

El álgebra de Lie de $U(n)$ es:

$$\mathfrak{u}(n) = \{ A \in M_{n \times n}(\mathbb{C}) \mid A^* = -A \}$$

$$\therefore \mathfrak{u}(n) \subseteq M_{n \times n}(\mathbb{C})$$

¿ $\mathfrak{u}(n)$ es subespacio complejo? NO.

$$A, B \in \mathfrak{u}(n) \implies A + B \in \mathfrak{u}(n)$$

$$a \in \mathbb{R}, A \in \mathfrak{u}(n):$$

$$(aA^*) = aA^* = -aA$$

$$aA \in \mathfrak{u}(n)$$

$$A \in \mathfrak{u}(n): (iA)^* = -iA^* = iA$$

$$\therefore iA \notin \mathfrak{u}(n)$$

Para la página 128:

$$\begin{aligned}\exp(\text{Ad}(\exp(tX))(tY)) &= \\ &= \exp(tX) \exp(tY) \exp(-tX)\end{aligned}$$

∴ lema 1.8:

$$= \exp(tY + t^2[X, Y] + \mathcal{O}(t^3))$$

Si $|t| < 1$, podemos usar inyectividad de \exp :

$$\begin{aligned}\text{Ad}(\exp(tX))(tY) &= \\ &= tY + t^2[X, Y] + \mathcal{O}(t^3)\end{aligned}$$

$$\begin{aligned}\therefore \text{Ad}(\exp(tX))(Y) &= \\ &= Y + t[X, Y] + \mathcal{O}(t^2)\end{aligned}$$

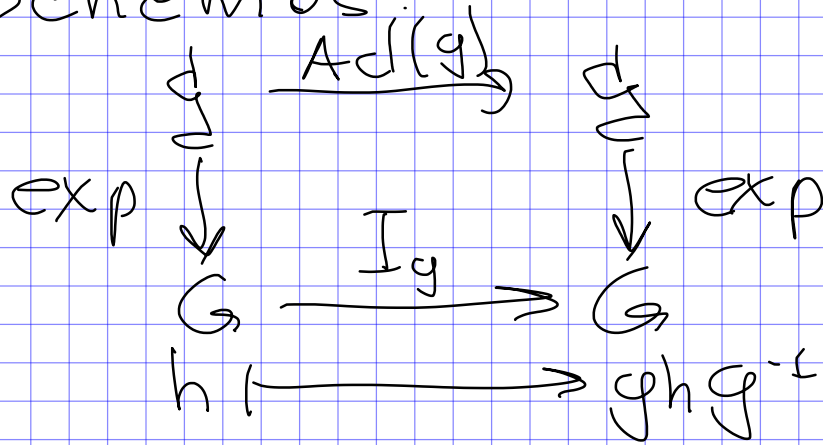
y tomamos $\frac{d}{dt}\Big|_{t=0}$:

$$d(\text{Ad})_e(X)(Y) = [X, Y]$$

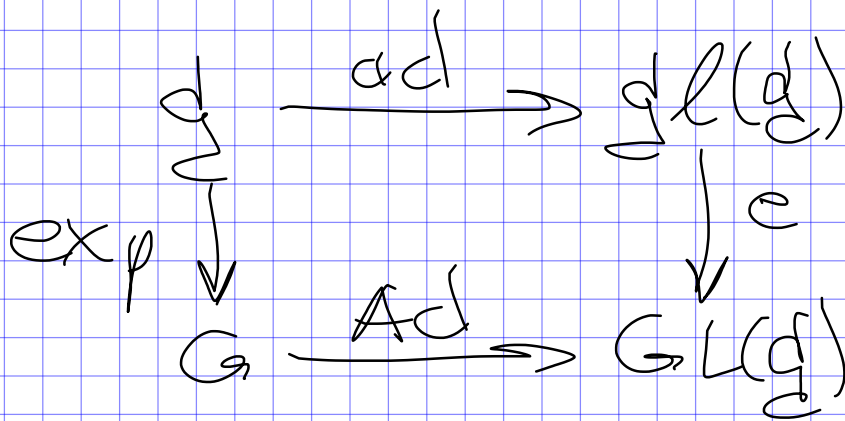
$$\therefore d(\text{Ad})_e(X) = \text{ad}(X)$$

$$d(\text{Ad})_e = \text{ad}$$

Obtenemos:



$$\begin{aligned}
 g \exp(x) g^{-1} &= \\
 &= \exp(\text{Ad}(g)(x))
 \end{aligned}$$



$$\begin{aligned}
 \text{Ad}(\exp(x)) &= \\
 &= e^{\text{ad}(x)}
 \end{aligned}$$

En la página 128 y por estos diagramas se tiene:

G grupo de Lie
conexo

$H \leq G$ subgrupo de Lie
conexo

Entonces:

$$H \trianglelefteq G \iff \mathfrak{h} \trianglelefteq \mathfrak{g}$$

\Rightarrow):

$$Y \in \mathfrak{h}, X \in \mathfrak{g}: [X, Y] \in \mathfrak{h}$$

Como $H \trianglelefteq G$, tenemos:

$$\exp(X)\exp(Y)\exp(-X) \in H$$

$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$. (por el primer diagrama)

$$\therefore \exp(\text{Ad}(\exp(X))(Y)) \in H$$

$$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}.$$

$\therefore \forall X \in \mathfrak{g}, Y \in \mathfrak{h}, t \in \mathbb{R}$:

$$\exp(t \text{Ad}(\exp(X))(Y)) \in H$$

Luego:

$$\text{Ad}(\exp(X))(Y) \in \mathfrak{h}$$

$$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}.$$

$$\therefore \text{Ad}(\exp(tX))(Y) \in \mathfrak{h}$$

$$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}, t \in \mathbb{R}$$

Derivando en $t=0$:

$$[X, Y] = d(\text{Ad})_e(X)(Y) \in \mathfrak{h}$$

↑ 2do diagrama

$$\therefore \forall X \in \mathfrak{g}, Y \in \mathfrak{h}.$$

$$\therefore \mathfrak{h} \trianglelefteq \mathfrak{g}.$$

⇐): Suponemos $\mathfrak{h} \trianglelefteq \mathfrak{g}$.

