Sobre el Teorema de Engel, observamos que:

$$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

que no es nilpotente.

Para la demostración del Teorema de Engel (página 160):

$$H^* : \mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}/\mathfrak{h}$$

$$H^*(Y+\mathfrak{h}) = ad_{\mathfrak{g}}(H)(Y) + \mathfrak{h}$$

$\therefore$ H^* es nilpotente $\forall H \in \mathfrak{h}$

Si $H_1, H_2 \in \mathfrak{h}$:

$$[H_1^*, H_2^*](Y+\mathfrak{h}) =$$

$$= [ad_{\mathfrak{g}}(H_1), ad_{\mathfrak{g}}(H_2)](Y) + \mathfrak{h}$$

$$= ad_{\mathfrak{g}}([H_1, H_2])(Y) + \mathfrak{h}$$

$$\therefore \quad [H_1^*, H_2^*] = [H_1, H_2]^*$$

El $X \in g$ se obtiene como:

$$H^*(X + h) = 0 \quad \forall H \in h$$

$$X + h \neq 0$$

$$\Rightarrow X \in g \setminus h$$

$$\Rightarrow ad_g(H)(X) \in h \quad \forall H \in h$$

Corolario 2.5 se lee:

$g \subseteq gl(V)$ subálgebra de Lie

$$\forall Z \in g \ \exists \ p \geq 1 \ni Z^p = 0$$

$$\Longrightarrow \exists \ s \geq 1 \ni \forall X_1, \ldots, X_s \in g : $$
$$X_1 \cdot \ldots \cdot X_s = 0$$

Para el corolario 2.6:

Sea $\mathfrak{g}$ nilpotente

$\therefore$ $ad_{\mathfrak{g}}(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$ es subálgebra de Lie, cuyos elementos:

$$ad_{\mathfrak{g}}(z)$$

son nilpotentes. Y podemos aplicar Engel.

Corolario anterior implica:

$$ad_{\mathfrak{g}}(X_1) \circ ad_{\mathfrak{g}}(X_2) \circ \dots \circ ad_{\mathfrak{g}}(X_s) = 0$$

$\forall s \geq \dim \mathfrak{g}$

Luego observamos:

$\mathcal{D}(\mathfrak{g})$: $[X, z] = ad_{\mathfrak{g}}(X)(z)$

$\mathcal{D}^2(\mathfrak{g})$: $[\underbrace{[Y_1, Y_2]}_{X_2}, [X_1, z]] =$

$$= ad_{\mathfrak{g}}(X_2)\, ad_{\mathfrak{g}}(X_1)(z)$$

$$\mathcal{D}^3(\mathfrak{g}) : \Big[\underbrace{\big[[\ ,\ ], [\ ,\ ] \big]}_{X_3}, \big[\underbrace{[\ ,\ ]}_{X_2}, [\ , z] \big] \Big]$$

$X_1 = [\ , z]$

$$= ad_{\mathfrak{g}}(X_3) \circ ad_{\mathfrak{g}}(X_2) \circ ad_{\mathfrak{g}}(X_1)(z)$$

en general:

$$\mathcal{D}^r(\mathfrak{g}) : ad_{\mathfrak{g}}(X_r) \circ \dots \circ ad_{\mathfrak{g}}(X_1)(z)$$

$$\therefore \mathcal{D}^s(\mathfrak{g}) = 0 \quad \forall s \geq \dim \mathfrak{g}.$$

$\Rightarrow \mathfrak{g}$ soluble. //

Pero (notación de la clase anterior)

$T(n)$ es soluble
no nilpotente.

Corolario 2.7:
$\mathcal{L}$ nilpotente
$$\Longleftrightarrow \quad \mathcal{C}^m(\mathcal{L}) = 0 \quad \forall m \geq \dim \mathcal{L}$$

