El álgebra de Lie de $U(n)$ es:

$$\mathfrak{u}(n) = \{A \in M_{n\times n}(\mathbb{C}) \mid A^* = -A\}$$

$\therefore$ $\mathfrak{u}(n) \subseteq M_{n\times n}(\mathbb{C})$

¿ $\mathfrak{u}(n)$ es subespacio complejo? NO.

$A, B \in \mathfrak{u}(n) \Rightarrow A+B \in \mathfrak{u}(n)$

$a \in \mathbb{R}$, $A \in \mathfrak{u}(n)$:

$$(aA^*) = aA^* = -aA$$

$$aA \in \mathfrak{u}(n)$$

$A \in \mathfrak{u}(n)$: $(iA)^* = -iA^* = iA$

$\therefore$ $iA \notin \mathfrak{u}(n)$

Para la página 128:

$$\exp(Ad(\exp(tX))(tY)) =$$

$$= \exp(tX)\exp(tY)\exp(-tX)$$

$\therefore$ lema 1.8:

$$= \exp(tY + t^2[X,Y] + O(t^3))$$

Si $|t| << 1$, podemos usar inyectividad de exp:

$$Ad(\exp(tX))(tY) =$$

$$= tY + t^2[X,Y] + O(t^3)$$

$$\therefore Ad(\exp(tX))(Y) =$$

$$= Y + t[X,Y] + O(t^2)$$

y tomamos $\frac{d}{dt}\Big|_{t=0}$:

$$d(Ad)_e(X)(Y) = [X,Y]$$

$$\therefore \quad d(Ad)_e(X) = ad(X)$$

$$d(Ad)_e = ad$$

Obtenemos:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "Ad(g)"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "I_g"] & G \\ h \arrow[r, mapsto] & ghg^{-1} \end{tikzcd}$$

$$g\exp(X)g^{-1} = \exp(Ad(g)(X))$$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "ad"] \arrow[d, "\exp"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "e"] \\ G \arrow[r, "Ad"] & GL(\mathfrak{g}) \end{tikzcd}$$

$$Ad(\exp(X)) = e^{ad(X)}$$

En la página 128 y por estos diagramas se tiene:

G grupo de Lie conexo

$H \leq G$ subgrupo de Lie conexo

Entonces:

$$H \trianglelefteq G \iff \mathfrak{h} \trianglelefteq \mathfrak{g}$$

$\Rightarrow)$:

$Y \in \mathfrak{h},\ X \in \mathfrak{g}: \quad [X, Y] \overset{?}{\in} \mathfrak{h}$

Como $H \trianglelefteq G$, tenemos:

$$\exp(X)\exp(Y)\exp(-X) \in H$$

$\forall X \in \mathfrak{g},\ Y \in \mathfrak{h}$. (por el primer diagrama)

$$\therefore \exp(Ad(\exp(X))(Y)) \in H$$

$\forall X \in \mathfrak{g},\ Y \in \mathfrak{h}$.

$\therefore \forall X \in \mathfrak{g},\ Y \in \mathfrak{h},\ t \in \mathbb{R}$:

$$\exp(t\,Ad(\exp(X))(Y)) \in H$$

Luego:

$$Ad(\exp(X))(Y) \in \mathfrak{h}$$

$\forall X \in \mathfrak{g},\ Y \in \mathfrak{h}$.

$$\therefore \quad Ad(\exp(tX))(Y) \in \mathfrak{h}$$

$\forall X \in \mathfrak{g},\ Y \in \mathfrak{h},\ t \in \mathbb{R}$

Derivando en $t = 0$:

$$[X, Y] = d(Ad)_e(X)(Y) \in \mathfrak{h}$$

$\uparrow$ 2do diagrama

$\therefore \forall X \in \mathfrak{g}, Y \in \mathfrak{h}.$

$\therefore \mathfrak{h} \trianglelefteq \mathfrak{g}.$

$\Leftarrow$): Suponemos $\mathfrak{h} \trianglelefteq \mathfrak{g}$.