Por probar:

$$ghg^{-1} \in H$$

$$\forall g \in G, h \in H.$$

Como G, H son conexos,
basta ver que:

$$\exp(X)\exp(Y)\exp(-X) \in H$$

$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$, pues $\exp(\mathfrak{g})$ y
 $\exp(\mathfrak{h})$ generan a G y H ,
resp.

Por el primer diagrama basta
ver que:

$$\exp(\text{Ad}(\exp(X))(Y)) \in \mathfrak{H}$$

$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$. Y esto ocurre

$$\Leftrightarrow \exp(\text{Ad}(\exp(X))(tY)) \in \mathfrak{H}$$

$$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}, t \in \mathbb{R}$$

$$\Leftrightarrow \text{Ad}(\exp(X))(Y) \in \mathfrak{h}$$

$$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$$

$$\Leftrightarrow \text{Ad}(\exp(tX))(Y) \in \mathfrak{h}$$

$$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}, t \in \mathbb{R}$$

$$\Leftrightarrow e^{t\text{ad}(X)}(Y) \in \mathfrak{h}$$

$$\uparrow \text{2do diagrama} \quad \forall X \in \mathfrak{g}, Y \in \mathfrak{h}, t \in \mathbb{R}$$

derivando en $t=0$:

$$\Rightarrow [X, Y] \in \mathfrak{h}$$

En esta parte suponemos

$$[X, Y] \in \mathfrak{h} \quad \forall X \in \mathfrak{g}, Y \in \mathfrak{h}.$$

$$\therefore \operatorname{ad}(X)(\mathfrak{h}) \subseteq \mathfrak{h} \quad \forall X \in \mathfrak{g}$$

$$\therefore \operatorname{ad}(X)^k(\mathfrak{h}) \subseteq \mathfrak{h} \quad \forall X \in \mathfrak{g}$$

$$\therefore e^{t \operatorname{ad}(X)}(Y) = \sum_{k=0}^{+\infty} \frac{t^k \operatorname{ad}(X)^k(Y)}{k!} \in \mathfrak{h}$$

$$\forall X \in \mathfrak{g}, Y \in \mathfrak{h} \\ t \in \mathbb{R}$$

Por las equivalencias anteriores esto implica $H \trianglelefteq G$.

Resumiendo ambos argumentos:

$$H \trianglelefteq G \iff \mathfrak{h} \trianglelefteq \mathfrak{g}$$

Como H y G son conexos:

$$H \trianglelefteq G$$

$$\iff \exp(X) \exp(Y) \exp(-X) \in H \\ \forall X \in \mathfrak{g}, Y \in \mathfrak{h}.$$

$$\Leftrightarrow \exp(\text{Ad}(\exp(X))(Y)) \in \mathfrak{h} \quad \forall X \in \mathfrak{g}, Y \in \mathfrak{h}.$$

(primer diagrama)

$$\Leftrightarrow \exp(t \text{Ad}(\exp(X))(Y)) \in \mathfrak{h} \quad \forall X \in \mathfrak{g}, Y \in \mathfrak{h}, t \in \mathbb{R}$$

$$\Leftrightarrow \text{Ad}(\exp(X))(Y) \in \mathfrak{h} \quad \forall X \in \mathfrak{g}, Y \in \mathfrak{h}.$$

$$\Leftrightarrow e^{\text{ad}(X)}(Y) \in \mathfrak{h} \quad \forall X \in \mathfrak{g}, Y \in \mathfrak{h}$$

(segundo diagrama)

$$\Leftrightarrow e^{t \text{ad}(X)}(Y) \in \mathfrak{h} \quad \forall X \in \mathfrak{g}, Y \in \mathfrak{h}, t \in \mathbb{R}$$

$$\Leftrightarrow \text{ad}(X)(Y) \in \mathfrak{h} \quad \forall X \in \mathfrak{g}, Y \in \mathfrak{h}$$

$$\Leftrightarrow \mathfrak{h} \subseteq \mathfrak{g}$$

Solamente resta ver $(*)$.

$(*) \Rightarrow$): se obtiene derivando en $t=0$.

$(*) \Leftarrow$): $\text{ad}(X)(Y) \in \mathfrak{h} \quad \forall X \in \mathfrak{g}, Y \in \mathfrak{h}$
 $\Rightarrow \text{ad}(X)^k(\mathfrak{h}) \subseteq \mathfrak{h} \quad \forall X \in \mathfrak{g}, k$
 $\Rightarrow \text{ad}(X)^k(Y) \in \mathfrak{h} \quad \forall X \in \mathfrak{g}, Y \in \mathfrak{h}, k$
 $\Rightarrow e^{t \text{ad}(X)}(Y) \in \mathfrak{h} \quad \forall X \in \mathfrak{g}, Y \in \mathfrak{h}, t \in \mathbb{R}.$