(\mathfrak{a}) \\ X &\longmapsto \mathrm{ad}(X) \end{aligned}$$

que es homomorfismo de álgebras de Lie (Jacobi).

$$\begin{tikzcd} \mathfrak{a} \arrow[r, "\mathrm{ad}"] & \mathfrak{gl}(\mathfrak{a}) \supseteq \mathrm{ad}(\mathfrak{a}) \arrow[d, "\exp"'] \arrow[d, leftrightarrow] \\ & GL(\mathfrak{a}) \supseteq \mathrm{Int}(\mathfrak{a}) \end{tikzcd}$$

grupo adjunto

$$\mathrm{Aut}(\mathfrak{a}) = \{ \varphi \in GL(\mathfrak{a}) \mid \varphi([X,Y]) = [\varphi(X), \varphi(Y)] \quad \forall X, Y \in \mathfrak{a} \}$$

es cerrado en $GL(\mathfrak{a})$.

Si $\mathfrak{a}$ es Abeliana:

$$\mathrm{Aut}(\mathfrak{a}) = GL(\mathfrak{a}).$$

$$D \in \partial(\mathfrak{a}) = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{a}))$$

$$\iff e^{tD} \in \mathrm{Aut}(\mathfrak{a})$$

$$\iff e^{tD}[X,Y] = [e^{tD}X, e^{tD}Y] \quad \forall X,Y \in \mathfrak{a} \quad \forall t \in \mathbb{R}$$

$$\overset{\frac{d}{dt}\big|_{t=0}}{\Longrightarrow} D[X,Y] =$$

$$= [DX,Y] + [X,DY] \quad \forall X,Y \in \mathfrak{a}$$

D es derivación.

Para $\Leftarrow$):

$$e^{tD}[X,Y] = \sum_{h=0}^{+\infty} \frac{t^h D^h}{h!}[X,Y]$$

$$= \sum_{h=0}^{+\infty} \frac{t^h}{h!} \sum_{i+j=h} \frac{h!}{i!\,j!}[D^iX, D^jY]$$

$$= \sum_{h=0}^{+\infty} \sum_{i+j=h} \frac{t^i}{i!} \frac{t^j}{j!} [D^iX, D^jY]$$

$$= \sum_{i,j=0}^{+\infty} \frac{t^i}{i!}\frac{t^j}{j!}[D^i X, D^j Y]$$

$$= [e^{tD}X, e^{tD}Y]$$

$$\therefore \quad \mathrm{Lie}(\mathrm{Aut}(\mathfrak{a})) = \partial(\mathfrak{a})$$

De hecho para cualquier tipo de álgebra A: $(\dim < +\infty)$

$$\mathrm{Lie}(\mathrm{Aut}(A)) = \partial(A).$$

Jacobi es equivalente a:

$$\mathrm{ad}(X)[Y,Z] = [\mathrm{ad}(X)(Y),Z] + [Y, \mathrm{ad}(X)(Z)]$$

$$\therefore \quad \mathrm{ad}(\mathfrak{a}) \subseteq \partial(\mathfrak{a})$$

$$\therefore \quad \mathrm{Int}(\mathfrak{a}) \leq \mathrm{Aut}(\mathfrak{a})$$

pues:

$\mathrm{Int}(\mathfrak{a}) =$ generado por $\exp(\mathrm{ad}(\mathfrak{a}))$

$\exp(\mathrm{ad}(\mathfrak{a})) \subseteq \exp(\partial(\mathfrak{a})) \subseteq \mathrm{Aut}(\mathfrak{a})$

Si $s \in \mathrm{Aut}(\mathfrak{a})$:

$$\begin{tikzcd}
\partial(\mathfrak{a}) \arrow[r, "d\sigma_e"] \arrow[d, "\exp"'] & \partial(\mathfrak{a}) \arrow[d, "\exp"] \\
\mathrm{Aut}(\mathfrak{a}) \arrow[r, "\sigma"] & \mathrm{Aut}(\mathfrak{a}) \\
g \arrow[r, mapsto] & sgs^{-1}
\end{tikzcd}$$

Vemos que:

$$\mathrm{Aut}(\mathfrak{a}), \partial(\mathfrak{a}) \subseteq \mathfrak{gl}(\mathfrak{a}) = \mathrm{End}(\mathfrak{a})$$

y σ es restricción de:

$$\begin{aligned}
\mathrm{End}(\mathfrak{a}) &\longrightarrow \mathrm{End}(\mathfrak{a}) \\
A &\longmapsto sAs^{-1}
\end{aligned}$$

que es $\mathbb{R}$-lineal.

$\therefore \quad d\sigma_e(X) = sXs^{-1}$

Esto va junto con el hecho de que:

$$Ae^{B}A^{-1} = e^{ABA^{-1}}$$

Por otro lado:

$$s\,ad(X)\,s^{-1}(Y) =$$
$$= s[X, s^{-1}Y] = [sX, Y]$$
$$= ad(s(X))$$

$$\therefore \quad d\sigma_e(ad(u)) = s\,ad(u)\,s^{-1}$$
$$= ad(u)$$

Por tanto:

$$s\exp(ad(u))\,s^{-1} =$$
$$= \sigma(\exp(ad(u)))$$
$$= \exp(d\sigma_e(ad(u)))$$
$$= \exp(ad(u))$$

Como $\exp(\mathrm{ad}(\mathfrak{a}))$ genera a $\mathrm{Int}(\mathfrak{a})$

$\Rightarrow s\,\mathrm{Int}(\mathfrak{a})\,s^{-1} = \mathrm{Int}(\mathfrak{a})$.

$\therefore \quad \mathrm{Int}(\mathfrak{a}) \trianglelefteq \mathrm{Aut}(\mathfrak{a})$.

$$\left(\begin{array}{l} \mathrm{ad}(X) = \\ \quad = [X, \ ] \end{array}\right)$$

Mapeo Ad:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad} = dI_e"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "I"] & G \end{tikzcd}$$

$$g \longmapsto \sigma g \sigma^{-1}$$

$\therefore \quad \mathrm{Ad}(\sigma) \in \mathrm{Aut}(\mathfrak{g})$

$\therefore \quad \mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g}) \subseteq GL(\mathfrak{g})$

rep. adjunta de G.

$\mathrm{Ad}(\sigma_1\sigma_2) = \mathrm{Ad}(\sigma_1)\,\mathrm{Ad}(\sigma_2)$ por la regla de la cadena.

Obs.: $\mathrm{Ad}(Z(G)) = \{ I_{\mathfrak{g}} \}$

$$Ad(\sigma) = dI_e = \frac{\partial C}{\partial g}(\sigma, e)$$

$$C(\sigma, g) = \sigma g \sigma^{-1}$$

$\therefore$ Ad es analítico.

En la página 128 se prueba que:

$$d(Ad)_e = ad$$

o se tenemos:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "ad"] \arrow[d, "exp"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "e"] \\ G \arrow[r, "Ad"] & GL(\mathfrak{g}) \end{tikzcd} \qquad ad(X) = [X, \ ]$$

es conmutativo. Recordar:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "Ad(\sigma)"] \arrow[d, "exp"'] & \mathfrak{g} \arrow[d, "exp"] \\ G \arrow[r, "I_\sigma"] & G \\ g \arrow[r, mapsto] & \sigma g \sigma^{-1} \end{tikzcd}$$