Por probar:

$$g h g^{-1} \in H$$

$\forall g \in G, h \in H.$

Como G, H son conexos, basta ver que:

$$\exp(X)\exp(Y)\exp(-X) \in H$$

$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$, pues $\exp(\mathfrak{g})$ y $\exp(\mathfrak{h})$ generan a G y H, resp.

Por el primer diagrama basta ver que:

$\exp(\mathrm{Ad}(\exp(X))(Y)) \in H$

$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$. Y esto ocurre

$\iff \exp(\mathrm{Ad}(\exp(X))(tY)) \in H$

$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}, t \in \mathbb{R}$

$\iff \mathrm{Ad}(\exp(X))(Y) \in \mathfrak{h}$

$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$

$\iff \mathrm{Ad}(\exp(tX))(Y) \in \mathfrak{h}$

$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}, t \in \mathbb{R}$

$\iff e^{t\,\mathrm{ad}(X)}(Y) \in \mathfrak{h}$

$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}, t \in \mathbb{R}$

2do diagrama

derivando en $t=0$:

$\Rightarrow [X, Y] \in \mathfrak{h}$

En esta parte suponemos

$[X, Y] \in \mathfrak{h} \quad \forall X \in \mathfrak{g}, Y \in \mathfrak{h}$.

$\therefore \quad ad(X)(\mathfrak{h}) \subseteq \mathfrak{h} \quad \forall X \in \mathfrak{g}$

$\therefore \quad ad(X)^k(\mathfrak{h}) \subseteq \mathfrak{h} \quad \forall X \in \mathfrak{g}$

$$\therefore \quad e^{t\,ad(X)}(Y) = \sum_{k=0}^{+\infty} \frac{t^k ad(X)^k(Y)}{k!} \in \mathfrak{h}$$

$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$
$t \in \mathbb{R}$

Por las equivalencias anteriores esto implica $H \trianglelefteq G$.

Resumiendo ambos argumentos:

$$H \trianglelefteq G \iff \mathfrak{h} \trianglelefteq \mathfrak{g}$$

Como H y G son conexos:

$H \trianglelefteq G$

$\iff \exp(X)\exp(Y)\exp(-X) \in H$

$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}.$

$\iff$ (primer diagrama) $\exp(\mathrm{Ad}(\exp(X))(Y)) \in H$ $\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$.

$\iff \exp(t\,\mathrm{Ad}(\exp(X))(Y)) \in H$ $\forall X \in \mathfrak{g}, Y \in \mathfrak{h}, t \in \mathbb{R}$

$\iff \mathrm{Ad}(\exp(X))(Y) \in \mathfrak{h}$ $\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$.

$\iff$ (segundo diagrama) $e^{\mathrm{ad}(X)}(Y) \in \mathfrak{h}$ $\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$

$\iff e^{t\,\mathrm{ad}(X)}(Y) \in \mathfrak{h}$ $\forall X \in \mathfrak{g}, Y \in \mathfrak{h}, t \in \mathbb{R}$

$\overset{(*)}{\iff} \mathrm{ad}(X)(Y) \in \mathfrak{h}$ $\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$

$\iff \mathfrak{h} \trianglelefteq \mathfrak{g}$

Solamente resta ver $\circledast$.

$\circledast$ $\Rightarrow$): se obtiene derivando en $t=0$.

$\circledast$ $\Leftarrow$): $\quad ad(X)(Y) \in h \quad \forall X \in g, Y \in h$

$\Rightarrow \quad ad(X)^k(h) \subseteq h \quad \forall X \in g, k$

$\Rightarrow \quad ad(X)^k(Y) \in h \quad \forall X \in g, Y \in h, k$

$\Rightarrow \quad e^{t\,ad(X)}(Y) \in h \quad \forall X \in g, Y \in h, t \in \mathbb{R}.$