Dem.:
$\mathcal{L}$ nilpotente
$$\overset{def}{\Longleftrightarrow} \quad ad_{\mathcal{L}}(X)^m = 0 \quad \forall X \in \mathcal{L}$$
$$\forall m \geq \dim \mathcal{L}$$

$$\mathcal{C}^m(\mathcal{L}) = [\underbrace{\mathcal{L}, [\mathcal{L}, \ldots, [\mathcal{L}, \mathcal{L}]}_{m+1 \text{ veces}} \cdots]$$

$$\mathcal{C}^m(\mathcal{L}) = 0$$
$$\overset{def}{\Longleftrightarrow} \quad ad_{\mathcal{L}}(X_1) \circ \cdots \circ ad_{\mathcal{L}}(X_m)(X) = 0$$
$$\forall X_1, \ldots, X_m, X \in \mathcal{L}$$
$$\forall m \geq \dim \mathcal{L}$$

$$\Longleftrightarrow \quad ad_{\mathcal{L}}(X_1) \circ \cdots \circ ad_{\mathcal{L}}(X_m) = 0$$
$$\forall X_1, \ldots, X_m \in \mathcal{L} \ , \ m \geq \dim \mathcal{L}$$

Y basta aplicar el Corolario

2.5 a $ad_{\ell}(\ell) \leq \mathfrak{gl}(\ell)$

Corolario 2.8 muestra que: ℓ nilpotente $\ell \neq 0 \Rightarrow$

$$m = \min\{k \mid \mathcal{C}^{k}(\ell) = 0\} \geq 1$$

$$\Rightarrow 0 \neq \mathcal{C}^{m-1}(\ell) \subseteq Z(\ell)$$

Por otro lado:

$$\ell = \mathcal{C}^{0}(\ell) \supseteq \mathcal{C}^{1}(\ell) \supseteq \cdots \supseteq \mathcal{C}^{r}(\ell) \supseteq \cdots$$

$$\mathcal{C}^{1}(\ell) = [\ell, \ell] \subseteq \ell$$

$$\mathcal{C}^{2}(\ell) = [\ell, [\ell, \ell]] \subseteq [\ell, \ell] = \mathcal{C}^{1}(\ell)$$

$$\vdots$$

$$\mathcal{C}^{r+1}(\ell) = [\ell, \mathcal{C}^{r}(\ell)] \subseteq [\ell, \mathcal{C}^{r-1}(\ell)] = \mathcal{C}^{r}(\ell)$$

Afirmación:

$$\mathcal{C}^{r}(\ell) \trianglelefteq \ell$$

Dem.: $[\mathcal{L}, \mathcal{C}^r(\mathcal{L})] = \mathcal{C}^{r+1}(\mathcal{L}) \subseteq \mathcal{C}^r(\mathcal{L})$ //

Consideramos:

$$\mathcal{C}^{r+1}(\mathcal{L}) \subseteq \mathcal{C}^r(\mathcal{L}) \subseteq \mathcal{L}$$

todos ideales y por tanto se induce un homomorfismo inyectivo:

$$\mathcal{C}^r(\mathcal{L}) / \mathcal{C}^{r+1}(\mathcal{L}) \hookrightarrow \mathcal{L} / \mathcal{C}^{r+1}(\mathcal{L})$$

$$X + \mathcal{C}^{r+1}(\mathcal{L}) \longmapsto X + \mathcal{C}^{r+1}(\mathcal{L})$$

Afirmación:

$$\mathcal{C}^r(\mathcal{L}) / \mathcal{C}^{r+1}(\mathcal{L}) \subseteq Z\left(\mathcal{L} / \mathcal{C}^{r+1}(\mathcal{L})\right)$$

Dem.: Sea $X \in \mathcal{L}$, $Z \in \mathcal{C}^r(\mathcal{L})$

$[X + \mathcal{C}^{r+1}(\mathcal{L}), Z + \mathcal{C}^{r+1}(\mathcal{L})] =$

$$= [X, \underset{\uparrow}{Z}] + \mathcal{C}^{r+1}(\mathcal{L}) = \mathcal{C}^{r+1}(\mathcal{L})$$

$\mathcal{C}^r(\mathcal{L}) \Rightarrow [X, Z] \in \mathcal{C}^{r+1}(\mathcal{L})$ //

Un par de resultados:

Sea $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{l} \to 0$ sucesión exacta corta de álgebra de Lie:

*) flechas son homomorfismos.

*) en cada vertice la imagen del mapeo entrante es el kernel del mapeo saliente.

Proposición:
Dado el diagrama exacto de arriba:

1) $\mathfrak{g}$ es soluble $\iff$ $\mathfrak{h}$ y $\mathfrak{l}$ son solubles.

2) Si $\mathfrak{h} \leq z(\mathfrak{g})$, entonces $\mathfrak{g}$ es nilpotente $\iff$ $\mathfrak{h}$ y $\mathfrak{l}$ son nilpotentes.